**Supplemental material for the paper:** Vanin F., Penna A., Beyer K. (2019), A three-dimensional macro-element for modelling of the in-plane and out-of-plane response of masonry walls, *submitted to Earthquake Engineering* & *Structural Dynamics, HHH.*

# A CO-ROTATIONAL FORMULATION OF COMPATIBILITY RELATIONS

The basic displacements can be computed from local displacements in a co-rotational formulation using Euler's formula for finite rotations in the three-dimensional space. A rotated vector $\boldsymbol{v'}$ is expressed as a function of the undeformed vector $\boldsymbol{v}$ and the set of three finite rotations $\alpha_1$, $\alpha_2$ and $\alpha_3$ as:

$$\boldsymbol{v'} = \boldsymbol{v}\cos\gamma + (\boldsymbol{n}\times\boldsymbol{v})\sin\gamma + (1-\cos\gamma)(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} \tag{A1}$$

The orientation vector of the two blocks (i.e. the direction vector from node $i$ to node $e$, and from node $e$ to node $j$, respectively) in the undeformed configuration can be expressed as $\boldsymbol{v} = [1,0,0]^T$. The direction vector of the first block in the deformed configuration, where $H$ is the height if the element, can be expressed as:

$$\boldsymbol{v'} = \frac{1}{H'_A}\begin{bmatrix} \frac{H}{2} + u_{eA} - u_i \\ v_{eA} - v_i \\ w_{eA} - w_i \end{bmatrix}, \qquad H'_A = \sqrt{(\frac{H}{2} + u_{eA} - u_i)^2 + (v_{eA} - v_i)^2 + (w_{eA} - w_i)^2} \tag{A2}$$

Furthermore, if one postulates that no torsional rotation is imposed to the block, the last term of Equation A1 vanishes. Solving Equation A1 for the remaining two rotations of the first block one obtains:

$$\gamma = \arccos v'_{(1)} \tag{A3}$$

$$\boldsymbol{n} = \frac{1}{\sqrt{1 - v'_{(1)}}}\begin{bmatrix} 0 \\ -v'_{(3)} \\ v'_{(2)} \end{bmatrix} \tag{A4}$$

Proceeding in the same way for the second block, the compatibility equations defining the basic displacements in co-rotational formulation take the following form:

$$d_1 = -\frac{H}{2} + H'_A \tag{A5}$$

$$\varphi_{z1} = -\varphi_{zi} + \frac{(v_{eA} - v_i)\arccos\frac{\frac{H}{2}+u_{eA}-u_i}{H'_A}}{\sqrt{H'^2_A - \left(\frac{H}{2} + u_{eA} - u_i\right)^2}} \tag{A6}$$

$$\varphi_{y1} = -\varphi_{yi} + \frac{(-w_{eA} + w_i)\arccos\frac{\frac{H}{2}+u_{eA}-u_i}{H'_A}}{\sqrt{H'^2_A - \left(\frac{H}{2} + u_{eA} - u_i\right)^2}} \tag{A7}$$

$$d_2 = \frac{-(u_{eA} - u_{eB})(H - u_i + u_j) + (v_{eA} - v_{eB})(v_i - v_j) + (w_{eA} - w_{eB})(w_i - w_j)}{\sqrt{(H + u_j - u_i)^2 + (v_j - v_i)^2 + (w_j - w_i)^2}} \tag{A8}$$

$$\varphi_{z2} = \frac{(-v_{eA} + v_i)\arccos\frac{\frac{H}{2}+u_{eA}-u_i}{H'_A}}{\sqrt{H'^2_A - \left(\frac{H}{2} + u_{eA} - u_i\right)^2}} + \frac{(-v_{eB} + v_j)\arccos\frac{\frac{H}{2}-u_{eB}+u_j}{H'_B}}{\sqrt{H'^2_B - \left(\frac{H}{2} - u_{eA} + u_i\right)^2}} \tag{A9}$$

$$\varphi_{y2} = \frac{(w_{eA} - w_i)\arccos\frac{\frac{H}{2}+u_{eA}-u_i}{H'_A}}{\sqrt{H'^2_A - \left(\frac{H}{2}+u_{eA}-u_i\right)^2}} + \frac{(w_{eB} - w_j)\arccos\frac{\frac{H}{2}-u_{eB}+u_j}{H'_B}}{\sqrt{H'^2_B - \left(\frac{H}{2}-u_{eA}+u_i\right)^2}} \tag{A10}$$

