Frequency Response Methods

Control Systems

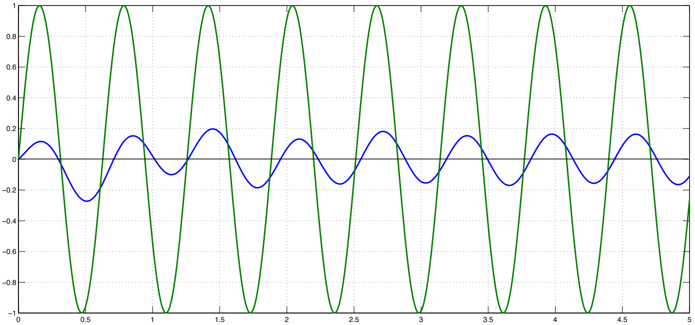
Fall 2023

Outline

- **Frequency Response of Linear Systems**
- **Bode Plots**
  - Sketching Bode Plots
  - Some Information in the Bode Plots (DC gain, Bandwidth, delay)
- **Nyquist Plot**
  - Sketching Nyquist Plot
  - Nyquist Stability Criterion
  - Robust Stability Margins (Gain, Phase and Modulus margins)
- **Control Synthesis in Frequency Domain**
  - Typical Shapes of Closed-Loop Transfer Functions
  - Robustness-Performance Trade-Off
  - Lead-Lag Compensator Design
  - PID Design by Loop Shaping

Frequency Response

The frequency response of a system is the steady state response to a sinusoidal input signal. For LTI stable systems, the steady-state response is sinusoidal with the same frequency as the input signal. It differs from the input wave form only in amplitude and phase angle.



Frequency Response

Find the frequency response of $G(s)$ to a sinusoidal signal $u(t) = \sin \omega t$.
Solution : We find first the response to $u(t) = e^{j\omega t}$.

$$u(t) = e^{j\omega t} = j \sin \omega t + \cos \omega t \quad \Rightarrow \quad U(s) = \frac{1}{s - j\omega}$$

$$\Rightarrow \quad Y(s) = G(s)U(s) = \frac{c_1}{s - p_1} + \cdots + \frac{c_n}{s - p_n} + \frac{c}{s - j\omega}$$

where p_i are the distinct poles of $G(s)$. Taking the inverse Laplace transform :

$$y(t) = c_1 e^{p_1 t} + \cdots + c_n e^{p_n t} + \mathcal{L}^{-1}\left\{\frac{c}{s - j\omega}\right\}$$

If the system is stable then all p_i have negative real parts and

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \mathcal{L}^{-1}\left\{\frac{c}{s - j\omega}\right\} = c\, e^{j\omega t}$$

where

$$c = \lim_{s \to j\omega} (s - j\omega) Y(s) = G(j\omega) \quad \Rightarrow \quad \lim_{t \to \infty} y(t) = G(j\omega) e^{j\omega t}$$

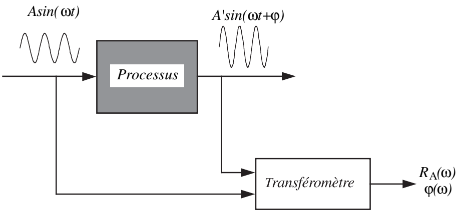
Frequency Response

Therefore, the steady state response to $u(t) = \sin \omega t$ is :

$$y_s(t) = I_m[G(j\omega)e^{j\omega t}] = |G(j\omega)| \sin(\omega t + \phi) \quad ; \quad \phi = \angle G(j\omega)$$

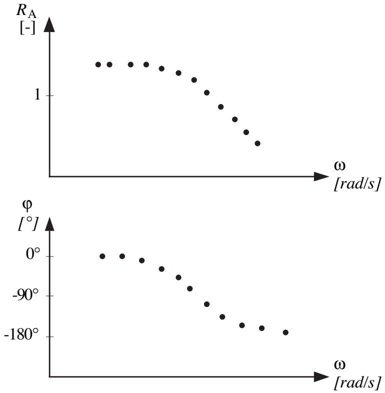
- $G(j\omega)$ can be obtained from a set of experiments with sinusoidal inputs of different frequencies.
- $G(j\omega)$ can be obtained using Fourier analysis if the system is excited with a white noise, a Pseudo Random Binary Sequence or a multi sinusoidal signal.
- $G(j\omega)$ gives very useful information about the system and can be used for controller design (loop shaping method).
- $G(j\omega)$ can be obtained by replacing $s = j\omega$ in $G(s)$. It is called the *frequency response function* and is defined even for unstable systems. In this case, it is not the steady-state response to a sinusoidal input.
- $G(j\omega) = \overline{G}(-j\omega)$, so $|G(j\omega)| = |G(-j\omega)|$ is an even function and $\angle G(j\omega) = -\angle G(-j\omega)$ an odd function. They are usually computed for $\omega \in [0\,,\,\infty[$.

Frequency analysis :



$$R_A(\omega) = \frac{A'}{A} = |G(j\omega)|$$

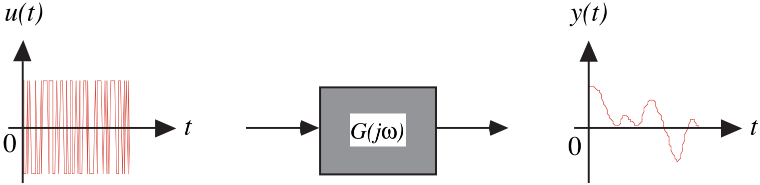
$$\varphi(\omega) = \arg[G(j\omega)]$$



Remarks : Frequency range should cover at least 2 or 3 decades with more than 10 frequency points per decade. The measurements should be done in steady state.

Identification of the Frequency Response

Fourier transform analysis :



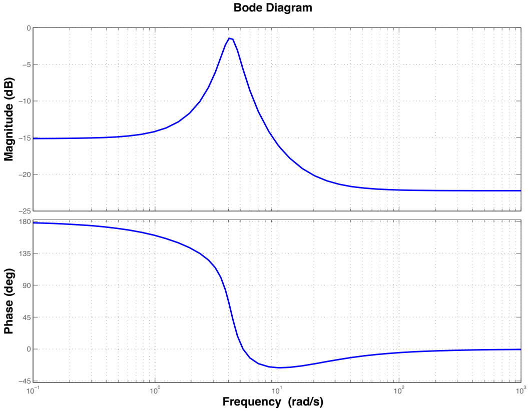
$$U(\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$\Rightarrow G(j\omega) = \frac{Y(\omega)}{U(\omega)}$$

Frequency Response Plots

Bode Plots : The amplitude and phase of $G(j\omega)$ are plotted in two figures with a logarithmic scale for frequency ω usually in rad/s (sometimes in Hz). The amplitude is in dB, i.e . $20\log_{10}|G(j\omega)|$. Use bode in Matlab.



Frequency Response Plots

Sketching Bode Plots : The transfer function $G(s)$ can be written as :

$$G(s) = K \prod_{i=1}^{m} (\tau_i s + 1) \prod_{i=m+1}^{m+n} \frac{1}{\tau_i s + 1}$$

then

$$20 \log |G(j\omega)| = 20 \log |K| + \sum_{i=1}^{m} 20 \log |j\tau_i \omega + 1| + \sum_{i=m+1}^{m+n} 20 \log \left| \frac{1}{j\tau_i \omega + 1} \right|$$

$$\arg G(j\omega) = \arg K + \sum_{i=1}^{m} \arg(j\tau_i \omega + 1) + \sum_{i=m+1}^{m+n} \arg \left(\frac{1}{j\tau_i \omega + 1} \right)$$

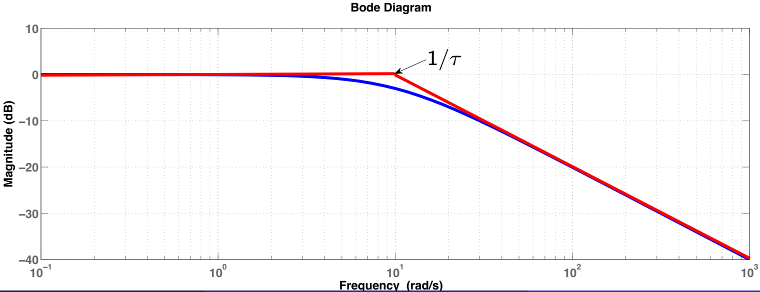
Remark : If we know how to plot the Bode diagram of a zero and a pole, then the sum of the magnitudes of all zeros and poles plus the gain K will give the total magnitude. The sum of the phases will give the phase diagram.

