# EP 501 Homework 4: Differentiation, Integration, and Multidimensional functions

November 12, 2019

**Instructions:**

- Submit all source code and publish Matlab results in .pdf form via Canvas. Please zip all contents of your solution into single file and then submit in a single zip file.
- Discussing the assignment with others is fine, but you must not copy anyone's code.
- I must be able to run your code and produce all results by executing a single top-level Matlab script, e.g. `assignment1.m` or similar.
- You may use any of the example codes from our course repository: `https://github.com/mattzett/EP501/`.
- Do not copy verbatim any other codes (i.e. any source codes other than from our course repository). You may use other examples as a reference but you must write you own programs (except for those I give you).

**Purpose of this assignment:**

- Deal with numerical differentiation to solve complex problems.
- Develop good coding and documentation practices, such that your programs are easily understood by others.
- Exercise good judgement in numerical problem setup.
- Demonstrate higher reasoning to synthesize a problem and devise a basic algorithm to solve it.

1. Vector derivatives and multidimensional plotting:

   (a) Plot the two components of the vector magnetic field defined by the piecewise function:

$$\mathbf{B}(x,y) = \begin{cases} \frac{\mu_0 I}{2\pi a^2}\sqrt{x^2+y^2}\left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right) & \left(\sqrt{x^2+y^2} < a\right) \\ \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right) & \left(\sqrt{x^2+y^2} \geq a\right) \end{cases} \tag{1}$$

   Assume the parameters in this equation have the values:

$$\begin{aligned} I &= 10 \qquad \text{(A)} \\ \mu_0 &= 4\pi \times 10^{-7} \qquad \text{(H/m)} \\ a &= 0.005 \qquad \text{(m)} \end{aligned}$$

   Use an image plot (`imagesc`, `pcolor`) for each component and have your plot show the region $-3a \leq x \leq 3a, -3a \leq y \leq 3a$. Make sure you add a colorbar and axis labels to your plot. You will need to define a range and resolution in $x$ and $y$, and create a meshgrid from that. Be sure to use a resolution fine enough to resolve important variations in this function.

   (b) Make a quiver plot of the magnetic field $\mathbf{B}$; add labels, etc.

   (c) Compute the numerical curl of $\mathbf{B}$, i.e. $\nabla \times \mathbf{B}$. Use centered differences on the interior grid points and first-order derivatives on the edges. Plot your result using `imagesc`, or `pcolor`.

   (d) Compute $\nabla \times \mathbf{B}$ analytically. Plot the alongside your numerical approximation. Demonstrate that they are suitably similar.

   (e) Compute and plot the scalar field:

$$\Phi(x,y,z) = \begin{cases} \frac{Q}{4\pi\epsilon_0 a} - \frac{Q}{8\pi\epsilon_0 a^3}\left(x^2+y^2+z^2-a^2\right) & \left(\sqrt{x^2+y^2+z^2} < a\right) \\ \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}} & \left(\sqrt{x^2+y^2+z^2} \geq a\right) \end{cases} \tag{2}$$

   Use the parameters:

$$\begin{aligned} Q &= 1 \qquad \text{(C)} \\ a &= 1 \qquad \text{(m)} \\ \epsilon_0 &= 8.854 \times 10^{-12} \qquad \text{(F/m)} \end{aligned}$$

   and plot this function in the region $-3a \leq x \leq 3a, -3a \leq y \leq 3a$ in the $z = 0$ plane. Be sure to use a resolution fine enough to resolve variations in this function (aside from those associated with the singularity).

   (e) Write a function to numerically compute the Laplacian of a scalar field, i.e. $\nabla^2\Phi$. Plot your result and include appropriate labels and colorbars.

2. Integration in multiple dimensions.

   (a) Numerically compute the electrostatic energy in the region $R \equiv -3a \leq x \leq 3a, -3a \leq y \leq 3a, -3a \leq z \leq 3a$, defined by the integral:

$$W_E = -\frac{1}{2}\iiint_R \left(\epsilon_0 \nabla^2 \Phi\right)\Phi \; dxdydz \tag{3}$$

   using a iterated trapezoidal or multi-dimensional trapezoidal method.

(b) Compute and plot the parametric path

$$\mathbf{r}(\phi) \equiv x(\phi)\hat{\mathbf{e}}_x + y(\phi)\hat{\mathbf{e}}_y = r_0 \cos\phi \; \hat{\mathbf{e}}_x + r_0 \sin\phi \; \hat{\mathbf{e}}_y \qquad (0 \leq \phi \leq 2\pi) \tag{4}$$

in the $x, y$ plane on the same axis as your magnetic field components from the previous problem (plot the path in the figure for each component). Take $r_0 = 2a$. You will need to define a grid in $\phi$ to do this.

(c) Plot the two components of the magnetic field $\mathbf{B}(x(\phi), y(\phi))$ at the $x, y$ points along $\mathbf{r}$ (I recommend `plotyy` for this) and visually compare against your image plots of the magnetic field.

(d) Numerically compute the tangent vector to the path $\mathbf{r}$ by performing the derivative: $d\mathbf{r}/d\phi$. Compare your numerical results against the analytical derivative (e.g. plot the two) and refine your grid in $\phi$ (if necessary) such that you get visually acceptable results - i.e. such that the path appears circular.

(e) Numerically compute the magnetic field integrated around the path $\mathbf{r}$, i.e.:

$$\frac{1}{\mu_0} \int_{\mathbf{r}} \mathbf{B} \cdot d\boldsymbol{\ell} \tag{5}$$

where the differential path length is given by:

$$d\boldsymbol{\ell} = \frac{d\mathbf{r}}{d\phi} d\phi \tag{6}$$