Analysis of Dynamic Systems

Control Systems

Fall 2023

Summary for modeling of dynamic systems

- A mathematical model gives the relation between output $y(t)$ and input $u(t)$ of a dynamic system : $y(t) = \mathcal{F}(u(\tau)) \quad \tau \leq t$
- This relation is usually given by differential equations.
- Transfer function of linear time-invariant systems with input signal $u(t)$ and output signal $y(t)$ is defined as :

$$\boxed{G(s) = \frac{Y(s)}{U(s)}}$$

  where all initial conditions are taken equal to zero.
- The nonlinear systems can be linearized around the operating point. The linear model is valid only for small variations around the operating point.

Analysis of Dynamic Systems

Objective :

Characterise the output (the response) of a linear time-invariant system to a given input signal.

There are several types of solution :

- Convolution technique, impulse response
- Laplace and inverse Laplace transform
- Solving numerically the differential equations (Matlab simulation)

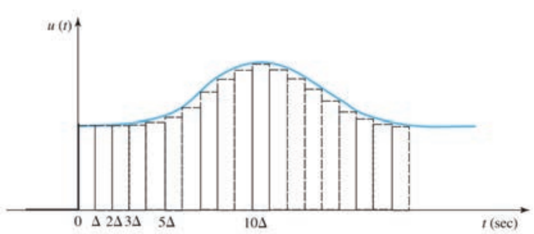
Outline

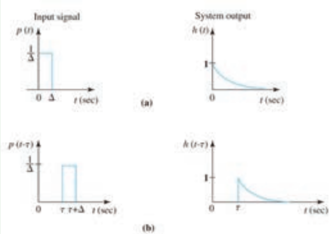
- Convolution
- Laplace Transform
  - Definition and properties
  - Transfer function
  - Inverse Laplace transform
  - Final value theorem, steady-state gain
  - Solving differential equations
- Effects of Pole and Zero Locations
- Time-domain Performance (rise time, overshoot, settling time)
- Block Diagram of Dynamic Systems
- Stability of LTI Systems
- Routh's Stability Criterion

Convolution

Main Idea : This method is based on the superposition principle. The input signal is expressed as a sum of signals, then the response of the system will be the sum of the individual responses to the respective signals.

Example (Convolution)





$$u(t) \approx \sum_{k=0}^{\infty} \Delta u(k\Delta) p(t - k\Delta) \quad \Rightarrow \quad y(t) \approx \sum_{k=0}^{\infty} \Delta u(k\Delta) h(t - k\Delta)$$

The response will be exact if Δ goes to zero !

Convolution

Impulse signal

The limit of $p(t)$ when Δ goes to zero is the impulse signal.

$$\delta(t) = \lim_{\Delta \to 0} p(t)$$

Note that $\delta(t) = 0$ when $t \neq 0$ and : $\displaystyle\int_{-\infty}^{\infty} \delta(t)dt = 1$

Impulse response : is the response of a system to an impulse signal and is shown by $g(t)$.

Sifting property

If $u(t)$ is continuous at $t = \tau$, then it has the sifting property :

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$$

Convolution

Convolution integral

For an LTI system, the response of the system to $\delta(t-\tau)$ is $g(t-\tau)$ so the response to $u(\tau)\delta(t-\tau)$ will be $u(\tau)g(t-\tau)$. Thus, the output of the system for a general input $u(t)$ (using sifting property) is :

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau = u(t) * g(t)$$

Causality

The output of a physical system at t does not depend the future values of the input signal. As a result, the upper bound of the integral can be limited to t. On the other hand, in most cases we take $t = 0$ as the time when the input starts, so the convolution integral can be written as :

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)d\tau$$

Convolution

If we know the impulse response $g(t)$ of an LTI system, we can find the response of the system to any input signal $u(t)$ using the convolution integral :

$$y(t) = \int_0^t u(\tau)g(t-\tau)d\tau = u(t) * g(t)$$

Example (Convolution)

The impulse response of a system is $g(t) = e^{-t}$ for $t \geq 0$. Compute, the response of the system to a unit step signal defined as :

$$u(t) = \mathbf{1}(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0 \end{cases}$$

Using the convolution integral, we have :

$$y(t) = \int_0^t u(\tau)e^{-t+\tau}d\tau = e^{-t+\tau}\Big|_{\tau=0}^{\tau=t}\mathbf{1}(t) = (1-e^{-t})\mathbf{1}(t)$$

Convolution

Exercise

The impulse response of a system is $g(t) = 6e^{-3t}$ for $t \geq 0$. Compute the response of the system to $u(t)$ defined as :

$$u(t) = \begin{cases} 0 & t < 0, t > 4 \\ 2 & 0 \leq t \leq 4 \end{cases}$$

(A) $y(t) = [-4e^{-3t} + 4e^{-3(t-4)}]\mathbf{1}(t)$

(B) $y(t) = [4(1 - e^{-3t})]\mathbf{1}(t)$

(C) $y(t) = 4(1 - e^{-3t})\mathbf{1}(t) - 4(1 - e^{-3(t-4)})\mathbf{1}(t-4)$

(D) $y(t) = [2(1 - e^{-3t}) - 2(1 - e^{-3(t-4)})]\mathbf{1}(t)$

(E) I do not know

Laplace Transform

Motivation : The Laplace transform converts a differential equation to an algebraic one.

Laplace Transform

The Laplace transformation for a function of time, $f(t)$, is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \mathcal{L}\{f(t)\}$$

Existence : The Laplace transform exists if the integral converges.
Fortunately, signals that are physically realizable always have a Laplace transform.