Sketching Bode Plots

Let's compute the magnitude of

$$G(j\omega) = \frac{1}{j\tau\omega + 1} \qquad \tau \in \mathbb{R} > 0$$

- $\omega \ll 1/\tau \Rightarrow |G(j\omega)| = 1 \Rightarrow 20\log|G(j\omega)| = 0$
- $\omega = 1/\tau \Rightarrow |G(j\omega)| = 1/\sqrt{2} \Rightarrow 20\log|G(j\omega)| = -3dB$
- $\omega \gg 1/\tau \Rightarrow |G(j\omega)| = \dfrac{1}{\tau\omega} \Rightarrow 20\log|G(j\omega)| = -20\log\omega - 20\log\tau$

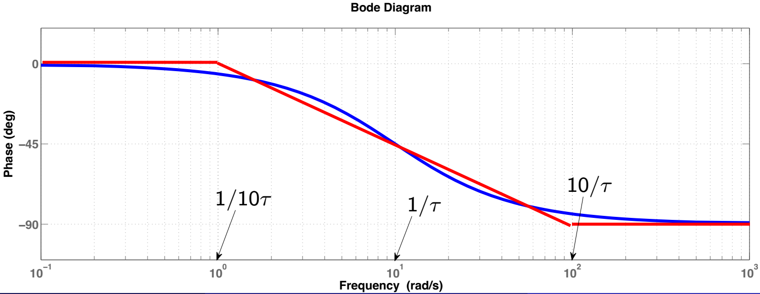


Sketching Bode Plots

Let's compute the phase of

$$G(j\omega) = \frac{1}{j\tau\omega + 1} \qquad \tau \in \mathbb{R} > 0 \qquad \Rightarrow \arg G(j\omega) = -\arctan \tau\omega$$

- $\omega \ll 1/\tau \Rightarrow \arg G(j\omega) = 0$
- $\omega = 1/\tau \Rightarrow \arg G(j\omega) = -\arctan 1 = -\pi/4$
- $\omega \gg 1/\tau \Rightarrow \arg G(j\omega) = \dfrac{-\pi}{2}$

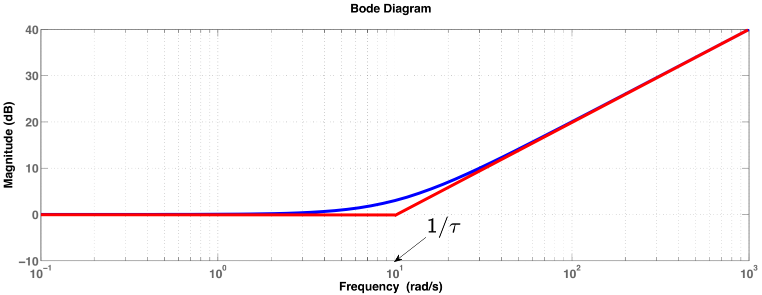


Sketching Bode Plots

Let's compute the magnitude of

$$G(j\omega) = j\tau\omega + 1 \qquad \tau \in \mathbb{R} > 0$$

- $\omega \ll 1/\tau \Rightarrow |G(j\omega)| = 1 \Rightarrow 20\log|G(j\omega)| = 0$
- $\omega = 1/\tau \Rightarrow |G(j\omega)| = \sqrt{2} \Rightarrow 20\log|G(j\omega)| = 3dB$
- $\omega \gg 1/\tau \Rightarrow |G(j\omega)| = \tau\omega \Rightarrow 20\log|G(j\omega)| = 20\log\omega + 20\log\tau$

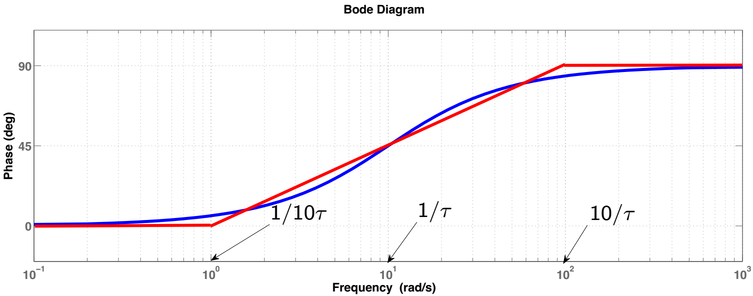


Sketching Bode Plots

Let's compute the phase of

$$G(j\omega) = j\tau\omega + 1 \qquad \tau \in \mathbb{R} > 0 \qquad \Rightarrow \arg G(j\omega) = \arctan \tau\omega$$

- $\omega \ll 1/\tau \Rightarrow \arg G(j\omega) = 0$
- $\omega = 1/\tau \Rightarrow \arg G(j\omega) = \arctan 1 = \pi/4$
- $\omega \gg 1/\tau \Rightarrow \arg G(j\omega) = \dfrac{\pi}{2}$

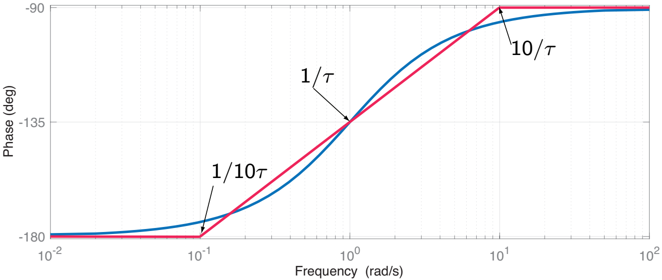


Sketching Bode Plots

Let's consider the Bode diagram of an unstable pole :

$$G(j\omega) = \frac{1}{j\tau\omega - 1} \qquad \tau \in \mathbb{R} > 0 \quad \Rightarrow \quad \left|\frac{1}{j\tau\omega - 1}\right| = \left|\frac{1}{j\tau\omega + 1}\right|$$

- $\omega \ll 1/\tau \Rightarrow G(j\omega) \approx -1 \quad \rightarrow \quad \arg G(j\omega) = -\pi$
- $\omega = 1/\tau \Rightarrow G(j\omega) = \frac{1}{j-1} = \frac{j+1}{-2} \quad \rightarrow \quad \arg G(j\omega) = -3\pi/4$
- $\omega \gg 1/\tau \Rightarrow G(j\omega) \approx \dfrac{1}{j\tau\omega} \quad \rightarrow \quad \arg G(j\omega) = -\dfrac{\pi}{2}$

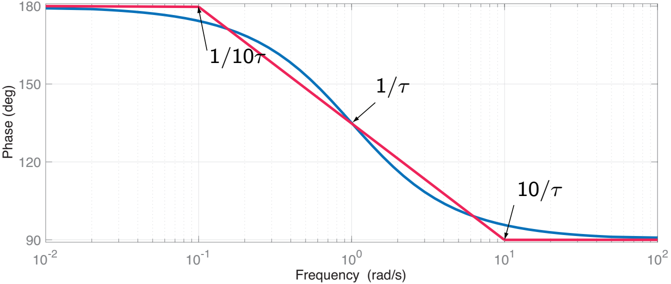


Sketching Bode Plots

Let's consider the Bode diagram of a RHP zero :

$$G(j\omega) = j\tau\omega - 1 \qquad \tau \in \mathbb{R} > 0 \quad \Rightarrow \quad |j\tau\omega - 1| = |j\tau\omega + 1|$$

- $\omega \ll 1/\tau \Rightarrow G(j\omega) \approx -1 \quad \rightarrow \quad \arg G(j\omega) = \pi$
- $\omega = 1/\tau \Rightarrow G(j\omega) = j - 1 \quad \rightarrow \quad \arg G(j\omega) = 3\pi/4$
- $\omega \gg 1/\tau \Rightarrow G(j\omega) \approx j\tau\omega \quad \rightarrow \quad \arg G(j\omega) = \dfrac{\pi}{2}$



Sketching Bode Plots

Let's compute the Bode magnitude diagram of

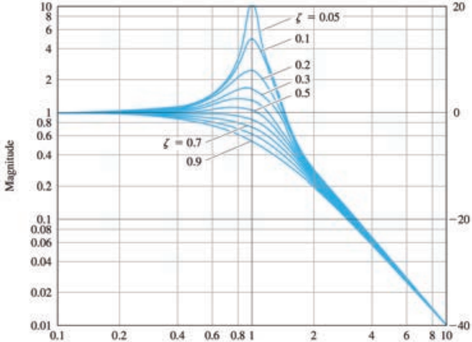
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \Rightarrow \quad G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

- $\omega \ll \omega_n \Rightarrow G(j\omega) \approx 1 \Rightarrow 20\log|G(j\omega)| = 0.$
- $\omega = \omega_n \Rightarrow G(j\omega_n) = \dfrac{1}{j2\zeta}$

  $$\Rightarrow 20\log|G(j\omega_n)| = 20\log\frac{1}{2\zeta}$$

- $\omega \gg \omega_n \Rightarrow G(j\omega) \approx \dfrac{-\omega_n^2}{\omega^2}$

  $$\Rightarrow 20\log|G(j\omega)| = -40\log\omega + 40\log\omega_n$$



Sketching Bode Plots

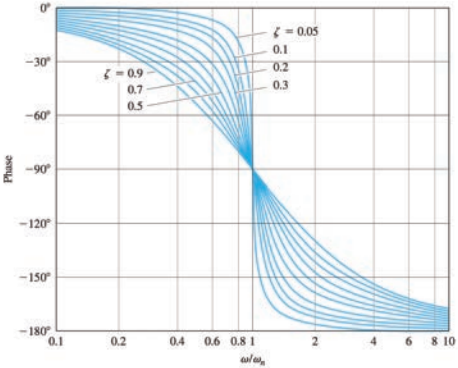
Let's compute the Bode phase diagram of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \Rightarrow \quad G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

- $\omega \ll \omega_n \Rightarrow G(j\omega) \approx 1 \Rightarrow \arg G(j\omega) = 0.$
- $\omega = \omega_n \Rightarrow G(j\omega_n) = \dfrac{1}{j2\zeta}$

  $$\Rightarrow \arg G(j\omega_n) = -\frac{\pi}{2}$$
- $\omega \gg \omega_n \Rightarrow G(j\omega) \approx \dfrac{-\omega_n^2}{\omega^2}$

  $$\Rightarrow \arg G(j\omega) = -\pi$$



Sketching Bode Plots

For magnitude Bode plot :

- Each pole decreases the slope by 20 dB/dec at the pole location.
- Each zero increases the slope by 20 dB/dec at the zero location.
- An integrator (a pole at zero) has a constant slope of -20dB/dec. It crosses the zero dB axis at 1 rad/s.
- Complex poles give resonance peak (larger for smaller ζ) and a change of slope of -40 dB/dec at ω_n.
- $j\tau\omega - 1$ and $j\tau\omega + 1$ have the same magnitude (no difference between stable and unstable poles).
- Time delay does not change the magnitude, i.e. $|e^{-jT\omega}G(j\omega)| = |G(j\omega)|$.
- $KG(j\omega)$ will shift up (if $K > 1$) or shift down (if $0 < K < 1$) the magnitude of $G(j\omega)$ for $20 \log K$.
- The physical systems have a negative slope at high frequencies. The slope at high frequency is $-20(n - m)$ dB, where $n - m$ is the relative degree of $G(s)$.

