Finding an Unknown Variable Based on the Equation for the Launch Angle: $\theta = \arctan\left(\frac{v^2 \pm \sqrt{v^4 - g(gx^2 + 2yv^2)}}{gx}\right)$

Elijah Sanderson

Long Trail School — May 2018

Physics Higher Level Internal Assessment

Contents

1 Introduction and Rationale

This investigation aims to manipulate and test an equation and its units in order to figure out what variable/quantity it represents. In other terms, given the equation $\theta = \arctan\left(\frac{v^2 \pm \sqrt{v^4 - g(gx^2+2yv^2)}}{gx}\right)$, which gives the angle of launch θ required to hit coordinates (x, y) when fired with initial velocity v, one can derive another equation which describes an unknown variable, thereby giving the purpose of this investigation (the specifics and mathematics behind this will be described in the following section).[1] The initial circumstance from which my idea for this exploration spawned was from studying the subtopic of projectile motion in year one. A specific problem asked for the angle of launch to hit specified coordinates (x, y) with a given velocity v. To my luck, I came across the aforementioned equation $\theta = \arctan\left(\frac{v^2 \pm \sqrt{v^4 - g(gx^2+2yv^2)}}{gx}\right)$. At the moment that I found this equation, I noticed it looks strangely similar to the quadratic formula and began to wonder what the equation would represent if I reverse-engineered it to something related to the quadratic equation. Moreover, would the new equation represent the same quantities as the original? And, if so, what would this new equation represent? Thus, the objective of my Internal Assessment was born.

2 Theory

As aforementioned, the theory which this investigation aims to test and uncover is if I were to reverse-engineer the equation for finding the launch angle θ for an object undergoing projectile motion, what would the new equation represent? To start unwrapping this conjecture, I began taking the original equation and manipulating it to resemble the quadratic formula.

$$\theta = \arctan\left(\frac{v^2 \pm \sqrt{v^4 - g(gx^2+2yv^2)}}{gx}\right) \Rightarrow \tan\theta = \frac{v^2 \pm \sqrt{v^4 - g(gx^2+2yv^2)}}{gx}$$

Next, I compared the two equations and created a new equation based on the properties of the quadratic formula. This yields that $x = \tan\theta$ in terms of converting from the quadratic formula to our equation for projectile motion.

$$\left[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right] \Rightarrow \left[\tan\theta = \frac{v^2 \pm \sqrt{v^4 - g(gx^2+2yv^2)}}{gx}\right]$$

$$\therefore x = \tan\theta$$

$$\begin{aligned} \therefore -b &= v^2 \\ b &= -v^2 \end{aligned}$$

$$\begin{aligned} \therefore 2a &= gx \\ a &= \frac{gx}{2} \end{aligned}$$

$$\begin{aligned} \therefore 4ac &= g(gx^2+2yv^2) \\ 4\left(\frac{gx}{2}\right)c &= g(gx^2+2yv^2) \\ 2xc &= gx^2+2yv^2 \\ c &= \frac{gx^2+2yv^2}{2x} \end{aligned}$$

[1] "Projectile Motion." *Wikipedia*, Wikimedia Foundation, 8 Apr. 2018, en.wikipedia.org/wiki/Projectile_motion.

Based on how we derive the quadratic formula, it is apparent that the values a, b, and c coincide with both equations in the manner below. With this, we can insert our derived a, b, and c values to arrive at a new form of the original equation.

$$\left[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right] \Rightarrow \left[y = ax^2 + bx + c\right]$$

$$\begin{aligned} \therefore\ ? &= \left(\frac{gx}{2}\right)\tan^2\theta + (-v^2)\tan\theta + \left(\frac{gx^2 + 2yv^2}{2x}\right) \\ ? &= \frac{gx}{2}\tan^2\theta - v^2\tan\theta + \frac{gx^2 + 2yv^2}{2x} \end{aligned}$$

At this point, I noticed that I do not know what this equation represents as when we derive the quadratic formula, we set the equation equal to 0 (we only care about where the parabola intercepts the x-axis). Despite this, however, I realized as this is based on projectile motion, each variable has its respective units which are true throughout the equation. Both sides of this equation should be in the same units and, thus, would give us a hit as to what this equation represents. With this revelation, we can substitute the respective units for each variable to understand what units the equation is in.

