SERII NUMERICE

1. Arătați că următoarele serii sunt convergente și calculați sumele lor:

a) $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}}$; Aplicăm criteriul raportului

Notăm cu $a_n = \frac{2}{3^{n+1}} > 0,\ n \in \mathbb{N}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ Fie seria cu termeni strict pozitivi $\sum_n^{\infty} a_n$. Presupunem că există L.

Dacă $L < 1$, seria este absolut convergentă

$L > 1$, seria este divergentă

$L = 1$, seria poate converge sau diverge, caz neconcludent

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2}{3^{n+2}}}{\frac{2}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2}}{\cancel{3^{n+2}}_{3}} \cdot \frac{\cancel{3^{n+1}}}{\cancel{2}} \right| =$$

$= \lim_{n\to\infty} \left| \frac{1}{3} \right| = \frac{1}{3}$; Deoarece limita este $\frac{1}{3} < 1$, seria $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}}$ este absolut convergentă.

$$\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} = \frac{2}{3^{0+1}} + \frac{2}{3^2} + \dots + \frac{2}{3^{n+1}}$$

$$= 2\left(\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n+1}} \right)$$

$$= 2\left(\frac{1}{3} \cdot \frac{\frac{1}{3^{n+1}} - 1}{\frac{1}{3} - \frac{3}{1}} \right) = \frac{2}{3}\left(\frac{1}{3^{n+1}} - 1 \right)\left(-\frac{3}{2} \right)$$

$$= \Big(1 - \underbrace{\frac{1}{3^{n+1}}}_{\to 0}\Big) \xrightarrow{n\to\infty} 1 - 0 = 1$$

b) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$; $a_n = \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$

$$a_n = \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{(n+1)\sqrt{n} - n\sqrt{n+1}} =$$

$$= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{(n+1)^2 n - n^2(n+1)} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{\cancel{n^3} + \cancel{2}n^2 + n - \cancel{n^3} - \cancel{n^2}} =$$

$$= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n^2 + n} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n(n+1)} =$$

$$= \frac{\cancel{(n+1)}\sqrt{n}}{n\cancel{(n+1)}} - \frac{\cancel{n}\sqrt{n+1}}{\cancel{n}(n+1)} = \frac{\cancel{\sqrt{n}}}{\sqrt{n}} - \frac{\cancel{\sqrt{n+1}}}{\sqrt{n+1}} =$$

$$= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \cancel{\ldots} +$$

$$+ \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$$

$$\lim_{n \to \infty} 1 - \frac{1}{\sqrt{n+1}} = 1 - 0 = 1$$

Din $\lim_{n \to \infty} S_n = 1$, rezultă că seria este convergentă,

cu suma $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = 1$

c) $\sum_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!}$; Aplicăm criteriul raportului

Fie seria cu termeni strict pozitivi $\sum_{n}^{\infty} a_n$.

Presupunem că există $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Dacă $L < 1$, seria converge

$L > 1$, seria diverge

$L = 1$, seria poate converge sau diverge, caz neconcludent

$$a_n = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2-(n+1)-1}{(n+2)!}}{\frac{n^2-n-1}{(n+1)!}} \right| = \cancel{\lim_{n\to\infty} \left| \frac{(n+1)^2-(n+1)-1}{(n+2)!} \cdot \frac{1}{\frac{n^2-n-1}{(n+1)!}} \right|} =$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2-(n+1)-1}{(n+2)!} \cdot \frac{(n+1)!}{n^2-n-1} \right| = \cancel{\lim_{n\to\infty}}$$

$$= \lim_{n\to\infty} \left| \frac{n^2+\cancel{2n}+\cancel{1}-n-\cancel{1}-1}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{n^2-n-1} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{n^2+n-1}{(n+2)(n^2-n-1)} \right| = \lim_{n\to\infty} \left| \frac{n^2+n-1}{n^3+n^2-3n-2} \right| =$$

$$= \cancel{\lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \left| \frac{\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}}{1+\frac{1}{n}} \right|}$$

$$= \lim_{n\to\infty} \left| \frac{\overset{0}{\frac{1}{n}}+\overset{0}{\frac{1}{n^2}}-\overset{0}{\frac{1}{n^3}}}{1+\underset{0}{\frac{1}{n}}-\underset{0}{\frac{3}{n^2}}-\underset{0}{\frac{2}{n^3}}} \right| = 0$$

Deoarece limita este $0<1$, seria $\sum_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!}$ este absolut convergentă

$$a_n = \frac{n^2-n-1}{(n+1)!} = \frac{n^2-1}{(n+1)!} - \frac{n}{(n+1)!} = \frac{(n+1)(n-1)}{(n+1)!} - \frac{n}{(n+1)!} =$$

$$= \frac{\cancel{(n+1)}\cancel{(n-1)}}{\cancel{(n+1)}\cdot n\cdot\cancel{(n-1)}\cdot(n-2)!} - \frac{\cancel{n}}{(n+1)\cdot\cancel{n}\cdot(n-1)!} =$$

$$= \frac{1}{n(n-2)!} - \frac{1}{(n+1)(n-1)!}$$

$$\sum_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!} = \sum_{n=1}^{\infty}\left(\frac{1}{n(n-2)!} - \frac{1}{(n+1)(n-1)!}\right)$$

