Dirac operator

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1 Introduction

The only substantial difference between the Dirac operator in `openQ*D` and `openQCD` is the coupling to the U(1) gauge field, which will be illustrated in these notes. Other aspects, for instance:

- definition of the block Dirac operator,
- implementation and numerical issues of the even-odd preconditioning,
- details of the gamma-matrix algebra,

require trivial generalization with respect to the pure SU(3) case, which is discussed in detail in [1].

2 Boundary conditions in time

In the time direction, the gauge field can be chosen to satisfy open (or type 0), SF (or type 1), open-SF (or type 2) or periodic (or type 3) boundary conditions. A detailed description of the boundary conditions in time for the gauge field can be found in [2, 3]. The boundary conditions in time for the quark fields depend on the boundary conditions for the gauge field, in the following way.

- If the gauge field satisfies *periodic* boundary conditions in time, the quark field $\psi(x)$ is defined for $0 \leq x < N_0$ and is extended to every value of x_0 by means of anti-periodic boundary conditions, i.e.

$$\psi(x + N_0\hat{0}) = -\psi(x) \ . \tag{1}$$

- If the gauge field satisfies *SF* or *open-SF* boundary conditions in time, the quark field $\psi(x)$ is defined for $0 \leq x \leq N_0$ and satisfies SF boundary conditions, i.e.

$$\psi(x)|_{x_0=0} = \psi(x)|_{x_0=N_0} = 0 \ . \tag{2}$$

- If the gauge field satisfies *open* boundary conditions in time, the quark field $\psi(x)$ is defined for $0 \leq x < N_0$ and satisfies SF boundary conditions, i.e.

$$\psi(x)|_{x_0=0} = \psi(x)|_{x_0=N_0-1} = 0 \ . \tag{3}$$

3 Boundary conditions in space

In the space direction, the gauge field can be chosen to be periodic or to satisfy $\mathrm{C}^\star$ boundary conditions. $\mathrm{C}^\star$ boundary conditions are implemented in `openQ*D` by means of an orbifold construction along direction x, which is described in detail in [4] (familiarity with this document is assumed in this section). The space direction k is said to be $\mathrm{C}^\star$ (resp. periodic), if the gauge and quark fields satisfy $\mathrm{C}^\star$ (resp. periodic) boundary conditions along direction k in the physical lattice. As explained in [4], $\mathrm{C}^\star$ boundary conditions are mapped into periodic or shifted boundary conditions on the extended lattice.

In all cases the quark field $\psi(x)$ is defined for $0 \leq x_k < N_k$ and $k = 1, 2, 3$. The way in which the quark field is extended to all values of x_k depends on the number of $\mathrm{C}^\star$ directions, in the following way.

- If all space directions are periodic, the quark field $\psi(x)$ is extended to all values of x_k for $k = 1, 2, 3$ by means of phase-periodic boundary conditions, i.e.

$$\psi(x + N_k \hat{k}) = e^{i\theta_k} \psi(x) \ . \tag{4}$$

- If the $k = 1$ direction is $\mathrm{C}^\star$, because of the orfifold construction, the quark field $\psi(x)$ is extended to all values of x_1 by means of periodic boundary conditions, i.e.

$$\psi(x + N_1 \hat{1}) = \psi(x) \ . \tag{5}$$

- If at least one $\mathrm{C}^\star$ boundary condition is chosen, and the $k = 2, 3$ direction is periodic, the quark field $\psi(x)$ is extended to all values of x_k by means of periodic boundary conditions, i.e.

$$\psi(x + N_k \hat{k}) = \psi(x) \ . \tag{6}$$

  Notice that $\mathrm{C}^\star$ boundary conditions in one direction are not compatible with phase-periodic boundary conditions in any other direction.

- If the $k = 2, 3$ direction is $\mathrm{C}^\star$, the quark field $\psi(x)$ is extended to all values of x_k by means of shifted boundary conditions, i.e.

