Simmons University

Department of Mathematics, Computing, and Statistics

Modern Algebra in Modern Music

Understanding sound through a mathematical lens

Author:
Elijah Sanderson

Supervisor:
Dr. Donna Beers

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Contents

1 Motivation

Since its dawn, music has been a staple of every culture and every civilization, both consisting of humans and non-humans. Every musical genre can be primarily associated with specific cultures and each culture has its own primary genre and style; western music has different stylistic and modal properties than those of eastern music. Despite these distinctions, and as our society progressed into the twentieth and twenty-first centuries, we have begun to have access to many different genres, styles, and rhythms from nearly every culture via the internet. Now, anyone can explore and dive into any kind of music from anywhere in the world. Although the art of music is incredibly widespread and vast, the physical and mathematical mechanics behind the phenomenon are not completely understood by the average person. The aim of this paper is to explore both the linear algebra and further analysis behind the mechanics of sound in a concise, coherent way. Hopefully, after reading paper one will have a much clearer and formal understanding of the topics presented.

2 Vector Spaces

2.1 What is a vector space?

Let us begin our exploration with some linear algebra. In mathematics and physics, a *vector* is a quantity that has both magnitude and direction. One example of a vector is velocity; velocity consists of how fast one is travelling (magnitude) and where one is travelling (direction). Some other examples include acceleration, force, and weight, as each consists of a magnitude and direction. With this insight, we can start discussing the algebraic structure known as a vector space.

In mathematics, a vector space is a collection of objects, called vectors, which can be added together or multiplied by numbers, often called scalars. When one thinks of common vectors, say velocity, one can easily add two velocities together to get a new velocity. Moreover, one can multiply the velocity by some constant to get a constant multiple of the original velocity. Below is the formal definition of a vector space.

Definition 2.1. A vector space over a field F is a set V together with two operations that satisfy the eight axioms listed below.

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

So, a vector space is a set of vectors of which the operations of addition and scalar multiplication are well-defined. With this, we can add more structure to create what is known as an *inner product space*, which is a perfect mathematical interpretation for all physical sounds.

2.1.1 Exercises

Determine whether or not each of the following are vector spaces and give a sufficient proof.

1. The set of all 3-by-3 magic squares with real entries.
2. The set of all polynomials of degree 3 with real coefficients.
3. The set of all continuous real-valued functions.

2.2 More structure: Inner product spaces

Let us now introduce the *inner product*. This additional structure associates any pair of vectors with a scalar quantity, often denoted as $\langle a, b \rangle$. This structure allows for the common geometrical notions of vectors in the plane, such as the angle between two vectors. Additionally, this creates a means to measure the orthogonality between two vectors. For example, a common inner product in Euclidean space is the dot, or scalar, product. Formally, the definition of an inner product space with an inner product is given below [2].

Definition 2.2. An *inner product space* is a vector space V over the field of scalars $\mathbf{F}$ together with a mapping

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbf{F}$$

called an *inner product* that satisfies the following conditions for all vectors $u, v, w \in V$ and all scalars $a \in \mathbf{F}$.

1. Linearity: $\langle au, v \rangle = a\langle x, y \rangle \quad \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
2. Conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$
3. Positive-definite: $\langle u, u \rangle > 0$, if $u \neq 0$

For a more conceptual understanding, let us give some examples of inner product spaces. As aforementioned, one common example of a Euclidean vector space is the real n-space $\mathbb{R}^n$ with the dot product.

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = x^T y = \sum_{i=1}^{n} x_i y_i + \cdots x_n y_n$$

Now, let's see if we can construct an inner product space that intuitively mimics our desired space of sounds. First, let us select our vector space to be $V = C[-\pi, \pi]$, the set of all continuous functions $f(t)$ defined from $-\pi \leq t \leq \pi$. Notice that we can define an inner product for this space.

$$f(t), g(t) \in V, \quad \langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t)g(t)\ dt$$

Now, we must find an orthogonal basis for this space, as that will decompose V into a set of fundamental functions of which every other function is a linear combination of. For this, let us cite the Orthogonal Decomposition Theorem.

