Starting parameters for lumfit

1 Logistic function

Standards: log of concentration $x_{(1)}, \ldots, x_{(n)}$ with corresponding log of summary (median, mean) MFI $y_{(1)}, \ldots, y_{(n)}$. We want to fit a logistic function to these standards:

$$y(x) = A_0 + \frac{A}{\left(1 + e^{-\frac{x - x_0}{s}}\right)^a} \tag{1}$$

1.1 Lower asymptote A_0 unknown

First, we choose three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, e.g. $x_1 = x_{(1)}, x_2 = x_{(\lceil n/2 \rceil)}, x_3 = x_{(n)}$. Next, find parameters A_0, A, s, x_0, and a such that the function passes through these points while minimizing some loss in regards to the remaining points (e.g. $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$):

$$\begin{aligned}
y_1 &= A_0 + \frac{A}{\left(1 + e^{-\frac{x_1 - x_0}{s}}\right)^a} \\
y_2 &= A_0 + \frac{A}{\left(1 + e^{-\frac{x_2 - x_0}{s}}\right)^a} \\
y_3 &= A_0 + \frac{A}{\left(1 + e^{-\frac{x_3 - x_0}{s}}\right)^a}
\end{aligned} \tag{2}$$

Reparameterize the model to get bounds on some of the parameters: let $\bar{w} = e^{\frac{x_0}{s}}$ and $v = e^{-\frac{1}{s}}$. Then

$$
\begin{aligned}
y_1 &= A_0 + \frac{A}{\left(1 + \bar{w}\, v^{x_1}\right)^a} \\
y_2 &= A_0 + \frac{A}{\left(1 + \bar{w}\, v^{x_2}\right)^a} \\
y_3 &= A_0 + \frac{A}{\left(1 + \bar{w}\, v^{x_3}\right)^a}.
\end{aligned}
\tag{3}
$$

Alternatively, we can set $w = e^{-\frac{x_1 - x_0}{x}}$ and keep $v = e^{-\frac{1}{s}}$. Then

$$
\begin{aligned}
y_1 &= A_0 + \frac{A}{(1 + w)^a} \\
y_2 &= A_0 + \frac{A}{\left(1 + w\, v^{x_2 - x_1}\right)^a} \\
y_3 &= A_0 + \frac{A}{\left(1 + w\, v^{x_3 - x_1}\right)^a}
\end{aligned}
\tag{4}
$$

To remove A_0:

$$
\begin{aligned}
y_2 - y_1 &= A\left[\frac{1}{\left(1 + \bar{w}\, v^{x_2}\right)^a} - \frac{1}{\left(1 + \bar{w}\, v^{x_1}\right)^a}\right] \\
&= A\left[\frac{1}{\left(1 + w\, v^{x_2 - x_1}\right)^a} - \frac{1}{(1 + w)^a}\right] \\
y_3 - y_1 &= A\left[\frac{1}{\left(1 + w\, v^{x_3 - x_1}\right)^a} - \frac{1}{(1 + w)^a}\right].
\end{aligned}
\tag{5}
$$

Next step is to remove [get rid of] A. Let $b = \frac{y_3 - y_1}{y_2 - y_1}$.

$$
\begin{aligned}
b &= \left[\frac{1}{\left(1 + \bar{w}\, v^{x_3}\right)^a} - \frac{1}{\left(1 + \bar{w}\, v^{x_1}\right)^a}\right] \Big/ \left[\frac{1}{\left(1 + \bar{w}\, v^{x_2}\right)^a} - \frac{1}{\left(1 + \bar{w}\, v^{x_1}\right)^a}\right] \\
&= \left[\frac{1}{\left(1 + w\, v^{x_3 - x_1}\right)^a} - \frac{1}{(1 + w)^a}\right] \Big/ \left[\frac{1}{\left(1 + w\, v^{x_2 - x_1}\right)^a} - \frac{1}{(1 + w)^a}\right]
\end{aligned}
\tag{6}
$$

Find w and v or $\bar{w}$ and v.