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k(k-2)!} - \frac{1}{(k+1)(k-1)!}\right) =$$

$$=\left(\frac{1}{\underset{1\cdot 0!\cdot(-1)}{1\cdot(1-2)!}} - \frac{\cancel{1}}{\underset{\cancel{2}\cdot 1}{(1+1)(1-1)!}}\right) + \left(\frac{\cancel{1}}{2\cdot 0!} - \frac{\cancel{1}}{3\cdot 1!}\right) + \left(\frac{\cancel{1}}{3\cdot 1!} - \frac{\cancel{1}}{4\cdot 2!}\right) + \cancel{\ldots} +$$

$$-1$$

$$+\left(\frac{\cancel{1}}{n(n-2)!} - \frac{1}{(n+1)(n-1)!}\right) = -1 - \frac{1}{(n+1)(n-1)!}$$

$$\lim_{n\to\infty} -1 - \frac{1}{(n+1)(n-1)!} = -1 - 0 = -1$$

Din $\lim\limits_{n\to\infty} S_n = -1$, suma $\sum\limits_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!} = -1$

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2. Demonstrați convergența următoarelor serii utilizând criteriul general de convergență a lui Cauchy.

a) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{2^n}$; $a_n = \frac{\cos n}{2^n}$, $n \in \mathbb{N}$

$|\cos t| \leq 1 \quad \forall t \in \mathbb{R} \Rightarrow |\cos t| \leq 1 \quad \forall t \in \mathbb{N}$

Fie $n, p \in \mathbb{N}^*$

$$|a_{n+p} - a_n| = |a_{n+1} + a_{n+2} + \dots + a_{n+p}| \le$$

$$\le |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| \le$$

$$\le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+p}} =$$

$$= \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{p-1}}\right) =$$

$$= \frac{1}{2^{n+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} =$$

$$= \frac{1}{2^{n+1}} \cdot \frac{1 - \frac{1}{2^p}}{\frac{1}{2}} =$$

$$= \frac{1}{2^n} \cdot \left(1 - \frac{1}{2^p}\right) < \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$$

$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N}^*$ astfel că $\frac{1}{2^n} < \varepsilon \ \forall n \ge n_\varepsilon$

$\Rightarrow |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon \quad \forall n, p \in \mathbb{N}^*, n \ge n_\varepsilon$

Conform criteriului general de convergență a lui Cauchy, seria este convergentă.

b) $\sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^n}{n^2}$; $a_n = \frac{\left(1 + \frac{1}{n}\right)^n}{n^2} > 0, \ \forall n \in \mathbb{N}^*$

$$a_n = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{n^2} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n^2} = \frac{(n+1)^n}{n^{n+2}}$$

Fie $n, p \in \mathbb{N}^*$

$$|a_{n+p} - a_n| = |a_{n+1} + a_{n+2} + \dots + a_{n+p}| \le |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| \le$$

$$\leq \left|\frac{(n+2)^{n+1}}{(n+1)^{n+3}}\right| + \left|\frac{(n+3)^{n+2}}{(n+2)^{n+4}}\right| + \dots + \left|\frac{(n+p+1)^{n+p}}{(n+p)^{n+p+2}}\right|$$

$! (n+2)^{n+1} \leq (n+1)^{n+2}, \forall n \geq 2 \Rightarrow$

$$\frac{(n+2)^{n+1}}{(n+1)^{n+2}} \leq 1 \Big| \cdot \frac{1}{n+1} \Rightarrow \frac{(n+2)^{n+1}}{(n+1)^{n+3}} \leq \frac{1}{n+1}$$

$$\Rightarrow \frac{(n+p+1)^{n+p}}{(n+p)^{n+p+2}} \leq \frac{1}{n+p} \Rightarrow$$

$$|a_{n+p} - a_n| \leq \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \leq \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1} \leq$$

$$\leq \frac{p}{n+1} \xrightarrow[n\to\infty]{} 0 \Rightarrow \forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}^* \text{ a.î. } \frac{p}{n+1} < \varepsilon \ \forall n \geq n_\varepsilon \Rightarrow$$

$$\Rightarrow |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon \ \forall n, p \in \mathbb{N}^*, n \geq n_\varepsilon$$

$\Rightarrow$ Conform criteriului general de convergență a lui Cauchy că seria este convergentă

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3. Studiați convergența absolută a următoarelor serii

a) $\sum_{n=1}^{\infty} \frac{\sin n^2}{3^n}$; $a_n = \frac{\sin n^2}{3^n}$

$$|a_n| = \left|\frac{\sin n^2}{3^n}\right| \leq \frac{1}{3^n} \xrightarrow[n\to\infty]{} 0 \Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ converge}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ este absolut convergentă

c)? b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$; $a_n = \frac{(-1)^n}{\ln n}$

$$|a_n| = \left|\frac{(-1)^n}{\ln n}\right| = \frac{1}{\ln n} \xrightarrow[n\to\infty]{} 0 \Rightarrow \sum_{n=2}^{\infty} |a_n| \text{ converge, deci } \sum_{n=2}^{\infty} \text{ este absolut convergentă}$$