2.3 Orthogonal Decomposition Theorem

The orthogonal decomposition of a vector $\vec{y} \in \mathbb{R}^n$ is the sum of a vector in a subspace W of $\mathbb{R}^n$ and a vector in the orthogonal complement $W^\perp$ to W. The theorem is stated below [1].

Theorem 2.3. *Let W be a subspace of $\mathbb{R}^n$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form*

$$\vec{y} = \vec{\hat{y}} + \vec{z}$$

where $\vec{\hat{y}} \in W$ and $\vec{z} \in W^\perp$. Additionally, if $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_p}\}$ is any orthogonal basis of W, then

$$\vec{\hat{y}} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}}\vec{u_1} + \frac{\vec{y} \cdot \vec{u_2}}{\vec{u_2} \cdot \vec{u_2}}\vec{u_2} + \cdots + \frac{\vec{y} \cdot \vec{u_p}}{\vec{u_p} \cdot \vec{u_p}}\vec{u_p}.$$

Geometrically speaking, $\vec{\hat{y}}$ is the orthogonal projection of $\vec{y}$ onto the subspace W and $\vec{z}$ is a vector orthogonal to $\vec{\hat{y}}$.

Applying the theorem to our inner product space of interest, we obtain the following infinite set as an orthogonal basis:

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}.$$

We can make this basis *orthonormal* by converting each vector into a unit vector. This gives us the orthonormal basis of our inner product space as

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \frac{\cos 3t}{\sqrt{\pi}}, \frac{\sin 3t}{\sqrt{\pi}}, \ldots\right\}.$$

Moreover, a linear combination of any of these members is precisely the Fourier series for the desired function, with each term being the projection of some function f onto $\cos(nt)$ or $\sin(nt)$, depending on which basis element you're projecting onto [3].

3 Fourier Analysis

3.1 What is Fourier analysis?

Generally, *Fourier analysis* is the study of the way general functions may be represented by sums of trigonometric functions. Today, however, Fourier analysis more commonly refers to the process of decomposing a function into its oscillatory components, while Fourier synthesis refers to the inverse process. The actual procedures of decomposing a function and re-composing a function are known as a *Fourier transformation* and a *Fourier series*, respectively.

As we are much more concerned with the re-composition process, we shall not delve too far into the Fourier transform but instead direct our attention toward the Fourier series, as referenced in the previous section. However, before we begin this venture, let us have a brief review of complex numbers and their behavior.

3.2 Complex Numbers

In mathematics, a *complex number* is a number that can be expressed in the form $a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit, satisfying $i^2 = -1$. For any complex number $z = a + bi$, a is the *real part* and b is the *imaginary part*, denoted by $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$, respectively. The set of complex numbers is denoted by $\mathbb{C}$ and is a superset for the real numbers, $\mathbb{R} \subset \mathbb{C}$.

As i is defined to satisfy $i^2 = -1$, we can easily see how the powers of i behave

$$\begin{aligned}
&i^2 = -1 \\
&i^3 = (-1)(i) = -i \\
&i^4 = (-i)(i) = -i^2 = -(-1) = 1 \\
&i^5 = (1)i = i \\
&i^6 = (i)(i) = i^2 = -1 \\
&\ \vdots
\end{aligned}$$

From this, we can see that i has the property that for any $k \in \mathbb{Z}$, $i^k = i^{k+4}$.

The *conjugate* of a complex number $z = a + bi$ is defined as $\overline{z} = a - bi$. Conjugates are very useful in simplifying complex fractions, as multiplying by the conjugate of the denominator will always yield a real denominator, so simplifying the fraction is simple.

Now, complex numbers not only have the Cartesian representation $a+bi$, but also a polar representation, given by Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$

for any real θ. An intuitive explanation for this can be seen by noticing that if $e^{i\theta} = a + bi$, then $a = \cos\theta$ and $b = \sin\theta$, which directly related to the sides of a right triangle in the Euclidean plane.

