Derivation of a Wave Function for Quantum Gravitation in 2-D*

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Abstract

Quantum gravity is an active field of research, and thus we still utilize a variety of analytic solutions to find the metric g for a given distribution of matter and energy, which is a solution to Einstein's equations. We use a probability amplitude Ψ which is a function of the space-like portion of g.

This poster provides an overview of the research we have been conducting on the subject. In particular, we have focused on:

Calculating complex integrals for possible wave functions.
Normalizing successfully calculated wave functions.

One common occurrence is that at short enough of a length scale, we find that space-time begins to act like it has only 2 dimensions, rather than 4 [1]. There are a variety of ways to observe this, one of which being diffusion. From here, we plan to use the calculated wave function to further investigate whether or not space-time acts like it has 2 dimensions on a small scale, rather than 4, using computational methods of diffusion.

Background

[2] [4] In general relativity, the theory of gravitation states that the observed gravitational effect between masses results from their warping of space-time. space-time is curved, so to find arc length, we need to create an updated arc length formula.

We update our definition so that the inner product (a bilinear, symmetric, positive definite mapping $V \times V \to \mathbb{R}$) of the tangent vector $\dot{x}(t)$ with itself is now $\dot{x}g\dot{x}$, where g is a matrix whose entries are functions of x and where $\dot{}$ denotes the derivative $\frac{d}{dt}$.

Definition: For an n-dimensional curved space $\left(x^1(t), x^2(t), \ldots, x^n(t)\right)$ where $x^i(t) \in \mathbb{R}$ and g is the metric, arc length between $t = a$ and $t = b$ is defined as

$$s = \int_a^b \sqrt{\dot{x}(t)g\dot{x}(t)}dt. \tag{1}$$

We may denote $\dot{x}(t)g\dot{x}(t)$ as $\langle \dot{x}(t), \dot{x}(t)\rangle_g$, which is defined as

$$\begin{bmatrix}\dot{x}^0(t) & \cdots & \dot{x}^{n-1}(t)\end{bmatrix} \begin{bmatrix} g_{0,0}(x) & \cdots & g_{n-1,0}(x) \\ \vdots & & \vdots \\ g_{0,n-1}(x) & \cdots & g_{n-1,n-1}(x)\end{bmatrix} \begin{bmatrix}\dot{x}^0(t) \\ \vdots \\ \dot{x}^{n-1}(t)\end{bmatrix} \tag{2}$$

Definition: Matrix g is called the metric and is required to be symmetric and have a nonzero determinant, so that $\langle x, y\rangle_g = \langle y, x\rangle_g$ and $\langle x, x\rangle_g = 0 \to x = 0$.

The metric for a given distribution of matter and energy is the solution of a system of PDEs, known as Einstein's equations.

Definition: Einstein's equations take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3}$$

where $G_{\mu\nu}$ is known as the Einstein tensor (a curvature metric tensor calculated from metric $g_{\mu\nu}$ and its derivatives), $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, Λ is the cosmological constant, and κ is Einstein's gravitational constant.

Although space-time is 4-dimensional, the metric distinguishes "timelike" directions. These are directions pointed out by tangent vectors whose dot product with themselves is negative. We can choose a time direction and write Einstein's equations so that they describe evolution over time of the spacelike block of the metric.

Definition: In the full 4-D metric

$$g = \begin{bmatrix} g_{0,0} & g_{0,1} & g_{0,2} & g_{0,3} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,3} \\ g_{2,0} & \vdots & \ddots & \vdots \\ g_{3,0} & g_{3,1} & \cdots & g_{3,3}\end{bmatrix}, \tag{4}$$

we take the 0-index to be in a timelike direction and the indices 1, 2, 3 in spacelike directions. Sometimes q is used to refer to the spacelike block instead of g.

Background, cont.