Sketching Bode Plots

For phase Bode plot :

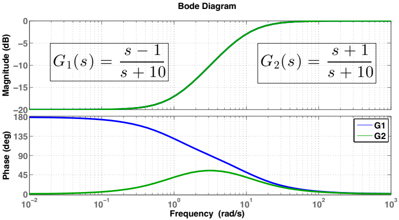
- Each real negative pole/zero decreases/increases the phase by 90°.
- Each real positive pole/zero increases/decreases the phase by 90°.
- An integrator (a pole at zero) has a constant phase of -90°.
- Real negative/positive complex poles decrease/increase the phase by 180° (90° at ω_n). Smaller ζ gives sharper change.
- A time delay adds a linear phase lag to the system, i.e.

$$\arg[e^{-jT\omega} G(j\omega)] = \arg G(j\omega) - T\omega$$

- $KG(j\omega)$, with $K > 0$, will not change the phase of $G(j\omega)$.
- The physical systems have a negative phase at high frequencies. The phase at high frequency is $-90(n-m)$ degree, where $n-m$ is the relative degree of $G(s)$.

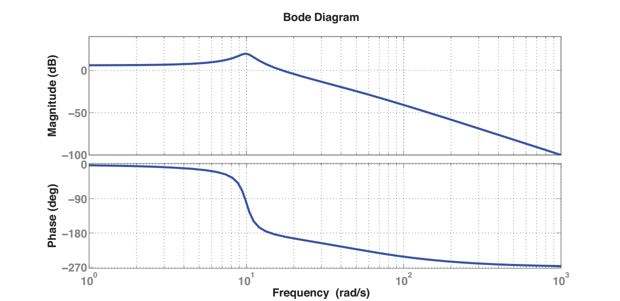
Some Information in the Bode Plots

- The bandwidth is the frequency at which the frequency response has declined 3dB from its low frequency value (the steady-state gain).
- A resonance mode in low frequency shows an oscillatory response (a pair of complex poles) ; higher the peak value larger overshoot can be expected.
- The relative degree of a continuous-time transfer function is equal to the slope of the magnitude Bode diagram in high frequencies divided by -20 or its phase divided by -90°.
- Linear decrease in the phase diagram, shows the existence of pure time delay.
- Systems with real positive zeros (non-minimum phase systems) have larger phase than systems with the same magnitude but with real negative zeros.



Some Information in the Bode Plots

Exercise



Question : How many poles ? How many zeros ?

A) 2 poles, no zero

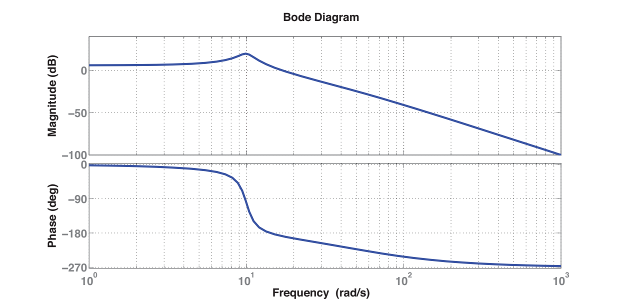
B) 3 poles, one zero

C) 3 poles, no zero

D) 4 poles, one zero

Some Information in the Bode Plots

Exercise



Question : Frequency of poles ?

A) 2 real poles around 10 rad/s and one pole around 5 rad/s

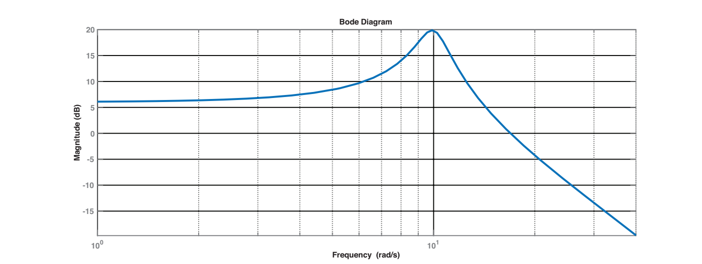
B) 2 complex poles around 10 rad/s and one pole around 20 rad/s

C) 2 complex poles around 10 rad/s and one pole around 100 rad/s

D) 2 complex poles around 10 rad/s and one pole around 50 rad/s

Some Information in the Bode Plots

Exercise (zoom on the magnitude plot)



Question 1 : What is the value of the steady-state gain ?

(A) 1 **(B)** 2 **(C)** 6 **(D)** -20

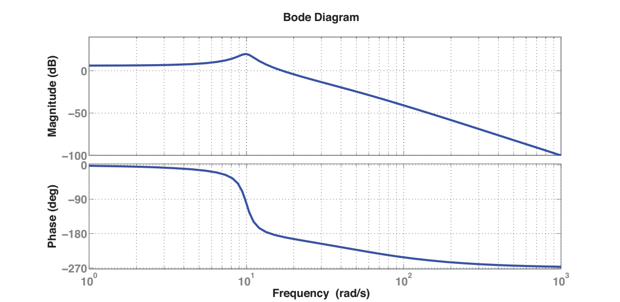
Question 2 : What is the bandwidth (rad/s) ?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 5

Question 3 : What is the damping factor ?

(A) $\zeta = 0.1$ **(B)** $\zeta = 0.3$ **(C)** $\zeta = 0.7$ **(D)** $\zeta = 0.05$

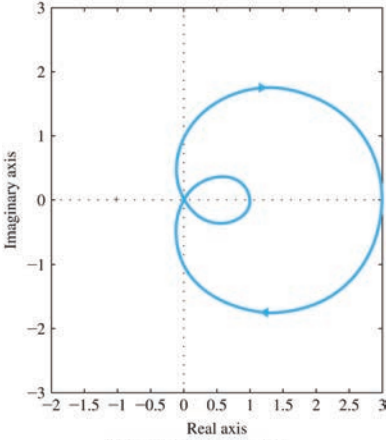
Exercise



Transfer function of the system ?

Nyquist Plot

Nyquist plot : The real and imaginary part of $G(j\omega)$ are plotted in a complex plane for $\omega \in]-\infty, \infty[$. It is very useful for stability and robustness analysis. Use `nyquist` in Matlab to plot it.



Nyquist Plot

Sketching Nyquist plot : The frequency function of simple systems can be sketched by finding some points on the plot.

Example

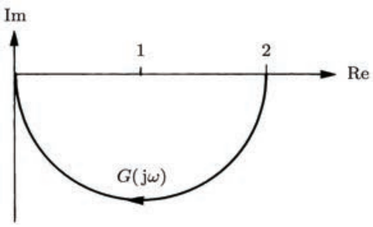
Plot the Nyquist diagram of $G(s) = \dfrac{4}{s+2}$. We have $G(j\omega) = \dfrac{4}{j\omega + 2}$.

- For $\omega \ll 2$, we get $R_e[G(j\omega)] \approx 2$, and $I_m[G(j\omega)] \approx 0$.
- For $\omega \gg 2$, we get $G(j\omega) \approx \frac{4}{j\omega}$. Then $R_e[G(j\omega)] \approx 0$, and $I_m[G(j\omega)] \approx 0$ and $\angle G(j\omega) \approx -\pi/2$.

Note that :

$$G(j\omega) = 1 + \frac{2 - j\omega}{2 + j\omega}$$

which represents a semicircle for $\omega > 0$.

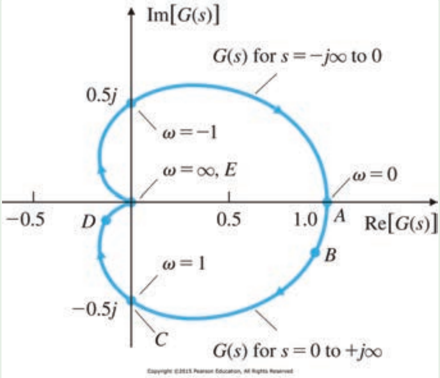


Nyquist Plot

Example

For $G(s) = \dfrac{1}{(s+1)^2}$, we have $G(j\omega) = \dfrac{1}{(j\omega+1)^2}$.

- **(A)** For $\omega \ll 1$, we get $R_e[G(j\omega)] \approx 1$, and $I_m[G(j\omega)] \approx 0$.
- **(C)** For $\omega = 1$, we get $G(j\omega) = -0.5j$, then $R_e[G(j\omega)] = 0$, and $I_m[G(j\omega)] = -0.5$.
- **(E)** For $\omega \gg 1$, we get $G(j\omega) \approx \frac{1}{-\omega^2}$. Then $R_e[G(j\omega)] \approx 0$, and $I_m[G(j\omega)] \approx 0$ and $\angle G(j\omega) \approx -\pi$.