Let some variable $Z = \frac{gx}{2}\tan^2\theta - v^2\tan\theta + \frac{gx^2+2yv^2}{2x}$.

$$\begin{aligned} Z &= \frac{gx}{2}\tan^2\theta - v^2\tan\theta + \frac{gx^2 + 2yv^2}{2x} \\ &= \frac{(\mathrm{ms^{-2}})(\mathrm{m})}{2}(1)^2 - (\mathrm{ms^{-1}})^2(1) + \frac{(\mathrm{ms^{-2}})(\mathrm{m})^2 + 2(\mathrm{m})(\mathrm{ms^{-1}})^2}{2(\mathrm{m})} \\ &= \frac{1}{2}(\mathrm{m^2s^{-2}}) - (\mathrm{m^2s^{-2}}) + \frac{(\mathrm{m^3s^{-2}}) + 2(\mathrm{m^3s^{-2}})}{2(\mathrm{m})} \\ &= \frac{1}{2}(\mathrm{m^2s^{-2}}) - (\mathrm{m^2s^{-2}}) + \frac{3}{2}(\mathrm{m^2s^{-2}}) \\ Z &= \mathrm{m^2s^{-2}} \end{aligned}$$

Based on the unit analysis, it is clear that the equation represents some quantity measured in units of m^2s^{-2}. These units relate to a number of quantities such as specific energy which "*is used to quantify, for example, stored heat or other thermodynamic properties of substances such as specific internal energy, specific enthalpy, specific Gibbs free energy, and specific Helmholtz free energy. It may also be used for the kinetic energy or potential energy of a body.*" [2] From this point, we can conduct an experiment to figure out exactly what quantity this new equation represents using the given variables.

[2] "Specific Energy." *Wikipedia*, Wikimedia Foundation, 9 Apr. 2018, en.wikipedia.org/wiki/Specific_energy.

3 Experiment Layout, Materials, and Procedures

In order to find the answer to this daunting question, I conducted an experiment to find the value of the variables in the equation $Z = \frac{gx}{2}\tan^2\theta - v^2\tan\theta + \frac{gx^2+2yv^2}{2x}$ for specific given angles of launch. My plan for the practical was to launch a ball of an arbitrary mass off a ramp/tube with a given angle and record its velocity at the time of launch as well as the horizontal distance the object covered while in flight. I constructed my experiment by bending a clear, vinyl tube, with inner radius larger than that of the ball, such that, when placed, the ball will launch as straight as possible. To record the velocity of launch, I used two photogates to give the time it took for the ball to pass through and used the definition of velocity, $v = \frac{s}{t}$, to calculate the speed of the ball. Finally, I placed a meter stick beginning at the end of the tube, and being parallel to the ground, to record the horizontal distance the ball covered throughout its entire journey as if it were launched from the 0cm point or at (0,0) in Cartesian coordinates. Below is a diagram of my experiment with respective labelling and on the following page are lists of my materials used and my procedures.