4. Studiati natura urmatoarelor serii cu termeni pozitivi:

a) $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n+4^n}$; $a_n = \frac{n^2}{2^n+4^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2^n+4^n} = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

b) $\sum\limits_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$; $a_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$; Aplicăm criteriul rădăcinii

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{\left(2+\frac{1}{n}\right)^n}} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{2+\frac{1}{n}} = \frac{1}{2} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n} \text{ converge}$$

c) $\sum\limits_{n=1}^{\infty} \frac{2^n}{n!}$; $a_n = \frac{2^n}{n!}$; Aplicăm criteriul raportului

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} =$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n(n+1)} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

d) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$; $a_n = \left(\frac{n}{n+1}\right)^{n^2}$; Aplicăm criteriul rădăcinii

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}} = e^{\lim\limits_{n\to\infty} \frac{-n}{n+1}} = e^{-1} = \frac{1}{e} < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

e) $\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$; $a_n = \frac{n}{n^2+n+1}$; Aplicăm criteriul comparației

$b_n = \frac{1}{n}$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+n+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^2+n+1}{n} =$$

$$= \lim_{n\to\infty} \frac{n^2+n+1}{n^2} = 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverge}$$

f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \sin\frac{1}{n}$; Aplicăm criteriul inegalității ; $\sin\frac{1}{n} \leq \frac{1}{n}$

$$a_n = \frac{1}{\sqrt{n}} \cdot \sin\frac{1}{n} = \frac{1}{n^{\frac{1}{2}}} \cdot \sin\frac{1}{n} \leq \frac{1}{n^{\frac{1}{2}}} \cdot \frac{1}{n} \Rightarrow a_n \leq \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converge $\Rightarrow \sum_{n=1}^{\infty} a_n$ converge

g) $\sum_{n=1}^{\infty} \left[\frac{1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-2)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}\right]^2$; $a_n = \left[\frac{1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-2)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}\right]^2$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\cancel{1 \cdot 4 \cdot 7 \cdot \ldots (3n-2)}(3n+1)}{\cancel{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}(3n+2)}\right]^2 \cdot \left[\frac{\cancel{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}}{\cancel{1 \cdot 4 \cdot 7 \ldots \cdot (3n-2)}}\right]^2 =$$

$$= \lim_{n\to\infty} \left(\frac{3n+1}{3n+2}\right)^2 = 1$$; Aplicam criteriul Raabe Duhamel

$$L = \lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left[\left(\frac{3n+2}{3n+1}\right)^2 - 1^{3n+1}\right] =$$

$$= \lim_{n\to\infty} n \cdot \frac{9n^2+12n+4-(9n^2+6n+1)}{9n^2+6n+1} =$$

$$= \lim_{n\to\infty} n \cdot \frac{6n+3}{9n^2+6n+1} = \lim_{n\to\infty} \frac{6n^2+3n}{9n^2+6n+1} = \frac{6}{9} = \frac{2}{3} < 1 \Rightarrow \text{diverge}$$

5. Discutați natura seriilor următoare după valorile parametrului $a > 0$.

a) $\sum\limits_{n=1}^{\infty} \frac{n!}{(an)^n}$; $a_n = \frac{n!}{(a_n)^n}$; Aplicam criteriul radacinii

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{(a_n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{a_n}$$; Aplicam criteriul raportului

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{[a\cdot(n+1)]^{n+1}} \cdot \frac{(a\cdot n)^n}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}\cdot\cancel{(n+1)}}{a^{n+1}\cdot(n+1)^{\cancel{n+1}}} \cdot \frac{\cancel{a^n}\cdot n^n}{\cancel{n!}} =$$

$$= \lim_{n\to\infty} \frac{n^n}{a\cdot(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{a} = \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n \cdot \frac{1}{a} =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{n}{-(n+1)}} \cdot \frac{1}{a} = \frac{1}{a} \cdot e^{\lim\limits_{n\to\infty} -\frac{n}{n+1}} = \frac{1}{a} \cdot e^{-1} =$$

$= \frac{1}{a\cdot e}$; Dacă $a > \frac{1}{e} \,\Big|\cdot e$

$ae > 1 \Rightarrow \frac{1}{ae} < 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converge

Dacă $a < \frac{1}{e} \,\Big|\cdot e$

$ae < 1 \Rightarrow \frac{1}{ae} > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverge

Dacă $a = \frac{1}{e} \Rightarrow \frac{1}{ae} = 1$, aplicăm Raabe-Duhamel

$$L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n^n}{a\cdot(n+1)^n} - \overset{a(n+1)^n}{1}\right) =$$

$$= \lim_{n\to\infty} n\left[\frac{n^n - a(n+1)^n}{a(n+1)^n}\right] = \lim_{n\to\infty} n\, \frac{\cancel{n^n}\left[1 - a\overbrace{\left(1+\frac{1}{n}\right)^n}^{\to e}\right]}{a\cdot\cancel{n^n}\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}} =$$

$$= \lim_{n\to\infty} n \cdot \frac{1-ae}{ae} = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverge}$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\dots(a+n)}$; $a_n = \dfrac{n!}{(a+1)(a+2)\dots(a+n)}$;