In quantum gravity, which aims to describe gravity in accordance with quantum mechanics, we start with the idea that the spacelike metric cannot be predicted exactly and that instead we will derive a probability amplitude Ψ which is a complex-valued function having as its domain allowed 3-metrics (i.e. Metrics of 3 dimensions).

Definition: Assume we can define an integration measure dq on the space of 3-metrics. Then,

$$\int_S \Psi[q]\overline{\Psi}[q]dq = 1, \tag{5}$$

where S is a superspace (space of allowed 3-metrics), and $\overline{\Psi}$ is the complex conjugate of Ψ. We also take note that

$$|\Psi[q]|^2 = \Psi[q]\overline{\Psi}[q] \tag{6}$$

is a probability density function for the spacelike q.

We do not yet fully understand how to quantize gravity as a research community, and we do not know how to obtain Ψ, although there are possible approaches for doing so. Deriving Ψ is one of the oldest and, in many ways, the most straightforward approach to quantum gravity, but there are many others such as string theory.

Deriving the Wave Function

We consider a model for a $(1+1)$ dimensional universe where the spatial universe is topologically a circle and the dynamical variable is the arc length ℓ of this circle, which varies as a function of time.

The classical behavior can be captured in terms of the Hamiltonian or total energy (kinetic and potential) of the universe, which is given by

$$H = \ell\Pi_\ell^2 + \Lambda\ell\,. \tag{7}$$

We modify the function by promoting classical variables to operators which act on quantum wave functions, producing the quantum Hamiltonian, represented by the family of orderings

$$\begin{aligned}\hat{H} &= -\hbar^2 l^{j_1}\hat{\Pi}_\ell l^{j_2}\hat{\Pi}_\ell l^{j_3} + \lambda\ell, \quad j_1 + j_2 + j_3 = 1 \\ &= -\hbar^2 l^{j_1}\frac{d}{d\ell}l^{1-(j_3+j_1)}\frac{d}{d\ell}l^{j_3} + \lambda\ell\end{aligned} \tag{8}$$

To come to a wave function, we begin with the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \tag{9}$$

the solutions of which produce quantum histories of the universe. Given initial data $\psi(\ell)$, the above equation can be solved by applying a time evolution operator $e^{-it\hat{H}/\hbar}$. We get an integral kernel

$$\begin{aligned}K^{\pm}(\ell_1, \ell_2; \tau) = \frac{\sqrt{\Lambda}}{\hbar}(\ell_1\ell_2)^{-\frac{J_-}{2}}(\sqrt{\Lambda}\tau)\exp\left(-\frac{\sqrt{\Lambda}}{\hbar}(\ell_1+\ell_2)\coth(\sqrt{\Lambda}\tau)\right) \\ \times I_{\pm|J_+|}\left(\frac{2\sqrt{\Lambda}}{\hbar}\sqrt{\ell_1\ell_2}\sqrt{\Lambda}\tau\right).\end{aligned} \tag{10}$$

Applying a change of variable $(\sqrt{\lambda}\tau) = \sinh(u)$ and formula 7 from Magnus [3], we integrate over all elapsed time. This results in a time-independent transition amplitude, an amplitude transitioning from an initial to a final circumference given any amount of elapsed time between two states.

Since our integral is symmetric in ℓ_1 and ℓ_2, we can assume $\ell_1 < \ell_2$. With $a = \ell_1,\ b = \ell_2,\ \nu = \pm\frac{|J_+|}{2},\ z = \frac{\sqrt{\Lambda}}{\hbar}$, and $t = u$, our wave function is found to be

$$\Psi(\ell_1,\, \ell_2) = \frac{1}{\hbar}(\ell_1\ell_2)^{-\frac{J_-}{2}} I_{\pm\frac{|J_+|}{2}}\left(\frac{\sqrt{\Lambda}}{\hbar}\ell_1\right) K_{\pm\frac{|J_+|}{2}}\left(\frac{\sqrt{\Lambda}}{\hbar}\ell_2\right) \tag{11}$$