Inverse Laplace Transform

The inverse Laplace transform is written as :

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}ds$$

Laplace Transform

Example (Laplace Transform of the Unit-Impulse Function)

$$F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

Example (Laplace Transform of the Unit-Step Function)

$$F(s) = \int_{0^-}^{\infty} \mathbf{1}(t) e^{-st} dt = \left. \frac{-e^{-st}}{s} \right|_0^{\infty} = \frac{1}{s}$$

Example (Laplace Transform of the Unit-Ramp Function)

$$F(s) = \int_{0^-}^{\infty} t e^{-st} dt = \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^{\infty} = \frac{1}{s^2}$$

where the technique of integration by part is used :

$$\int u dv = uv - \int v du$$

Laplace Transform

Example (Laplace Transform of the Exponential Function)

Let's take $f(t) = e^{-at}\mathbf{1}(t)$, then

$$F(s) = \int_{0^-}^{\infty} e^{-at} e^{-st} dt = \left.\frac{-e^{-(s+a)t}}{s+a}\right|_0^{\infty} = \frac{1}{s+a}$$

Example (Laplace Transform of the Sinusoid Function)

Let's take $f(t) = e^{-j\omega t}\mathbf{1}(t) = (\cos\omega t - j\sin\omega t)\mathbf{1}(t)$, then

$$F(s) = \frac{1}{s+j\omega} = \frac{s-j\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} - j\frac{\omega}{s^2+\omega^2}$$

Therefore :

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2} \quad ; \quad \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t^2
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$\dfrac{1}{s+a}$	e^{-at}
8	$\dfrac{1}{(s+a)^2}$	te^{-at}
9	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2!}t^2e^{-at}$
10	$\dfrac{1}{(s+a)^m}$	$\dfrac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$

Laplace Transform

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
12	$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(at - 1 + e^{-at})$
13	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$
14	$\dfrac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\dfrac{a^2}{s(s+a)^2}$	$1 - e^{-at}(1+at)$
16	$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt} - ae^{-at}$
17	$\dfrac{a}{s^2+a^2}$	$\sin at$
18	$\dfrac{s}{s^2+a^2}$	$\cos at$
19	$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\dfrac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\dfrac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1 - e^{-at}\left(\cos bt + \dfrac{a}{b}\sin bt\right)$

Laplace Transform Properties

Superposition

The Laplace transform is a linear transformation so the superposition applies :

$$\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$$

Time Delay

Suppose that $f(t) = 0$ for $t < 0$ and $\lambda > 0$ is constant. Then, the Laplace transform of $f(t - \lambda)$ is :

$$\begin{aligned}
\mathcal{L}[f(t-\lambda)] &= \int_0^\infty f(t-\lambda)e^{-st}dt \\
&= \int_{-\lambda}^\infty f(t')e^{-s(t'+\lambda)}dt' \\
&= e^{-s\lambda}\int_0^\infty f(t')e^{-st'}dt' = e^{-s\lambda}F(s)
\end{aligned}$$

Laplace Transform Properties

Differentiation

The Laplace transform of the derivative of a signal is related to its Laplace transform and its initial condition :

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty}\left(\frac{df}{dt}\right)e^{-st}dt = e^{-st}f(t)\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt$$

Since $e^{-st}f(t) \to 0$ as $t \to \infty$, we obtain :

$$\mathcal{L}\{\dot{f}\} = sF(s) - f(0^-)$$

For the second derivative, we have :

$$\mathcal{L}\{\ddot{f}\} = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

In the same way, the Laplace transform of the m-th derivative of f reads :

$$\mathcal{L}\{f^m(t)\} = s^mF(s) - s^{m-1}f(0^-) - s^{m-2}\dot{f}(0^-) - \cdots - f^{(m-1)}(0^-)$$

Laplace Transform Properties

Shift in Frequency

The Laplace transform of the multiplication of a signal $f(t)$ by an exponential expression is

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$

Integration

The Laplace transform of the integral of a signal $f(t)$ is

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Convolution

Convolution in the time domain corresponds to multiplication in the frequency domain. Assume that $\mathcal{L}\{f_1(t)\} = F_1(s)$ and $\mathcal{L}\{f_2(t)\} = F_2(s)$, then :

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$$

Laplace Transform

Transfer Function

The transfer function of a system is the Laplace transform of its unit-impulse response, because :

$$y(t) = g(t) * u(t) \quad \Rightarrow \quad Y(s) = G(s)U(s) \quad \Rightarrow \quad G(s) = \frac{Y(s)}{U(s)}$$

where $G(s) = \mathcal{L}\{g(t)\}$.

Example

The impulse response of a system is given as : $g(t) = (2e^{-t} + 3e^{-2t})\mathbf{1}(t)$. Find the transfer function of the system.

Solution : The Laplace transform of $\mathbf{1}(t)$ is $1/s$ and therefore, $\mathcal{L}\{e^{-t}\mathbf{1}(t)\} = 1/(s+1)$:

$$G(s) = \mathcal{L}\{g(t)\} = \frac{2}{s+1} + \frac{3}{s+2} = \frac{5s+7}{(s+1)(s+2)}$$

Laplace Transform

Laplace Transform

The impulse response of a system is given as : $g(t) = (3e^{-t})\mathbf{1}(t-2)$. Find the transfer function of the system.

(A) $G(s) = \dfrac{3}{s+1}$ **(B)** $G(s) = \dfrac{3e^{-2s}}{s+1}$

(C) $G(s) = \dfrac{3e^{-2st}}{s+1}$ **(D)** $G(s) = \dfrac{0.406e^{-2s}}{s+1}$

Inverse Laplace Transform

Objective : Given $F(s)$, find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

Partial-Fraction Expansion : If $F(s)$ is rational, it can be expanded as a sum of simpler terms that can be found in the tables.