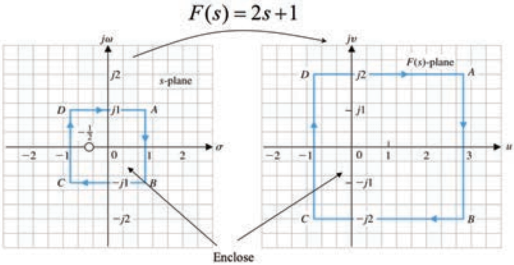


Nyquist Stability Criterion

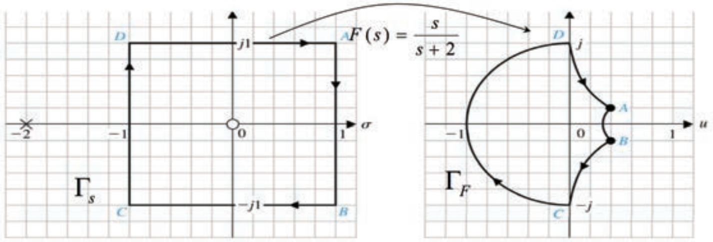
Remark : The Nyquist stability criterion is based on the *Cauchy's Argument Principle* for complex functions. Note that $G(s)$ is a complex function from $\mathbb{C}$ to $\mathbb{C}$ and follows this principle.

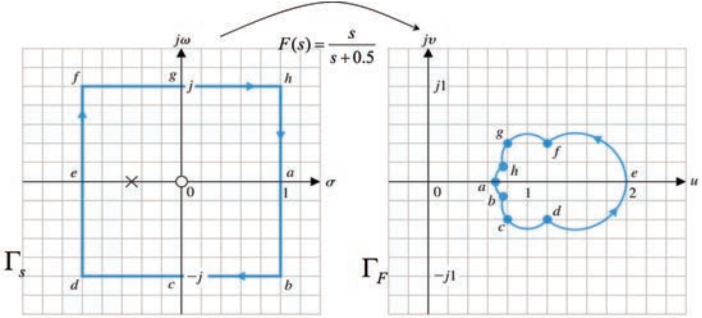
Principle of the Argument

If a contour Γ_s in the s-plane encircles Z zeros and P poles of $F(s)$ and does not pass through any poles or zeros of $F(s)$ and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the $F(s)$-plane encircles the origin of the $F(s)$-plane, $N = Z - P$ times in the clockwise direction.



Principle of the Argument



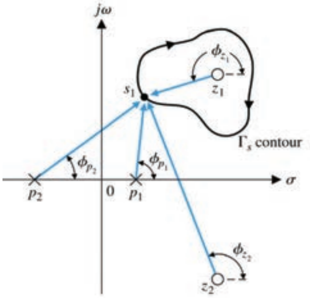


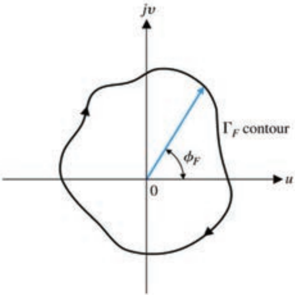
Principle of the Argument

Why ? Let's compute the argument of $F(s) = K\dfrac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}$:

$$\begin{aligned}\arg(F(s)) &= \arg(s-z_1) + \arg(s-z_2) - \arg(s-p_1) - \arg(s-p_2) \\ &= \phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2}\end{aligned}$$

When s traverses Γ_s in a clockwise direction, for a full rotation, the net angle change of ϕ_{z_2}, ϕ_{p_1} and ϕ_{p_2} is zero. However, it is 360° for ϕ_{z_1}.





Nyquist Stability Criterion

Objective : Knowing $G(s)$ and $D_c(s)$ we want to investigate the stability of the unit feedback system (the number of RHP zeros of $1 + G(s)D_c(s)$).

Basic Principle :

- The number of clockwise encirclements of the origin by the image of Γ_s under the mapping $F(s)$ is equal to $N = Z - P$, where Z and P are respectively the number of zeros and poles of $F(s)$ inside Γ_s.
- Choose Γ_s as a contour that covers the RHP of the complex plane.
- Choose $F(s) = 1 + L(s)$ where $L(s) = G(s)D_c(s)$ is the open-loop transfer function.
- As a result Z is the number of zeros of $1 + L(s)$ on the RHP. These zeros are the unstable poles of the closed-loop system. We wish $Z = 0$ to have closed-loop stability.
- P is the number of RHP poles of $F(s) = 1 + L(s)$. Note that the poles of $1 + L(s)$ are the poles of $G(s)D_c(s)$ and therefore P is known.

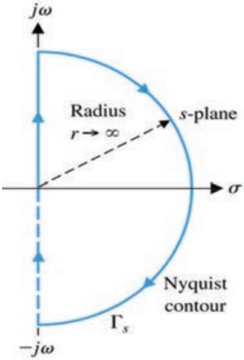
Nyquist Stability Criterion

Procedure

1. Compute P the number of RHP poles of $L(s) = G(s)D_c(s)$ (usually $P = 0$ for stable plant models).
2. Plot the image of Γ_s under the mapping $1 + L(s)$ and count the number of clockwise encirclements of the origin N.
3. Compute $Z = N + P$ the number of unstable poles of the closed-loop.
4. Usually we plot the image of Γ_s under the mapping $L(s)$ and count the number of clockwise encirclements of the critical point (-1,0).

Theorem

The zeros of $1 + L(s)$ have all negative real parts (i.e. the closed-loop system is stable), iff the image of Γ_s by the mapping $s \mapsto L(s)$ encircles counterclockwise the critical point (-1,0), P times, where P is the number of RHP poles of $L(s)$.



- The Nyquist contour Γ_s should not pass through the poles of $L(s) = G(s)D_c(s)$ on the imaginary axis.
- Since $L(s)$ is strictly proper (degree of den > deg of num), $L(\infty) = 0$. So the image of semicircle with infinity radius will be the origin. Therefore, Γ_L is usually plotted just for the imaginary axis.
- If $P = 0$ and Γ_L does not encircle -1, the closed loop system is stable.

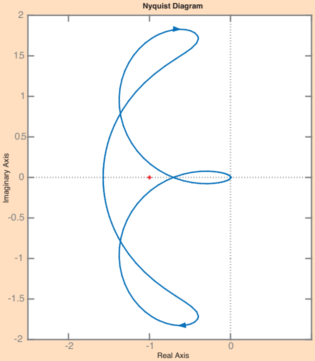
Nyquist Stability Criterion

Exercise

Le diagram Nyquist de

$$G(s) = \frac{0.0026(s-22)(s-2)(s+0.1053)}{(s-0.0952)(s^2+0.2266s+0.0793)}$$

est donnée dans cette figure. Utiliser le Théorème de Nyquist pour étudier la stabilité de ce système en boucle fermée avec un régulateur proportionnel pour les valeurs suivantes :
(a) $k_P = 1$, (b) $k_P = 2$ and (c) $k_P = 0.5$.

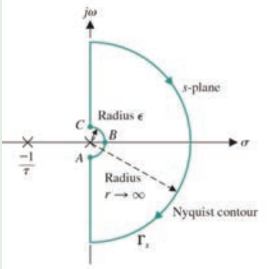


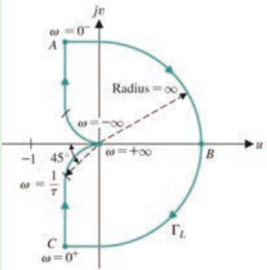
Nyquist Plot

Example (Sketching Nyquist plot)

Plot the Nyquist diagram of $G(s) = \dfrac{1}{s(\tau s + 1)}$.

- The Nyquist contour Γ_s should not pass through the poles of $G(s)$. So it should be modified to have a small detour around zero.
- **(A)** For $\omega = 0^-$, $G(j\omega) \approx 1/(j\omega) = j\infty$.
- **(C)** For $\omega = 0^+$, $G(j\omega) \approx 1/(j\omega) = -j\infty$.
- **(B)** When s makes a detour on Γ_s, we have $s = \epsilon e^{j\theta}$ with θ going from $-\pi/2$ to $\pi/2$. The image of $G(s)$ makes a semicircle with an ∞ radius. For $\theta = 0$, the image pass through the real axis in point B.

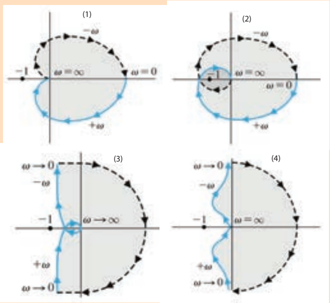




Nyquist Plot

Sketching Nyquist plot

Find the corresponding transfer function for each Nyquist plot :



(A) $L_1(s) = \dfrac{K(\tau_3 s + 1)}{s(\tau_1 s + 1)(\tau_2 s + 1)}$

(B) $L_2(s) = \dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$

(C) $L_3(s) = \dfrac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$

(D) $L_4(s) = \dfrac{K}{s^2(\tau_1 s + 1)}$

(E) $L_5(s) = \dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$

$K > 0, \tau_1 > 0, \tau_2 > 0, \tau_3 > 0$

Robust Stability Margins

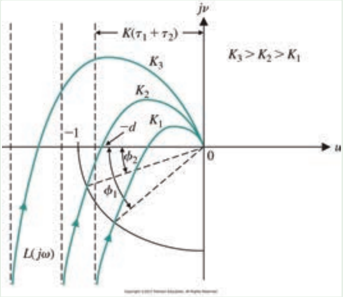
- We can check the stability of a closed-loop system given the controller D_c and the plant model G using the Routh criterion or using the Nyquist stability criterion.
- Using the Nyquist stability criterion we can define the concept of relative stability by a measure of the closeness of the Nyquist plot to the critical point.