Aplicăm criteriul raportului

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\overset{(n+1)}{\cancel{(n+1)!}}}{\cancel{(a+1)(a+2)\dots(a+n)}(a+n+1)} \cdot \frac{\cancel{(a+1)(a+2)\dots(a+n)}}{\cancel{n!}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{a+n+1} = 1 \text{ ; Aplicăm Raabe-Duhamel}$$

$$L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{a+n+1}{n+1} - \overset{n+1}{\cancel{1}}\right) =$$

$$= \lim_{n\to\infty} n \cdot \frac{a+\cancel{n}+\cancel{1}-\cancel{n}-\cancel{1}}{n+1} = \lim_{n\to\infty} \frac{an}{n+1} = a$$

Dacă $a > 1$, atunci $\sum_{n=1}^{\infty} a_n$ converge

$a \in (0,1)$, atunci $\sum_{n=1}^{\infty} a_n$ diverge

Pentru $a = 1 \Rightarrow a_n = \dfrac{n!}{2\cdot 3\cdot\ldots\cdot(n+1)} = \dfrac{n!}{(n+1)!} = \dfrac{1}{n+1}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+1} < \sum_{n=1}^{\infty} \frac{1}{n} \text{ ; } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverge} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverge}$$

c) $\displaystyle\sum_{n=1}^{\infty} a^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}$; $a_n = a^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}$; Aplicăm criteriul raportului

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a^{\cancel{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}+\frac{1}{n+1}}}{a^{\cancel{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}}} =$$

$$= \lim_{n\to\infty} a^{\frac{1}{n+1}\to 0} = a^0 = 1 \text{ ; Aplicăm Raabe-Duhamel}$$

$$L = \lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{1}{a^{\frac{1}{n+1}}} - 1\right) =$$

$$= -\lim_{n\to\infty} n\left(1-\frac{1}{a^{\frac{1}{n+1}}}\right) = -\lim_{n\to\infty} n\cdot\frac{a^{\frac{1}{n+1}}-1}{a^{\frac{1}{n+1}}} =$$

$$= -\lim_{n\to\infty} n\cdot\left(\frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\frac{1}{n+1}\cdot\frac{1}{a^{\frac{1}{n+1}}}\right) = -\lim_{n\to\infty} n\left(\ln a\cdot\frac{1}{n-1}\cdot\frac{1}{a^{\frac{1}{n+1}}}\right) =$$

$$= -\ln a\cdot\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{a^{\frac{1}{n+1}}} = -\ln a = \ln a^{-1} = \ln\frac{1}{a}$$

Dacă $a > \frac{1}{e} \Rightarrow \frac{1}{a} < e \Rightarrow \ln\frac{1}{a} < \ln e = 1 \Rightarrow \sum_{n\geq 1}^{\infty} a_n$ diverge

Dacă $a < \frac{1}{e} \Rightarrow \frac{1}{a} > e \Rightarrow \ln\frac{1}{a} > \ln e = 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converge

Dacă $a = \frac{1}{e} \Rightarrow a_n = \left(\frac{1}{e}\right)^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} = \left(e^{-1}\right)^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} =$

$= \left(e^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}\right)^{-1}$; Aplicam criteriul logaritmului

$$L = \lim_{n\to\infty}\frac{\ln\frac{1}{a_n}}{\ln n} = \lim_{n\to\infty}\frac{\ln a_n^{-1}}{\ln n} = \lim_{n\to\infty}\frac{\ln e^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}}{\ln n} =$$

$$= \lim_{n\to\infty}\frac{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n} = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

f. d) $\sum_{n=1}^{\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)^a$; $a_n = \left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)^a$

$$= \left(\frac{\sqrt[3]{n+1}^{\,3}-\sqrt[3]{n}^{\,3}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2+n}+\sqrt[3]{n^2}}\right)^a =$$

$$= \left(\frac{\cancel{n}+1-\cancel{n}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2+n}+\sqrt[3]{n^2}}\right)^a =$$

$$= \left(\frac{1}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2+n}+\sqrt[3]{n^2}}\right)^a$$

$$a_n = \left[\frac{1}{\sqrt[3]{n^2}\left(\sqrt[3]{\frac{1}{n}+1}\right)^2 + \sqrt[3]{\frac{1}{n}+1} + 1}\right]^a \text{; Fie } b_n = \left(\frac{1}{\sqrt[3]{n^2}}\right)^a$$

$$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{\cancel{\left(\sqrt[3]{n^2}\right)^a}} \cdot \frac{\cancel{\left(\sqrt[3]{n^2}\right)^a}\left(\left(\sqrt[3]{\frac{1}{n}+1}\right)^2 + \sqrt[3]{\frac{1}{n}+1} + 1\right)^a}{1} =$$

$$= 3^a \in (0,\infty),\ \forall\, a > 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ și } \sum_{n=1}^{\infty} b_n \text{ au aceeași natură}$$

$$b_n = \frac{1}{n^{\frac{2}{3}a}} \text{; dacă } \frac{2}{3}a > 1 \,\Big|\cdot \frac{3}{2} \Rightarrow a > \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ converge} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

$$\text{dacă } \frac{2}{3}a \le 1 \Rightarrow a \le \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverge} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverge}$$