Step 1 : Factorize the numerator and the denominator of $F(s)$

$$F(s) = \frac{b_1 s^m + b_2 s^{m-1} \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

where z_i and p_i are referred to **zeros** and **poles** of $F(s)$.

Step 2 : For the simplest case of distinct poles, we have :

$$F(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + \frac{C_n}{s - p_n}$$

where $C_i = (s - p_i)F(s)\big|_{s=p_i}$.

Step 3 : Note that $\mathcal{L}^{-1}\{1/(s - p_i)\} = e^{p_i t}\mathbf{1}(t)$, thus :

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{n} C_i e^{p_i t}\mathbf{1}(t)$$

Inverse Laplace Transform

Example (Partial-Fraction Expansion)

Compute $y(t)$ if $Y(s)$ is given by :

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Solution : We write $Y(s)$ in terms of its partial-fraction expansion.

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3}$$

where $\quad C_1 = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3} \quad , \quad C_2 = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2}$

and $C_3 = -1/6$. Then, using the Laplace transform tables we obtain :

$$y(t) = \frac{8}{3}\mathbf{1}(t) - \frac{3}{2}e^{-t}\mathbf{1}(t) - \frac{1}{6}e^{-3t}\mathbf{1}(t)$$

Laplace Transform

Theorem (Final Value Theorem)

If $sY(s)$ has no pole in the right half-plane and on the imaginary axis, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

Proof :

Using the derivative relationship, we have :

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = \int_0^\infty e^{-st}\frac{dy}{dt}dt$$

Now we find the limit when $s \to 0$:

$$\lim_{s\to 0}[sY(s) - y(0)] = \lim_{s\to 0}\left(\int_0^\infty e^{-st}\frac{dy}{dt}dt\right) = \lim_{t\to\infty}[y(t) - y(0)]$$

which leads to $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$.

Laplace Transform

Example

In a mass-spring-damper system, what is the final position of the mass if a constant force F_0 is applied to the mass.

Solution : The Laplace transform of the differential equation of the system reads :

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s) \quad \Rightarrow Y(s) = \frac{1}{Ms^2 + bs + k}R(s)$$

The constant force is equivalent to a step function of magnitude F_0 as $r(t) = F_0\mathbf{1}(t)$. As a result $R(s) = F_0/s$ and

$$Y(s) = \frac{1}{Ms^2 + bs + k}\frac{F_0}{s} \quad \Rightarrow \quad \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{F_0}{k}$$

Final Value Theorem

Exercise

Find the final value of $y(t)$ if $Y(s) = \dfrac{2(s+1)}{(s+5)(s+6)}$

A) $y(\infty) = 1/15$ **B)** $y(\infty) = -1/15$

C) $y(\infty) = 0$ **D)** $y(\infty) = 1/5$

Exercise

Find the final value of $y(t)$ if $Y(s) = \dfrac{2}{s(s-1)}$

A) $y(\infty) = 2$ **B)** $y(\infty) = -2$

C) $y(\infty) = \infty$ **D)** $y(\infty) = 0$

Final Value Theorem

Steady-State Gain

The steady-state gain or DC gain is the ratio of the output of a system to its input (presumed constant) after all transients have decayed. If we assume that the input signal is a unit step, then $U(s) = 1/s$ and we can find the DC gain (which will be the final value of $y(t)$) of a system $G(s)$ as follows :

$$\text{DC gain} = y(\infty) = \lim_{s \to 0} sG(s)\frac{1}{s} = \lim_{s \to 0} G(s)$$

Example

Find the DC gain of the following system

$$G(s) = \frac{3(s+2)}{s^2 + 2s + 10}$$

Solution : The DC gain is $G(0) = 0.6$.

Laplace Transform

Properties of Laplace Transforms

Number	Laplace Transform	Time Function	Comment
—	$F(s)$	$f(t)$	Transform pair
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)
3	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling
4	$F(s+a)$	$e^{-at}f(t)$	Shift in frequency
5	$s^m F(s) - s^{m-1}f(0) - s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
6	$\frac{1}{s}F(s)$	$\int_0^t f(\zeta)\,d\zeta$	Integration
7	$F_1(s)F_2(s)$	$f_1(t) * f_2(t)$	Convolution
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem

Laplace Transform

Solving a differential equation

- Take the Laplace transform of the differential equation.
- Find the Laplace transform of the output.
- Find the time response using the inverse Laplace transform of the output.

Example

Find the solution of $\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 3\mathbf{1}(t)$, where $y(0) = 1$ and $\dot{y}(0) = 2$.

- Taking the Laplace transform :

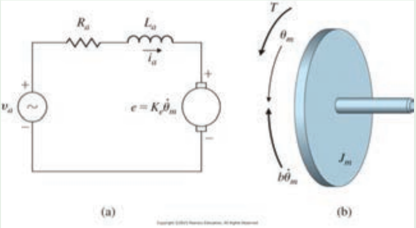
$$s^2 Y(s) - s - 2 + 5[sY(s) - 1] + 4Y(s) = 3/s$$

- Solving for $Y(s)$: $Y(s) = \dfrac{s(s+7)+3}{s(s+1)(s+4)} = \dfrac{0.75}{s} + \dfrac{1}{s+1} + \dfrac{-0.75}{s+4}$
- Taking the inverse Laplace transform of $Y(s)$:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = (0.75 + e^{-t} - 0.75e^{-4t})\mathbf{1}(t)$$

Analysis of a Dynamic System

Example (Transfer Function of a DC Motor)



DC motor Equations :
(Ignoring L_a)

$$v_a = R_a i_a + K_e \dot{\theta}_m$$
$$K_t i_a = J_m \ddot{\theta}_m + b\dot{\theta}_m$$

$$\boxed{\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{s(\tau s + 1)}} \quad \text{where :} \quad K = \frac{K_t}{bR_a + K_t K_e} \quad ; \quad \tau = \frac{R_a J_m}{bR_a + K_t K_e}$$

What is the transfer function between the angular speed $\Omega_m(s)$ and $V_a(s)$?

