Optimization: A Method to Expand Binomials raised to the nth Power, where $n \in \mathbb{R}$

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Contents

1 Introduction and Rationale

This investigation aims to find another way to express a binomial without using the binomial expansion. In other words, given that $(a+b)^n$, where $a, b \in \mathbb{R}^+, n \in \mathbb{R}$, is there a possible way to represent this expression in the following way: $a^u + b^v$, where $u, v \in \mathbb{R}$? The initial circumstance from which my idea for this exploration spawned from was a mistake during the process of my Internal Assessment (IA) for HL Physics. In short terms, a major part of my Physics IA consisted of unit analysis and discovery in relation to projectile motion. During the process of manipulating and rewriting the equations to figure out which units corresponded to an unknown variable I reached this expression which contained these units in this order:

$$\left(m^4 s^{-4} + m^2 s^{-2}\right)^{\frac{1}{2}}$$

At this point, I was stuck because, according to my knowledge, there is no way one can rewrite an expression such as $(a+b)^n$ in terms of only a and b, where $-1 < n < 1$. This was the spark necessary to begin the investigation which would become my IA.

2 Theory

As aforementioned, the theory which this investigation aims to test and uncover is whether it is possible to represent a binomial in terms of only the two arguments to some indices for all cases of n, rather than using the binomial expansion which only gives an expression for when $n \leq -1$ and $n \geq 1$. To begin to unwrap this conjecture, I began with considering a specific case with a set index of the RHS and then comparing the result to a similar case with an increased index.

Given $(a+b)^n = a^u + b^v$, let $a = 3$, $b = 5$, and $n = 1$.

$$\begin{aligned}(3+5)^1 &= 3^u + 5^v \\ 8 &= 3^u + 5^v\end{aligned}$$

Next, I isolated v in the above expression so it can be graphed on a plane of axes u and v. The graph of the following equation is shown on the following page.

$$\begin{aligned}8 &= 3^u + 5^v \\ 5^v &= 8 - 3^u \\ \ln(5^v) &= \ln(8 - 3^u) \\ v\ln(5) &= \ln(8 - 3^u) \\ v &= \frac{\ln(8 - 3^u)}{\ln(5)} \qquad (1)\end{aligned}$$

Additionally, I changed the value of n so there are three equations each with their respective value of n (1, 2, and 3) as shown below.

$$\begin{aligned} 8^1 &= 3^u + 5^{v_1} \\ 5^{v_1} &= 8 - 3^u \\ \ln\left(5^{v_1}\right) &= \ln\left(8 - 3^u\right) \\ v_1 \ln(5) &= \ln\left(8 - 3^u\right) \\ \therefore v_1 &= \frac{\ln\left(8 - 3^u\right)}{\ln(5)} \end{aligned} \qquad \begin{aligned} 8^2 &= 3^u + 5^{v_2} \\ 5^{v_2} &= 8^2 - 3^u \\ \ln\left(5^{v_2}\right) &= \ln\left(8^2 - 3^u\right) \\ v_2 \ln(5) &= \ln\left(8^2 - 3^u\right) \\ \therefore v_2 &= \frac{\ln\left(8^2 - 3^u\right)}{\ln(5)} \end{aligned} \qquad \begin{aligned} 8^3 &= 3^u + 5^{v_3} \\ 5^{v_3} &= 8^3 - 3^u \\ \ln\left(5^{v_3}\right) &= \ln\left(8^3 - 3^u\right) \\ v_3 \ln(5) &= \ln\left(8^3 - 3^u\right) \\ \therefore v_3 &= \frac{\ln\left(8^3 - 3^u\right)}{\ln(5)} \end{aligned}$$