---

6. Demonstrați convergența seriilor următoare:

a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n}}$; $a_n = \frac{1}{\sqrt[3]{n}}$; Aplicăm criteriul lui Leibniz

$\sum_n (-1)^n \cdot a_n$, $a_n \searrow 0 \Rightarrow \sum_n$ converge

$$\frac{a_{n+1}}{a_n} = \frac{1}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{1} = \sqrt[3]{\frac{n}{n+1}} < 1 \Rightarrow a_n \text{ monoton descresc}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n}} \text{ converge}$$

b) $$\sum_{n=1}^{\infty} (-1)^n \frac{2n + (-1)^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2\cancel{n}}{\cancel{n}\sqrt{n}} + \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2}{\sqrt{n}} + \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \text{ ; } a_n = \frac{2}{\sqrt{n}} \text{ monoton descresc la } 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{2}{\sqrt{n}} \text{ converge}$$

$$\cancel{\sum_{n=1} \frac{1}{\sqrt{n}} \text{ converge}} \quad \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \text{ converge}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2}{\sqrt{n}} + \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2}{\sqrt{n}} + \frac{1}{n^{\frac{3}{2}}} \text{ converge}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n + (-1)^n}{n\sqrt{n}} \text{ converge}$$

c) $\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}$, $\theta \in \mathbb{R}$; $\sum_{n=1}^{\infty} \cos(n\theta) \cdot \frac{1}{n}$; $\frac{1}{n}$ sir monoton descr la 0

$$\sum_{n=1}^{\infty} \cos(n\theta) = \cos\theta + \cos(2\theta) + \ldots + \cos(n\theta)$$

$$\Rightarrow \sum_{n=1}^{\infty} \cos(n\theta) = \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} + n\cdot\theta\right) \Big| \cdot \sin\frac{\theta}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \cos(n\theta) \cdot \sin\frac{\theta}{2} = \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} + n\theta\right) \cdot \sin\frac{\theta}{2} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left[\cos\left(\frac{\pi}{2} + n\theta - \frac{\theta}{2}\right) - \cos\left(\frac{\pi}{2} + n\theta + \frac{\theta}{2}\right)\right] =$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right) - \cancel{\cos\left(\frac{\pi}{2} + \frac{3\theta}{2}\right)}\right] +$$

$$+ \frac{1}{2}\left[\cancel{\cos\left(\frac{\pi}{2} + \frac{3\theta}{2}\right)} - \cancel{\cos\left(\frac{\pi}{2} + \frac{5\theta}{2}\right)}\right] +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \frac{1}{2}\left[\cancel{\cos\left(\frac{\pi}{2} + \frac{(2n-1)\theta}{2}\right)} - \cos\left(\frac{\pi}{2} + \frac{(2n+1)\theta}{2}\right)\right] =$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right) - \cos\left(\frac{\pi}{2} + \frac{(2n+1)\theta}{2}\right)\right] =$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \sin\left[\frac{\pi}{2} + \frac{\theta}{2} + \frac{\pi}{2} + \frac{(2n+1)\theta}{2}\right] \cdot \sin\left[\cancel{\frac{\pi}{2}} + \frac{(2n+1)\theta}{2} - \cancel{\frac{\pi}{2}} - \frac{\theta}{2}\right] =$$

$$= \sin\left[\pi + (n+1)\theta\right]\cdot \sin n\theta = -\sin\left[(n+1)\theta\right]\cdot \sin n\theta$$

$$\Rightarrow \sum_{n=1}^{\infty} \cos n\theta = \frac{-\sin\left[(n+1)\theta\right]\cdot \sin n\theta}{\sin\frac{\theta}{2}} \Rightarrow$$

$$\Rightarrow \left|\sum_{n=1}^{\infty} \cos n\theta\right| = \left|\frac{-\sin\left[(n+1)\theta\right]\cdot \sin n\theta}{\sin\frac{\theta}{2}}\right| \leq \frac{1}{\sin\frac{\theta}{2}} \text{ pentru } \sin\frac{\theta}{2} \neq 0$$

Pentru $\sin\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = k\pi \Rightarrow \theta = 2k\pi,\ k \in \mathbb{Z} \Rightarrow$

$$\Rightarrow \left|\cos\theta + \cos 2\theta + \ldots + \cos n\theta\right| = \left|1+1+\ldots+1\right| = n$$

$\sum_{n=1}^{\infty} \cos n\theta$ este o serie convergentă, iar șirul sumelor parțial mărginit și $\left(\frac{1}{n}\right)_{n\geq 1}$ șir monoton descresc la 0

