Elastic Rod Scissor Linkages

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We describe how to model the static equilibria of linkages formed by elastic rods connecting at scissor joints. Our linkage model builds on the popular discrete elastic rods model of [1].

1 Rod Linkage Graph

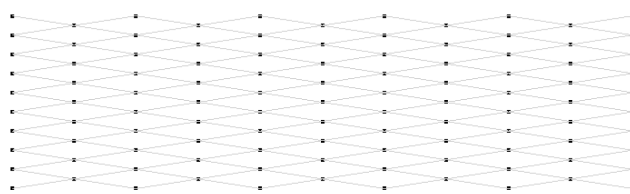


Figure 1 *An example graph defining a rod linkage.*

The linkage's initial configuration is defined by an embedded graph (i.e., line mesh) with vertices and edges (V, E). Each edge of this graph is referred to as a rod *segment*, and generally these segments continue across the vertices to form a complete rod as shown in Figure 1.

Each graph vertex represents either a scissor joint or a rod's free end. Vertex valences can be one (free end), two (two rod ends pinned together), three (one rod's end pinned to another's interior), or four (two rods pinned together in their interior). We prohibit valences above four. Figure 1 shows examples of each of these valences.

Each rod segment is modeled as a *distinct* discrete elastic rod with n_s subdivisions. We will see how to properly couple the segments making up a full rod so that they behave just like one large elastic rod. We label the n^{th} segment s_n for $n \in \{0 \dots |E| - 1\}$ and the joints $\jmath_i$ for $i \in \{0 \dots |J| - 1\}$, where $J \subset V$ is the set of vertices of valence 2–4.

2 Rod and Joint Representation

Recall that a discrete elastic rod's configuration is defined by its centerline positions and its material frame angles (expressed relative to the rod's hidden reference frame state). The material frame is used to express the orientation/twist of the rod's cross sections, which is particularly important for flat, anisotropic rods.

The rod segments meeting at a joint can be partitioned into two sets, one for each full rod passing through the joint. This partitioning is done by determining which segments connect to form the straightest path across the vertex in the rest configuration. We need to glue together the segments making up a rod and allow the two incident rods pivot around the vertex. We accomplish this gluing by having the joint impose the same terminal edge vector for the segments connecting to form a single rod. Also, the joint constrains the orientation of all incident segments' centerlines: the second material axis $\mathbf{d}_2$ must be normal to the plane spanned by both incident centerline edges. See Figure 2 for an example joint configuration.

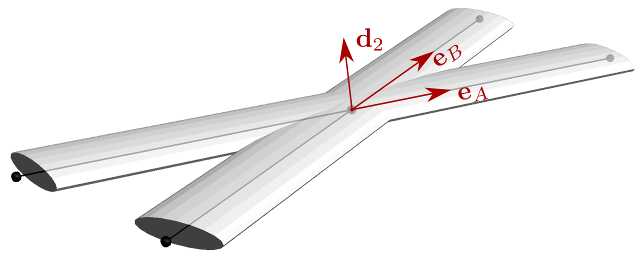


Figure 2 *The geometry of rod scissor linkages. Here we visualize the linkage graph for a single scissor linkage (comprising four rod segments) and the corresponding rod geometry. The joint parameters determine the edge vectors and material frame for the terminal edges of all incident rod segments.*

3 Linkage Degrees of Freedom

Our rod linkage model incorporates the constraints imposed by the joints by constructing a reduced set of variables that parametrize all admissible rod configurations. These reduced variables consist of (in order):

- For each rod segment $s \in E$:
  - Centerline positions for all interior and free-end nodes of s. The x, y, and z coordinates of the first interior/free node come first, followed by the coordinates for all subsequent nodes.
  - Material frame angles θ for all interior and free-end edges of s.
- Parameters for each joint $\jmath \in J$: position and edge vectors $\mathbf{p}, \mathbf{e}_A, \mathbf{e}_B \in \mathbb{R}^3$ flattened in x, y, z order.