Example

Consider

$$L(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

with different value of K. It is clear that by increasing K the system becomes less stable and even unstable.

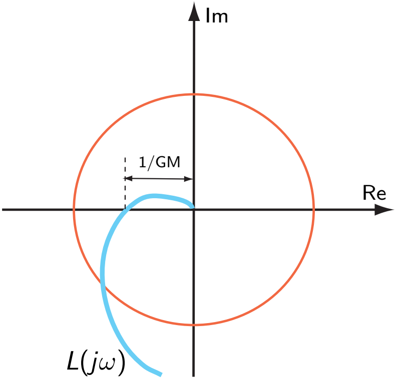


Robust Stability Margins

Gain Margin

The gain margin GM is equal to the inverse of the distance between the origin and the crossing point of $L(j\omega)$ and the negative real axis.

- For closed-loop stable systems, the gain margin shows the amount that gain can increase to make the system marginally stable (with poles on the imaginary axis).
- Usually expressed in dB. A value between 4dB and 12dB is considered safe.
- The frequency at the crossing point, ω_{cr}, is called the critical frequency.
- If there are multiple intersections with the negative real axis the smallest value will be chosen (closest to -1).



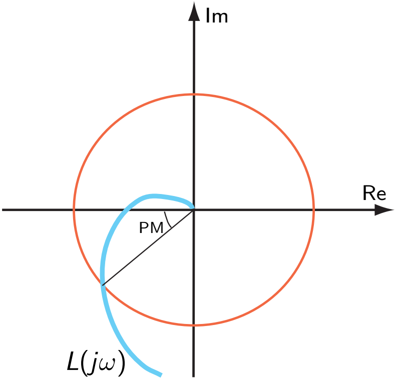
$$\angle L(j\omega_{cr}) = -\pi \,, \; \text{GM} = |L(j\omega_{cr})|^{-1}$$

Robust Stability Margins

Phase Margin

The phase margin PM is the angle of $L(j\omega_c)$ with the negative real axis, where ω_c is the frequency of $L(j\omega)$ when crossing the unit circle.

- For a closed-loop stable system, the phase margin shows the amount that phase can decrease to make the system marginally stable (with poles on the imaginary axis).
- Typically, a phase margin between 30° and 60° is considered safe.
- The frequency at the crossing point, ω_c, is called the crossover frequency.
- If there are multiple intersections with the unit circle the smallest phase will be chosen (closest to -1).

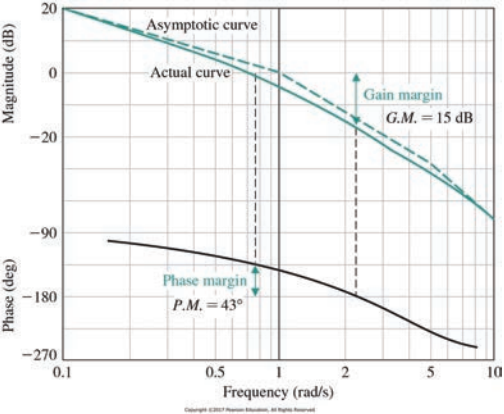


$$|L(j\omega_c)| = 1\,,\ \text{PM} = 180 + \angle L(j\omega_c)$$

Robust Stability Margins

Gain and Phase margin in the Bode diagram of $L(j\omega)$:

- **Gain margin :** On the phase plot find ω_{cr} at which the phase is -180°. The gain margin is $-20\log|L(j\omega_{cr})|$.
- **Phase margin :** On the magnitude plot find ω_c at which the amplitude is 0dB. The phase margin is $180 + \angle L(j\omega_c)$.

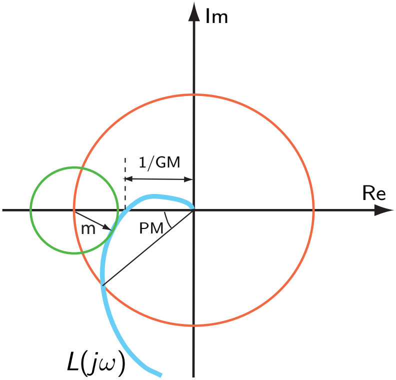


Modulus Margin or Vector Margin

For a closed-loop stable system, the modulus margin is the radius of the smallest circle centred at the critical point and tangent to $L(j\omega)$:

$$m = \inf_{\omega} |1 + L(j\omega)|$$

- A typical value for the modulus margin is $m = 0.5$ (-6dB).
- A modulus margin of m, guarantees a gain margin greater than $\dfrac{1}{1-m}$ and a phase margin of at least $2 \arcsin \frac{m}{2}$.
- A modulus margin of 0.5 guarantees a gain margin of at least 6dB and a phase margin of at least 29°.

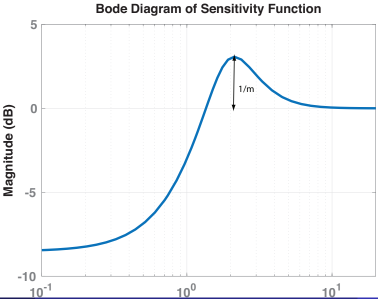


Robust Stability Margins

Modulus margin from the Bode diagram of the sensitivity function :
The modulus margin is the inverse of the maximum value of the magnitude Bode plot of $\mathcal{S}(j\omega)$, because :

$$\mathcal{S}(j\omega) = \frac{1}{1 + L(j\omega)} \quad \Rightarrow \quad |1 + L(j\omega)| = \frac{1}{|\mathcal{S}(j\omega)|}$$

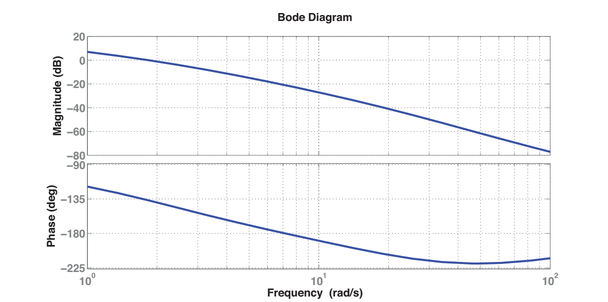
$$\Rightarrow \quad m = \inf_{\omega} |1 + L(j\omega)| = \inf_{\omega} \frac{1}{|\mathcal{S}(j\omega)|} = \frac{1}{\sup_{\omega} |\mathcal{S}(j\omega)|}$$



Stability Margins

Exercise

The frequency response of the open-loop transfer function is given by :



Find the following information from the diagram :

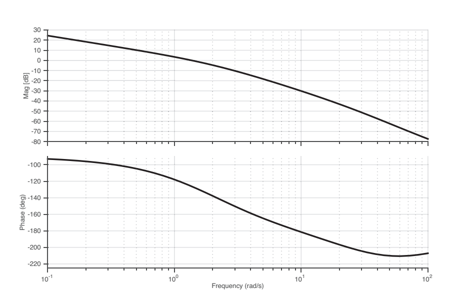
	(A)	(B)	(C)	(D)
ω_{cr}	1.8	7	1	10^4
ω_c	1.8	10	7	45
GM	20dB	-20dB	20	0.1
PM	60°	90°	135°	45°

Stability margins

Question : Exam 2015

Le diagramme de Bode de $G(s) = \dfrac{s+100}{s(s+2)(s+30)}$ est donné. Avec un régulateur proportionnel $D_c(s) = 10$, calculer la marge de gain et la marge de phase approximative (0.2 point).

(A) GM $\approx$ 30dB, PM $\approx 55^\circ$
(B) GM $\approx$ 20dB, PM $\approx 30^\circ$
(C) GM $\approx$ 10dB, PM $\approx 15^\circ$
(D) GM $\approx$ 30dB, PM $\approx 30^\circ$

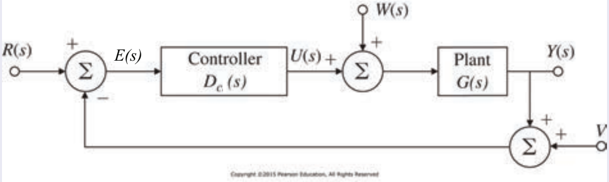


Loop Shaping Method

Typical shapes of closed-loop transfer functions

Transfer function between reference and output : is typically a low-pass filter.

$$\mathcal{T}(s) = \frac{G(s)D_c(s)}{1 + G(s)D_c(s)}$$



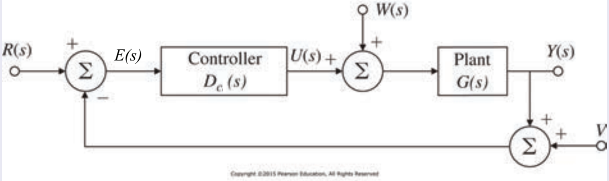
- It should be close to one in low frequencies to have a good tracking property ($y(t) \approx r(t)$).
- It should be small in high frequencies to reject the measurement noise.
- The closed-loop bandwidth ω_{BW} is the frequency where the amplitude is 3 dB less than the steady-state value.
- A peak in the magnitude of $\mathcal{T}$ gives an overshoot in the step response.
- It is strictly proper, i.e. $\mathcal{T}(j\infty) = 0$, because $G(s)$ is strictly proper.