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \text{ este convergentă}$$

ȘIRURI DE FUNCȚII

1. Studiați convergența punctuală și convergența uniformă pentru următoarele șiruri de funcții

a) $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{nx}{1+n^2x^2},\ n \in \mathbb{N}^*$

$$\lim_{n\to\pm\infty} f_n(x) = \lim_{n\to\pm\infty} \frac{nx}{1+n^2x^2} = 0$$

Criteriul cu lim din supremum: $f_n \xrightarrow[\Delta]{u} f \Leftrightarrow \lim\limits_{n\to\infty} \sup\limits_{x\in\Delta} |f_n(x) - f(x)| = 0$

$$f_n(x) = \frac{nx}{1+n^2x^2},\ x \in \mathbb{R},\ x \in \mathbb{N}^*$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0,\ \forall x \in \mathbb{R} \Rightarrow f(x) = 0$$

$$|f_n(x) - f(x)| = \frac{|x| \cdot n}{1 + n^2x^2}$$

$$\sup \frac{|x| \cdot n}{1 + n^2x^2} = \sup |f_n(x)|$$

$$f_n'(x) = \left(\frac{nx}{1+n^2x^2}\right)' = \frac{n(1+n^2x^2) - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n + n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2} =$$

$$= \frac{n - n^3x^2}{(1+n^2x^2)^2} \; ; \; f_n'(x) = 0 \Rightarrow n - n^3x^2 = 0 \Rightarrow n(1 - n^2x^2) = 0 \Rightarrow$$

$$\Rightarrow 1 - n^2x^2 = 0 \Rightarrow n^2x^2 = 1 \Rightarrow x^2 = \frac{1}{n^2} \Rightarrow x = \pm \frac{1}{n}, \; n \in \mathbb{N}^*$$

$$f_n\left(\frac{1}{n}\right) = \frac{\cancel{n} \cdot \frac{1}{\cancel{n}}}{1 + \cancel{n^2} \cdot \frac{1}{\cancel{n^2}}} = \frac{1}{2}$$

$$f_n\left(-\frac{1}{n}\right) = \frac{\cancel{n} \cdot \left(-\frac{1}{\cancel{n}}\right)}{1 + \cancel{n^2} \cdot \left(-\frac{1}{\cancel{n^2}}\right)} = -\frac{1}{2}$$

x	$-\infty$		$-\frac{1}{n}$		$\frac{1}{n}$		$+\infty$
$f_n'(x)$		$- - -$	0	$+ + +$	0	$- - -$	
$f_n(x)$	0	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$	0

$$\sup |f_n(x) - f(x)| = \frac{1}{2}$$

$$\lim_{n\to\infty} \sup |f_n(x) - f(x)| = \frac{1}{2} \neq 0 \Rightarrow f_n(x) \xrightarrow[R]{\,p\,} 0$$

$$f_n \;\cancel{\xrightarrow[R]{\,u\,}}\; 0$$

b) $f_n : (-1, 1] \to \mathbb{R}, \; f_n(x) = \dfrac{x^2}{1 + x^{2n}}, \; n \in \mathbb{N}^*$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{1 + x^{2n}} = \lim_{x\to\infty} \frac{\cancel{x^n}}{x^{2n}\left(1 + \frac{1}{x^{2n}}\right)} = \lim_{n\to\infty} \frac{1}{x^n\left(1 + \frac{1}{x^{2n}}\right)} =$$

$$= \lim_{x\to\infty} \frac{1}{x^n + \frac{1}{x^n}} \; ; \quad x^n = \begin{cases} \infty, & x > 1 \\ 1, & x = 1 \\ 0, & x \in (-1,1) \\ \nexists, & x \leq -1 \end{cases} \qquad \frac{1}{x^n} = \begin{cases} 0, & x > 1 \\ 1, & x = 1 \\ \infty, & x \in (-1,1) \end{cases}$$

$$x^n + \frac{1}{x^n} = \begin{cases} 2, & x = 1 \\ \infty, & x \in (-1,1) \end{cases} \qquad \frac{1}{x^n + \frac{1}{x^n}} = \begin{cases} \frac{1}{2}, & x = 1 \\ 0, & x \in (-1,1) \end{cases}$$

$$\Rightarrow f_n(x) \xrightarrow[(-1,1]]{R} f(x) \text{ unde } f(x) = \begin{cases} \frac{1}{2}, & x = 1 \\ 0, & x \in (-1,1) \end{cases}$$

$$\left.\begin{array}{l} f \text{ nu e cont în } x = 1 \\ f \text{ e cont în } (-1,1) \end{array}\right\} \Rightarrow f_n(x) \xrightarrow[(-1,1]]{u\!\!\!/} f$$

c) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n(1-x^n)$, $n \in \mathbb{N}^*$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n(1-x^n) = \begin{cases} 0\cdot 1 = 0, & x \in [0,1) \\ 1\cdot(1-1) = 0, & x = 1 \end{cases}$$

$$\Rightarrow f_n(x) \xrightarrow[[0,1]]{f} 0 \; ; \quad f(x) = 0, \; x \in [0,1]$$