We implement two types of joints, constrained and unconstrained. An unconstrained joint is specified by two vectors, indicating the direction and length of the two centerline edges pivoting around the vertex. A constrained joint has a fixed direction for both of these centerline edges, and only their lengths are variables. In both cases, the joint parameters completely determine the degrees of freedom (centerline positions and θs) for the terminal edges of all rod segments incident the joint. See Figure 2 for a visualization of how the joint's degrees of freedom $\mathbf{e}_A$ and $\mathbf{e}_B$ determine the geometry of the incident rod edges. The material axis angle θ for a given incident rod segment edge is determined from that edge's current reference frame $(\underline{\mathbf{d}_1}, \underline{\mathbf{d}_2})$ and the relationship:

$$\mathbf{d}_2 = \mathbf{n} := \frac{\mathbf{e}_A \times \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|}.$$

4 Elastic Energy, Gradients, and Hessians

Our model's joints store no energy, so the elastic energy of the full linkage is computed by summing the energy from each rod segment. We can then compute derivatives of this energy with respect to our degrees of freedom by assembling the (transformed) derivatives of the discrete elastic rod model.

4.1 Gradients

Since $\mathbf{e}_A$ and $\mathbf{e}_B$ directly determine the segments' incident edge vectors, it's straightforward to compute by chain rule the gradient contribution due to the changing centerline. The only challenging part is computing with the contributions from the changing material frame angles induced by the constraint $\mathbf{d}_2 = \mathbf{n}$. For this, we use chain rule and the relationship:

$$\left(\frac{\partial \theta^j}{\partial \mathbf{e}_A}\right)^T = -\mathbf{d}_1^j \cdot \frac{\partial \mathbf{n}}{\partial \mathbf{e}_A} + \underline{\mathbf{d}_1^j} \cdot \frac{\partial P_{\widehat{\mathbf{t}}^j}^{\mathbf{t}^j} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{e}_A}$$

$$\left(\frac{\partial \theta^j}{\partial \mathbf{e}_B}\right)^T = -\mathbf{d}_1^j \cdot \frac{\partial \mathbf{n}}{\partial \mathbf{e}_B} + \underline{\mathbf{d}_1^j} \cdot \frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}^j} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{e}_B}.$$

The first terms capture the rotation of $\mathbf{d}_2 = \mathbf{n}$, and the second subtract off the rotation of the reference frame due to parallel transport. Note that, for a given terminal edge "j," only one of the second terms will appear, depending on whether the edge is part of rod A or B.

We already obtained an expression for the parallel transport term when deriving the finite transport twist energy gradient for discrete elastic rods:

$$\underline{\mathbf{d}_1^j} \cdot \frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}^j} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{e}_j} = \frac{I - \mathbf{t}^j \otimes \mathbf{t}^j}{\|\mathbf{e}^j\|} \left(\frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_1^j} \times \widehat{\mathbf{t}^j}\big) + \big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)\widehat{\underline{\mathbf{d}_1^j}}}{1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j} - \frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)}{(1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j)^2} \widehat{\mathbf{t}^j} + \widehat{\underline{\mathbf{d}_1^j}} \times \underline{\mathbf{d}_1^j} \right).$$

We compute the normal rotation term as follows:

$$-\mathbf{d}_1^j \cdot \frac{\partial \mathbf{n}}{\partial \mathbf{e}_A} = -\mathbf{d}_1^j \cdot \frac{(I - \cancelto{\perp}{\mathbf{n}} \otimes \mathbf{n})[\mathbf{e}_B]_\times^T}{\|\mathbf{e}_A \times \mathbf{e}_B\|} = \frac{\left(\mathbf{d}_1^j \times \mathbf{e}_B\right)^T}{\|\mathbf{e}_A \times \mathbf{e}_B\|} = -\frac{\mathbf{t}^j \cdot \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|}\mathbf{n}^T.$$

$$-\mathbf{d}_1^j \cdot \frac{\partial \mathbf{n}}{\partial \mathbf{e}_B} = -\mathbf{d}_1^j \cdot \frac{(I - \cancelto{\perp}{\mathbf{n}} \otimes \mathbf{n})[\mathbf{e}_A]_\times}{\|\mathbf{e}_A \times \mathbf{e}_B\|} = \frac{\left(\mathbf{e}_A \times \mathbf{d}_1^j\right)^T}{\|\mathbf{e}_A \times \mathbf{e}_B\|} = \frac{\mathbf{t}^j \cdot \mathbf{e}_A}{\|\mathbf{e}_A \times \mathbf{e}_B\|}\mathbf{n}^T.$$