Loop Shaping Method

Typical shapes of closed-loop transfer functions

Transfer function between reference and error : is typically a high-pass filter.

$$\mathcal{S}(s) = \frac{1}{1 + G(s)D_c(s)}$$



- It should be very small in low frequencies to have a good tracking property ($y(t) \approx r(t)$).
- It will be close to one in high frequencies because $G(j\infty) = 0$.
- The peak value of $|\mathcal{S}(j\omega)|$ is a measure of closed-loop robustness (smaller peak means more robust).
- The attenuation band or disturbance rejection band ω_{DRB} is the frequency where the amplitude is -3 dB.

Loop Shaping Method

Example

For a typical second-order open-loop system,

$$L(s) = G(s)D_c(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},$$ we have the following curves :

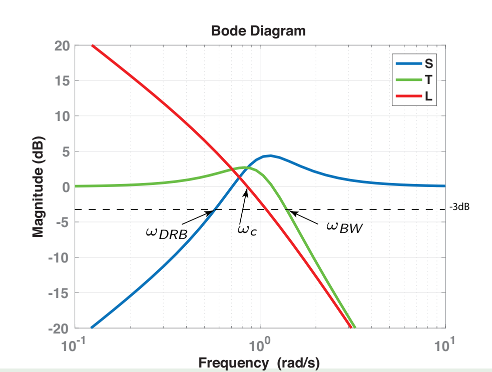
$$\omega_{BW} = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$$

$$\omega_c = \omega_n\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$$

$$\omega_{DRB} = \omega_n\sqrt{-1-2\zeta^2+\sqrt{2+4\zeta^2+4\zeta^4}}$$

In general, we have : $\omega_c \leq \omega_{BW} \leq 2\omega_c$
and

$$\text{PM} \approx 100\zeta$$



Remark : For a first order open-loop system $L(s) = K/s$, we have

$$\omega_{BW} = \omega_c = \omega_{DRB} = K$$

Robustness-Performance Trade-Off

Tracking performance for sinusoidal signals : Consider $r(t) = \sin(\omega t)$. If the closed-loop system is stable, the tracking error at steady state will be given by :

$$\lim_{t\to\infty} e(t) = \left|\frac{1}{1 + D_c(j\omega)G(j\omega)}\right| \sin(\omega t + \phi) = |\mathcal{S}(j\omega)| \sin(\omega t + \phi)$$

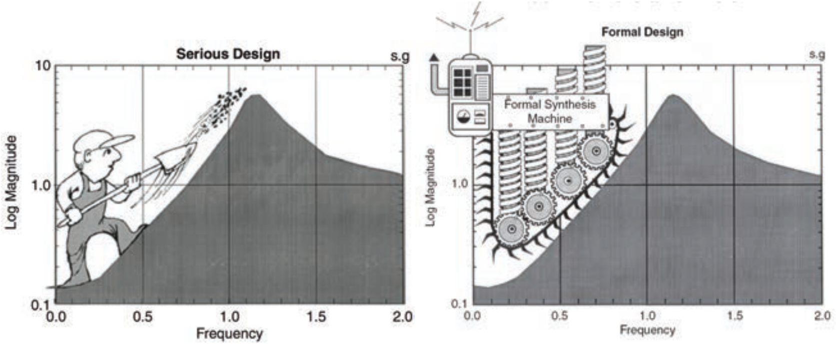
- If $|\mathcal{S}(j\omega)| \ll 1$ for $\omega \in [0\,,\,\omega_0]$, we will have good tracking performance for sinusoidal signals with frequencies $< \omega_0$.
- So larger ω_0 gives better tracking performance (larger bandwidth).
- Can we make ω_0 as large as we wish ?

Theorem (Bode Sensitivity Integral)

The sensitivity function of a closed-loop stable system with a strictly proper $L(s)$ that has P RHP poles satisfies the following equality :

$$\int_0^\infty \ln|\mathcal{S}(j\omega)| d\omega = \pi \sum_{i=1}^{P} R_e\{p_i\}$$

Robustness-Performance Trade-Off



- The integral of $\ln|\mathcal{S}(e^{j\omega})|$ is constant. So small tracking errors in low frequencies will be paid by large errors (signal amplification) at high frequencies.
- Improving the tracking performance can increase the maximum of $|\mathcal{S}(\omega)|$ and reduce the modulus margin $m = (\sup_\omega |\mathcal{S}(\omega)|)^{-1}$.

Controller Design

Controller synthesis : Given a plant model, design a controller to satisfy a set of performance and robustness specifications :

- **Steady-state performance :**
  - Zero steady-state error for step, ramp or parabolic signals ;
  - Desired steady state gain (position, velocity or acceleration) ;
  - Error bounds for tracking or rejection of sinusoidal signals in a given frequency range.
- **Transient performance :**
  - Desired rise-time in step response (tracking) ;
  - Desired settling-time in step response (tracking and regulation) ;
  - Maximum overshoot ;
  - Desired bandwidth ;
  - Desired reference model.
- **Robustness specifications :**
  - Gain margin, phase margin, modulus margin ;
  - Sensitivity to measurement noise ;
  - Sensitivity to unmodelled dynamics in high frequencies.

Loop-Shaping method

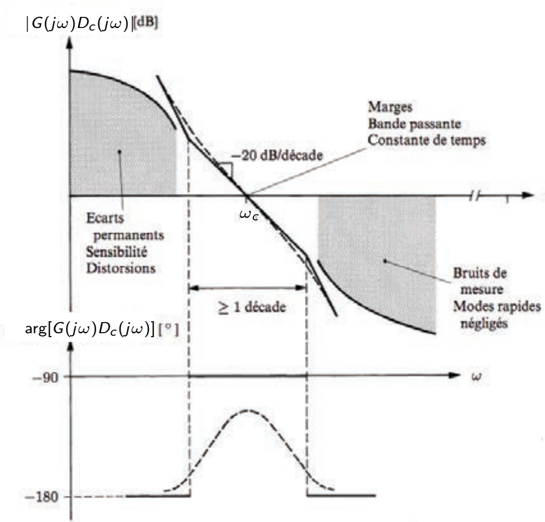
Loop-Shaping method : For a given stable plant G, the controller $D_c(s)$ is designed such that the magnitude of open loop transfer function $L(j\omega) = G(j\omega)D_c(j\omega)$ has a desired shape.

- The steady-state performance can be translated to some lower bounds on the magnitude frequency response of the open-loop transfer function.
- The transient performance is closely related to the closed-loop bandwidth which is close to the crossover frequency ω_c.
- A desired reference model can be used to compute a desired open-loop transfer function.
- The gain and phase margin can be easily tuned in the Bode diagram.
- The sensitivity to measurement noise and unmodelled dynamics can be represented by some upper bound on the magnitude frequency response of the open-loop transfer function.

Loop-Shaping method

The Frequency Response of the Desired Open-Loop Transfer Function

- Large at low frequencies ;
- Small at high frequencies ;
- Slope of -20 dB/dec when crossing the 0dB axis.



Loop-Shaping method

Why "large at low frequencies"? For good steady-state performance.

- Integrators in open-loop transfer function increase the gain at low frequencies :

$$\left|\frac{1}{(j\omega)^{\ell}}\right| \gg 1 \qquad \text{for} \qquad \omega \ll 1$$

- Small steady-state error for step, ramp, etc. requires large steady-state gain :

$$K_p = \lim_{s\to 0} D_c(s)G(s)\,;\; K_v = \lim_{s\to 0} sD_c(s)G(s)\,;\; K_a = \lim_{s\to 0} s^2 D_c(s)G(s)$$

  Large gain at $\omega = 0$ leads to large gain at low frequencies.

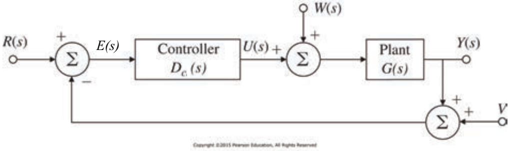
- Small steady-state error for tracking low frequency sinusoidal signals :

$$|\mathcal{S}(j\omega)| = \left|\frac{1}{1 + D_c(j\omega)G(j\omega)}\right| < \epsilon \quad \text{for} \quad 0 < \omega < \omega_l$$

$$\Rightarrow \quad |1 + D_c(j\omega)G(j\omega)| > \epsilon^{-1} \quad \Rightarrow \quad |D_c(j\omega)G(j\omega)| > \epsilon^{-1}$$

Why "small at high frequencies"?

- **Reducing the effect of measurement noise :**



Noise is a high frequency signal, so $|\mathcal{T}(j\omega)|$ should be small at high frequencies.

$$\left|\frac{D_c(j\omega)G(j\omega)}{1+D_c(j\omega)G(j\omega)}\right| \approx |D_c(j\omega)G(j\omega)| < \epsilon$$

- **Improving the robustness :** Fast high frequency modes are usually not modelled. Neglected time delay $e^{-T\omega}$ leads to a phase lag of $T\omega$ which is large at high frequencies. So modelling error is large at high frequencies. Therefore, to be far from the critical point at high frequencies, we should have : $|D_c(j\omega)G(j\omega)| \ll 1$

Loop-Shaping method

Why "slope of -20 dB/dec when crossing the 0dB axis"?

- It corresponds locally, around ω_c, to an integrator :

$$D_c(j\omega)G(j\omega) \approx \frac{\omega_c}{j\omega} \quad \Rightarrow \quad \arg\left(D_c(j\omega_c)G(j\omega_c)\right) \approx -90$$

  So it leads to a good phase margin of around 90°.