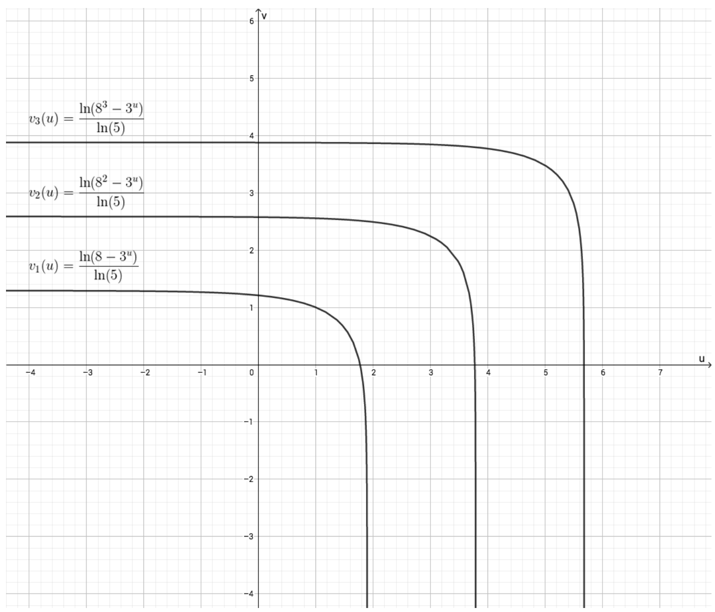

Figure 1: *Graphs of v_1, v_2, and v_3.*

Graph	u-intercept	v-intercept
v_1	$\log_3(8-1) \simeq 1.77$	$\log_5(8-1) \simeq 1.21$
v_2	$\log_3(8^2-1) \simeq 3.77$	$\log_5(8^2-1) \simeq 2.57$
v_3	$\log_3(8^3-1) \simeq 5.68$	$\log_5(8^3-1) \simeq 3.88$

Table 1: *The u- and v-intercepts of v_1, v_2, and v_3.*

According to the graph above, as n increases by a fixed amount, the u– and v–intercepts increase by an amount proportional to the index n. Meaning there may exist a linear relationship between n and the other indices u and v. The preamble to find this relationship, I calculated Δv and Δu between each of the graphs' intercepts to verify the linearity of the relationship between the graphs with different values of n.

$$\begin{aligned}\Delta v_{3,2} &= v_3 - v_2 \\ &\simeq 1.30 \\ \Delta u_{3,2} &= u_3 - u_2 \\ &\simeq 1.91\end{aligned} \qquad \begin{aligned}\Delta v_{2,1} &= v_2 - v_1 \\ &= 1.37 \\ \Delta u_{2,1} &= u_2 - u_1 \\ &\simeq 1\end{aligned} \qquad \begin{aligned}\Delta v_{3,1} &= v_3 - v_1 \\ &\simeq 2.67 \\ \Delta u_{3,1} &= u_3 - u_1 \\ &\simeq 3.91\end{aligned}$$

$$\begin{aligned}\therefore \frac{\Delta v_{3,2}}{\Delta u_{3,2}} &= \frac{1.30}{1.91} \\ &\simeq 0.682\end{aligned} \qquad \begin{aligned}\therefore \frac{\Delta v_{2,1}}{\Delta u_{2,1}} &= \frac{1.37}{2} \\ &\simeq 0.682\end{aligned} \qquad \begin{aligned}\therefore \frac{\Delta v_{3,1}}{\Delta u_{3,1}} &= \frac{2.67}{3.91} \\ &\simeq 0.682\end{aligned}$$

As you can see, every ratio of the changes between each interval of u and v result in the same value. This validates my claim that there exists some linear relationship between v_1, v_2, and v_3. In order to attempt to generalize this claim, I used the method of optimization to find the rectangle under each curve with the largest area.

3 Optimization

If I were to find the area of an arbitrary rectangle under the area of any curve, I would use the area formula $A = bh$. Hence, I would need to multiply the length u by the height v, as they make up the two pairs of parallel sides of the rectangle. I will do so using the first case where $n = 1$. Using equation 1, we have the following:

$$\begin{aligned}A &= uv \\ A &= \frac{u \ln\left(8 - 3^u\right)}{\ln(5)}\end{aligned}$$

As I am optimizing the function to find the maximum possible area, I need to first find the first derivative of A.

$$A' = \frac{d}{du}\left[\frac{u \ln\left(8 - 3^u\right)}{\ln(5)}\right]$$

$$A' = \frac{\left(8 - 3^u\right)\ln\left(8 - 3^u\right) - 3^u u \ln(3)}{\left(8 - 3^u\right)\ln(5)}$$

Next, I need to set the equation equal to zero to find the value of u which will give the maximum area of the rectangle.

$$\begin{aligned} 0 &= \frac{(8-3^u)\ln(8-3^u) - 3^u u \ln(3)}{(8-3^u)\ln(5)} \\ 3^u u \ln(3) &= (8-3^u)\ln(8-3^u) \\ 3^u \ln(3^u) &= (8-3^u)\ln(8-3^u) \end{aligned}$$