We can write

$$b - \frac{\left[\frac{1}{\left(1+\bar{w}\,v^{x_3}\right)^a} - \frac{1}{\left(1+\bar{w}\,v^{x_1}\right)^a}\right]}{\left[\frac{1}{\left(1+\bar{w}\,v^{x_2}\right)^a} - \frac{1}{\left(1+\bar{w}\,v^{x_1}\right)^a}\right]} = b - \frac{\left[\frac{1}{\left(1+w\,v^{x_3-x_1}\right)^a} - \frac{1}{(1+w)^a}\right]}{\left[\frac{1}{\left(1+w\,v^{x_2-x_1}\right)^a} - \frac{1}{(1+w)^a}\right]} = 0 \tag{7}$$

Since the value of b is known from the data (standards), we can assign various values to v and calculate w or $\bar{w}$ for each of these values so that they satisfy (7) (the root of equation (7) in regard to w or $\bar{w}$). This can be found numerically - having pairs (w, v) or $(\hat{w}, v)$, find A:

$$A = \frac{y_3 - y_1}{\frac{1}{\left(1+\bar{w}\,v^{x_3}\right)^a} - \frac{1}{\left(1+\bar{w}\,v^{x_1}\right)^a}} = \frac{y_3 - y_1}{\frac{1}{\left(1+w\,v^{x_3-x_1}\right)^a} - \frac{1}{(1+w)^a}}, \tag{8}$$

or similarly it can be done through $(y_2 - y_1)/(\dots)$.

Then we can find A_0 through y_1 or y_2 or y_3, e.g.

$$A_0 = y_3 - \frac{A}{(1+\bar{w}\,v^{x_3})^a} = y_3 - \frac{A}{(1+w\,v^{x_3-x_1})^a}. \tag{9}$$

For some situations, a special case when $a = 1$ (the curve is symmetric about a middle point) can be desired; for others, fitting asymmetry parameter a can be beneficial. Still, subsequent optimization performs well in terms of fitting a even when the starting value is 1, so we set $a = 1$ for this stage. Then we can rewrite the difference

$$y_2 - y_1 = A\left[\frac{1}{1+\bar{w}\,v^{x_2}} - \frac{1}{1+\bar{w}\,v^{x_1}}\right] = A\,\frac{1+\bar{w}\,v^{x_2} - 1 - \bar{w}\,v^{x_2}}{\left(1+\bar{w}\,v^{x_2}\right)\left(1+\bar{w}\,v^{x_1}\right)}$$

$$= A\,\bar{w}\,\frac{v^{x_1} - v^{x_2}}{\left(1+\bar{w}\,v^{x_2}\right)\left(1+\bar{w}\,v^{x_1}\right)}$$

$$= A\,\bar{w}\,v^{x_1}\,\frac{1 - v^{x_2-x_1}}{\left(1+\bar{w}\,v^{x_2}\right)\left(1+\bar{w}\,v^{x_1}\right)} = A\,w\,\frac{1 - v^{x_2-x_1}}{(1+w)\left(1+w\,v^{x_2-x_1}\right)} \tag{10}$$

$$y_3 - y_1 = A\,\bar{w}\,v^{x_1}\,\frac{1 - v^{x_3-x_1}}{\left(1+\bar{w}\,v^{x_3}\right)\left(1+\bar{w}\,v^{x_1}\right)} = A\,\frac{w}{1+w}\,\frac{1 - v^{x_3-x_1}}{1+w\,v^{x_3-x_1}} \tag{11}$$

By dividing (11) by (10), we get

$$b = \frac{\left(1 - v^{x_3-x_1}\right)}{\left(1+\bar{w}\,v^{x^3}\right)}\,\frac{\left(1+\bar{w}\,v^{x_2}\right)}{\left(1 - v^{x_2-x_1}\right)} = \frac{\left(1 - v^{x_3-x_1}\right)\left(1+w\,v^{x_2-x_1}\right)}{\left(1+w\,v^{x_3-x_1}\right)\left(1 - v^{x_2-x_1}\right)}. \tag{12}$$