We have $\Omega_m(s) = s\Theta_m(s)$, then $\boxed{\dfrac{\Omega_m(s)}{V_a(s)} = \dfrac{K}{\tau s + 1}}$

Analysis of a Dynamic System

Example (Time-domain analysis)

- Find the angular speed of the motor if a constant voltage 5 V is applied to the motor from $t > 0$.

$$\Omega_m(s) = \frac{K}{\tau s + 1} V_a(s) \quad , \quad V_a(s) = \frac{5}{s}$$

$$\Rightarrow \quad \Omega_m(s) = \frac{5K}{s(\tau s + 1)} = \frac{5K}{s} + \frac{-5K}{s + 1/\tau}$$

$$\Rightarrow \quad \omega_m(t) = \mathcal{L}^{-1}\{\Omega_m(s)\} = (5K - 5Ke^{-t/\tau})\mathbf{1}(t)$$

- Find the final value of the angular speed.

$$\lim_{t \to \infty} \omega_m(t) = \lim_{s \to 0} s\Omega_m(s) = 5K$$

- Find the steady-state gain of the system. DC gain $= \left. \dfrac{K}{\tau s + 1} \right|_{s=0} = K$

Consider a first order system :

$$H(s) = \frac{K}{\tau s + 1} \qquad \text{one pole at } -1/\tau$$

Impulse response :

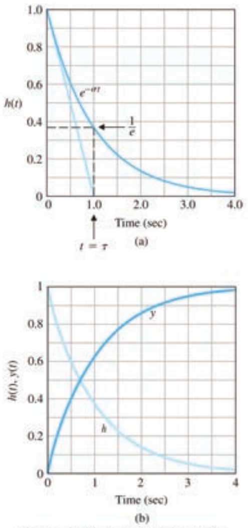
$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{K}{\tau} e^{-t/\tau} \mathbf{1}(t)$$

Step response :

$$y(t) = \mathcal{L}^{-1}\{H(s)\frac{1}{s}\} = K(1 - e^{-t/\tau})\mathbf{1}(t)$$

Time constant : $\tau > 0$ is the time constant of the system. At $t = \tau$ the step response attains 63% of the final value. Smaller τ gives faster response. For $\tau < 0$ the response becomes unbounded.

For $K = 1$ and $\tau = 1$



Effects of Pole and Zero Locations

Consider a second order system :

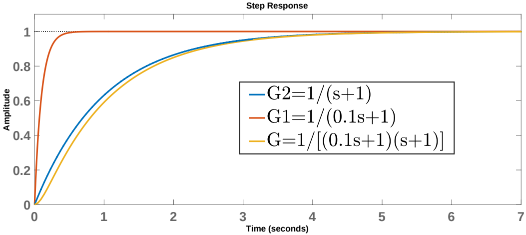
$$H(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad \text{with two real poles at } -1/\tau_1 \quad \text{and} \quad -1/\tau_2$$

Step response : $y(t) = \mathcal{L}^{-1}\{H(s)\frac{1}{s}\} = [C_0 + C_1 e^{-t/\tau_1} + C_2 e^{-t/\tau_2}]\mathbf{1}(t)$

Fast and slow poles : if $\tau_1 \ll \tau_2$ then the pole at $-1/\tau_1$ is much faster than the pole at $-1/\tau_2$ (e^{-t/τ_1} decays much faster than e^{-t/τ_2}).

Dominant poles : The slow poles represents dominant dynamics of a system. The fast poles have less effects and can be ignored.

For $K = 1$ and $\tau_1 = 0.1$ and $\tau_2 = 1$



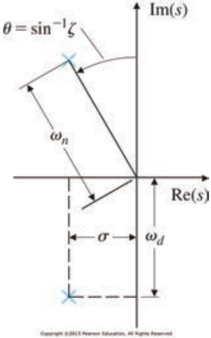
Effects of Pole and Zero Locations

Consider a second order system :

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

where ω_n is the natural frequency and $\zeta \geq 0$ is the damping ratio.

Poles : $\boxed{-\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\sigma \pm j\omega_d}$
where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.



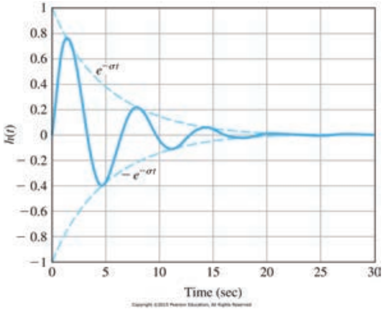
Damping ratio :

$\zeta > 1$	Two real poles (damped)
$\zeta = 1$	Repeated real poles (critical damping)
$0 < \zeta < 1$	Complex conjugate poles (underdamped)
$\zeta = 0$	Two imaginary poles (undamped)

Effects of Pole and Zero Locations

Impulse response :

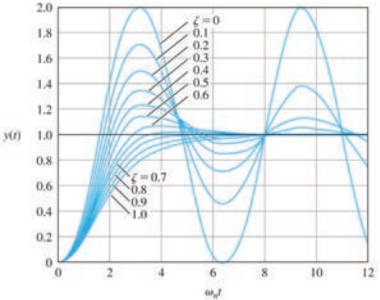
$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) \mathbf{1}(t)$$



Step response :