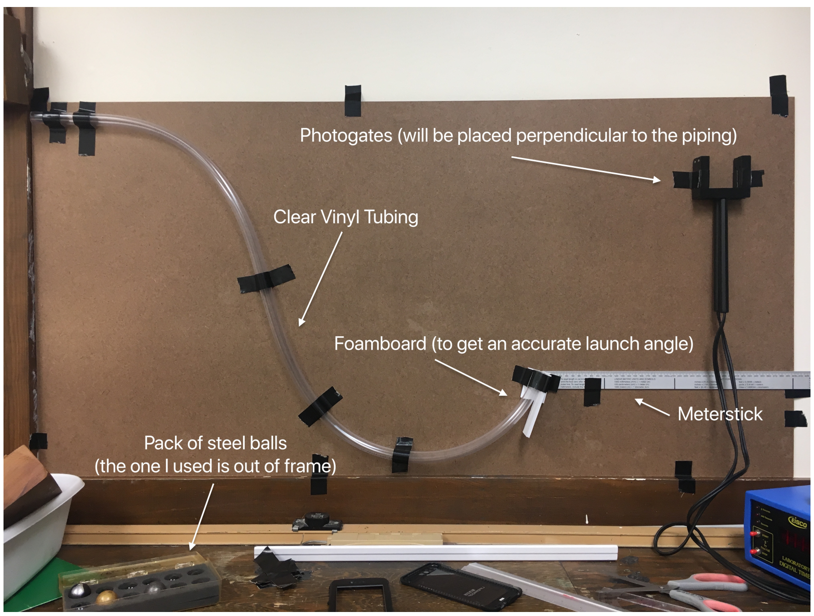

Figure 1: *Diagram of my experiment. Conducted Spring of 2018*

Materials Used

- steel ball of radius 0.6cm
- large cutout of particle board (I used 120cm x 60cm)
- clear vinyl tubing of inner radius 1.5cm
- two photogates
- a high-FPS camera
- aluminum meter stick
- duct tape
- protractor
- foam board of arbitrary dimensions

Procedures

1. Bend the vinyl tubing to form a "U" shape with it being level at the top left and tape it to the particle board.
2. Cut out a small piece of foam board and tape it to the end of the tube such that it will mimic a tangent line to the curve.
3. Use the protractor to set the angle between the foam board and the horizontal to 30°.
4. With the tubing set in place with the correct angle, tape the meterstick to the particle board such that the "0cm" mark is just under the end of the tubing.
5. Tape the two photogates together to minimize the spacing between the sensors
6. Have a volunteer place the ball in the tubing and hold the photogates such that they are perpendicular to both the particle board and the launch angle of the tubing while you record the ball falling past the meter stick with the camera.
7. With the data presented, divide the distance between the sensors of the photogates by the time taken for the ball to pass through to obtain the velocity of launch, and, with the camera, view the exact frame where the ball is crossing the meter stick and record the distance shown.
8. Repeat steps 1-8 three (3) times for each angle of 30°, 45°, 60°, and 75°.

4 Data Analysis

Via following my procedures, I was able to obtain the following raw data with uncertainties:

Raw Data

Trial	Angle of Launch $\theta \pm 1°$	Time $t \pm 0.001$s	Velocity $v \pm 0.001\text{ms}^{-1}$	Distance $x \pm 0.001$m
1	$30° \to \frac{\pi}{6}$	$9.23 \cdot 10^{-3}$	1.63	0.430
2	$30° \to \frac{\pi}{6}$	$9.28 \cdot 10^{-3}$	1.62	0.390
3	$30° \to \frac{\pi}{6}$	$9.06 \cdot 10^{-3}$	1.66	0.385
4	$45° \to \frac{\pi}{4}$	$9.47 \cdot 10^{-3}$	1.58	0.350
5	$45° \to \frac{\pi}{4}$	$9.41 \cdot 10^{-3}$	1.59	0.355
6	$45° \to \frac{\pi}{4}$	$9.38 \cdot 10^{-3}$	1.60	0.348
7	$60° \to \frac{\pi}{3}$	$9.99 \cdot 10^{-3}$	1.50	0.293
8	$60° \to \frac{\pi}{3}$	$10.1 \cdot 10^{-3}$	1.48	0.278
9	$60° \to \frac{\pi}{3}$	$9.79 \cdot 10^{-3}$	1.53	0.30
10	$75° \to \frac{5\pi}{12}$	$11.4 \cdot 10^{-3}$	1.31	0.123
11	$75° \to \frac{5\pi}{12}$	$12.0 \cdot 10^{-3}$	1.25	0.125
12	$75° \to \frac{5\pi}{12}$	$11.2 \cdot 10^{-3}$	1.34	0.140

With this data, I can input all quantities for all respective variables into my equation for the unknown value Z and graph the results on a Cartesian plane with axes θ and Z. Note that, 1), the inputted values of the angles were converted to radians to make calculations simpler and, if possible, in terms of π and, 2), I took the average of the values of Z for each respective value of θ and added this data to the table.