Therefore:

$$\frac{\partial \theta^j}{\partial \mathbf{e}_A} = -\frac{\mathbf{t}^j \cdot \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|}\mathbf{n} + s_{jA}\frac{(I - \mathbf{t}_A \otimes \mathbf{t}_A)}{\|\mathbf{e}_A\|} \left(\frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_1^j} \times \widehat{\mathbf{t}^j}\big) + \big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)\widehat{\underline{\mathbf{d}_1^j}}}{1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j} - \frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)}{(1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j)^2} \widehat{\mathbf{t}^j} + \widehat{\underline{\mathbf{d}_1^j}} \times \underline{\mathbf{d}_1^j} \right)$$

$$\frac{\partial \theta^j}{\partial \mathbf{e}_B} = \frac{\mathbf{t}^j \cdot \mathbf{e}_A}{\|\mathbf{e}_A \times \mathbf{e}_B\|}\mathbf{n} + s_{jB}\frac{(I - \mathbf{t}_B \otimes \mathbf{t}_B)}{\|\mathbf{e}_B\|} \left(\frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_1^j} \times \widehat{\mathbf{t}^j}\big) + \big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)\widehat{\underline{\mathbf{d}_1^j}}}{1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j} - \frac{\big(\widehat{\underline{\mathbf{d}_1^j}} \cdot \mathbf{t}^j\big)\big(\underline{\mathbf{d}_2^j} \cdot \widehat{\mathbf{t}^j}\big)}{(1 + \widehat{\mathbf{t}^j} \cdot \mathbf{t}^j)^2} \widehat{\mathbf{t}^j} + \widehat{\underline{\mathbf{d}_1^j}} \times \underline{\mathbf{d}_1^j} \right),$$

where

$$s_{jX} = \begin{cases} 0 & \text{if terminal edge } j \text{ isn't part of rod } X, \\ 1 & \text{if terminal edge } j\text{'s orientation agrees with joint edge vector } \mathbf{e}_X, \text{ or} \\ -1 & \text{if terminal edge } j\text{'s orientation disagrees with joint edge vector } \mathbf{e}_X. \end{cases}$$

4.2 Hessian

Next, we compute the Hessian with respect to our reduced variables. We express the elastic energy as

$$E(\mathbf{v}(\mathbf{r})),$$

where vectors $\mathbf{v}$ and $\mathbf{r}$ collect the linkage's full and reduced variables. So $\mathbf{v}$ contain the centerline positions $\mathbf{x}$ and material frame angles θ for all rod segments, and $\mathbf{r}$ contains the unconstrained centerline positions and material frame angles as well as the joint parameters $\mathbf{e}_A$ and $\mathbf{e}_B$ controlling the constrained segment variables. The gradient and Hessian with respect to the reduced variables can now be computed using the chain rule:

$$\frac{\partial E}{\partial r_i} = \frac{\partial E}{\partial v_k}(\mathbf{v}(\mathbf{r}))\frac{\partial v_k}{\partial r_i}$$

$$\frac{\partial^2 E}{\partial r_i \partial r_j} = \frac{\partial v_k}{\partial r_i}\frac{\partial^2 E}{\partial v_k \partial v_l}\frac{\partial v_l}{\partial r_j} + \frac{\partial E}{\partial v_k}\frac{\partial^2 v_k}{\partial r_i \partial r_j}.$$

For all the unconstrained variables (segments' interior/free end quantities), $\frac{\partial v_k}{\partial r_i}$ is essentially just a Kronecker delta (1 if reduced variable r_i corresponds to unconstrained variable v_k, 0 otherwise). The Hessians of these

v_k are zero, and all that remains is the first term, which is just a permutation of the sub-block of the unreduced Hessian corresponding to the unconstrained variables. We implement this by simply re-indexing the sparse matrix entries.