Let $x = 3^u$ and $y = 8 - 3^u$.

$$\begin{aligned} x\ln(x) &= y\ln(y) \\ \ln(x^x) &= \ln(y^y) \\ e^{\ln(x^x)} &= e^{\ln(y^y)} \\ x^x &= y^y \\ x &= y \quad \text{(see proof at end of paper)} \\ 3^u &= 8 - 3^u \\ 2(3^u) &= 8 \\ 3^u &= 4 \\ u &= \log_3(4) \\ u &= \log_3\left(\frac{3+5}{2}\right) \simeq 1.262 \end{aligned}$$

To find the height of the rectangle v, I inputted the value of u into the original equation for v.

$$\begin{aligned} v &= \frac{\ln\left(8 - 3^{(\log_3(4))}\right)}{\ln(5)} \\ v &= \frac{\ln(8-4)}{\ln(5)} \\ v &= \frac{\ln(4)}{\ln(5)} \\ v &= \log_5(4) \\ v &= \log_5\left(\frac{3+5}{3}\right) \simeq 0.861 \end{aligned}$$

If I repeat the same calculations for when $n = 2$ and 3, then I will obtain the following results:

Let $n = 2$.

$$\begin{aligned}
A' = 0 &= \frac{(8^2 - 3^u)\ln(8^2 - 3^u) - 3^u u \ln(3)}{(8^2 - 3^u)\ln(5)} \\
3^u \ln(3^u) &= (64 - 3^u)\ln(64 - 3^u) \\
3^u &= 64 - 3^u \\
3^u &= 32 \\
u &= \log_3(32) \\
u &= \log_3\left(\frac{(3+5)^2}{2}\right) \simeq 3.16
\end{aligned}$$

$$\begin{aligned}
\therefore v &= \frac{\ln\left(64 - 3^{(\log_3(32))}\right)}{\ln(5)} \\
v &= \frac{\ln(64 - 32)}{\ln(5)} \\
v &= \frac{\ln(32)}{\ln(5)} \\
v &= \log_5\left(\frac{(3+5)^2}{2}\right) \simeq 2.15
\end{aligned}$$

Let $n = 3$.

$$\begin{aligned}
A' = 0 &= \frac{(8^3 - 3^u)\ln(8^3 - 3^u) - 3^u u \ln(3)}{(8^3 - 3^u)\ln(5)} \\
3^u \ln(3^u) &= (512 - 3^u)\ln(512 - 3^u) \\
3^u &= 512 - 3^u \\
3^u &= 128 \\
u &= \log_3(128) \\
u &= \log_3\left(\frac{(3+5)^3}{2}\right) \simeq 4.42
\end{aligned}$$

$$\begin{aligned}
\therefore v &= \frac{\ln\left(512 - 3^{(\log_3(128))}\right)}{\ln(5)} \\
v &= \frac{\ln(512 - 128)}{\ln(5)} \\
v &= \frac{\ln(128)}{\ln(5)} \\
v &= \log_5\left(\frac{(3+5)^2}{2}\right) \simeq 3.02
\end{aligned}$$

After the calculations, I acquired values for the lengths u and heights v based on different values of n, as shown in the table below. I can then use these values to find the line joining the top right corners of each rectangle under each of the three curves (see Figure 2 and subsequent calculations).

n	u	v
1	$\log_3\left(\frac{3+5}{2}\right) \simeq 1.262$	$\log_5\left(\frac{3+5}{3}\right) \simeq 0.861$
2	$\log_3\left(\frac{(3+5)^2}{2}\right) \simeq 3.16$	$\log_5\left(\frac{(3+5)^2}{2}\right) \simeq 2.15$
3	$\log_3\left(\frac{(3+5)^3}{2}\right) \simeq 4.42$	$\log_5\left(\frac{(3+5)^2}{2}\right) \simeq 3.02$

Table 2: *This table represents the lengths u and heights v of the rectangle with the maximum possible area with varying values of n.*