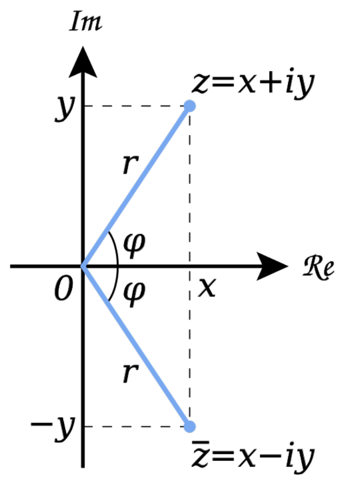


Figure 1: *Geometric representation of z and its conjugate $\overline{z}$ in the complex plane*

3.3 Fourier Series

Now that we have a foundation of understanding of complex numbers, let us dive into the Fourier series. Much like how a Taylor series approximates any function with polynomials, a *Fourier series* is a series of harmonically related sinusoids, or sines and cosines, combined by a weighted summation. This summation approximates any function defined on some closed interval and is the basis for the inner product space defined in section 2.2. The formal definition is given below [4].

Definition 3.1. The Fourier series of a function $f(t)$ defined on $-\pi \leq t \leq \pi$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt]$$

whose coefficients are given by the inner product formulae

$$a_k = \langle f, \cos kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \; dt, \qquad k = 0, 1, 2, 3, \ldots$$
$$b_k = \langle f, \sin kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \; dt, \qquad k = 1, 2, 3, \ldots$$

Let us work through a basic example of computing the Fourier series for an elementary function.

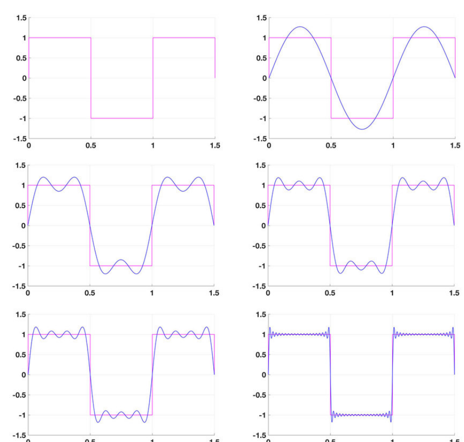
Example 3.1. *Let us compute the Fourier series for the function $f(t) = t$. We can calculate the Fourier coefficients directly:*

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos kt \; dt = \frac{1}{\pi} \left(\frac{t \sin kt}{k} + \frac{\cos kt}{k^2} \right) \Big|_{-\pi}^{\pi} = 0$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin kt \; dt = \frac{1}{\pi} \left(-\frac{t \cos kt}{k} + \frac{\sin kt}{k^2} \right) \Big|_{-\pi}^{\pi} = \frac{2(-1)^{k+1}}{k}$$

Thus, the resulting Fourier series is

$$t \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kt = 2 \left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + - \cdots \right)$$

Graphically, one can easily see the accuracy of the Fourier series by plotting subsequent partial sums. For example, the plot at right shows six partial sums of the Fourier series for a square wave plotted against the wave itself.



3.3.1 Exercises

Compute the Fourier series for each of the functions given below.

$$1.\ f(t) = t^2 \qquad 2.\ f(t) = \sin^2 t \qquad 3.\ f(t) = e^{-t}$$

As shown, the Fourier series can be used to decompose any function into a linear combination of a (possibly infinite) amount of sinusoidal waves. This is essential in the analysis of sounds as now we have a method to take any function that represents an arbitrary sound and determine the basic sines and cosines that comprise it. Moreover, with defining our inner product space from the vector space $V = C[-\pi, \pi]$ as

$$f(t), g(t) \in V, \quad \langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t)g(t)\ dt,$$

we know the orthonormal basis of this space as

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \frac{\cos 3t}{\sqrt{\pi}}, \frac{\sin 3t}{\sqrt{\pi}}, \ldots \right\}.$$

Altogether, we have found and shown that the set of all possible sounds forms an inner product space with an infinite orthonormal basis of single-frequency sinusoidal waves. This represents the mathematics behind all music composition and production in addition to how we interpret sound.