$$|f_n(x) - f(x)| = |\underbrace{x^n}_{>0}\underbrace{(1-x^n)}_{>0}| = x^n(1-x^n)$$

$$\sup_{x\in[0,0]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} x^n(1-x^n) = \sup_{x\in[0,1]} f_n(x)$$

$$f_n'(x) = [x^n(1-x^n)]' = nx^{n-1}(1-x^n) + x^n(-nx^{n-1}) =$$
$$= nx^{n-1} - nx^{2n-1} - nx^{2n-1} = nx^{n-1} - 2nx^{2n-1} = nx^{n-1}(1-2x^n)$$

$$f_n'(x) = 0 \Rightarrow nx^{n-1}(1-2x^n) = 0 \Rightarrow x = 0 \text{ sau } 1 - 2x^n = 0$$

$$1 = 2x^n$$
$$x^n = \frac{1}{2}$$
$$x = \frac{1}{\sqrt[n]{2}}$$

$$f\left(\frac{1}{\sqrt[n]{2}}\right) = \frac{1}{(\sqrt[n]{2})^n}\left(1 - \frac{1}{\sqrt[n]{2}^{\,n}}\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

$$f(0) = 0^n(1 - 0^n) = 0 \; ; \; f(1) = 1^n(1 - 1^n) = 0$$

x	$-\infty$	0			$\frac{1}{\sqrt[n]{2}}$			1	$+\infty$
$f_n'(x)$		0	$+$	$+$	0	$-$	$-$		
$f(x)$		0	$\nearrow$		$\frac{1}{4}$	$\searrow$		0	

$$\max(f(x)) = \frac{1}{4} \Rightarrow \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{4} \neq 0; \; f_n \xrightarrow[{[0,1]}]{u} 0$$

d) $f_n \; [0, 2\pi] \to \mathbb{R}, \; f_n(x) = n \sin\frac{x}{n}, \; n \in \mathbb{N}^*$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\cdot \sin\frac{x}{n} = \lim_{n\to\infty} \underbrace{\frac{\sin\frac{x}{n}}{\frac{x}{n}}}_{\to 1} \cdot x = x \Rightarrow f_n(x) \xrightarrow{p} x \Rightarrow f(x) = x$$

$$|f_n(x) - f(x)| = \left| n\cdot \sin\frac{x}{n} - x\right| = x - n\cdot\sin\frac{x}{n}$$

$$n\cdot \sin\frac{x}{n} \le n\cdot\frac{x}{n} \le x$$

$$g(x) = x - n\cdot \sin\frac{x}{n}$$

$$g'(x) = 1 - n\cdot\cos\frac{x}{n}\cdot\frac{1}{n} = 1 - \cos\frac{x}{n} \ge 0$$

$$g'(x) = 0 \Rightarrow 1 - \cos\frac{x}{n} = 0 \Rightarrow \cos\frac{x}{n} = 1$$

$$\Rightarrow \frac{x}{n} = 2k\pi, \; k\in\mathbb{Z}, \; n\in\mathbb{N}^* \Rightarrow x = 2nk\pi \Rightarrow x \in \{0; 2\pi\}, \; x\in[0, 2\pi]$$

x	0					2π
$g'(x)$	0	$+$	$+$	$+$	$+$	$+$ 0
$g(x)$	0	$\nearrow$				$g(2\pi)$

$$g(0) = 0 - n\sin\frac{0}{n} = 0$$

$$g(2\pi) = 2\pi - n\sin\frac{2\pi}{n}$$

$$= 2\pi - \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\cdot 2\pi$$

$$= 2\pi\left(1 - \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$$

$$\sup_{x\in[0,2\pi]} |f_n(x)-f(x)| = \sup_{x\in[0,2\pi]} \left(x - n\sin\frac{x}{n}\right) - g(2\pi)$$

$$\lim_{n\to\infty} \sup_{x\in[0,2\pi]} |f_n(x)-f(x)| = \lim_{n\to\infty} 2\pi\left(1 - \underbrace{\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}}_{1}\right) = 0$$

$$\Rightarrow f_n(x) \xrightarrow[{[0,2\pi]}]{u} x$$

SERII DE PUTERI

1. Determinați raza de convergență și domeniul de convergență pentru următoarele serii de puteri:

a) $\sum_{n\geq 1} \frac{n+1}{n!} x^n$; $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$; $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$a_n = \frac{n+1}{n!}$$

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{n!} \cdot \frac{(n+1)!}{n+2} = \frac{n+1}{\cancel{n!}} \cdot \frac{\cancel{n!}(n+1)}{n+2} = \frac{(n+1)^2}{n+2} = \frac{n^2+2n+1}{n+2}$$

$R = \lim_{n\to\infty} \left|\frac{n^2+2n+1}{n+2}\right| = \infty > 0 \Rightarrow$ seria de puteri converge absolut pentru $\forall\, x \in (-\infty, \infty)$, $x \in \mathbb{R}$

b) $\sum_{n\geq 1} \frac{n^2}{2^n+3^n} (x-1)^n$; $y = x-1$

$\sum_{n\geq 1} \frac{n^2}{2^n+3^n} \cdot y^n$; $a_n = \frac{n^2}{2^n+3^n}$

$$\frac{a_n}{a_{n+1}} = \frac{n^2}{2^n+3^n} \cdot \frac{2^{n+1}+3^{n+1}}{(n+1)^2}$$