- Moreover, with this -20 dB/dec slope, the closed-loop bandwidth ω_{BW} will be very close to the crossover frequency ω_c. We have :

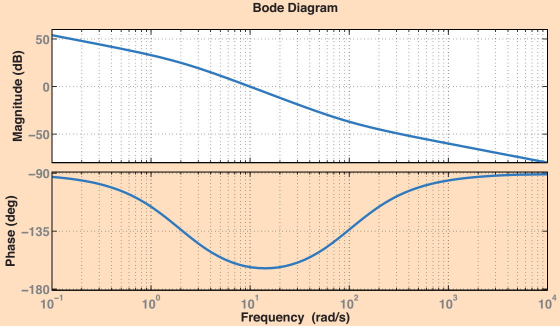
$$\left|\frac{D_c(j\omega_{BW})G(j\omega_{BW})}{1 + D_c(j\omega_{BW})G(j\omega_{BW})}\right| = -3dB$$

$$\left|\frac{D_c(j\omega_c)G(j\omega_c)}{1 + D_c(j\omega_c)G(j\omega_c)}\right| = \frac{1}{|1 + D_c(j\omega_c)G(j\omega_c)|} \approx \frac{1}{\sqrt{2}} = -3dB$$

- A slope of -40 dB/dec corresponds locally to a double integrator and leads to a very small phase margin and is not acceptable.

Exercise

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller k_P such that



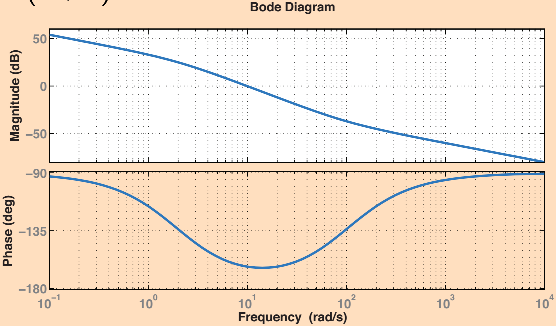
The phase margin is equal to 45° :

A) $k_P = 25$ or $k_P = -37$ **B)** $k_P = -25$ or $k_P = 37$
C) $k_P = 0.056$ or $k_P = 70$ **D)** $k_P = 17.8$ or $k_P = 70$

Loop Shaping Method

Exercise

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller k_P such that



The crossover frequency is equal to 30 rad/s :

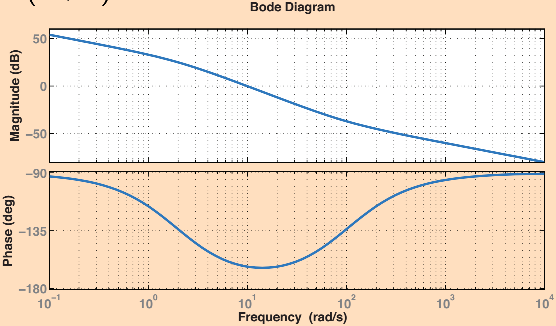
A) $k_P = 20$ **B)** $k_P = 10$

C) $k_P = 0.1$ **D)** $k_P = -20$

Loop Shaping Method

Exercise

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller K_p such that



The steady-state error for tracking a ramp signal is 0.01 :

A) $k_P = 2$ **B)** $k_P = 10$

C) $k_P = 20$ **D)** $k_P = 100$

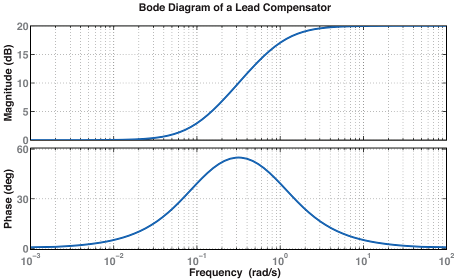
Lead-Lag Compensator

Lead compensator :

is a first order transfer function that has some positive phase ($\alpha > 1$).

$$C(s) = \frac{1 + \alpha\tau s}{1 + \tau s}$$

- It has one pole at $-1/\tau$ and one zero at $-1/(\alpha\tau)$. So its zero has smaller frequency than its pole.



- The maximum positive phase appears at $\omega_m = \frac{1}{\tau\sqrt{\alpha}}$

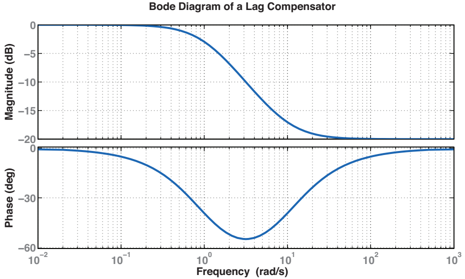
Lead-Lag Compensator

Lag compensator :

is a first order transfer function that has some negative phase $(0 < \alpha < 1)$.

$$C(s) = \frac{1 + \alpha\tau s}{1 + \tau s}$$

- It has one pole at $-1/\tau$ and one zero at $-1/(\alpha\tau)$. So its pole has smaller frequency than its zero.



- The maximum negative phase appears at $\omega_m = \frac{1}{\tau\sqrt{\alpha}}$

Lead-Lag Compensator

Loop-Shaping controller structure : This controller consists of a proportional gain, one or more integrators and some lead-lag compensators.

$$D_c(s) = K \frac{1}{s^\ell} \prod_{i=1}^{m} \frac{1 + \alpha_i \tau_i s}{1 + \tau_i s}$$

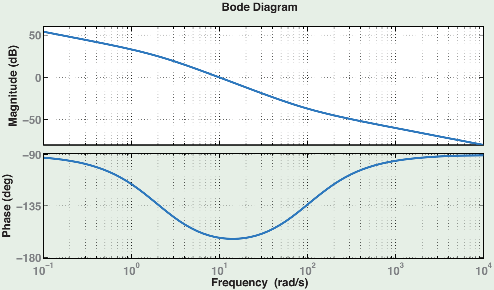
Controller design steps :

1. Select ℓ according to the steady-state performance specifications and number of integrators in the plant model.
2. Compute K according to the desired steady-state error for a given type of reference or disturbance signal.
3. Design one or more lead-lag compensators to achieve the desired crossover frequency, desired phase margin or noise attenuation at high frequencies.

Lead-Lag Compensator

Example

Consider $G(s) = \dfrac{(s+100)}{s(s+2)}$. How much gain and phase to have a phase margin of 60° and a crossover frequency of 30 rad/s ?



We need a lead compensator to give a contribution of 20dB in the magnitude and 40° in the phase at $\omega_c = 30$ rad/s :

$$|G(j\omega_c)| \approx -20dB \qquad \arg(G(j\omega_c)) \approx -160^\circ$$

Lead-Lag Compensator

Magnitude contribution $\sqrt{c}$ at a given frequency ω_l :

$$c = \left|\frac{1 + j\alpha\tau\omega_l}{1 + j\tau\omega_l}\right|^2 = \frac{1 + (\alpha\tau\omega_l)^2}{1 + (\tau\omega_l)^2}$$

Phase contribution ϕ at a given frequency ω_l :

$$\begin{aligned}\phi &= \arg\left(\frac{1 + j\alpha\tau\omega_l}{1 + j\tau\omega_l}\right) = \arg\left(\frac{(1 + j\alpha\tau\omega_l)(1 - j\tau\omega_l)}{1 + (\tau\omega_l)^2}\right) \\ &= \arg(1 + \alpha(\tau\omega_l)^2 + j(\alpha\tau\omega_l - \tau\omega_l)) \quad \Rightarrow \quad p = \tan\phi = \frac{\alpha\tau\omega_l - \tau\omega_l}{1 + \alpha(\tau\omega_l)^2}\end{aligned}$$

Lead-Lag compensator design : Given ω_l, c and p, find α and τ.
Eliminating $\tau\omega_l$ from the above equations, we can obtain α by solving :

$$(p^2 - c + 1)\alpha^2 + 2p^2c\alpha + p^2c^2 + c^2 - c = 0$$

For a lead compensator $\alpha > 1$ and for a lag compensator $0 < \alpha < 1$ is acceptable. Then τ can be obtained from : $\tau = \frac{1}{\omega_l}\sqrt{\frac{1 - c}{c - \alpha^2}}$

Lead-Lag Compensator

Example

Compute a lead compensator for the previous example to add a phase of 40° and amplitude of 20dB at $\omega_l = \omega_c = 30$ rad/s.

$$\sqrt{c} = 20dB = 10 \Rightarrow c = 100 \quad ; \quad p = \tan 40^\circ = 0.84$$

We should solve :

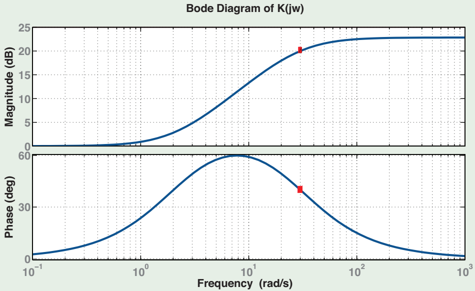
$$(p^2 - c + 1)\alpha^2 + 2p^2 c\alpha + p^2 c^2 + c^2 - c = 0 \quad \Rightarrow \quad \alpha = 13.86$$

Therefore :

$$\tau = \frac{1}{\omega_c}\sqrt{\frac{1-c}{c-\alpha^2}} = 0.0345$$

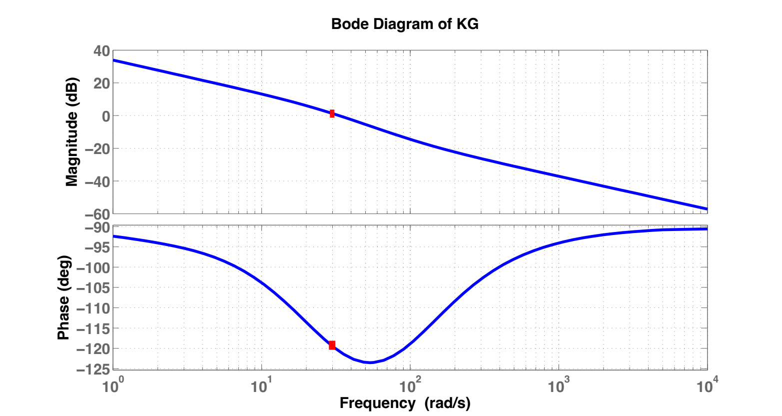
and

$$D_c(s) = \frac{1+\alpha\tau s}{1+\tau s} = \frac{1+0.4788s}{1+0.0345s}$$



Example

We can check the final result by drawing the Bode Plots of $G(j\omega)D_c(j\omega)$:



The phase margin is equal to 60° and the crossover frequency is 30 rad/s.