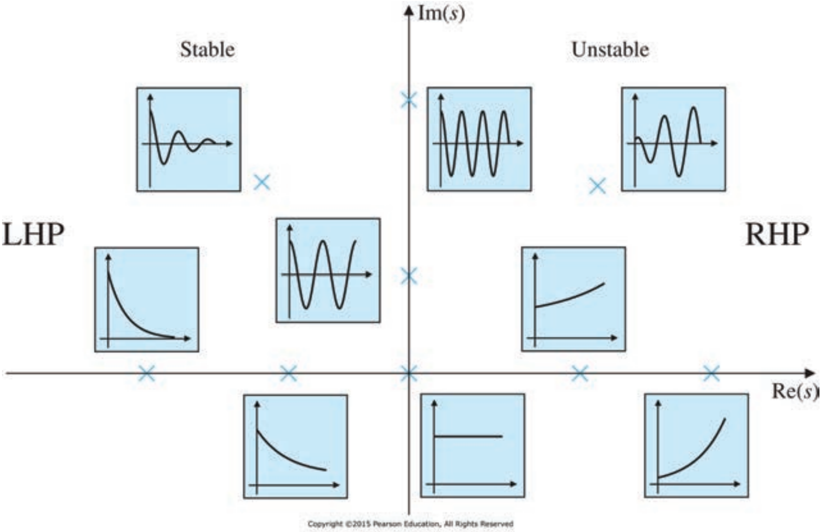
$$y(t) = [1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \theta)] \mathbf{1}(t)$$

where $\theta = \sin^{-1} \zeta$.



Effects of Pole and Zero Locations

Impulse responses associated to the poles



Effects of Pole and Zero Locations

Effects of zeros : Consider the following systems

$$\boxed{H(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}} \quad \Rightarrow \quad \boxed{h(t) = 2e^{-t} - 2e^{-2t}}$$

$$\boxed{H_\alpha(s) = \frac{2(s+\alpha)}{(s+1)(s+2)} = \frac{2(\alpha-1)}{s+1} - \frac{2(\alpha-2)}{s+2}}$$

$$\Rightarrow \quad \boxed{h_\alpha(t) = 2(\alpha-1)e^{-t} - 2(\alpha-2)e^{-2t}}$$

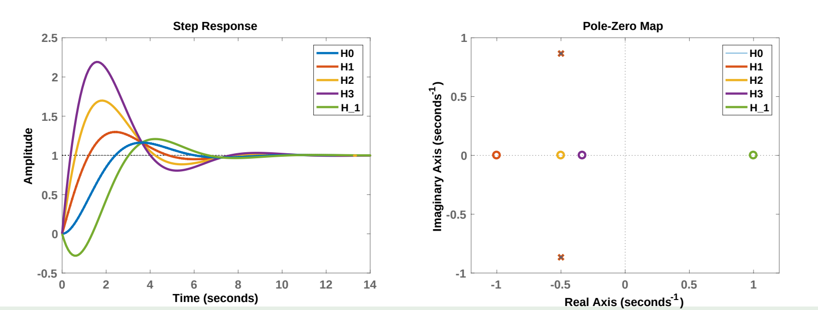
- The system has two exponential modes e^{-t} and e^{-2t} related only to the poles of the system.
- The response of the system is the weighted sum of the modes.
- The zero of the system affects only the weights.
- For $\alpha = 1$ or $\alpha = 2$, one of the modes can be canceled.
- For α close to 1, the effect of the mode e^{-t} will be reduced.

Effects of Pole and Zero Locations

Example (Effect of zeros)

Consider the step response of a second order system with one zero :

$$H_\alpha(s) = \frac{\alpha s + 1}{s^2 + s + 1} \quad \text{for} \quad \alpha = 0, 1, 2, 3, -1$$

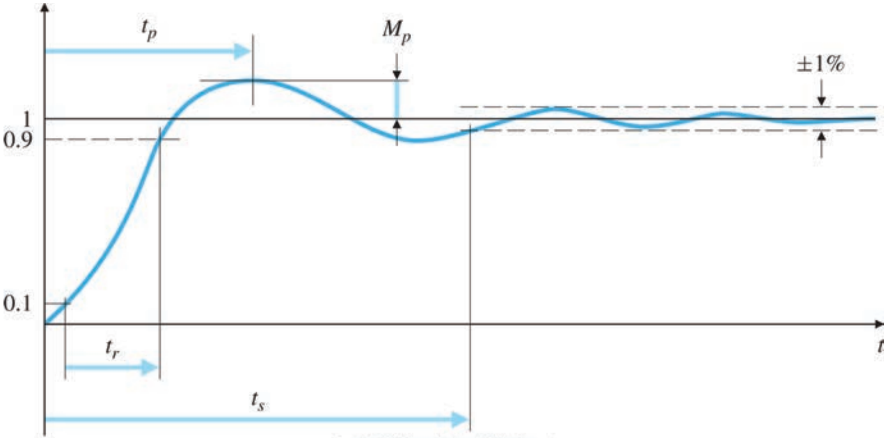


Remark : For RHP zero, the step response goes to the opposite direction.

Time-Domain Specifications

A good controlled system should

- have zero steady-state error ;
- and attain the steady state as fast as possible (good transient response : small rise-time, settling time and overshoot).



Time-Domain Specifications

Rise Time : is the time it takes the output to reach the vicinity of its final value (usually from 10% to 90%).

Overshoot : is the maximum of the output minus its final value divided by the final value.

Settling Time : is the time it takes the system transient to decay.