Let

$$b_v = b\,\frac{1-v^{x_2-x_1}}{1-v^{x_3-x_1}} = \frac{1+\bar{w}\,v^{x_2}}{1+\bar{w}\,v^{x_3}} = \frac{1+w\,v^{x_2-x_1}}{1+w\,v^{x_3-x_1}} \tag{13}$$

From that, we can get w or $\bar{w}$ directly:

$$\begin{aligned}
&b_v\left(1+w, v^{x_3-x_1}\right) = 1+w\,v^{x_2-x_1}\\
&b_v + b_v\,w\,v^{x_3-x_1} = 1+w\,v^{x_2-x^1}\\
&b_v - 1 = w\,v^{x_2-x_1} - w\,b_v\,v^{x_3-x_1} = w\,v^{x_2-x_1}\left(1-b_v\,v^{x_3-x_2}\right) = \bar{w}\,v^{x_2}\left(1-b_v\,v^{x_3-x_2}\right)\\
&w = \frac{b_v-1}{1-b_v\,v^{x_3-x_2}}\,v^{x_1-x_2} && (14)\\
&\bar{w} = \frac{b_v-1}{1+b_v\,v^{x_3-x_2}}\,v^{-x_2} = \frac{b_v-1}{v^{x_2}-b_v\,v^{x_3}}. && (15)
\end{aligned}$$

Further,

$$A = (y_3-y_1)\,\frac{1+w}{w}\,\frac{1+w\,v^{x_3-x_1}}{1-v^{x_3-x_1}} = (y_3-y_1)\,\frac{1+\bar{w}\,v^{x_1}}{\bar{w}\,v^{x_1}}\,\frac{1+\bar{w}\,v^{x_3}}{1-v^{x_3-x_1}}, \tag{16}$$

followed by (9).

1.2 Lower asymptote known

In this case, A_0 in (1) is a known (given) value. Let

$$y_0(x) = y(x) - A_0. \tag{17}$$

Then instead of (3) and (4) we get:

$$\begin{aligned}
y_{01} &= \frac{A}{\left(1+\bar{w}\,v^{x_1}\right)^a}\\
y_{02} &= \frac{A}{\left(1+\bar{w}\,v^{x_2}\right)^a}\\
y_{03} &= \frac{A}{\left(1+\bar{w}\,v^{x_3}\right)^a} && (18)
\end{aligned}$$

Or:

$$y_{01} = \frac{A}{\left(1+w\right)^a}$$
$$y_{02} = \frac{A}{\left(1+w\,v^{x_2-x_1}\right)^a}$$
$$y_{03} = \frac{A}{\left(1+w\,v^{x_3-x_1}\right)^a} \tag{19}$$

Let

$$B_2 = \frac{y_{02}}{y_{01}}, \qquad B_3 = \frac{y_{03}}{y_{01}} \quad \text{and} \tag{20}$$
$$b_2 = B_2^{\frac{1}{a}} = \left(\frac{y_{02}}{y_{01}}\right)^{\frac{1}{a}}, b_3 = B_3^{\frac{1}{a}} = \left(\frac{y_{03}}{y_{01}}\right)^{\frac{1}{a}}. \tag{21}$$

We can see that

$$b_2 = \frac{1+w}{1+w\,v^{x_2-x_1}}$$
$$b_3 = \frac{1+w}{1+w\,v^{x_3-x_1}}, \text{ and therefore} \tag{22}$$
$$b_2 + b_2\,w\,v^{x_2-x_1} = 1+w$$
$$b_3 + b_3\,w\,v^{x_3-x_1} = 1+w, \tag{23}$$

from which we can easily find w for given v or find v for given w.