$$d_3 = -\frac{H}{2} + H'_B \tag{A11}$$

$$\varphi_{z3} = \varphi_{zj} + \frac{(v_{eB} - v_j)\arccos\frac{\frac{H}{2}-u_{eB}+u_j}{H'_B}}{\sqrt{H'^2_B - \left(\frac{H}{2}-u_{eB}+u_j\right)^2}} \tag{A12}$$

$$\varphi_{y3} = \varphi_{yj} + \frac{(w_{eB} - w_j)\arccos\frac{\frac{H}{2}-u_{eB}+u_j}{H'_B}}{\sqrt{H'^2_B - \left(\frac{H}{2}-u_{eB}+u_j\right)^2}} \tag{A13}$$

$$s_y = \frac{-(v_{eA} - v_{eB})(H - u_i + u_j) - (u_{eA} - u_{eB})(v_i - v_j) +}{\sqrt{(H + u_j - u_i)^2 + (v_j - v_i)^2 + (w_j - w_i)^2}} \tag{A14}$$

$$s_y = \frac{-(w_{eA} - w_{eB})(H - u_i + u_j) - (u_{eA} - u_{eB})(w_i - w_j) +}{\sqrt{(H + u_j - u_i)^2 + (v_j - v_i)^2 + (w_j - w_i)^2}} \tag{A15}$$

# B INCREMENTAL EQUILIBRIUM MATRIX

The incremental compatibility matrix $\boldsymbol{\Gamma}_{\boldsymbol{C}}$ can be derived directly from the relations expressing the basic displacements as a function of local displacements. Enforcing the principle of contragradiency, the equilibrium matrix $\boldsymbol{\Gamma}_{\boldsymbol{E}}$, expressing the equilibrium conditions in the deformed configuration, can be obtained as the transpose of the incremental compatibility matrix. The expressions in Equation 2 can be obtained by Taylor-series expansion in Annex A, under the assumption that the orientation of the central interface remains defined by the undeformed configuration. This simplifies the equilibrium matrix, as defined in Equation B16, where the displacement-dependent terms define the $P-\Delta$ effect.

$$\boldsymbol{\Gamma}_{\boldsymbol{E}} = \left(\frac{\partial \boldsymbol{u}_{\boldsymbol{basic}}}{\partial \boldsymbol{u}_{\boldsymbol{local}}}\right)^{\boldsymbol{T}} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{H}(v_i - v_{eA}) & -\frac{2}{H} & 0 & 0 & \frac{2}{H} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{H}(w_i - w_{eA}) & 0 & \frac{2}{H} & 0 & 0 & -\frac{2}{H} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{H} & 0 & \frac{2}{H}(v_j - v_{eB}) & -\frac{2}{H} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{H} & \frac{2}{H}(w_j - w_{eB}) & 0 & \frac{2}{H} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{H}(v_{eA} - v_i) & \frac{2}{H} & 0 & 0 & -\frac{2}{H} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\frac{2}{H}(w_{eA} - w_i) & 0 & -\frac{2}{H} & 0 & 0 & \frac{2}{H} & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{H} & 0 & \frac{2}{H}(v_{eB} - v_j) & \frac{2}{H} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{H} & \frac{2}{H}(w_{eB} - w_j) & 0 & -\frac{2}{H} & 0 & 1 & 0
\end{bmatrix} \tag{B16}$$

# C EFFECT OF DISTRIBUTED LOADS ALONG THE ELEMENT

Accounting for distributed element forces along the element axis, which can induce second-order effects, requires the definition of the two vectors $\boldsymbol{q_0}$ and $\boldsymbol{p_0}$. The former adds an equivalent second-effect of the distributed load to the sectional forces in the basic system, while the latter ensures equilibrium conditions for the four bodies in Figure C1 , i.e. nodes $i$ and $j$ and the blocks A and B. Since the distributed load is defined in the undeformed local system, its direction does not rotate with the macro-element. If the load is constant along the axis of the element, the vector $\boldsymbol{q_0}$ can be defined as in Equation C17. If the load is triangular, with its maximum value in correspondence to the first node, Equation C18 holds. Such loading can be useful when modelling the overturning of a gable element subjected mainly to its self-weight. In both expressions only the axial component of the distributed load affects the geometrical nonlinearity, coherently with the assumed $P-\Delta$ formulation.

$$\boldsymbol{q_0}^{(rect)} = \left[ -\frac{q_x H}{4}, 0, 0, \quad 0, 0, 0, \quad \frac{q_x H}{4}, 0, 0, \quad 0, 0, 0 \right] \tag{C17}$$

$$\boldsymbol{q_0}^{(tr)} = \left[ -\frac{5 q_x H}{24}, 0, 0, \quad 0, 0, 0, \quad \frac{q_x H}{24}, 0, 0, \quad 0, 0, 0 \right] \tag{C18}$$