Example (First order system)

Compute, rise time, overshoot and settling time for a first order system :

$$H(s) = \frac{1}{\tau s + 1}$$

Step response : $y(t) = \mathcal{L}^{-1}\left[H(s)\frac{1}{s}\right] = (1 - e^{-t/\tau})\mathbf{1}(t)$

- The rise time from 0.1 to 0.9 is $t_r = (\ln 0.9 - \ln 0.1)\tau = 2.2\tau$.
- There is no overshoot, so $M_p = 0$.
- The 2% settling time is $t_s = -\tau \ln 0.02 = 3.9\tau$ and the 1% settling time is $t_s = -\tau \ln 0.01 = 4.6\tau$

Time-Domain Specifications

Example (Second order oscillatory system)

Compute, rise time, overshoot and settling time for a second order system :

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step response : $y(t) = \mathcal{L}^{-1}\left[H(s)\frac{1}{s}\right] = [1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \theta)]\mathbf{1}(t)$

- The rise time for $0.3 \leq \zeta \leq 0.8$ can be approximated as : $t_r \approx \dfrac{1.8}{\omega_n}$
- The pick time t_p and the overshoot M_p can be computed as :

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad ; \quad M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

- The 2% and 1% settling time are respectively : $t_s = \dfrac{3.9}{\zeta\omega_n} \quad ; \quad t_s = \dfrac{4.6}{\zeta\omega_n}$

Identification of Simple Models by Step Response

Step response :

- System should be in a stationary state (*système au repos*).
- Noise level should be measured.
- System should be excited with a step with amplitude α.
- How α should be selected ?
- Sampling period should be chosen as small as possible.

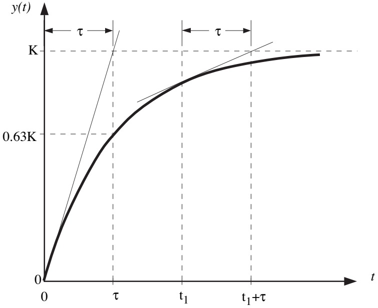
First order model :

First order model :

$$G(s) = \frac{\gamma}{\tau s + 1}$$

$$y(t) = K(1 - e^{-t/\tau}) \Rightarrow \gamma = K/\alpha$$

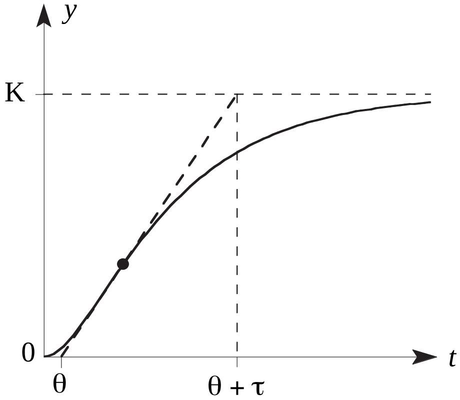
$$y'(t) = \frac{K}{\tau} e^{-t/\tau}$$



Identification of Simple Models

First order model with delay : Higher order models with damped modes can be approximated by a first-order model with delay.

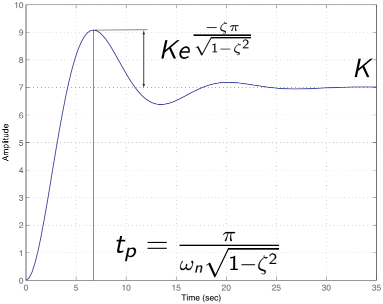
$$G(s) \approx \frac{\gamma e^{-\theta s}}{\tau s + 1} \quad \Rightarrow \quad \gamma = K/\alpha$$



Identification of Simple Models

Identify a second-order model from step response :

$$G(s) = \frac{\gamma\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Identification procedure :

- Measure K, t_p and $y(t_p)$.
- Compute $\gamma = K/\alpha$.
- Compute the overshoot

$$M_p = \frac{y(t_p) - K}{K}$$

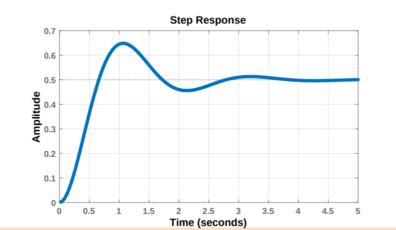
- Compute the damping factor ζ from :

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \Rightarrow \quad \zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}}$$

- Compute the natural frequency $\quad \omega_n = \dfrac{\pi}{t_p\sqrt{1-\zeta^2}}$

Identifying a second order model

The unit step response of a system is given by :



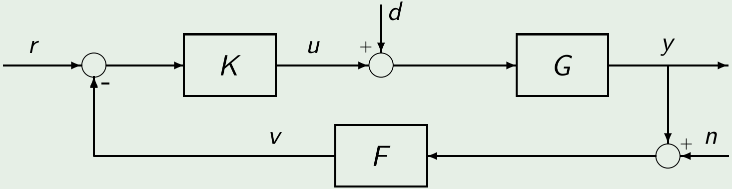
1. Compute the overshoot M_p.
   (A) 0.65 **(B)** 0.325 **(C)** 0.3 **(D)** 1.3
2. Compute the damping factor *zeta*.
   (A) 0.36 **(B)** 0.64 **(C)** 0.7 **(D)** 0.25
3. Compute the natural frequency ω_n.
   (A) 3.5 **(B)** 3.05 **(C)** 2.5 **(D)** 1.1

How to simplify a block diagram

First method : Write the equation for the output of each block as a function of the outputs of the other blocks and external inputs. Then eliminate all internal variables.