$$\psi(x + N_k \hat{k}) = \psi(x + \tfrac{N_1}{2} \hat{1}) \ . \tag{7}$$

4 Definition of the Dirac operator

The Dirac operator can be written as

$$D = m_0 + D_\mathrm{w} + \delta D_\mathrm{sw} + \delta D_\mathrm{b} \ . \tag{8}$$

where D_w is the (unimproved) Wilson-Dirac operator, δD_sw is the Sheikholeslami–Wohlert (SW) term, and δD_b is the boundary $O(a)$-improvement term. In presence of electromagnetism, the Dirac operator depends on the electric charge of the quark field. Let q be the physical electric charge in units of e (i.e. $q = 2/3$ for the up quark, and $q = -1/3$ for the down quark). In the compact formulation of QED, all electric charges must be integer multiples of an elementary charge $q_{\rm el}$, which appears as a parameter in the U(1) gauge action (for more details, see [2]). It is useful to introduce the integer parameter

$$\hat{q} = \frac{q}{q_{\rm el}} \in \mathbb{Z} \ . \tag{9}$$

The Wilson-Dirac operator can be written as

$$D_{\rm w} = \sum_{\mu=0}^{3} \frac{1}{2} \left\{ \gamma_\mu (\nabla_\mu + \nabla_\mu^*) - \nabla_\mu^* \nabla_\mu \right\} \ , \tag{10}$$

where the covariant derivatives are defined as

$$\nabla_\mu \psi(x) = U(x,\mu) e^{i\hat{q}A(x,\mu)} \psi(x+\hat{\mu}) - \psi(x) \ , \tag{11}$$

$$\nabla_\mu^* \psi(x) = \psi(x) - U(x-\hat{\mu},\mu)^\dagger e^{-i\hat{q}A(x-\hat{\mu},\mu)} \psi(x-\hat{\mu}) \ . \tag{12}$$

Notice that the integer parameter $\hat{q}$ appears in the hopping term of the Wilson-Dirac operator. It is understood that $\nabla_\mu \psi(x) = \nabla_\mu^* \psi(x) = 0$ if x belongs to a time boundary.

The SW term is given by

$$\delta D_{\rm sw} = c_{\rm sw}^{\rm SU(3)} \sum_{\mu,\nu=0}^{3} \frac{i}{4} \sigma_{\mu\nu} \widehat{F}_{\mu\nu} + q \, c_{\rm sw}^{\rm U(1)} \sum_{\mu,\nu=0}^{3} \frac{i}{4} \sigma_{\mu\nu} \widehat{A}_{\mu\nu} \ . \tag{13}$$

The SU(3) field tensor $\widehat{F}_{\mu\nu}(x)$ is constructed as in eqs. (2.13) and (2.14) in [1]. The U(1) field tensor $\widehat{A}_{\mu\nu}(x)$ is defined as

$$\hat{A}_{\mu\nu}(x) = \frac{i}{4q_{\rm el}} \, \mathrm{Im} \left\{ z_{\mu\nu}(x) + z_{\mu\nu}(x-\hat{\mu}) + z_{\mu\nu}(x-\hat{\nu}) + z_{\mu\nu}(x-\hat{\mu}-\hat{\nu}) \right\} \ , \tag{14}$$

$$z_{\mu\nu}(x) = e^{i\{A(x,\mu)+A(x+\hat{\mu},\nu)-A(x+\hat{\nu},\mu)-A(x,\nu)\}} \ . \tag{15}$$

The normalization is chosen in such a way that $-ie_0 \hat{A}_{\mu\nu}(x)$ is the canonically-normalized field tensor in the naive continuum limit. Notice that the field tensors are anti-hermitian.

The definition of δD_b depends on the boundary conditions in time. For periodic boundary conditions $\delta D_b = 0$. In all other cases

$$\delta D_b \psi(x) = \{ (c_{\rm F} - 1) \delta_{x_0,1} + (c_{\rm F}' - 1) \delta_{x_0,T-1} \} \psi(x) \ , \tag{16}$$

where $T = N_0$ for SF and open-SF boundary conditions, and $T = N_0 - 1$ for open boundary conditions. For SF and open boundary conditions the relation $c_{\mathrm{F}} = c'_{\mathrm{F}}$ is imposed.

At tree-level of perturbation theory, on-shell $O(a)$-improvement is achieved by setting $c_{\mathrm{sw}}^{\mathrm{SU}(3)} = c_{\mathrm{sw}}^{\mathrm{U}(1)} = c_{\mathrm{F}} = c'_{\mathrm{F}} = 1$.

References

[1] M. Luscher, *Implementation of the lattice Dirac operator*, code documentation, `doc/openQCD-1.6/dirac.pdf`.

[2] A. Patella, *Gauge actions*, code documentation, `doc/gauge_action.pdf`.

[3] M. Luscher, *Gauge actions in openQCD simulations*, code documentation, `doc/openQCD-1.6/gauge_action.pdf`.

[4] A. Patella, *$C^\star$ boundary conditions*, code documentation, `doc/cstar.pdf`.