The set of reactions $\boldsymbol{p_0}$ ensuring equilibrium is defined in Equations C19 and C20 for rectangular and triangular distributed loads, respectively.

$$\boldsymbol{p_0}^{(rect)} = \left[ \frac{q_x H}{4}, \frac{q_y H}{4}, \frac{q_z H}{4}, 0, 0, 0, \quad \frac{q_x H}{4}, \frac{q_y H}{4}, \frac{q_z H}{4}, 0, 0, 0, \quad \frac{q_x H}{4}, \frac{q_y H}{4}, \frac{q_z H}{4}, \frac{q_x H}{4}, \frac{q_y H}{4}, \frac{q_z H}{4} \right] \tag{C19}$$

$$\boldsymbol{p_0}^{(tr)} = \left[ \frac{5 q_x H}{24}, \frac{5 q_y H}{24}, \frac{5 q_z H}{24}, 0, 0, 0, \quad \frac{q_x H}{24}, \frac{q_y H}{24}, \frac{q_z H}{24}, 0, 0, 0, \quad \frac{q_x H}{6}, \frac{q_y H}{6}, \frac{q_z H}{6}, \frac{q_x H}{12}, \frac{q_y H}{12}, \frac{q_z H}{12} \right] \tag{C20}$$

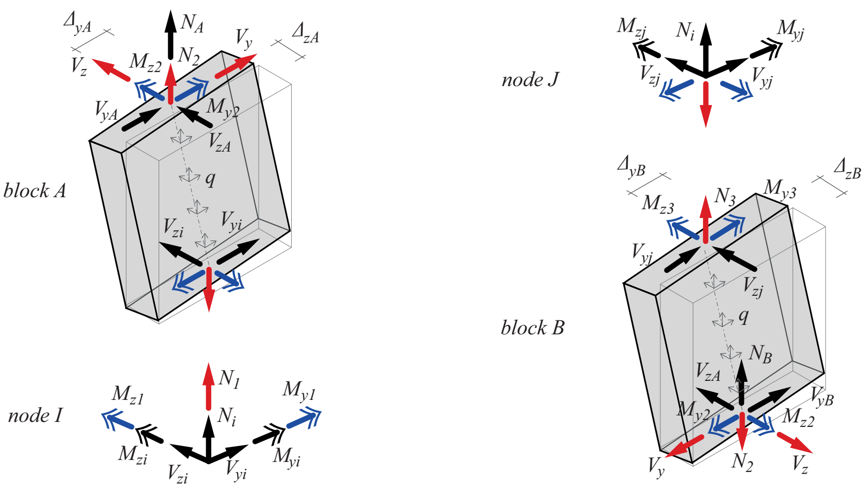


**FIGURE C1** Derivation of local forces from equilibrium conditions applied in the deformed configuration.

# D NONLINEAR ELASTIC FLEXURAL MODEL

The flexural interface described in Section 3, if a no-tension material, linear elastic in compression, is postulated, is essentially a nonlinear elastic model. It is governed by the position of the neutral axis in the section, and the consequent variation of the integration domain of the strain field defined in Equation 16. In addition to the elastic solution solution provided in Equation 18 for a fully compressed section, and the null solution for the case in which the section is subjected to only traction, four cases need to be defined (Figure 3). Solutions of case 1, 2 and 4 are provided in Equations D21-D23. The solution of case 3, for a triangular integration domain, can be obtained by subtraction of the solution of case 2 (Equation D22) from the linear elastic case (Equation 18).

$$\begin{cases} N_{el} &= -\dfrac{Et}{24|\chi_z|}\left[t^2\chi_y^2 + 3(L|\chi_z| - 2\varepsilon_0)^2\right] \\ M_{z,el} &= \dfrac{Et}{24\chi_z|\chi_z|}\left(4\varepsilon_0^3 + t^2\varepsilon_0\chi_y^2 - 3L^2\varepsilon_0\chi_z^2 + L^3|\chi_z^3|\right) \\ M_{y,el} &= \dfrac{Et}{24|\chi_z|}\chi_y t^2(L|\chi_z| - 2\varepsilon_0) \end{cases} \quad \text{(case 1)} \tag{D21}$$