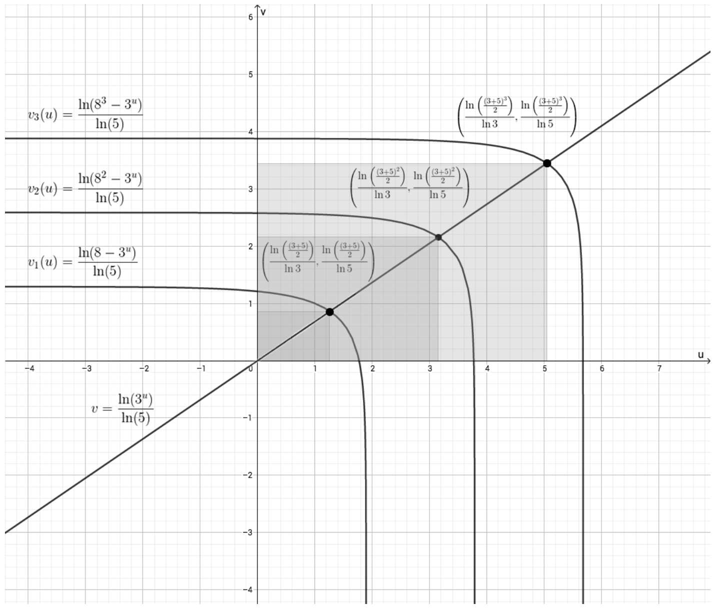

Figure 2: *Graphs of v_1, v_2, v_3, the line $v = \frac{\ln(3^u)}{\ln(5)}$, and each rectangle of largest area.*

$$\left[m = \frac{\Delta y}{\Delta x}\right]$$

$$m = \frac{\frac{\ln(\frac{8^2}{2}) - \ln(\frac{8}{2})}{\ln(5)}}{\log_3(\frac{8^2}{2}) - \log_3(\frac{8}{2})}$$

$$= \frac{\frac{2\ln(8) - \ln(2) - \ln(8) + \ln(2)}{\ln(5)}}{2\log_3(8) - \log_3(2) - \log_3(8) + \log_3(2)}$$

$$= \frac{\frac{\ln(8)}{\ln(5)}}{\log_3(8)}$$

$$= \frac{\log_5(8)}{\log_3(8)}$$

$$= \frac{\log_5(8)}{\log_3(8)}\left(\frac{\log_8(3)}{\log_8(3)}\right)$$

$$= \log_5(8)\log_8(3)$$

$$m = \log_5(3) \simeq 0.683$$

Therefore, the line which connects all top right corners of the largest possible rectangle under each curve for this specific case is:

$$v = u\log_5(8)\log_8(3)$$

Hence, the point of intersection of this line and each curve gives the values of u and v which make the primal equation ($8^n = 3^u + 5^v$) true.

4 Conclusion

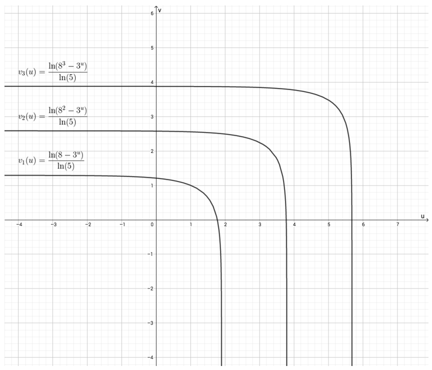

Figure 3: *Graphs of v_1, v_2, and v_3.*

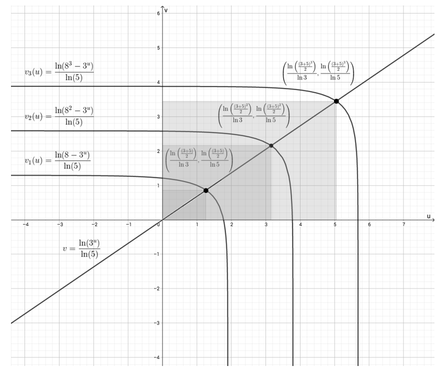

Figure 4: *Graphs of v_1, v_2, v_3, the line $v = \frac{\ln(3^u)}{\ln(5)}$, and each rectangle of largest area.*

As you can see, when the index n increases, the only effect it has on the equations of u and v is itself; it does not affect any other term. Furthermore, it can be assumed that this holds true for the other variables a and b, as nothing changes in their case.