Processed Data

Trial	Angle of Launch $\theta \pm 1°$	Retrieved Value of Z
1	$30° \rightarrow \frac{\pi}{6}$	1.28
2	$30° \rightarrow \frac{\pi}{6}$	1.03
3	$30° \rightarrow \frac{\pi}{6}$	0.924
AVG	$30° \rightarrow \frac{\pi}{6}$	1.08
4	$45° \rightarrow \frac{\pi}{4}$	0.934
5	$45° \rightarrow \frac{\pi}{4}$	0.951
6	$45° \rightarrow \frac{\pi}{4}$	0.850
AVG	$45° \rightarrow \frac{\pi}{4}$	0.912
7	$60° \rightarrow \frac{\pi}{3}$	1.85
8	$60° \rightarrow \frac{\pi}{3}$	1.66
9	$60° \rightarrow \frac{\pi}{3}$	1.83
AVG	$60° \rightarrow \frac{\pi}{3}$	1.78
10	$75° \rightarrow \frac{5\pi}{12}$	2.59
11	$75° \rightarrow \frac{5\pi}{12}$	3.31
12	$75° \rightarrow \frac{5\pi}{12}$	3.54
AVG	$75° \rightarrow \frac{5\pi}{12}$	3.15

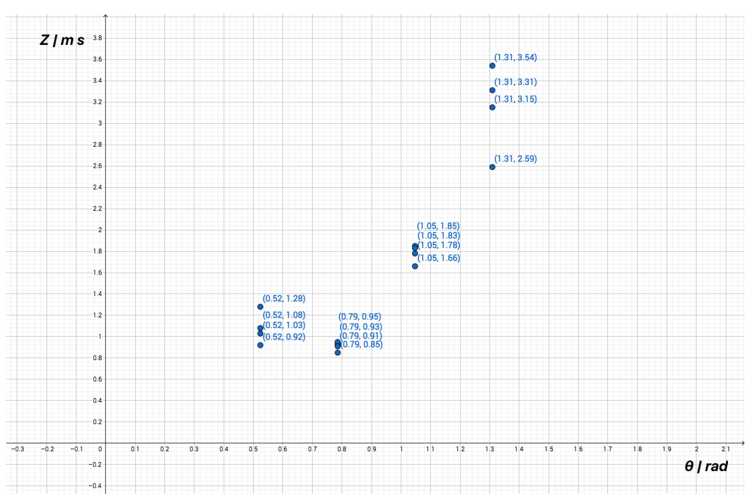

Figure 2: *Graph of the above list of points on the θ and Z axes*

Now that I have a clear representation of what my data looks like graphically, I ran a regression to find a clear equation for this unknown value of Z. I chose a sinusoidal regression as this was the only trigonometric regression available to me and as it was the most accurate in terms of the correlation coefficient.

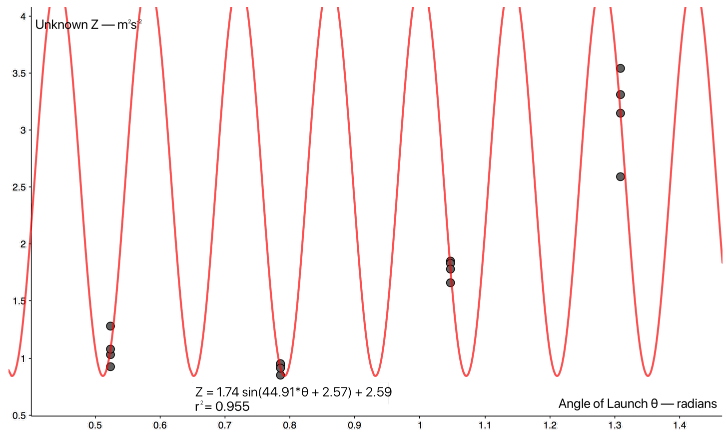

Figure 3: *Graph of the sinusoidal regression* $Z = 1.74\sin(44.91\theta + 2.57) + 2.59$ *on the* θ *and* Z *axes*

As shown above, the regression revealed the trigonometric equation $Z = 1.74\sin(44.91\theta + 2.57) + 2.59$ which, applied to this data, has a correlation coefficient of $r^2 = 0.955$. Based on this new data, it is quite clear that the equation retrieved from the regression is the equation of Z. Of course, taking into consideration my uncertainties in the aforementioned data sets negates the remaining 4.5% error left from the correlation coefficient as they allow for some error. Despite this, I tried the regression once more with only the averaged data and retrieved a similar equation but with a correlation coefficient of $r^2 = 1$, as shown below.