Example (Find $y(s)$)

We have $y = G(d+u) \quad ; \quad u = K(r-v) \quad ; \quad v = F(y+n)$



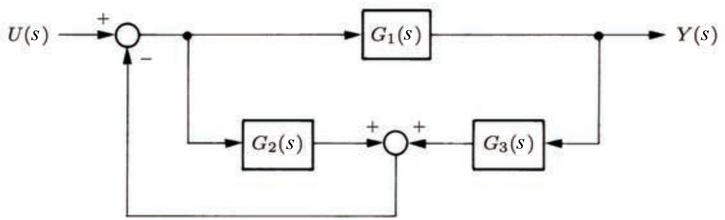
$$\Rightarrow \quad y = G(d + K(r - F(y+n))) \quad \Rightarrow \quad y = Gd + GKr - GKFy - GKFn$$

$$y = \frac{G}{1+GKF}d + \frac{GK}{1+GKF}r - \frac{GKF}{1+GKF}n$$

Exercise

Exercise

Déterminer la fonction de transfert $Y(s)/U(s)$ du système de la figure



A $\dfrac{G_1}{1 + G_2 + G_1 G_3}$

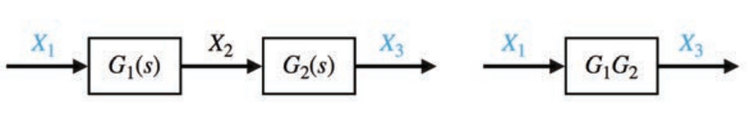
B $\dfrac{G_1}{G_2 + G_3}$

C $\dfrac{G_1}{G_1 + G_2 + G_3}$

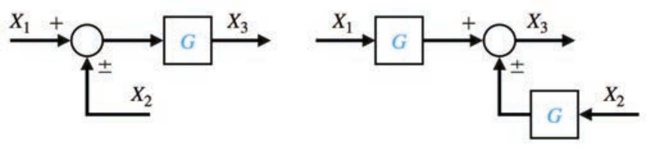
D None of the above

Second method : Use some simple rules :

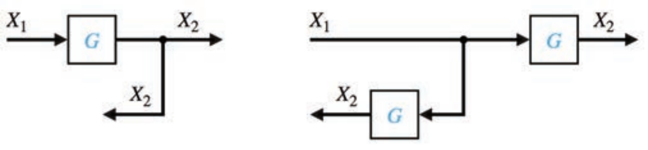
- Combining blocks in cascades



- Moving a summing point behind a block

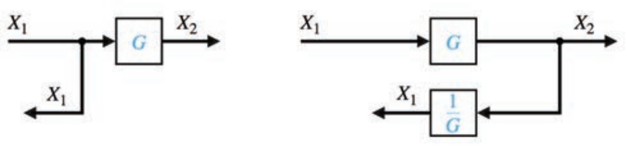


- Moving a pickoff point ahead of a block

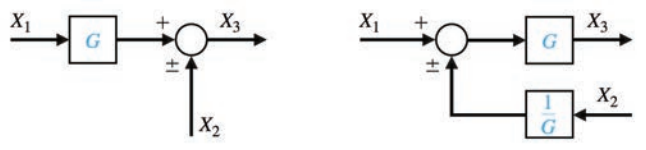


Block Diagram Reduction

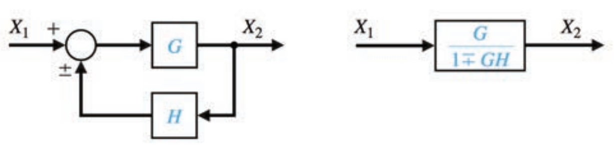
- Moving a pickoff point behind of a block



- Moving a summing point ahead of a block



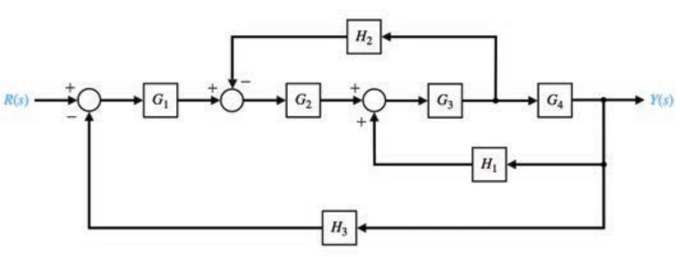
- Eliminating a feedback loop



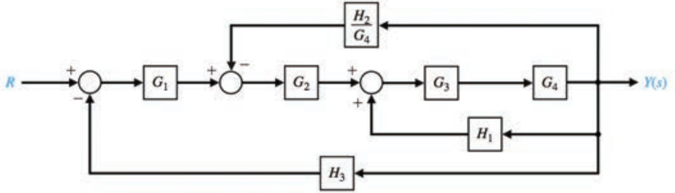
Block Diagram Reduction

Example

Find the transfer function between $Y(s)$ and $R(s)$.

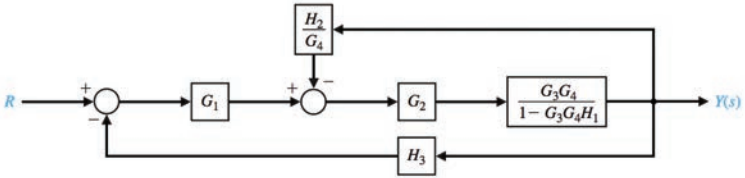


Step 1 :

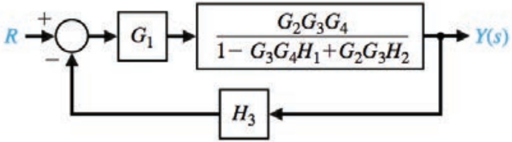


Block Diagram Reduction

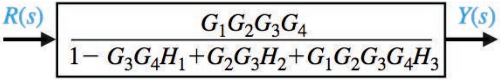
Step 2 :



Step 3 :



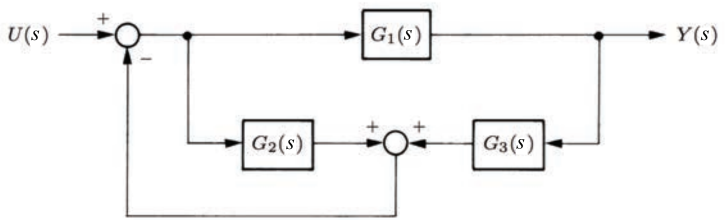
Step 4 :



Written Exercise

Exercise

Déterminer la fonction de transfert $Y(s)/U(s)$ du système de la figure



A $$\frac{G_1}{1 + G_2 + G_1 G_3}$$

B $$\frac{G_1}{G_2 + G_3}$$

C $$\frac{G_1}{G_1 + G_2 + G_3}$$

D None of the above

Stability of Dynamic Systems

Bounded signals :

A continuous-time signal $w(t)$ is bounded if there exists a finite number C such that $|w(t)| < C$ for all t.