Loop-Shaping Method

Example

Given

$$G(s) = 0.51 \times 10^{-5} \frac{(s + 9760)(80 - s)}{s(s + 2.05)}$$

Compute a controller to have :

- Zero steady-state error for tracking a step signal.
- A steady-state error of 0.05 for tracking a ramp signal.
- A crossover frequency of $\omega_c = 14.3$.
- A phase margin of at least 60°.

Loop-Shaping Method

Example

Step-by-Step procedure :

- First specification requires a type 1 open-loop transfer function. Since the plant model has an integrator, it is no need to add an integrator in $D_c(s)$. Therefore, $\ell = 0$ and : $D_c(s) = K\dfrac{1+\alpha\tau s}{1+\tau s}$
- The steady-state error for a ramp signal is $1/K_v$ that can be computed as :

$$K_v = \frac{1}{0.05} = \lim_{s\to 0} sD_c(s)G(s) = K\frac{0.51\times 10^{-5}\times 9760\times 80}{2.05}$$

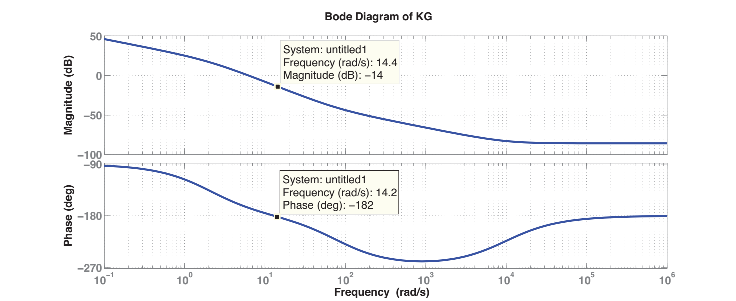
$$\Rightarrow K = 10.29$$

Loop-Shaping Method

Example

Step-by-Step procedure :

- Draw the Bode Plots of $KG(j\omega)$



A crossover frequency of 14.3 and a phase margin of 60° require a magnitude contribution of 14dB and a phase contribution of 62° at $\omega_c = 14.3$.

$$\sqrt{c} = 14dB = 5 \quad \Rightarrow c = 25 \quad \text{and} \quad p = \tan(62^\circ) = 1.88$$

Example

Step-by-Step procedure :

- First compute α by solving :

$$(p^2 - c + 1)\alpha^2 + 2p^2 c\alpha + p^2 c^2 + c^2 - c = 0 \quad \Rightarrow \quad \alpha = 16.81$$

- Then compute τ from

$$\tau = \frac{1}{\omega_c}\sqrt{\frac{1-c}{c-\alpha^2}} = 0.0213$$

- The final controllers, which is a PD controller, reads :

$$D_c(s) = K\frac{1+\alpha\tau s}{1+\tau s} = K\left(1 + \frac{(\alpha-1)\tau s}{1+\tau s}\right) = 10.3\left(1 + \frac{0.3375s}{1+0.02135s}\right)$$

- This controller leads to a phase margin of 60° and a gain margin of 14.6dB.

Loop-Shaping Method

- A lead compensator has a stabilizing effect and can be used to improve the phase margin.
- The desired phase and magnitude contribution may not be achievable with one lead compensator. In this case, the second order equation will not have an admissible solution for α. A remedy is to implement two lead compensators such that each compensator gives half of the required contributions.
- A lead compensator increases the open-loop gain at high frequencies, so it reduce the performance for noise rejection.
- A lag compensator can be used at very high frequencies to reduce the open loop gain (with the same procedure, however the contribution in magnitude and phase are negative).
- A lead compensator can be used to increase the bandwidth but it leads to higher control signal that may not be implementable (because of saturation).
- A lag compensator can be used to reduce the bandwidth and avoid the saturation of the control signal.

Loop-Shaping Method

PID controller design :

- Only magnitude Bode Plots is used.
- A good phase margin is obtained by ensuring a slope of -20 dB/dec for the open-loop transfer function.

PD controller design by Loop-Shaping :

$$D_c(s) = K_p(1 + T_d s)$$

- K_p is chosen to satisfy the steady-state performance.
- T_d is computed to obtain the desired crossover frequency.

PI controller design by Loop-Shaping :

$$D_c(s) = K_p(1 + \frac{1}{T_i s}) = \frac{K_p}{s}(s + \frac{1}{T_i})$$

- T_i is chosen such that the dominant pole of the plant model is canceled. This will not change the slope of $L = GD_c$ around the crossover frequency.
- K_p is chosen to obtain the desired crossover frequency.

Loop-Shaping Method

Example (PI controller design)

Given $G(s) = 0.05\dfrac{80 - s}{s + 2.05}$, compute a PI controller to have :

- Zero steady-state error for tracking a step signal.
- A crossover frequency of $\omega_c = 14.3$.
- A slope of -20 dB/dec around the crossover frequency.

Solution :

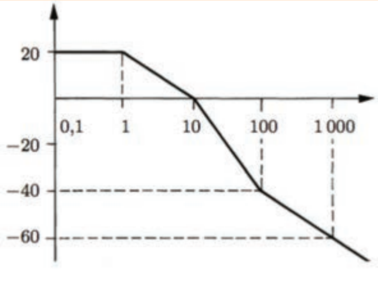
- $T_i = 1/2.05$ to cancel the plant model pole and have a slope of -20 dB/dec.
- K_p is chosen to have $\omega_c = 14.3$: $\quad |D_c(j\omega_c)G(j\omega_c)| = 1$

$$\Rightarrow \quad \left| 0.05\frac{K_p}{j\omega_c}(80 - j\omega_c) \right| = 1 \quad \Rightarrow \quad K_p = 3.52$$

Loop-Shaping Method

Exercise

The magnitude Bode diagram of $G(s)$ is given :



Specifications :

- Crossover frequency of $\omega_c = 30$ rad/s.
- Zero steady-state error (for a step reference).

Which type of controller is adequate ?
(A) P **(B)** PD **(C)** PI **(D)** PID

1. What are reasonable values of T_i and T_d ?
   (A) $T_i = T_d = 1$ **(B)** $T_i = 1; T_d = 30$
   (C) $T_i = T_d = 100$ **(D)** $T_i = T_d = 30$
2. What is the value of K_p (series structure) ?
   (A) $K_p = 30$ **(B)** $K_p = 0.33$ **(C)** $K_p = 1$ **(D)** $K_p = 3$

Loop-Shaping Method

- The loop shaping method can be applied only to stable plant models.
- Second order compensators can be used to cancel the resonance modes in the plant model :

$$C(s) = \frac{s^2 + 2\zeta_1\omega_1 s + \omega_1^2}{s^2 + 2\zeta_2\omega_2 s + \omega_2^2}$$

  For low-damped resonance modes this compensator is a Notch Filter.
- If $G(s)$ is stable and minimum phase, then the loop shaping controller can be computed as :

$$D_c(s) = L_d(s)G^{-1}(s)$$

  where $L_d(s)$ is the desired open-loop transfer function. Note that $D_c(s)$ will be proper if the relative degree of $L_d(s)$ is greater than or equal to that of the plant model.
- A very simple choice for $L_d(s)$, with relative degree 1, is :

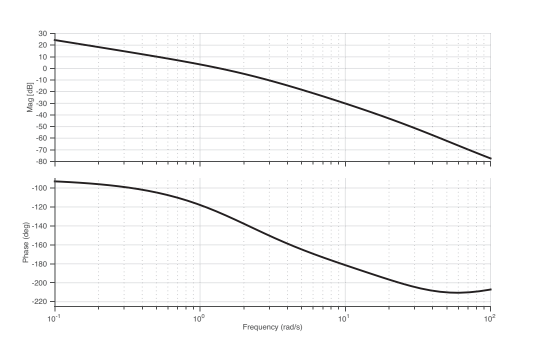
$$L_d(s) = \frac{\omega_c}{s} \quad \Rightarrow \quad L_d(j\omega) = \frac{\omega_c}{j\omega}$$

  The magnitude of $L_d(j\omega)$ is large at low frequencies, small at high frequencies and has a slope of -20 dB/dec when crossing the zero dB at the desired band width $\omega_{BW} \approx \omega_c$.

Loop-Shaping Method

Exercise

The Bode diagram of $G(s) = \dfrac{s+100}{s(s+2)(s+30)}$ is given :



Design a PID controller with the Loop-Shaping method to follow a ramp with no steady-state error, obtain a crossover frequency of 15 rad/s and a phase margin of around 60 degrees.