The terminal edge centerline positions are simply linear functions of the joint parameters, so again the term involving the Hessian of v_k vanishes. Note: these rows/columns of $\frac{\partial v_k}{\partial r_i}$ have multiple nonzero entries. Finally, the terminal edge material frame angles are nonlinear functions of the joint parameters, so for these we do need to compute second Hessian term:

$$\frac{\partial E}{\partial \theta^j} \frac{\partial^2 \theta^j}{\partial \mathbf{e}_X \partial \mathbf{e}_Y}.$$

Like when we derived the elastic rod energy Hessians, we simplify our expressions by evaluating the Hessian only at the configuration from which the source reference frame was set.

$$\begin{aligned}
\frac{\partial^2 \theta^j}{\partial \mathbf{e}_A \partial \mathbf{e}_A} &= -\frac{\mathbf{n}}{\|\mathbf{e}_A \times \mathbf{e}_B\|} \otimes \underbrace{\left(\frac{I - \mathbf{t}_A \otimes \mathbf{t}_A}{\|\mathbf{e}_A\|} \mathbf{e}_B\right) s_{jA}}_{\|\mathbf{e}_A \times \mathbf{e}_B\| \frac{\mathbf{n} \times \mathbf{t}^j}{\|\mathbf{e}_A\|^2} \delta_{jA}} + \frac{\mathbf{t}^j \cdot \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2} \left(I - 2\mathbf{n} \otimes \mathbf{n}\right) [\mathbf{e}_B]_\times + s_{jA}^2 \frac{[\mathbf{t}^j]_\times}{2\|\mathbf{e}^j\|^2} \\
&= -\frac{\mathbf{n} \otimes (\mathbf{n} \times \mathbf{t}^j)}{\|\mathbf{e}_A\|^2} \delta_{jA} + \frac{\mathbf{t}^j \cdot \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2} \left(I - 2\mathbf{n} \otimes \mathbf{n}\right) [\mathbf{e}_B]_\times + \delta_{jA} \frac{[\mathbf{t}^j]_\times}{2\|\mathbf{e}_A\|^2} \\
&= -\frac{\mathbf{d}_2^j \otimes \mathbf{d}_1^j + \mathbf{d}_1^j \otimes \mathbf{d}_2^j}{2\|\mathbf{e}_A\|^2} \delta_{jA} + \frac{\mathbf{t}^j \cdot \mathbf{e}_B}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2} \left(I - 2\mathbf{d}_2^j \otimes \mathbf{d}_2^j\right) [\mathbf{e}_B]_\times,
\end{aligned}$$

where $\delta_{jX} := s_{jX}^2$ is 1 if joint parameter $\mathbf{e}_A$ controls terminal edge j and 0 otherwise.

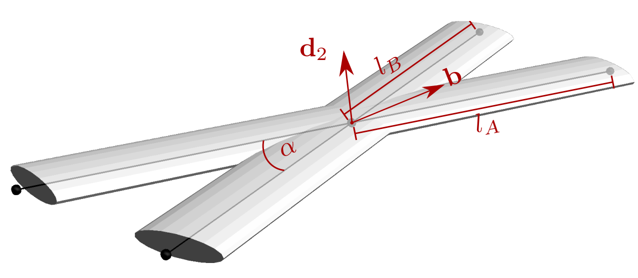
Note, the last step of this derivation can be interpreted either as symmetrizing this Hessian block or plugging in the identity $[\mathbf{t}^j]_\times = \mathbf{d}_2^j \otimes \mathbf{d}_1^j - \mathbf{d}_1^j \otimes \mathbf{d}_2^j$. Also, we used the fact $s_{jA}\mathbf{t}_A = \mathbf{t}^j$. Likewise, we find:

$$\begin{aligned}
\frac{\partial^2 \theta^j}{\partial \mathbf{e}_B \partial \mathbf{e}_B} &= \frac{\mathbf{n}}{\|\mathbf{e}_A \times \mathbf{e}_B\|} \otimes \underbrace{\left(\frac{I - \mathbf{t}_B \otimes \mathbf{t}_B}{\|\mathbf{e}_B\|} \mathbf{e}_A\right) s_{jB}}_{-\|\mathbf{e}_A \times \mathbf{e}_B\| \frac{\mathbf{n} \times \mathbf{t}^j}{\|\mathbf{e}_B\|^2} \delta_{jB}} + \frac{\mathbf{t}^j \cdot \mathbf{e}_A}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2} \left(I - 2\mathbf{n} \otimes \mathbf{n}\right) [\mathbf{e}_A]_\times + s_{jB}^2 \frac{[\mathbf{t}^j]_\times}{2\|\mathbf{e}^j\|^2} \\
&= -\frac{\mathbf{d}_2^j \otimes \mathbf{d}_1^j + \mathbf{d}_1^j \otimes \mathbf{d}_2^j}{2\|\mathbf{e}_B\|^2} \delta_{jB} + \frac{\mathbf{t}^j \cdot \mathbf{e}_A}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2} \left(I - 2\mathbf{d}_2^j \otimes \mathbf{d}_2^j\right) [\mathbf{e}_A]_\times.
\end{aligned}$$

Finally, we compute the cross terms:

$$\left(\frac{\partial^2 \theta^j}{\partial \mathbf{e}_B \partial \mathbf{e}_A}\right)^T = \frac{\partial^2 \theta^j}{\partial \mathbf{e}_A \partial \mathbf{e}_B} = -\frac{\mathbf{n} \otimes \mathbf{t}^j}{\|\mathbf{e}_A \times \mathbf{e}_B\|} - \frac{(\mathbf{t}^j \cdot \mathbf{e}_B)(I - 2\mathbf{n} \otimes \mathbf{n})[\mathbf{e}_A]_\times}{\|\mathbf{e}_A \times \mathbf{e}_B\|^2}.$$

5 Rotation-based Joint Representation



The parameterization used above for the joints, though simplifying the dependence of rod state on the joint parameters, has several drawbacks. First, it will become ill-conditioned/degenerate as $\mathbf{e}_A$ and $\mathbf{e}_B$

become nearly parallel. Second, the joint's orientation is coupled with the opening angle, so it's no possible to pin down rigid motion by constraining a joint without also fixing the opening angle. Third, the opening angle is not an explicit parameter in the system, making it difficult to formulate more global deployment actuations (writing energies/constraints in terms of the angles at all joints).

We therefore opt to parametrize each joint by a position, $\mathbf{p}$, a global rotation/frame (represented as an angle-scaled axis vector, $\boldsymbol{\omega}$), an opening angle α between rods A and B, and the deformed length of the edge vectors. Since a joint can be constrained by fixing its $\mathbf{p}$, $\boldsymbol{\omega}$, and α variables, we no longer need to use a separate parametrization for "constrained joints" as described above; we can impose joint constraints just by fixing variables in the optimization.

The joint's incident edge vectors $\mathbf{e}_A$ and $\mathbf{e}_B$ and joint normal are then:

$$\mathbf{e}_A = R(\boldsymbol{\omega})\hat{\mathbf{t}}_A l_A, \quad \mathbf{e}_B = R(\boldsymbol{\omega}) \underbrace{\Big(\hat{\mathbf{t}}_A \cos(\alpha) + (\hat{\mathbf{n}} \times \hat{\mathbf{t}}_A)\sin(\alpha)\Big)}_{:=\hat{\mathbf{t}}_B(\alpha)} l_B, \quad \mathbf{n} = R(\boldsymbol{\omega})\hat{\mathbf{n}},$$

where $\hat{\mathbf{t}}_A$ and $\hat{\mathbf{n}}$ are the "source"/initial rod A tangent and joint normal vectors (also stored in each joint). These can be interpreted as defining a reference rotation matrix $R_0 := \left(\ \hat{\mathbf{t}}_A \ \middle|\ \hat{\mathbf{n}} \times \hat{\mathbf{t}}_A \ \middle|\ \hat{\mathbf{n}}\ \right)$, so that $\boldsymbol{\omega}$ is a vector in the tangent space of $SO(3)$ at R_0 (see the documentation in the `RotationOptimization` repository).

