# Modeling of Dynamic Systems

Control Systems

Fall 2023

# Outline

1. Dynamic Systems
2. Differential Equations of Physical Systems
    - Mechanical systems
    - Electrical systems
3. Transfer Functions of Linear Systems
    - Rotational systems
    - DC motor example
4. Linear Approximation of Physical Systems
    - Linearization of a water tank model
    - Linearization of a pendulum

# Dynamic Systems

### System :

A system is an abstraction of a physical reality that contains only the important elements for the study. The system is an object in which different sorts of variables interact and produce observable signals.

### Inputs :

The system is usually driven by external independent input variables. The inputs that can be manipulated by the user are called command inputs. The inputs that are not manipulated are called disturbance inputs.

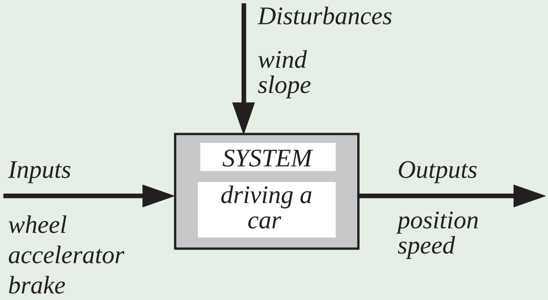
### Outputs :

The useful information about the system are provided by the output variables. These are dependent variables and cannot be directly manipulated.

## Dynamic Systems

### Example

**Driving a car :** The most useful informations are the position and the speed of the car. These variables can be controlled by accelerator and brake pedal and steering wheel. The road condition and wind are disturbance inputs.



# Dynamic Systems

## Mathematical model :

A mathematical model represents the relation between inputs and outputs of a system by some algebraic and/or differential equations.

## Dynamic systems :

If the outputs of a system are functions of the present and past inputs, the system is called dynamic.

- A dynamic system has a memory or inertia to respond to the inputs.
- The mathematical model of a dynamic system is given by some differential equations.

## Time-Invariant systems :

Assume that $y(t)$ is the output caused by $u(t)$. The system is called time-invariant if $y(t-\tau)$ is the response to $u(t-\tau)$.

# Linear Systems

## Linear models :

Assume that the output of a system when excited by $u_1(t)$ and $u_2(t)$ is, respectively $y_1(t)$ and $y_2(t)$. Then the system is linear if and only if it follows the superposition principles : It has additivity and homogeneity properties.

Additivity : The excitation $u_1(t) + u_2(t)$ results in a response $y_1(t) + y_2(t)$.

Homogeneity : The output of the system to $\alpha u_1(t)$ is equal to $\alpha y_1(t)$.

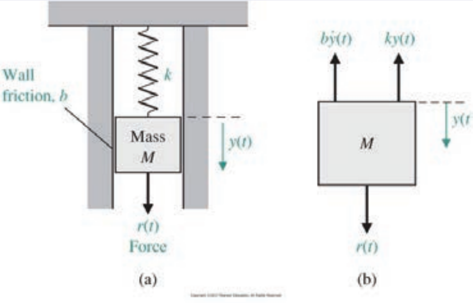
## Question

Are these systems A) linear or B) nonlinear ?

1. $\dot{y}(t) + 2y(t) = 3u(t-1)$
2. $\dot{y}(t) + 2y^2(t) = 3u(t-1)$
3. $y(t) = mu(t) + y_0$

# Modeling of dynamic systems

## Mechanical Systems

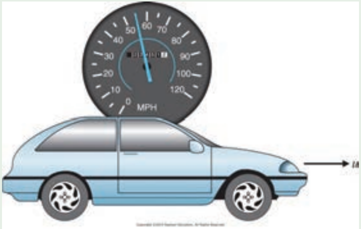


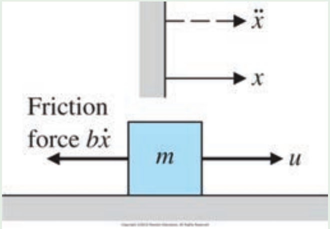

**Newton's second law :** $r(t) = Ma(t) + bv(t) + ky(t)$

Using $v(t) = \frac{dy(t)}{dt}$ and $a(t) = \frac{dv(t)}{dt}$ we obtain a second order ODE :