Normalizing the Wave Function

The next logical step in using our wave function as a probability amplitude is to normalize it. To do this, suppose we fix l_1 and let l_2 be an arc-length at a later time. For ease of notation, let $\nu = \pm\frac{|J_+|}{2}$. The integral of the probability density function is

$$\int_0^{l_1}\left[\tfrac{1}{\hbar}(l_1l_2)^{-\frac{J_-}{2}}I_\nu\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_2\right)K_\nu\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_1\right)\right]^2 l_2^{J_-}dl_2 + \int_{l_1}^{\infty}\left[\tfrac{1}{\hbar}(l_1l_2)^{-\frac{J_-}{2}}I_\nu\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_1\right)K_\nu\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_2\right)\right]^2 l_2^{J_-}\,dl_2. \tag{12}$$

First integral

Denote the first integral as $\mathcal{I}_1$. Algebraic manipulation leaves only a squared modified Bessel function $I_\nu^2\left(\frac{\sqrt{\Lambda}}{\hbar}l_2\right)$ in the integrand, which is easily integrable by reverting to the Bessel function of the first kind through the formula $I_\alpha(x) = i^{-\alpha}J_\alpha(ix)$, where $x \in \mathbb{C}$. This is key as now we can easily convert to a generalized hypergeometric function thanks to this formula from Yudell [5]

$$J_\nu^2(z) = \frac{(z/2)^{2\nu}}{[\Gamma(\nu+1)]^2}\,{}_1F_2\left(\nu+\tfrac{1}{2}; \nu+1, 2\nu+1; -z^2\right). \tag{13}$$

After conversion, and a simple u-substitution with $w = \frac{\Lambda}{\hbar^2}l_2^2$, we are left with a product of a hypergeometric function and a power of w. Here, we can easily employ an integral formula, derived form the series definition of the generalized hypergeometric function, given by

$$\int z^{\alpha-1}\,{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\,dz = \tfrac{z^\alpha}{\alpha}\,{}_{p+1}F_{q+1}(\alpha, a_1, \ldots, a_p; \alpha+1, b_1, \ldots, b_q; z). \tag{14}$$

Thus, we are left with the following result for $\mathcal{I}_1$,

$$\frac{l_1^{-J_-}}{\hbar\sqrt{\Lambda}}K_\nu^2\left(\frac{\sqrt{\Lambda}}{\hbar}l_1\right)\frac{1}{2^{2\nu+1}[\Gamma(\nu+1)]^2}\frac{w^{\nu+1}}{\nu+1}\,{}_1F_2(\nu+\tfrac{1}{2}; \nu+2, 2\nu+1; w), \quad w = \tfrac{\Lambda}{\hbar^2}l_1^2. \tag{15}$$

Finally, noticing that we can use (8) to revert back to a squared Bessel function, we realize this integral as

$$\mathcal{I}_1 = \frac{\sqrt{\Lambda}\,l_1^{-J_-}l_2^2}{2\hbar(\nu+1)}\,K_\nu^2\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_1\right)J_\nu^2\left(\tfrac{\sqrt{\Lambda}}{\hbar}l_1\right). \tag{16}$$

Second integral

The second integral deems much more complicated, first by the integration argument being contained within a modified Bessel function of the second kind, as there is no simple way to transcribe to a standard Bessel function; second by the upper limit of integration being infinity. With this, we employed Mathematica to obtain a representation for the functional result, and are currently finalizing the integration to achieve the result analytically.

Future work

As stated above, we are currently finalizing the analytical result of the second integral of the probability density function. Furthermore, we will use the resulting, then-normalized wave function to discover methods which show the universe behaving as if it had but 2 dimensions on a small scale.

One intriguing result is that many very different approaches to quantum gravity seem to show that at short length scales, space-time stops behaving as if it has 4 dimensions, and instead acts as if it only has 2. Since there are many different ways to measure dimension, there are many ways to see this. With this research, we will focus on measuring dimension by studying the process of diffusion.

References

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