Visualisation of quantum algorithms

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Introduction

1 Qubits

choice of representation etc.

The Hadamard gate 'H' transforms the state $|0\rangle$ to the state $(|0\rangle + |1\rangle)/\sqrt{2}$ and transforms $|1\rangle$ to $(|0\rangle - |1\rangle)/\sqrt{2}$.
The Hadamard gate is also it's own inverse so:

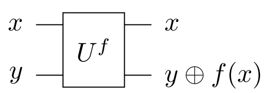
$$H \cdot |0\rangle + |1\rangle)/\sqrt{2} = |0\rangle$$
$$H \cdot (|0\rangle - |1\rangle)/\sqrt{2} = |1\rangle$$

Hadamard $-\boxed{H}-$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

2 Deutsch-Josza algorithm

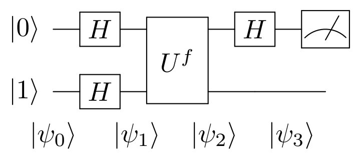
2.1 Deutsch algorithm

The Deutsch problem is the first known algorithm to demonstrate the possibility for quantum computers to perform better than their classical counterparts and is described as follows. Given a function $f(x) : \{0, 1\} \to \{0, 1\}$ determine whether $f(0) = f(1)$. On a classical computer the function $f(x)$ must be evaluated twice to solve this. The Deutsch algorithm is capable of determining if $f(0) = f(1)$ with only a single application of a quantum gate U^f that implements the function $f(x)$.



Here $\oplus$ denotes addition modulo 2. The circuit depicted below implements Deutsch's algorithm, if $f(0) = f(1)$ then the first qubit when measured will return 0. In the obverse case where $f(0) \neq f(1)$ then the first qubit will always be found to be 1.

A quantum circuit implementing the Deutsch algorithm



The steps in the algorithm are as follows; first the input state $|\psi_0\rangle = |01\rangle$ is prepared and then operated upon by two Hadamard gates to obtain the state:

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Next the gate U^f is applied to $|\psi_1\rangle$, the four possibilities for the resultant state $|\psi_2\rangle$ are enumerated in the following table:

$\|\psi_2\rangle$	$f(0) = 0$	$f(0) = 1$
$f(1) = 0$	$\left[\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}\right]\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$	$-\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$
$f(1) = 1$	$\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$	$-\left[\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}\right]\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$

Now applying the hadamard transform to the first qubit one obtains:

$\|\psi_3\rangle$	$f(0) = 0$	$f(0) = 1$
$f(1) = 0$	$\|0\rangle\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$	$-\|1\rangle\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$
$f(1) = 1$	$\|1\rangle\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$	$-\|0\rangle\left[\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}\right]$

Then upon measuring the first qubit one obtains as expected.

measurement	$f(0) = 0$	$f(0) = 1$
$f(1) = 0$	0	1
$f(1) = 1$	1	0

2.2 Deutsch-Josza algorithm

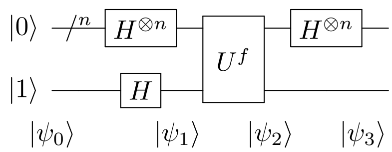
The Deutsch-Josza algorithm is a quantum algorithm which solves a more general version of Deutsch's problem and is described as follows:
For a natural number $n \in \mathbb{N}$ and a blackbox implementing a function $f(x) : \{0, 1, 2, ..., 2^n - 1\} \rightarrow \{0, 1\}$ from one of the following two sets of functions:

The constant functions: $C = \{f(x) = 0,\ f(x) = 1\}$

The balanced functions: $B = \{f(x) | \sum_{x=0}^{2^n-1} f(x) = 2^{(n-1)}\}$

Determine from which of these two sets the function implemented by the blackbox is an element of.

In the classical case at worst one would need to test $2^{n-1} + 1$ inputs to solve this problem. It is however possible to solve this problem with only a single application of a unitary transform U_f that implements the function $f(x)$ as shown below.



The Deutsch-Josza algorithm is similar to the algorithm in the previous section used to solve the original Deutsch problem. First n qubits are prepared in the state $|0\rangle$ and these are referred to as the query register, an additional qubit is prepared in the state $|1\rangle$ and is referred to as the answer register. The system is now in the state:

$$|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$$

Next the Hadamard transform is applied to the query register and the answer register to obtain the state:

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]^n \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

After the gate U^f is applied there are three cases to be considered:

$$f(x) = 0,\ |\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]^n \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(x) = 1,\ |\psi_2\rangle = \left(-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\right)^n \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(x) \in B,\ |\psi_2\rangle = \left(\text{a permutation of} \left(-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\right)^{\frac{n}{2}} \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]^{\frac{n}{2}}\right) \cdot \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Now applying the Hadamard transform to the query register for the above cases one obtains:

$$f(x) = 0,\ |\psi_2\rangle = |0\rangle^n \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(x) = 1,\ |\psi_2\rangle = (-|0\rangle)^n \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(x) \in B,\ |\psi_2\rangle = \left(\text{a permutation of} \left(-|0\rangle\right)^{\frac{n}{2}} |0\rangle^{\frac{n}{2}}\right) \cdot \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$