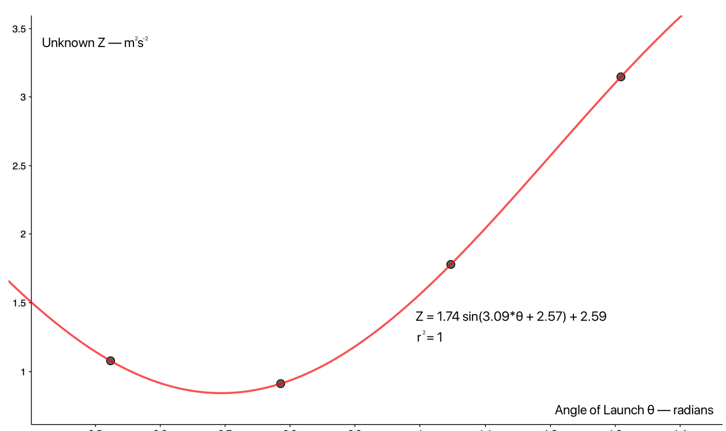

Figure 4: *Graph of the sinusoidal regression* $Z = 1.74\sin(3.09\theta + 2.57) + 2.59$ *on the* θ *and* Z *axes*

Therefore, according to this data, the equation $Z = 1.74\sin(3.09\theta + 2.57) + 2.59$ is the optimal equation needed to calculate the value of Z for any projectile being launched from a specific angle. However, one aspect of the equation for Z that this regression does not take into account is the fact that the function is undefined for two instances, one when $\theta = n\frac{\pi}{2}$, where $n = 1, 3, 5, 7, \ldots$, as $\tan(n\frac{\pi}{2})$ is undefined and when $x = 0$ as this input would result in the last term being $\frac{g(0)^2+2yv^2}{2(0)}$ which is also undefined. In order to solve this conundrum, I took the average value of v and x for each respective angle, plugged these into my equation for Z, and compared the resultant values to those which I acquired through experimentation. Furthermore, to test if my equation approaches infinity as the angle approaches 90° or $\frac{\pi}{2}$, I obtained the same data for when the angle of launch $\theta = 85° = \frac{17\pi}{36}$ which is as close to 90 degrees as I can realistically get.

Averaged Data

Angle of Launch $\theta \pm 1°$	Velocity $v \pm 0.001\text{ms}^{-1}$	Distance $x \pm 0.001\text{m}$
$30° \to \frac{\pi}{6}$	1.63	0.402
$45° \to \frac{\pi}{4}$	1.59	0.351
$60° \to \frac{\pi}{3}$	1.51	0.290
$75° \to \frac{5\pi}{12}$	1.30	0.129
$85° \to \frac{17\pi}{36}$	1.11	0.128

Respective Equations for Z

$$\theta = 30° = \frac{\pi}{6} \Rightarrow Z = \frac{0.402g}{2}\tan^2\theta - 1.63^2\tan\theta + \frac{0.402g}{2}$$
$$\theta = 45° = \frac{\pi}{4} \Rightarrow Z = \frac{0.351g}{2}\tan^2\theta - 1.59^2\tan\theta + \frac{0.351g}{2}$$
$$\theta = 60° = \frac{\pi}{3} \Rightarrow Z = \frac{0.290g}{2}\tan^2\theta - 1.51^2\tan\theta + \frac{0.290g}{2}$$
$$\theta = 75° = \frac{5\pi}{12} \Rightarrow Z = \frac{0.129g}{2}\tan^2\theta - 1.30^2\tan\theta + \frac{0.129g}{2}$$
$$\theta = 85° = \frac{17\pi}{36} \Rightarrow Z = \frac{0.128g}{2}\tan^2\theta - 1.11^2\tan\theta + \frac{0.128g}{2}$$