BIBO stability :

A system is BIBO stable if for any bounded input signal, the output is bounded.

Theorem

An LTI system is BIBO stable if and only if there exists a finite number C such that its impulse response $g(t)$ satisfies :

$$\int_0^{\infty} |g(t)| dt \leq C$$

Gripen Accident !

Stability of Dynamic Systems

Theorem (Continuous-time)

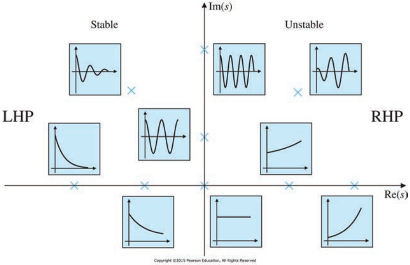
A continuous-time LTI system represented by a proper rational transfer function $G(s)$ is BIBO stable iff all its poles have negative real parts.

Proof sketch : $G(s)$ can be written as (z_i and p_i may be complex numbers) :

$$G(s) = \frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)^{\ell_i}}$$

then for $\ell_i = 1$

$$g(t) = \mathcal{L}^{-1}\{H(s)\} = \sum_{i=1}^{n} c_i e^{p_i t}$$



Therefore, $g(t)$ will be bounded if all poles are in the left-hand s-plane. What happens for the multiple poles ? The terms $t^{\ell_i - 1}e^{p_i t}$ converge to zero when $t \to \infty$.

Stability of Dynamic Systems

Routh Stability Criterion : Check the stability, without computing the roots. Given $A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, we can construct the Routh array :

$$
\begin{array}{c|cccc}
n & 1 & a_2 & a_4 & \cdots \\
n-1 & a_1 & a_3 & a_5 & \cdots \\
n-2 & b_1 & b_2 & b_3 & \cdots \\
n-3 & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
2 & * & * & & \\
1 & * & & & \\
0 & * & & &
\end{array}
$$

$$b_1 = -\frac{\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} \qquad b_2 = -\frac{\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$c_1 = -\frac{\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} \qquad c_2 = -\frac{\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

Routh Criterion :

If all elements of the first column, $1, a_1, b_1, c_1, \ldots$ are positive, then all the roots of the polynomial are in the LHP. The number of sign changes in the column shows the number of poles in the RHP.

Routh Criterion

Example

Determine whether any of the roots of the following polynomial are in the RHP :

$$A(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 11s + 10$$

Solution :

5	1	3	11
4	2	4	10
3	b_1	b_2	0
2	c_1	c_2	
1	d_1	0	
0	e_1		

$$b_1 = -\frac{4-6}{2} = 1 \qquad b_2 = -\frac{10-22}{2} = 6$$

$$c_1 = -\frac{2b_2 - 4b_1}{b_1} = -8 \qquad c_2 = -\frac{0 - 10b_1}{b_1} = 10$$

$$d_1 = -\frac{b_1 c_2 - b_2 c_1}{c_1} = -\frac{10+48}{-8} = \frac{58}{8}$$

$$e_1 = -\frac{-d_1 c_2}{d_1} = 10$$

The first column $[1\,;\,2\,;\,1\,;\,-8\,;\,58/8\,;\,10]$ has two sign changes, so the polynomial $A(s)$ has two roots in the RHP.

Routh Criterion

Exercise

Determine whether any of the roots of the following polynomial are in the RHP :

$$A(s) = s^3 + 7s^2 + 25s + 35$$

(A) One RHP root

(B) No RHP root

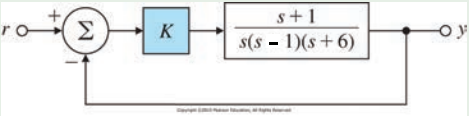
(C) Two RHP roots

(D) Three RHP roots

Routh Criterion

Example

Determine the range of K over which this closed-loop system is stable.



Solution : First we compute the transfer function between r and y :

$$\frac{Y(s)}{R(s)} = \frac{\frac{K(s+1)}{s(s-1)(s+6)}}{1+\frac{K(s+1)}{s(s-1)(s+6)}} = \frac{K(s+1)}{s^3+5s^2+(K-6)s+K}$$

3	1	$K-6$
2	5	K
1	$-\frac{K-5K+30}{5}$	0
0	K	

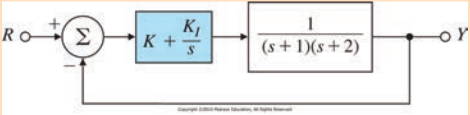
$\Rightarrow \quad \frac{4K-30}{5} > 0 \quad$ and $\quad K > 0$

The closed-loop system is stable if $K > 7.5$

Routh Criterion

Exercise

Determine the range of (K, K_I) over which this closed-loop system is stable.



(A) $K > \frac{1}{3}K_I - 2 \quad \text{and} \quad K_I > 0$

(B) $K > \frac{1}{2}K_I - 3 \quad \text{and} \quad K_I > 0$

(C) $K > \frac{1}{3}K_I - 1 \quad \text{and} \quad K_I > 1$

(D) None of the above

(E) I do not know