$$v = \left(\frac{1+w-b_2}{b_2\,w}\right)^{\frac{1}{x_2-x_1}} \text{ or } v = \left(\frac{1+w-b_3}{b_3\,w}\right)^{\frac{1}{x_3-x_1}} \tag{24}$$
$$w = \frac{b_2-1}{1-b_2\,v^{x_2-x_1}} \qquad \text{or } w = \frac{b_3-1}{1-b_3\,v^{x_3-x_1}} \tag{25}$$

$$A = y_{02}\left(1+w\,v^{x_2-x_1}\right)^a \text{ or}$$
$$A = y_{03}\left(1+w\,v^{x_3-x_1}\right)^a \tag{26}$$

Alternatively, we can use $\bar{w}$ instead of w. Now instead of (22) we get

$$b_2 = \frac{1+\bar{w}\,v^{x_1}}{1+\bar{w}\,v^{x_2}}$$

$$b_3 = \frac{1+\bar{w}\,v^{x_1}}{1+\bar{w}\,v^{x_3}}, \text{ and therefore} \tag{27}$$

$$\begin{aligned} b_2 + b_2\,\bar{w}\,v^{x_2} &= 1+\bar{w}\,v^{x_1} \\ b_3 + b_3\,\bar{w}\,v^{x_3} &= 1+\bar{w}\,v^{x_1}. \end{aligned} \tag{28}$$

In this case, finding v is not as convenient but finding $\bar{w}$ is:

$$\bar{w} = \frac{b_2-1}{v^{x_1}-b_2\,v^{x_2}} \text{ or } \bar{w} = \frac{b_3-1}{v^{x_1}-b_3\,v^{x_3}}, \text{ and consequently} \tag{29}$$

$$\begin{aligned} A &= y_{02}\left(1+\bar{w}\,v^{x_3}\right)^a \\ A &= y_{03}\left(1+\bar{w}\,v^{x_2}\right)^a. \end{aligned} \tag{30}$$

Note that these expressions yield different values of A (???).

Now, make the function pass through the designated three points.
Using (29), we can write

$$\frac{b_2-1}{\cancel{v^{x_1}}\left(1-b_2\,v^{x_2-x_1}\right)} = \frac{b_3-1}{\cancel{v^{x_1}}\left(1-b_3\,v^{x_3-x_1}\right)}$$

Let $V \equiv v^{x_2-x_1}$ and $m \equiv \frac{x_3-x_1}{x_2-x_1}$. Then

$$\frac{b_2-1}{1-b_2\,V} = \frac{b_3-1}{1-b_3\,V^m}, \text{ and thus} \tag{31}$$

$$1-b_3\,V^m = \frac{b_3-1}{b_2-1}\left(1-b_2\,V\right) \tag{32}$$

$$b_3\,V^m - b_2\frac{b_3-1}{b_2-1}\,V + \left(\frac{b_3-1}{b_2-1}-1\right) = 0$$

$$V^m - \frac{b_2}{b_3}\,\frac{b_3-1}{b_2-1}\,V + \frac{1}{b_3}\,\frac{b_3-b_2}{b_2-1} = 0. \tag{33}$$

For $m \neq 2$, equation (42) could be solved numerically, but for $m = 2$, there is a nice solution:

$$V^2 - \frac{b_2}{b_3}\,\frac{b_3 - 1}{b_2 - 1}\,V + \frac{b_3 - b_2}{b_3(b_2 - 1)} = 0 \text{ can be written as} \tag{34}$$

$$(V-1)\left(V - \frac{b_3 - b_2}{b_3(b_2 - 1)}\right) = 0. \tag{35}$$

Solution $V = 1$ is not relevant since for every x, $1^x = 1$. Therefore the only applicable solution is

$$V = \frac{b_3 - b_2}{b_3(b_2 - 1)}. \tag{36}$$

Then

$$\begin{aligned} v &= V^{\frac{1}{x_2 - x_1}} \\ w &= \frac{b_2 - 1}{1 - b_2\,V} \qquad \text{or } w = \frac{b_3 - 1}{1 - b_3\,V^m} \\ A &= y_{02}\,(1 + w\,V)^a \text{ or } A = y_{03}\,\left(1 + w\,V^m\right)^a. \end{aligned} \tag{37}$$

Remark:

If we multiply equation (43) by $b_3\,(b_2 - 1)$, we get

$$b_3\,(b_2 - 1)\,V^2 - b_2\,(b_3 - 1)\,V + b_3 - b_2 = 0. \tag{38}$$

When $V = 1$, the equation (47) is true for any b_2 and b_3 and thus 1 is one of the solutions of the equation (43). For the other solution, note that if x_1 and x_2 are the roots of the equation $x^2 - bx + c = 0$, then $(x - x_1)(x - x_2) = 0$ or $x^2 - (x_1 + x_2)x + x_1 x_2 = 0$ and therefore $b = x_1 + x_2$ and $c = x_1 x_2$. If $x_1 = 1$, it is immediately follows that $x_2 = c$.

*** WITH ALTERNATIVE NOTATION:

Changing notation to match the source code: $c_{21} \equiv b_2$, $c_{31} \equiv b_3$,

$$\frac{c_{21} - 1}{1 - c_{21}\,v^{x_2 - x_1}} = \frac{c_{31} - 1}{1 - c_{31}\,v^{x_3 - x_1}}. \tag{39}$$

Let $V \equiv v^{x_2 - x_1}$ and $m \equiv \frac{x_3 - x_1}{x_2 - x_1}$. Then

$$\frac{c_{21} - 1}{1 - c_{21}\,V} = \frac{c_{31} - 1}{1 - c_{31}\,V^m}, \text{ and thus} \tag{40}$$

$$1 - c_{31}\,V^m = \frac{c_{31}-1}{c_{21}-1}\,(1 - c_{21}\,V) \tag{41}$$

$$c_{31}\,V^m - c_{21}\frac{c_{31}-1}{c_{21}-1}\,V + \left(\frac{c_{31}-1}{c_{21}-1} - 1\right) = 0$$

$$V^m - \frac{c_{21}}{c_{31}}\,\frac{c_{31}-1}{c_{21}-1}\,V + \frac{1}{c_{31}}\,\frac{c_{31}-c_{21}}{c_{21}-1} = 0. \tag{42}$$

For $m \neq 2$, equation (42) could be solved numerically, but for $m = 2$, there is a nice solution:

$$V^2 - \frac{c_{21}}{c_{31}}\,\frac{c_{31}-1}{c_{21}-1}\,V + \frac{c_{31}-c_{21}}{c_{31}(c_{21}-1)} = 0 \text{ can be written as} \tag{43}$$

$$(V-1)\left(V - \frac{c_{31}-c_{21}}{c_{31}(c_{21}-1)}\right) = 0. \tag{44}$$

Solution $V = 1$ is not relevant since for every x, $1^x = 1$. Therefore the only applicable solution is

$$V = \frac{c_{31}-c_{21}}{c_{31}(c_{21}-1)}. \tag{45}$$

Then

$$\begin{aligned} v &= V^{\frac{1}{x_2-x_1}} \\ w &= \frac{c_{21}-1}{1-c_{21}\,V} \qquad \text{or } w = \frac{c_{31}-1}{1-c_{31}\,V^m} \\ A &= y_{02}\,(1 + w\,V)^a \text{ or } A = y_{03}\,\big(1 + w\,V^m\big)^a. \end{aligned} \tag{46}$$

Remark:

If we multiply equation (43) by $c_{31}\,(c_{21}-1)$, we get

$$c_{31}\,(c_{21}-1)\,V^2 - c_{21}\,(c_{31}-1)\,V + c_{31} - c_{21} = 0. \tag{47}$$

When $V = 1$, the equation (47) is true for any c_{21} and c_{31} and thus 1 is one of the solutions of the equation (43). For the other solution, note that if x_1 and x_2 are the roots of the equation $x^2 - bx + c = 0$, then $(x-x_1)(x-x_2) = 0$ or $x^2 - (x_1+x_2)x + x_1x_2 = 0$ and therefore $b = x_1 + x_2$ and $c = x_1x_2$. If $x_1 = 1$, it is immediately follows that $x_2 = c$.