As before, the joint's normal $\mathbf{n}$ determines the material axis $\mathbf{d}_2$ for all incident rod edges (controlling their θ parameters). The edge vectors $\mathbf{e}_A$ and $\mathbf{e}_B$ control the centerline positions of the incident rod edges, placing them at, e.g., $\mathbf{p} \pm \frac{1}{2}\mathbf{e}_A$. The *nonzero* blocks of the gradient of these edge vectors and θ^j are:

$$\begin{aligned}
\frac{\partial \mathbf{e}_A}{\partial \boldsymbol{\omega}} &= \frac{\partial (R(\boldsymbol{\omega})\hat{\mathbf{t}}_A)}{\partial \boldsymbol{\omega}} l_A, \quad \frac{\partial \mathbf{e}_A}{\partial l_A} = R(\boldsymbol{\omega})\hat{\mathbf{t}}_A, \\
\frac{\partial \mathbf{e}_B}{\partial \boldsymbol{\omega}} &= \frac{\partial \left(R(\boldsymbol{\omega})\hat{\mathbf{t}}_B(\alpha)\right)}{\partial \boldsymbol{\omega}} l_B, \quad \frac{\partial \mathbf{e}_B}{\partial \alpha} = R(\boldsymbol{\omega}) \underbrace{\Big(- \hat{\mathbf{t}}_A \sin(\alpha) + (\hat{\mathbf{n}} \times \hat{\mathbf{t}}_A)\cos(\alpha)\Big)}_{:=\hat{\mathbf{t}}_B^{\perp}(\alpha)} l_B, \quad \frac{\partial \mathbf{e}_B}{\partial l_B} = R(\boldsymbol{\omega})\hat{\mathbf{t}}_B(\alpha)
\end{aligned}$$

$$\left(\frac{\partial \theta^j}{\partial \boldsymbol{\omega}}\right)^T = -\mathbf{d}_1^j \cdot \frac{\partial \mathbf{n}}{\partial \boldsymbol{\omega}} + \underline{\mathbf{d}_1^j} \cdot \left(s_{jA} \frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}_A} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{t}_A} \frac{\partial \mathbf{t}_A}{\partial \boldsymbol{\omega}} + s_{jB} \frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}_B} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{t}_B} \frac{\partial \mathbf{t}_B}{\partial \boldsymbol{\omega}} \right), \quad \frac{\partial \theta^j}{\partial \alpha} = s_{jB} \underline{\mathbf{d}_1^j} \cdot \left(\frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}_B} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{t}_B} \frac{\partial \mathbf{t}_B}{\partial \alpha} \right),$$

where the terms involving $\underline{\mathbf{d}_1^j}$ subtract off the rotation of the reference frame due to parallel transport. Note that this parallel transport term vanishes if the source frame has been updated. The terms here like $\frac{\partial (R(\boldsymbol{\omega})\hat{\mathbf{t}}_A)}{\partial \boldsymbol{\omega}}$ can be evaluated with the gradient-of-rotated-vector function provided by `RotationOptimization`. The Hessian blocks of the edge vectors are:

$$\boxed{\begin{aligned}
\frac{\partial^2 \mathbf{e}_A}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} &= \frac{\partial^2 (R(\boldsymbol{\omega})\hat{\mathbf{t}}_A)}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} l_A, \quad \frac{\partial^2 \mathbf{e}_A}{\partial \boldsymbol{\omega} \partial l_A} = \frac{\partial (R(\boldsymbol{\omega})\hat{\mathbf{t}}_A)}{\partial \boldsymbol{\omega}}, \\
\frac{\partial^2 \mathbf{e}_B}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} &= \frac{\partial^2 \left(R(\boldsymbol{\omega})\hat{\mathbf{t}}_B(\alpha)\right)}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} l_B, \quad \frac{\partial^2 \mathbf{e}_B}{\partial \boldsymbol{\omega} \partial l_B} = \frac{\partial \left(R(\boldsymbol{\omega})\hat{\mathbf{t}}_B(\alpha)\right)}{\partial \boldsymbol{\omega}}, \quad \frac{\partial^2 \mathbf{e}_B}{\partial \boldsymbol{\omega} \partial \alpha} = \frac{\partial \left(R(\boldsymbol{\omega})\hat{\mathbf{t}}_B^{\perp}(\alpha)\right)}{\partial \boldsymbol{\omega}} l_B, \\
\frac{\partial^2 \mathbf{e}_B}{\partial l_B \partial \alpha} &= R(\boldsymbol{\omega})\hat{\mathbf{t}}_B^{\perp}(\alpha), \quad \frac{\partial^2 \mathbf{e}_B}{\partial \alpha \partial \alpha} = -R(\boldsymbol{\omega})\hat{\mathbf{t}}_B(\alpha) l_B = -\mathbf{e}_B
\end{aligned}}$$