> ***Conjecture:*** *Firstly, an arbitrary binomial of the form $(a+b)^n$, where $a, b \in \mathbb{R}^+$, can be rewritten as an expression of the form $a^u + b^v$, where $u, v \in \mathbb{R}$, for all real values of n. Secondly, $(a+b)^n$ can be rewritten as $a^u + b^v$ without the utilization of the binomial expansion for when $n \leq -1, n \geq 1$. Finally, given the equation $(a+b)^n = a^u + b^v$, the equations for u and v are, respectively, $u = \log_a(\frac{(a+b)^n}{2})$ and $v = \log_b(\frac{(a+b)^n}{2})$.*

In order to prove this for where $n \in \mathbb{R}$, I can use simple algebra and logic. The purpose behind the conjecture is to figure out a way to expand or represent a binomial that is raised to a power that is not an integer greater than 1 or less than -1. My theory is that if one were to separate $(a+b)^n$ into two separate terms a^u and b^v, then the values of each term need to be equal and exactly half of the original expression so that the sum of the terms gives the correct answer. Then, in order to obtain the equation in the form $(a+b)^n = a^u + b^v$, I would need to take separate logarithms with bases equal to the two terms in the binomial and raise those terms to the power to their respective logarithm on the RHS (if I go backwards the bases and the logs would cancel each other out so nothing is actually changing). Finally, I can make two substitutions for the powers of the terms on the RHS and obtain the equation to prove my conjecture for all real values of n.

$$
\begin{aligned}
(a+b)^n &= (a+b)^n \\
&= \frac{(a+b)^n}{2} + \frac{(a+b)^n}{2} \\
&= a^{\log_a\left(\frac{(a+b)^n}{2}\right)} + b^{\log_b\left(\frac{(a+b)^n}{2}\right)}
\end{aligned}
$$

$$\therefore (a+b)^n = a^u + b^v, u = \log_a\left(\frac{(a+b)^n}{2}\right), v = \log_b\left(\frac{(a+b)^n}{2}\right)$$

5 Proof that if $x^x = y^y$ then $x = y$, for $x \neq 0, 1, y \neq 0$

$$\begin{aligned} x^x &= y^y \\ \ln(x^x) &= \ln(y^y) \\ x\ln(x) &= y\ln(y) \end{aligned}$$

$$\frac{x}{y} = \frac{\ln(y)}{\ln(x)}$$

As the above statement equates two ratios, then the pair of numerators and denominators should be equal, as well. After some simplification, I am able to graph each equation to see an obvious occurrence that the two equations are, indeed, inverses of each other.

Numerators:

$$\begin{aligned} \therefore x &= \ln(y) \\ e^x &= e^{\ln(y)} \\ e^x &= y \end{aligned}$$

Denominators:

$$\begin{aligned} \therefore y &= \ln(x) \\ e^y &= e^{\ln(x)} \\ e^y &= x \end{aligned}$$

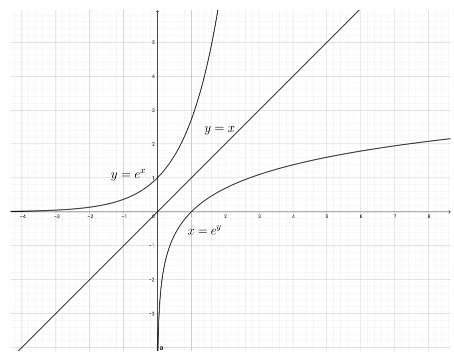

Figure 5: *Graphs of $y = e^x$, $x = e^y$, and $y = x$.*

By definition, "an inverse function is a function that 'reverses' another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa. I.e. $f(x) = y$ if and only if $g(y) = x$."[1][2] Moreover, from a graphic standpoint, inverse functions are mirror images of each other across the line $y = x$. Hence, in order to convert a function to its inverse, one needs to "swap" the variables x and y as each input becomes the others output and vice versa. Therefore, based on the phenomenon of inverse functions, if $x^x = y^y$ then $x = y$.

[1] Keisler, Howard Jerome. "Differentiation" (PDF). Retrieved 2018-02-20. §2.4

[2] Scheinerman, Edward R. (2013). *Mathematics: A Discrete Introduction.* Brooks/Cole. p. 173. ISBN 978-0840049421.

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Keisler, Howard Jerome. "Differentiation" (PDF). Retrieved 2018-02-20. §2.4

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