Free Groups

A way to Define Every Group

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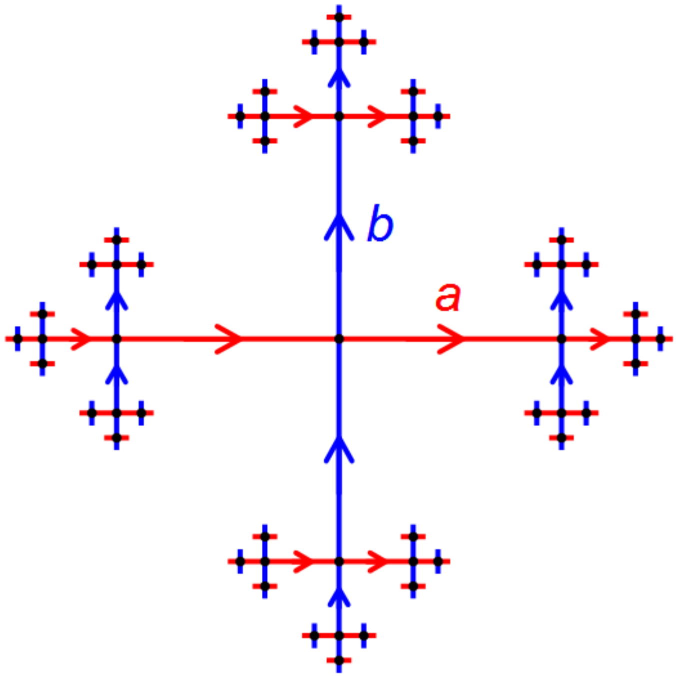

Figure 1: Diagram showing what the Cayley graph for the free group on two generators would look like.

Abstract

First studied in 1924, (Nielsen, 1924) a free group (F_S) is a type of group that is considered to be free of relations. That is, it is constructed by a set of elements from a group G, denoted S and its inverse set, S^{-1} in such a way that $F_S/N = G$ where N is a normal subgroup of F_S. Free groups have applications in group theory, category theory, topology, and other areas of mathematics.

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Free Groups

1.1 | What is a Free Group?

A free group F over a set S consists of all possible distinct element combinations from members of S, and S^{-1}.The elements in S are all generators of F. We call some group G free if the entirety of G can be constructed using only a finite string of elements from S and S^{-1}.

A particularly interesting result from free groups is proven in Theorem 26.2 in *Contemporary Abstract Algebra (9th Edition)* by Joseph Gallian. This result, called the Universal Mapping Property, states that every group is a homomorphic image of a free group, with the proof given below.

1.2 | Universal Mapping Property

Theorem: Every group is a homomorphic image of a free group (Gallian, 2017)

Proof. Let G be a group and let S be a set of generators for G. (Such a set exists, because we may take S to be G itself.) Now let F be the free group on S. Unfortunately, since our notation for any word in $W(S)$ also denotes an element of G, we have created a notational problem for ourselves. So, to distinguish between these two cases, we will denote the word $x_1x_2\cdots x_n$ in $W(S)$ by $(x_1x_2\cdots x_n)_F$ and the product $x_1x_2\cdots x_n$ in G by $(x_1x_2\cdots x_n)_G$. As before, $\overline{x_1x_2\cdots x_n}$ denotes the equivalence class in F containing the word $(x_1x_2\cdots x_n)_F$ in $W(S)$. Notice that $\overline{x_1 x_2 \cdots x_n}$ and $(x_1 x_2 \cdots x_n)_G$ are entirely different elements, since the operations on F and G are different.

Now consider the mapping from F into G given by

$$\phi(\overline{x_1 x_2 \cdots x_n}) = (x_1 x_2 \cdots x_n)_G$$

All we are doing is taking a product in F and viewing it as a product in G. For example, if G is the cyclic group of order 4 generated by a, then

$$\phi(aaaaa) = (aaaaa)G = a$$

Clearly, ϕ is well-defined, for inserting or deleting expressions of the form xx^{-1} or $x^{-1}x$ in elements of $W(S)$ corresponds to inserting or deleting the identity in G. To check that ϕ is operation-preserving, observe that

$$\begin{aligned} \phi(\overline{x_1 x_2 \cdots x_n})(\overline{y_1 y_2 \cdots y_m}) &= \phi(\overline{x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m}) \\ &= (x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m)_G \\ &= (x_1 x_2 \cdots x_n)_G (y_1 y_2 \cdots y_m)_G \end{aligned}$$

Finally, ϕ is onto G because S generates G. $\square$

1.3 | Any group is isomorphic to a quotient group of some free group

Now we shall prove the property that any group is isomorphic to the quotient of a free group by a normal subgroup. Thus, we seek the existence of an isomorphism $\gamma : F/N \to G$. First, let G be a group. By the Universal Mapping Property, there exists a free group F and a homomorphism ϕ such that $\phi(F) = G$. Now, let $N = \text{Ker}(\phi)$ (note that N is normal since the kernel of any group is a normal subgroup). From the First Isomorphism Theorem, there exist a unique isomorphism $\gamma : F/N \to \phi(F)$. Thus, γ is an isomorphism from the quotient group F/N of the free group F to a group G, rewritten as $\gamma : F/N \to G$. Therefore, any group is isomorphic to a quotient group of some free group.

$$\begin{tikzcd} & F_S \arrow[dd, dashed, "\varphi"] \\ S \arrow[ur] \arrow[dr, "f"'] & \\ & G \end{tikzcd}$$

Figure 1.1: Directed graph demonstrating the construction of a free group.

1.4 | Example of a Free Group

A simple example of a free group is $G = (\mathbb{Z}, +)$. Given a set $S = \{1\}$, implying $S^{-1} = \{-1\}$ we see that F, the set generated by all possible symbol combinations of $S \cup S^{-1}$ will be denoted as follows

$$F = \{..., -2, -1, 0, 1, 2, ...\} = \mathbb{Z}$$

which has a one-to-one mapping to the elements in G, as the elements of G are also equal to the set of integers. Thus, the group G is considered free over S.

We see that S, F, and G satisfy this graph, with:

$$S = \{1\}, F = \mathbb{Z}, G = G, \text{ and } \phi(x) = x$$

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Conclusions

2.1 | Future Work and Final Remarks

If we were to embark on this project in the future, we would aim to investigate more types of free groups and examine other theorems of the structure. For example, the Nielsen–Schreier theorem states that every subgroup of a free group is itself free. The proof of this theorem involves usage of the algebraic topology of fundamental groups and covering spaces, with looking at the topological implications of group properties. Another intriguing topic are SQ-universal groups, which are defined to be a countable group, meaning the cardinality is the same as some subset of $\mathbb{N}$, where every countable group can be embedded in one of its quotient groups (any free group with two elements is said to be SQ universal). Finally, the Grushko theorem states that the rank, that is, the smallest cardinality of a generating set, of a free product of two groups is equal to the sum of the ranks of the two free factors. One very interesting consequence of the theorem is that if a subset B of a free group F on n elements generates F and has n elements, then B generates F freely. Overall, it was very interesting to study this subtopic of abstract algebra and to closely see and understand the implications of free groups. With this, there are a multitude of interesting properties, implications, and consequences of free groups which would be very intriguing to study on a further note, with the possibility of developing this into a fully-developed research project.

References

Joseph Gallian. *Contemporary Abstract Algebra*, 2017. ISSN 978-1-305-65796-0.

Jakob Nielsen. Die isomorphismengruppe der freien gruppen. *Mathematische Annalen*, 9 1924. ISSN 1432-1807. doi: 10.1007/BF01556078.