$$\begin{cases} N_{el} &= -\dfrac{E}{48|\chi_y\chi_z|}\Big(-2\varepsilon_0 + t|\chi_y| + L|\chi_z|\Big)^3 \\ M_{z,el} &= \dfrac{E}{384|\chi_y\chi_z|\chi_z}\Big(-2\varepsilon_0 + t|\chi_y| - 3L|\chi_z|\Big)\Big(-2\varepsilon_0 + t|\chi_y| + L|\chi_z|\Big)^3 \\ M_{y,el} &= \dfrac{E}{384|\chi_y\chi_z|\chi_y}\Big(2\varepsilon_0 + 3t|\chi_y| - L|\chi_z|\Big)\Big(-2\varepsilon_0 + t|\chi_y| + L|\chi_z|\Big)^3 \end{cases} \quad \text{(case 2)} \tag{D22}$$

$$\begin{cases} N_{el} &= -\dfrac{EL}{24|\chi_y|}\left[L^2\chi_z^2 + 3(t|\chi_y| - 2\varepsilon_0)^2\right] \\ M_{z,el} &= \dfrac{EL}{24|\chi_y|}\chi_z L^2(t|\chi_y| - 2\varepsilon_0) \\ M_{y,el} &= \dfrac{EL}{24\chi_y|\chi_y|}\left[(\varepsilon_0 + t|\chi_y|)(2\varepsilon_0 - t|\chi_y|)^2 + \varepsilon_0 L^2\chi_z^2\right] \end{cases} \quad \text{(case 4)} \tag{D23}$$

# E CONSISTENT MASS MATRIX

A consistent formulation of the mass matrix can be derived by imposing that the kinetic energy of the element with distributed mass and the element with lumped mass be equal; the kinematic energy is defined as a function of the velocity vector $\dot{\boldsymbol{u}}(x)$ of every point $x$ along the element axis and the mass associated to each point as:

$$E_{kin} = \frac{1}{2}\rho A \int_0^H \dot{\boldsymbol{u}}(x)^T \dot{\boldsymbol{u}}(x) dx \tag{E24}$$

In Equation E24 it is postulated that the mass is equally applied in all directions, and that all mass is concentrated along the element axis, since in the OpenSees implementation the element has no information on the actual geometry of the section. Such distributed mass is expressed by the quantity $\rho A$ (i.e. mass per unit length), constant along the element height. For gable elements, the latter hypothesis does not hold, and a lumped matrix as in Equation E28 can be applied.

The velocity vector $\dot{\boldsymbol{u}}(x)$ can be derived as a function of the nodal displacements and a shape function $\boldsymbol{N_A}(x)$ defined for every point $x$. For simplicity of notation, the point position can be defined by a normalised coordinate $\xi_A$ or $\xi_B$ for block A and B (Figure E2 a). The expressions of such shape functions are rather simple, since the macro-element is composed by rigid blocks with linear displacement fields. For block A, the function $\boldsymbol{N_A}(\xi_A)$ can be defined as:

$$\dot{\boldsymbol{u}}(\xi_A) = \begin{bmatrix} 1-\xi_A & 0 & 0 & 0\ 0\ 0 & & \xi_A & 0 & 0 & 0\ 0\ 0 \\ 0 & 1-\xi_A & 0 & 0\ 0\ 0 & [\boldsymbol{0}]_{\boldsymbol{3x6}} & 0 & \xi_A & 0 & 0\ 0\ 0 \\ 0 & 0 & 1-\xi_A & 0\ 0\ 0 & & 0 & 0 & \xi_A & 0\ 0\ 0 \end{bmatrix} \dot{\boldsymbol{u}}_{\boldsymbol{local}} = \boldsymbol{N_A}(\xi_A)\, \dot{\boldsymbol{u}}_{\boldsymbol{local}} \tag{E25}$$

Substituting this expression in Equation E24, the kinetic energy of block A can be expressed as:

$$E_{kin,A} = \frac{1}{2}\dot{\boldsymbol{u}}_{\boldsymbol{local}}{}^T \left[\rho A \frac{H}{2} \int_0^1 \boldsymbol{N_A}(\xi_A)^T\ \boldsymbol{N_A}(\xi_A)\ d\xi_A\right] \dot{\boldsymbol{u}}_{\boldsymbol{local}} = \frac{1}{2}\dot{\boldsymbol{u}}_{\boldsymbol{local}}{}^T \boldsymbol{M_{local,A}} \dot{\boldsymbol{u}}_{\boldsymbol{local}} \tag{E26}$$

Solving the integral and summing the contributions of both rigid blocks A and B, the consistent mass matrix of the macro-element can be therefore derived as:

$$\boldsymbol{M_{local}}^{(cons.)} = \rho A H \begin{bmatrix} \frac{1}{6}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{12}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{0}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{1}{6}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{12}[\boldsymbol{I}]_{\boldsymbol{3x3}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{0}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} \\ \frac{1}{12}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{6}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{1}{12}[\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{6}[\boldsymbol{I}]_{\boldsymbol{3x3}} \end{bmatrix} \tag{E27}$$