Comparing Experimental and Theoretical Values of Z

Angle of Launch $\theta \pm 1°$	Experimental Value of Z	Theoretical Value of Z
$30° = \frac{\pi}{6}$	1.08	1.08
$45° = \frac{\pi}{4}$	0.912	0.903
$60° = \frac{\pi}{3}$	1.78	1.76
$75° = \frac{5\pi}{12}$	3.15	3.15
$85° = \frac{17\pi}{36}$	68.2	68.2

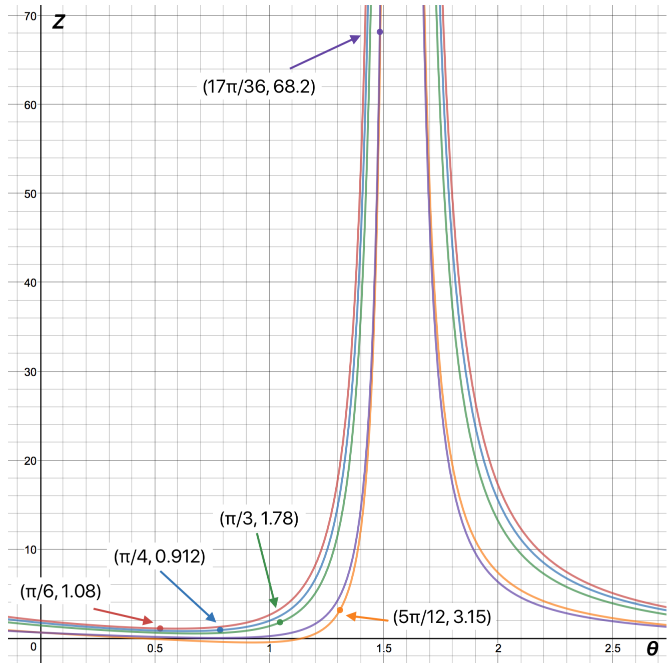

Figure 5: *Graphs of my experimental data points (as shown in previous table) and corresponding equations for Z in the same color.*

Based on the table above, it is clear that, with error accounted for, my predictions of what Z would be for each angle are correct and, thus, verify my equation for Z.

5 Conclusion and Reflection

As my original intent of this investigation was to try to figure out what the variable Z represents in the equation $Z = \frac{gx}{2}\tan^2\theta - v^2\tan\theta + \frac{gx^2+2yv^2}{2x}$, there was no guarantee that I would arrive at any answer or solution. Based on the methods and processes I used and the data I retrieved, it is quite clear that for any angle θ such that $0 < \theta < \frac{\pi}{2}$, and with the corresponding measurements of the initial velocity v and distance travelled x, the theoretical equation for Z obtained above stands true. Despite this, I was unable to determine what value Z represents aside from having the units $\mathrm{m^2s^{-2}}$. However, the fact that I did not come to a solution to this problem does not signify that I did not gain any knowledge from the process and analysis as a whole. Through my workings out I tried, in multiple ways, to find out what Z represents and, from attempting this, my problem solving skills have matured incredibly. Both my HL Physics Internal Assessment and that of HL Mathematics had the goal of finding an answer to an unsolved problem, whether it be about a connection between the equation for angle of launch in projectile motion or how to expand a binomial raised to a power that is not a positive integer. Moreover, despite my failure to arrive at a solution to this problem in my IA itself, I plan to continue this research into my university years to finally come to an answer.

In the future I plan to execute this experiment again, but with keeping the initial velocity v and distance travelled x constant such that the only quantity which is changing is the launch angle θ. That way, I will be able to construct only one equation for Z for all of the data and, hopefully, see a clear correlation/trend which I did not see in this generation.

References

"Projectile Motion." *Wikipedia*, Wikimedia Foundation, 8 Apr. 2018, en.wikipedia.org/wiki/Projectile_motion.

"Specific Energy." *Wikipedia*, Wikimedia Foundation, 9 Apr. 2018, en.wikipedia.org/wiki/Specific_energy.