Convex Proof

2.5 Power of a nonnegative function

Rule 5. *If f is convex and nonnegative (i.e., $f(x) \geq 0, \ \forall x$) and $k \geq 1$, then f^k is convex.*

Proof: We prove this in the case where f is twice differentiable. Let $g = f^k$. Then

$$\nabla g(x) = k f^{k-1} \nabla f(x)$$
$$\nabla^2 g(x) = k\left((k-1) f^{k-2} \nabla f(x) \nabla f^T(x) + f^{k-1} \nabla^2 f(x)\right).$$

We see that $\nabla^2 g(x) \succeq 0$ for all x (why?). □

Does this result hold if you remove the nonnegativity assumption on f?

2.3 Pointwise maximum

Rule 3. *If $f_1, \ldots, f_m$ are convex functions, then their pointwise maximum*

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\},$$

with $dom(f) = dom(f_1) \cap \ldots \cap dom(f_m)$ is also convex.

Figure 1: An illustration of the pointwise maximum rule

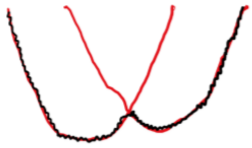
Proof: Pick any $x, y \in dom(f), \lambda \in [0,1]$. Then,

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= f_j(\lambda x + (1-\lambda) y) \text{ (for some } j \in \{1, \ldots, m\}) \\ &\leq \lambda f_j(x) + (1-\lambda) f_j(y) \\ &\leq \lambda \max\{f_1(x), \ldots, f_m(x)\} + (1-\lambda) \max\{f_1(y), \ldots, f_m(y)\} \\ &= \lambda f(x) + (1-\lambda) f(y). \ \square \end{aligned}$$

- It is also easy to prove this result using epigraphs. Recall that f convex $\Leftrightarrow$ $epi(f)$ is convex. But $epi(f) = \cap_{i=1}^{m} epi(f_i)$, and we know that the intersection of convex sets is convex.

- One can similarly show that the pointwise minimum of two concave functions is concave.

- But the pointwise minimum of two convex functions may not be convex.



cartesian product of convex sets in convex

Let $(x_1, y_1), (x_2, y_2) \in S \times T$ and $t \in [0, 1]$. Then

$$(1-t)(x_1, y_1) + t(x_2, y_2) = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2).$$

Since S is convex and $x_1, x_2 \in S$, so is $(1-t)x_1 + tx_2 \in S$; since T is convex and $y_1, y_2 \in T$, so is $(1-t)y_1 + ty_2 \in T$. Therefore

$$((1-t)x_1 + tx_2, (1-t)y_1 + ty_2) \in S \times T,$$

which shows that $(1-t)(x_1, y_1) + t(x_2, y_2) \in S \times T$.

Suppose that $S \subset \mathbb{R}^m$ is a convex set and $T \subset \mathbb{R}^n$ is a convex set. Show that the set

$$S \times T = \{(x_1, \ldots, x_{m+n} \in \mathbb{R}^{m+n}) : (x_1, \ldots, x_m) \in S; (x_{m+1}, \ldots, x_{m+n}) \in T\}$$

is a convex subset of $\mathbb{R}^{m+n}$.

linear combination of convex sets is convex

Suppose S is a convex subset of $\mathbb{R}^n$, and suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is any linear transformation. Prove that the set $\{T(x) \mid x \in S\}$ is also convex.

Say, $y_1, y_2 \in T(S)$. So, we can write them as, $y_1 = T(x_1),\ y_2 = T(x_2)$ where $x_1, x_2 \in S$.

$$ty_1 + (1-t)y_2 \text{ where } t \in [0,1]$$

$$= tT(x_1) + (1-t)T(x_1)$$

$$= T(tx_1 + (1-t)x_2) \in T(S) \text{ because } tx_1 + (1-t)x_2 \in S$$

sum of two convex sets is convex

> Let A and B be two convex subsets in $\mathbb{R}^n$. Define a set C given by
>
> $$C = A + B = \{a + b : a \in A \text{ and } b \in B\}.$$
>
> Is C a convex set?

For $e, f \in C$, let $a, b \in A$ and $c, d \in B$ s.t. $e = a + c$ and $f = b + d$,

$$te + (1-t)f = t(a+c) + (1-t)(b+d) = (ta + (1-t)b) + (tc + (1-t)d) \in A + B = C$$

For all $t \in [0,1]$. $\square$

epigraph of a function f is convex then f is a convex function

We want to show that if the epigraph of a function f is convex then f is a convex function.

Proof:

Suppose $\{(x,y) \in X \times Y : y \geq f(x)\}$ is convex. To show that f is convex, pick any arbitrary $x', x'' \in X$, and any $\lambda \in (0,1)$. Notice that both $(x', f(x'))$ and $(x'', f(x''))$ belongs to the epigraph i.e.,

- $(x', f(x')) \in \{(x,y) \in X \times Y : y \geq f(x)\}$
- $(x'', f(x'')) \in \{(x,y) \in X \times Y : y \geq f(x)\}$

By convexity of the epigraph, it follows that

$\lambda(x', f(x')) + (1-\lambda)(x'', f(x'')) = (\lambda x' + (1-\lambda)x'', \lambda f(x') + (1-\lambda)f(x''))$
also belongs to the epigraph. Therefore,

$f(\lambda x' + (1-\lambda)x'') \leq \lambda f(x') + (1-\lambda)f(x'')$

Hence, $f : X \to Y$ is convex.