$$r(t) = M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t)$$

## Example (Cruise Control Model)

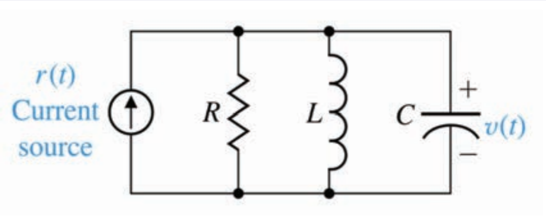



**Newton's second law :**

$$u = m\ddot{x} + b\dot{x} \quad \text{or} \quad u = m\dot{v} + bv$$

## Electrical Systems




**Kirchhoff's current law :**

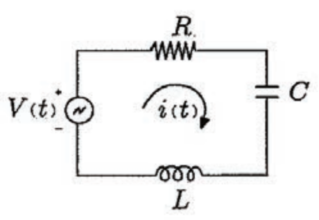
$$r(t) = \frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt$$

If we use $v(t) = L\frac{di_L(t)}{dt}$ we obtain a second order ODE :

$$r(t) = \frac{L}{R}\frac{di_L(t)}{dt} + LC\frac{d^2 i_L(t)}{dt^2} + i_L(t)$$

# Modeling of dynamic systems

## Electrical Systems




**Kirchhoff's voltage law :**

$$v(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)dt + L\frac{di(t)}{dt}$$

If we use $i(t) = \frac{dq_C(t)}{dt}$ we obtain a second order ODE :

$$v(t) = R\frac{dq_C(t)}{dt} + \frac{q_C(t)}{C} + L\frac{d^2q_C(t)}{dt^2}$$

# Transfer Functions

## Transfer Function

The **transfer function** of a linear time-invariant (LTI) system is the ratio of the Laplace transform of the output variable to that of the input variable, with all initial conditions assumed to be zero.

## Example

- What is the transfer function between the force (input variable) and the position (output variable) in the mass-spring-damper system ?

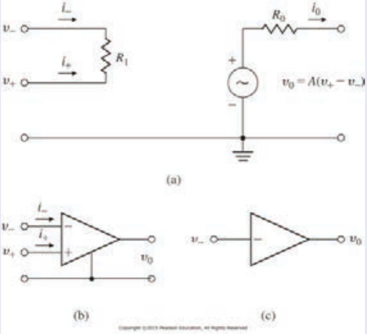
$$Ms^2Y(s) + bsY(s) + kY(s) = R(s) \quad \Rightarrow \quad \boxed{\frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}}$$

- Find the transfer function between the current source and the voltage of the capacitor in a parallel RLC network.

$$\frac{V(s)}{R} + CsV(s) + \frac{1}{Ls}V(s) = I(s) \quad \Rightarrow \quad \boxed{\frac{V(s)}{I(s)} = \frac{RLs}{RLCs^2 + Ls + R}}$$

# Modeling of dynamic systems

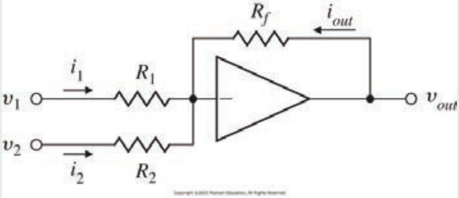
## Electrical Systems (Operational Amplifier)




**Operational Amplifier :** is a very high gain ($A$) amplifier with very large input impedance ($R_1$) and very small output impedance ($R_0$).
**Ideal Op-Amp :** In an ideal Op-Amp we have $A = \infty$, $R_0 = 0$ and $R_1 = \infty$. When only one input is shown, it means that the other is connected to ground.

## Example (Operational Amplifier : Summer)

Find the mathematical model of the following electrical circuit :




**Solution :** For the ideal Op-Amp we have $v_- = 0$ and thus

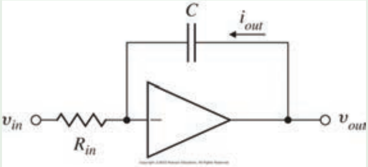
$$i_1 = \frac{v_1}{R_1} \quad ; \quad i_2 = \frac{v_2}{R_2} \quad ; \quad i_1 + i_2 + i_{out} = 0$$

Therefore :

$$\boxed{v_{out} = -\left[\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2\right]}$$

## Example (Operational Amplifier : Integrator)

Find the mathematical model of the following electrical circuit :




**Solution :** For the ideal Op-Amp, we have $v_{-} = 0$ and thus

$$i_{in} + i_{out} = 0 \quad \Rightarrow \quad \frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0$$