To simplify the Hessian of θ^j, we assume that the source frame has been updated (i.e., we evaluate at $\mathbf{t}^j = \widehat{\mathbf{t}^j}$). Recall the derivative with respect to the edge tangent of (minus) the reference frame rotation:

$$\frac{\partial}{\partial \mathbf{t}^j}\bigg|_{\mathbf{t}^j = \widehat{\mathbf{t}^j}} \left[\underline{\mathbf{d}_1^j} \cdot \left(\frac{\partial P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}^j} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{t}^j} \right) \right] = (\underline{\mathbf{d}_2^j} \otimes - \mathbf{t}^j) \cdot (-\mathbf{t}^j \otimes \underline{\mathbf{d}_1^j}) + \mathbf{d}_1^j \cdot \frac{\partial^2 P_{\widehat{\mathbf{t}^j}}^{\mathbf{t}^j} \widehat{\underline{\mathbf{d}_2^j}}}{\partial \mathbf{t}^j \partial \mathbf{t}^j}\Bigg|_{\mathbf{t}^j = \widehat{\mathbf{t}^j}} = \underline{\mathbf{d}_2^j} \otimes \underline{\mathbf{d}_1^j} - \frac{\underline{\mathbf{d}_2^j} \otimes \underline{\mathbf{d}_1^j} + \underline{\mathbf{d}_1^j} \otimes \underline{\mathbf{d}_2^j}}{2} = \frac{[\mathbf{t}^j]_\times}{2}.$$

$$\boxed{\begin{aligned}
\frac{\partial^2 \theta^j}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} &= -\left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\omega}}\right)^T \frac{\partial \mathbf{d}_1^j}{\partial \boldsymbol{\omega}} - \mathbf{d}_1^j \cdot \frac{\partial^2 \mathbf{n}}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}} + \delta_{jA} \left(\frac{\partial \mathbf{t}_A}{\partial \boldsymbol{\omega}}\right)^T \frac{[\mathbf{t}^j]_\times}{2} \frac{\partial \mathbf{t}_A}{\partial \boldsymbol{\omega}} + \delta_{jB} \left(\frac{\partial \mathbf{t}_B}{\partial \boldsymbol{\omega}}\right)^T \frac{[\mathbf{t}^j]_\times}{2} \frac{\partial \mathbf{t}_B}{\partial \boldsymbol{\omega}} \\
&= -\operatorname{sym}\left(\left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\omega}}\right)^T \frac{\partial \mathbf{d}_1^j}{\partial \boldsymbol{\omega}}\right) - \mathbf{d}_1^j \cdot \frac{\partial^2 \mathbf{n}}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}}, \\
\frac{\partial^2 \theta^j}{\partial \alpha \partial \boldsymbol{\omega}} &= \delta_{jB} \left(\frac{\partial \mathbf{t}_B}{\partial \alpha}\right)^T \frac{[\mathbf{t}^j]_\times}{2} \frac{\partial \mathbf{t}_B}{\partial \boldsymbol{\omega}} = -\frac{\delta_{jB}}{2} \left(\frac{\partial \mathbf{t}_B}{\partial \boldsymbol{\omega}}\right)^T \left(\mathbf{t}_B \times \frac{\partial \mathbf{t}_B}{\partial \alpha}\right), \\
\frac{\partial^2 \theta^j}{\partial \alpha \partial \alpha} &= \delta_{jB} \left(\frac{\partial \mathbf{t}_B}{\partial \alpha}\right)^T \frac{[\mathbf{t}^j]_\times}{2} \frac{\partial \mathbf{t}_B}{\partial \alpha} = 0.
\end{aligned}}$$

References

[1] Miklós Bergou, Basile Audoly, Etienne Vouga, Max Wardetzky, and Eitan Grinspun. Discrete viscous threads. *ACM Trans. Graph.*, 29(4):116:1–116:10, July 2010.