Since for some explicit analyses the performance of the solver can be significantly improved by the use of a diagonal mass matrix, as an additional option available to the user, a lumped mass formulation was implemented. One criterion that may be applied for determining an optimal scheme of mass lumping is the equivalence of the inertial forces originating from a constant acceleration profile along the macro-element, between the two cases of distributed (Figure E2 b) and concentrated masses (Figure E2 c). If a limit analysis approach is used to determine the capacity of two overturning mechanisms, for both cantilever elements and walls spanning between floor/ceiling supports, such equivalence can be expressed in terms of the load multiplier corresponding to the activation of the mechanisms. This condition is satisfied for both sets of boundary conditions when half of the element mass is assigned to the central node (Equation E28), which is the standard approach implemented in OpenSees.

$$\boldsymbol{M_{local}}^{(lumped,pier)} = \frac{1}{4}\rho A H \begin{bmatrix} [\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{0}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{0}]_{\boldsymbol{3x3}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{I}]_{\boldsymbol{3x3}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{I}]_{\boldsymbol{3x3}} \end{bmatrix} \tag{E28}$$

If the element is defined as gable element, following the same approach, a lumped mass matrix can be defined as:.

$$\boldsymbol{M_{local}}^{(lumped,gable)} = \rho A H \begin{bmatrix} \frac{5}{12}[\boldsymbol{I}]_{3x3} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{0}]_{3x3} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{1}{12}[\boldsymbol{I}]_{3x3} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & [\boldsymbol{0}]_{3x3} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{3}[\boldsymbol{I}]_{3x3} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{1}{6}[\boldsymbol{I}]_{3x3} \end{bmatrix} \tag{E29}$$

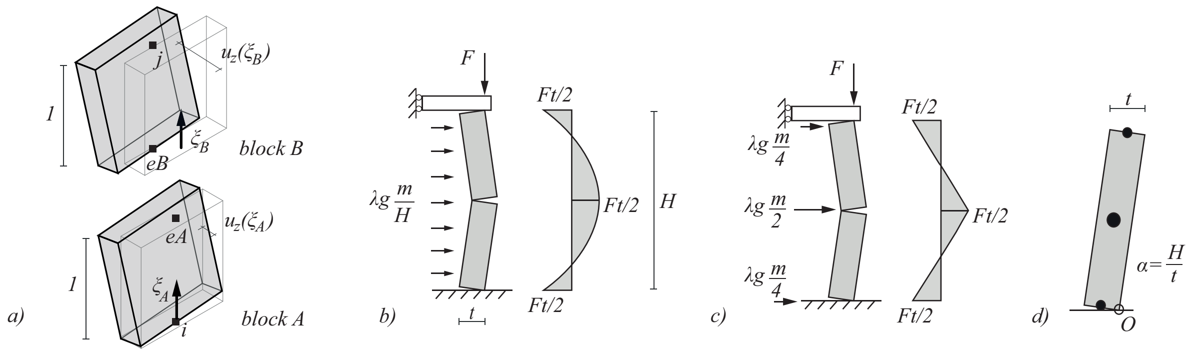


**FIGURE E2** ”Shape functions” for derivation of a consistent mass matrix (a). Equivalent limit analysis solutions for a wall spanning between floor levels with distributed mass (b) and lumped masses (c). Cantilever overturning mechanism with lumped masses (d).

However, the mass matrix in Equation E28 does not yield the exact rotational moment of inertia for a cantilever overturning mechanism such as the one in Figure E2 d. Comparing the exact moment of inertia $I_O$, relative to a rotation around the point $O$, to the solutions of the consistent (Equation E27) and lumped mass matrix (Equation E28), the error can be computed as a function of the height to thickness ratio $\alpha = H/t$ of the element:

$$\frac{I_O{}^{(cons.)}}{I_O} = 1 - \frac{1}{2}\frac{1}{1+\alpha^2} \tag{E30}$$

$$\frac{I_O{}^{(cons.)}}{I_O} = \frac{9}{8}\left(1 - \frac{1}{2}\frac{1}{1+\alpha^2}\right) \tag{E31}$$

For the consistent formulation the only error introduced depends on the assumption of concentrating the mass along the element axis; it remains however reasonably small for common slenderness ratios. The lumped mass matrix introduces a further error of a factor 9/8. Although for this particular mechanism the error would be reduced if the central node were assigned 2/3 of the mass instead of 1/2, the matrix in Equation E28 is still preferred as a standard option since, as discussed, it provides exact results in terms of load multiplier under constant lateral acceleration.