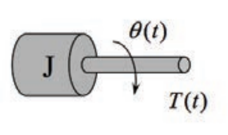
Therefore, with $T = R_{in}C$ :

$$V_{in}(s) + TsV_{out}(s) = 0 \quad \Rightarrow \quad \boxed{\frac{V_{out}(s)}{V_{in}(s)} = -\frac{1}{Ts}}$$

## Rotational Systems

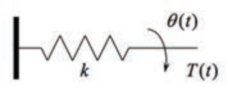
**Inertia :**

$$T(t) = J\frac{d^2\theta(t)}{dt^2}$$



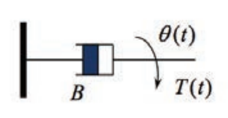

**Torsional spring :**

$$T(t) = K\theta(t)$$



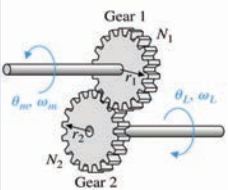

**Friction :**

$$T(t) = B\frac{d\theta(t)}{dt}$$



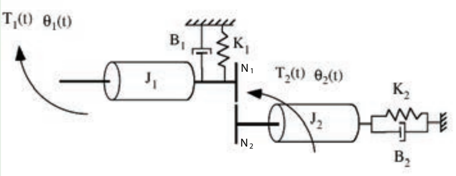

**Gear train :**

$$\text{Gear ratio} = n = N_1/N_2 \qquad \theta_L(t) = n\theta_m(t)$$



# Transfer Functions

## Example (Rotational System)




$$\begin{aligned} T_1(t) - nT_2(t) &= J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + K_1\theta_1(t) \\ T_2(t) &= J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + K_2\theta_2(t) \\ \frac{\theta_2(t)}{\theta_1(t)} &= \frac{N_1}{N_2} = n \end{aligned}$$

If we take $T_1(t)$ as input and $\theta_1(t)$ as output we have :

$$T_1(t) = (J_1 + n^2 J_2)\frac{d^2\theta_1(t)}{dt^2} + (B_1 + n^2 B_2)\frac{d\theta_1(t)}{dt} + (K_1 + n^2 K_2)\theta_1(t)$$

# Transfer Functions

## Example (Rotational System)

If we take $T_1(t)$ as input and $\theta_1(t)$ as output we have :

$$T_1(t) = (J_1 + n^2 J_2)\frac{d^2\theta_1(t)}{dt^2} + (B_1 + n^2 B_2)\frac{d\theta_1(t)}{dt} + (K_1 + n^2 K_2)\theta_1(t)$$

Taking the Laplace transform :

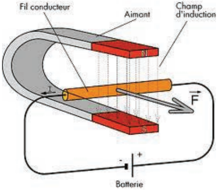
$$T_1(s) = (J_1 + n^2 J_2)s^2\theta_1(s) + (B_1 + n^2 B_2)s\theta_1(s) + (K_1 + n^2 K_2)\theta_1(s)$$

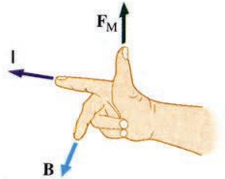
The transfer function between $T_1$ and $\theta_1$ reads :

$$\boxed{\frac{\theta_1(s)}{T_1(s)} = \frac{1}{(J_1 + n^2 J_2)s^2 + (B_1 + n^2 B_2)s + (K_1 + n^2 K_2)}}$$

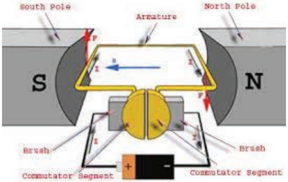
# How a DC motor works

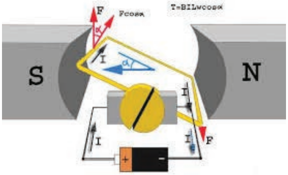
- A wire carrying a current placed in a magnetic field experiences the Lorentz force.




- An armature with current $i_a$ in a magnetic field generates a torque $T$ which is proportional to the current : $T \propto i_a$

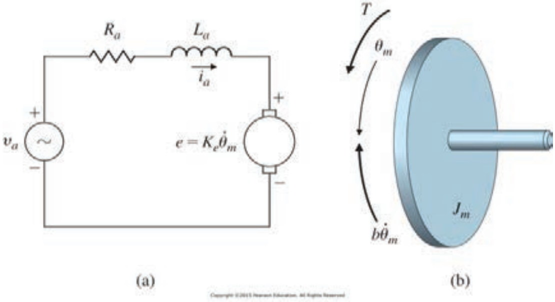



- A commutator changes the current direction on the right time to have a continuous rotation.

