IPX Reference

Version 1.0

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Functionality

IPX solves linear programming (LP) problems in the form

$$\begin{aligned}
\underset{\mathtt{x}}{\text{minimize}} \quad & \mathtt{obj}^T\mathtt{x} && \text{(1a)}\\
\text{subject to} \quad & A\mathtt{x}\{\geq,\leq,=\}\mathtt{rhs}, && \text{(1b)}\\
& \mathtt{lb} \leq \mathtt{x} \leq \mathtt{ub}. && \text{(1c)}
\end{aligned}$$

The matrix A has `num_constr` rows and `num_var` columns. Associated with (1b) are dual variables $\mathtt{y}$ with the sign convention that

$$\begin{aligned}
\mathtt{y}[i] \geq 0 \quad & \text{if constraint is of type } \geq, && \text{(2a)}\\
\mathtt{y}[i] \leq 0 \quad & \text{if constraint is of type } \leq, && \text{(2b)}\\
\mathtt{y}[i] \text{ free} \quad & \text{if constraint is of type } =. && \text{(2c)}
\end{aligned}$$

Associated with $\mathtt{lb} \leq \mathtt{x}$ and $\mathtt{x} \leq \mathtt{ub}$ are dual variable $\mathtt{zl} \geq 0$ and $\mathtt{zu} \geq 0$ respectively. Entries of $-\mathtt{lb}$ and `ub` can be infinity, in which case the dual is fixed at zero.

Interior Point Method

The interior point method (IPM) computes a primal-dual point $(\mathtt{x}, \mathtt{slack}, \mathtt{xl}, \mathtt{xu}, \mathtt{y}, \mathtt{zl}, \mathtt{zu})$ that approximately satisfies

$$\begin{aligned}
& A\mathtt{x} + \mathtt{slack} = \mathtt{rhs}, \quad \mathtt{x} - \mathtt{xl} = \mathtt{lb}, \quad \mathtt{x} + \mathtt{xu} = \mathtt{ub}, && \text{(3a)}\\
& A^T\mathtt{y} + \mathtt{zl} - \mathtt{zu} = \mathtt{obj}, && \text{(3b)}
\end{aligned}$$

and that is guaranteed to satisfy $\mathtt{xl} \geq 0$, $\mathtt{xu} \geq 0$, (2) and

$$\begin{aligned}
\mathtt{slack}[i] \leq 0 \quad & \text{if constraint is of type } \geq, && \text{(4a)}\\
\mathtt{slack}[i] \geq 0 \quad & \text{if constraint is of type } \leq, && \text{(4b)}\\
\mathtt{slack}[i] = 0 \quad & \text{if constraint is of type } =. && \text{(4c)}
\end{aligned}$$

In theory, the IPM iterates will in the limit satisfy (3a) and (3b), and the primal objective will equal the dual objective

$$\mathtt{rhs}^T\mathtt{y} + \mathtt{lb}^T\mathtt{zl} - \mathtt{ub}^T\mathtt{zu}. \tag{5}$$

(Entries for which `-lb` or `ub` is infinity are understood to be dropped from the sum.)

Crossover

The crossover method recovers an optimal basis from the interior solution. A basis is defined by variable and constraint statuses

$$\texttt{vbasis}[j] \in \{\texttt{IPX\_basic}, \texttt{IPX\_nonbasic\_lb}, \texttt{IPX\_nonbasic\_ub}, \texttt{IPX\_superbasic}\}, \tag{6}$$
$$\texttt{cbasis}[i] \in \{\texttt{IPX\_basic}, \texttt{IPX\_nonbasic}\}. \tag{7}$$

The columns of A for which $\texttt{vbasis}[j] = \texttt{IPX\_basic}$ and the columns of the identity matrix for which $\texttt{cbasis}[i] = \texttt{IPX\_basic}$ form a square, nonsingular matrix of dimension `num_constr`. The corresponding basic solution $(\texttt{x}, \texttt{slack}, \texttt{y}, \texttt{z})$ is obtained by setting

$$\texttt{z}[j] = 0 \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_basic}, \tag{8a}$$
$$\texttt{x}[j] = \texttt{lb}[j] \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_nonbasic\_lb}, \tag{8b}$$
$$\texttt{x}[j] = \texttt{ub}[j] \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_nonbasic\_ub}, \tag{8c}$$
$$\texttt{x}[j] = 0 \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_superbasic}, \tag{8d}$$
$$\texttt{y}[i] = 0 \quad \text{if } \texttt{cbasis}[i] = \texttt{IPX\_basic}, \tag{8e}$$
$$\texttt{slack}[i] = 0 \quad \text{if } \texttt{cbasis}[i] = \texttt{IPX\_nonbasic} \tag{8f}$$

and computing the remaining components such that $A\texttt{x} + \texttt{slack} = \texttt{rhs}$ and $A^T\texttt{y} + \texttt{z} = \texttt{obj}$. The basis is primal feasible if $\texttt{lb} \le \texttt{x} \le \texttt{ub}$ and (4) hold; the basis is dual feasible if (2) holds and

$$\texttt{z}[j] \ge 0 \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_nonbasic\_lb}, \tag{9a}$$
$$\texttt{z}[j] \le 0 \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_nonbasic\_ub}, \tag{9b}$$
$$\texttt{z}[j] = 0 \quad \text{if } \texttt{vbasis}[j] = \texttt{IPX\_superbasic}. \tag{9c}$$

The IPX crossover consists of a primal and dual push phase. Depending on the accuracy of the interior solution and the numerical stability of the LP problem, the obtained basis may not be primal and/or dual feasible. In this case reoptimization with an external simplex code is required.