The amount of mathematics that stem from this, including signal processing and image processing, is far too vast to cover in a paper of this length. However, it is with great hope that this gives a brief, intuitive, and convenient analysis of the backbone of this arduous subject among mathematics.

4 Reflection

Overall, my module met the criteria well. Nearly all areas were represented throughout, especially the connection to a real-life example, that being the composition of sound waves from sinusoids, and of the additionally theory, that being Fourier series and harmonics. In terms of the linear algebra-specific content, I should have spent more time on what type of vector space is produced by sounds, specifically diving into inner product spaces and their mathematical interpretations.

Looking toward the class presentation specifically, I believe it went fairly well. The biggest issue that came about was the budgeting of time, as the allotted time had ceased prior to the final activity. If I were to give it once more, I would spend more time going through examples step-by-step with the class in hopes that it would better deliver the targeted concepts.

References

[1] Viktor Bengtsson. *Orthogonal Decompositon*. URL: https://mathworld.wolfram.com/OrthogonalDecomposition.html.

[2] P.K. Jain and Khalil Ahmad. *Functional Analysis*. New Age International, 1995. Chap. 5.1 Definitions and basic properties of inner product spaces and Hilbert spaces.

[3] David C. Lay. *Linear Algebra and Its Applications*. Pearson, 2020.

[4] Peter J. Olver. *Introduction to Partial Differential Equations*. Springer, 2016. Chap. 3.2 Fourier Series.

A Exercise Solutions

A.1 Vector Spaces

1. The set of all 3-by-3 magic squares with real entries forms a vector space.
2. The set of all polynomials of degree 3 does not form a vector space as it fails closure under vector addition.
3. The set of all continuous real-valued functions forms a vector space.

A.2 Fourier Series

1. $f(t) = t^2$

   Computing the coefficients a_k, b_k yields the following.

   $$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(kt)\ dt = \frac{4(-1)^k}{k^2} \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin(kt)\ dt = 0 \end{aligned}$$

   Therefore, our real Fourier series for $f(t) = t^2$ is given by

   $$t^2 \sim \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kt)$$

2. $f(t) = \sin^3 t$

   Computing the coefficients a_k, b_k yields the following.

   $$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3(t)\cos(kt)\ dt = 0 \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3(t)\sin(kt)\ dt = \frac{12\sin(\pi k)}{\pi(k+1)(k-1)(k+3)(k-3)} \end{aligned}$$

Since $\sin(\pi k) = 0$ for all $k \in \mathbb{N}$, $b_k = 0$ unless $k = \{-3, -1, 1, 3\}$, as these would cause an indeterminate form. To solve this, we examine the limit of b_k as k approaches each value.

$$\lim_{k \to -3} b_k = \frac{1}{4} \qquad \lim_{k \to -1} b_k = -\frac{3}{4} \qquad \lim_{k \to 1} b_k = \frac{3}{4} \qquad \lim_{k \to 3} b_k = -\frac{1}{4}$$

Therefore we have a finite real Fourier series for $f(t) = \sin^3 t$ given by

$$\sin^3 t \sim \frac{1}{4}\sin(-3t) - \frac{3}{4}\sin(-t) + \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t)$$

3. $f(t) = e^{-t}$

   Computing the coefficients a_k, b_k yields the following.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-t}\cos(kt)\ dt = \frac{2\sinh(\pi)(-1)^k}{\pi(k^2+1)}$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-t}\sin(kt)\ dt = \frac{2\sinh(\pi)(-1)^k k}{\pi(k^2+1))}$$

   Therefore, our real Fourier series for $f(t) = t^2$ is given by

$$e^{-x} \sim \sum_{k=1}^{\infty} \frac{2\sinh(\pi)(-1)^k}{\pi(k^2+1)}\cos(kt) + \frac{2\sinh(\pi)(-1)^k k}{\pi(k^2+1)}\sin(kt)$$