# How a DC motor works

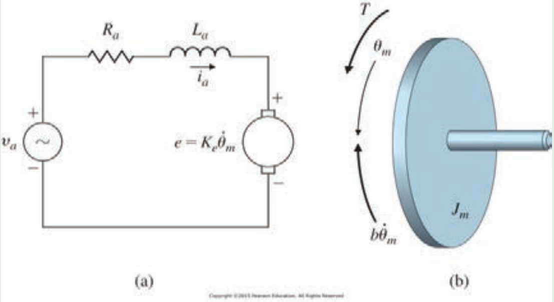
- When an armature rotates, the armature conductors move through a magnetic field and hence electromotive force is induced in them as in a generator.
- The EMF voltage is proportional to the constant magnetic flux $\Phi$ generated by the magnetic field and the armature speed $\omega_m$ and acts in opposite direction to the applied voltage (Lenz's law). So it is called the Back EMF voltage : $e(t) \propto \omega_m(t)$




$$e = K_e \dot{\theta}_m$$
$$T = K_t i_a$$

# Transfer Functions

## Example (Transfer Function of a DC Motor)




**DC motor Equations :** Electromechanical equations are

$$
\begin{aligned}
v_a &= R_a i_a + L_a \frac{di_a}{dt} + K_e \dot{\theta}_m \\
K_t i_a &= J_m \ddot{\theta}_m + b\dot{\theta}_m
\end{aligned}
$$

## Example (Transfer Function of a DC Motor)

The relative effect of the inductance $L_a$ is negligible compared with the mechanical motion. We compute $i_a$ from the first equation and replace it in the second equation :

$$i_a = \frac{v_a - K_e\dot{\theta}_m}{R_a} \quad \Rightarrow \quad J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}v_a$$

Taking the Laplace transform, we compute the transfer function :

$$\left(J_m s^2 + bs + \frac{K_t K_e}{R_a}s\right)\Theta_m(s) = \frac{K_t}{R_a}V_a(s) \quad \Rightarrow \quad \boxed{\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{s(\tau s + 1)}}$$

$$\text{where :} \qquad K = \frac{K_t}{bR_a + K_t K_e} \quad ; \quad \tau = \frac{R_a J_m}{bR_a + K_t K_e}$$

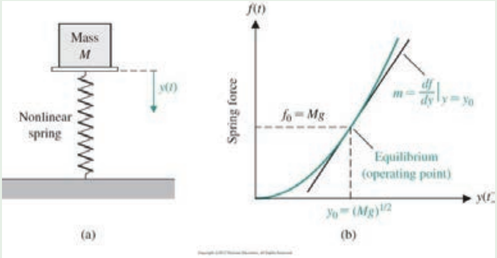
# Linear Approximation

**Linear approximation :** If a system is nonlinear, it can be approximated by a linear model assuming small-signal conditions.
**Taylor series :** A nonlinear function $y = f(x)$ can be linearly approximated with a first order Taylor series around an operating point $x_0$ as :

$$y \approx f(x_0) + (x - x_0) \left. \frac{df}{dx} \right|_{x=x_0}$$

Then the variables $\Delta y = y - f(x_0)$ and $\Delta x = x - x_0$ have a linear relation.

## Example



# Linear Approximation

## Example

Find a linearized model for $\dot{y}(t) + 2y^2(t) = 3u(t-1)$ with $y(0) = y_0$ :

**Step 1 :** Use Taylor series to approximate $y^2(t)$ :

$$y^2(t) \approx y_0^2 + 2y_0[y(t) - y_0]$$

**Step 2 :** Define $\Delta y = y(t) - y_0$ and $\Delta u(t) = u(t) - u_0$. Therefore,

$$\dot{\Delta y}(t) + 2y_0^2 + 4y_0 \Delta y(t) = 3\Delta u(t-1) + 3u_0$$

**Step 3 :** Simplify the equation noting that $2y_0^2 = 3u_0$ (why ?) :

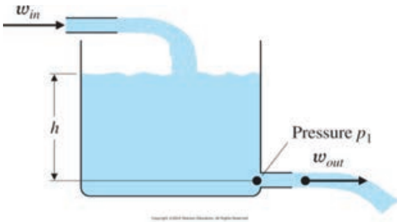
$$\dot{\Delta y}(t) + 4y_0 \Delta y(t) = 3\Delta u(t-1)$$

**Step 4 :** Take the Laplace transform $\mathcal{L}[\Delta y(t)] = Y(s)$ :

$$sY(s) + 4y_0 Y(s) = 3e^{-s}U(s) \quad \Rightarrow \quad \boxed{\frac{Y(s)}{U(s)} = \frac{3e^{-s}}{s + 4y_0}}$$

# Dynamic model of a water tank

**Objective :** Determine a model describing the height of the water in a tank with $w_{in}$ the mass flow rate into the tank and $w_{out}$ the mass flow rate out of the tank.




**Mass conservation :**

$$\dot{m} = w_{in} - w_{out}$$

- $m = A\rho h$
- $A =$ area of the tank
- $\rho =$ density of water
- $m =$ mass of water in the tank

Suppose that $w_{out} = K\sqrt{h}$, where $K$ depends on the output valve. Then the nonlinear model of the system is :

$$\boxed{A\rho\dot{h} = w_{in} - K\sqrt{h}}$$

# Dynamic model of a water tank

## Example (Linearization of the water tank model)

Linearize the water tank model around the operating height $h_0$.

- Use Taylor series to approximate $\sqrt{h}$ :

$$\sqrt{h} \approx \sqrt{h_0} + \frac{1}{2\sqrt{h_0}}(h - h_0)$$

- Define $\Delta h = h - h_0$ and $\Delta w_{in} = w_{in} - w_{in}^{\circ}$. Thus,

$$A\rho\Delta\dot{h} = \Delta w_{in} + w_{in}^{\circ} - K\sqrt{h_0} - \frac{K}{2\sqrt{h_0}}\Delta h$$

- Simplify the equation noting that $w_{in}^{\circ} = K\sqrt{h_0}$ and take the Laplace transform $\mathcal{L}[\Delta h] = H(s), \mathcal{L}[\Delta w_{in}] = W_{in}(s)$ :

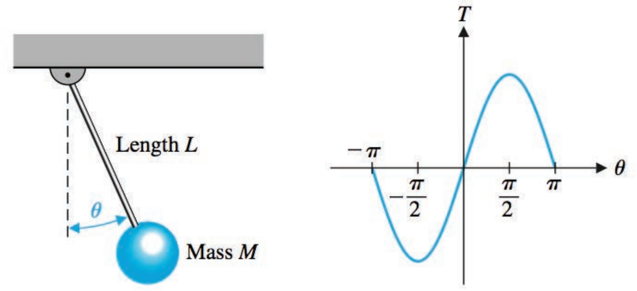
$$A\rho s H(s) + \frac{K}{2\sqrt{h_0}}H(s) = W_{in}(s) \quad \Rightarrow \quad \boxed{\frac{H(s)}{W_{in}(s)} = \frac{2\sqrt{h_0}}{2\sqrt{h_0}A\rho s + K}}$$

# Linear Approximation

## Example (Linearization of a pendulum)

The torque on the mass in a pendulum is : $T = MgL\sin\theta$

Find a linear model around the equilibrium point $\theta_0 = 0$.




$$T \approx T_0 + MgL(\theta - \theta_0)\left.\frac{d\sin\theta}{d\theta}\right|_{\theta=\theta_0} = T_0 + MgL(\theta-\theta_0)\cos 0$$

Since $T_0 = 0$ we obtain : $\boxed{T = MgL\theta}$

# Linear Approximation

## Linearization of a nonlinear model

Compute a linear transfer function model around $u_0 = 4$ and $y_0 = 2$ for the nonlinear model given by :

$$\ddot{y}(t) + 4\dot{y}(t) + y^2(t) = 2\sqrt{u(t)}$$

**(A)** $G(s) = \dfrac{2}{s^2 + 4s + 4}$ **(B)** $G(s) = \dfrac{1}{s^2 + 4s + 4}$

**(C)** $G(s) = \dfrac{0.5}{s^2 + 4s + 4}$ **(D)** $G(s) = \dfrac{0.25}{s^2 + 4s + 4}$

# Summary for modeling of dynamic systems

- A mathematical model gives the relation between output $y(t)$ and input $u(t)$ of a dynamic system : $y(t) = \mathcal{F}(u(\tau)) \quad \tau \leq t$
- This relation is usually given by some differential equations.
- Transfer function of linear time-invariant systems with input signal $u(t)$ and output signal $y(t)$ is defined as :

$$\boxed{G(s) = \frac{Y(s)}{U(s)}}$$

  where all initial conditions are taken equal to zero.
- The nonlinear systems can be linearized around the operating point. The linear model is valid only for small variations around the operating point.