$R = \lim_{n\to\infty} \left| \frac{\cancel{n^2} \cdot \cancel{3^{n+1}}\, 3 \left[\left(\frac{2}{3}\right)^{n+1} + 1\right]}{\cancel{n^2}\underbrace{\left(1+\frac{1}{n}\right)^2}_{0} \cdot \cancel{3^n}\left[\underbrace{\left(\frac{2}{3}\right)^n}_{0}+1\right]} \right| = 3 > 0 \Rightarrow$ seria converge absolut pt $\forall\, y$ cu proprietatea că $|y| < R$

$\Rightarrow$ converge pt. $y \in (-3, 3) \Rightarrow (x-1) \in (-3,3) \Rightarrow x \in (-2, 4)$

$\Rightarrow$ Domeniul este $(-2,4)$

c) $\sum_{n\geq 1} \frac{n!}{n^n}(x+2)^n$; $y = x+2$; $a_n = \frac{n!}{n^n}$

$x = y-2$

$$R = \lim_{n\to\infty}\left|\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n!}}{\cancel{n^n}}\cdot\frac{\cancel{n^n}\cdot n\left(1+\frac{1}{n}\right)^{n+1}}{\cancel{n!}(n+1)}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{n}\left(1+\frac{1}{n}\right)^n\cancel{\left(1+\frac{1}{n}\right)}}{\cancel{n}\cancel{\left(1+\frac{1}{n}\right)}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 0$$

$|y| < e$

$\Rightarrow$ seria converge absolut pt $y \in (-e, e) \Rightarrow (x+2) \in (-e, e) \;\Big|\, -2$

$x \in (-e-2, e-2)$

FORMULA LUI TAYLOR. SERII TAYLOR

1. Aplicați funcțiilor următoare formula lui Taylor cu rest Lagrange, de ordinul indicat n, în punctul indicat $a \in \mathbb{R}$.

Fie J un interval și $f: J \to \mathbb{R}$ o funcție derivabilă de $n+1$ ori pe J $n \in \mathbb{N}$. Atunci $\forall x \in J$ și $a \in J$ cu $x \neq a$, $\exists\, c \in (a, x)$ a.î. $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots +$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}$$

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}\cdot(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \text{arctg}\, x$, $n = 2$, $a = 0$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c)}{3!}(x-0)^3$$

$f'(x) = (\text{arctg}\, x)' = \frac{1}{1+x^2}$; $f(0) = \text{arctg}\, 0 = 0$; $f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \left(\frac{1}{1+x^2}\right)' = -\frac{2x}{(1+x^2)^2} \Rightarrow f''(0) = 0$

$$f'''(x) = \left(-\frac{2x}{(1+x^2)^2}\right)' = -\frac{2(1+x^2)^2 - 2x\cdot 2\cdot(1+x^2)\cdot 2x}{(1+x^2)^4}$$

$$f'''(x) = -2\cancel{(1+x^2)}\cdot\frac{1+x^2-4x^2}{(1+x^2)^{\cancel{4}\,3}} = -2\cdot\frac{1-3x^2}{(1+x^2)^3} = \frac{6x^2-2}{(x^2+1)^3}$$

$$f(x) = 0 + \frac{1}{1}\cdot x + \frac{0}{2}x^2 + \frac{6c^2-2}{(c^2+1)^3}\cdot\frac{1}{3!}x^3$$

$$f(x) = x + \frac{\cancel{2}(3c^2-1)}{(c^2+1)^3}\cdot\frac{1}{\cancel{6}}x^2 = x + \frac{3c^2-1}{3(c^2+1)^3}\cdot x^2$$

b) $f:\mathbb{R}\to\mathbb{R}$, $f(x) = e^{-2x}$, $n=3$, $a=1$

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(c)}{4!}(x-1)^4$$

$$f'(x) = (e^{-2x})' = -2e^{-2x};\ f(1) = e^{-2};\ f'(1) = -2e^{-2}$$

$$f''(x) = (e^{-2x})'' = (-2e^{-2x})' = -2\cdot(-2)e^{-2x} = 4e^{-2x};\ f''(1) = 4e^{-2}$$

$$f'''(x) = 4\cdot(-2)e^{-2x} = -8e^{-2x};\ f'''(1) = -8e^{-2}$$

$$f^{(4)}(x) = 16e^{-2x};\ f^{(4)}(c) = 16e^{-2c}$$

$$f(x) = \frac{1}{e^2} - \frac{2}{e^{-2}}(x-1) + \frac{\cancel{4}^{\,2}}{e^2}\cdot\frac{1}{\cancel{2}}(x-1)^2 - \frac{\cancel{8}^{\,4}}{e^2}\cdot\frac{1}{\cancel{6}_3}(x-1)^3 + \frac{\cancel{16}^{\,2}}{e^{2c}}\cdot\frac{1}{\cancel{24}_3}(x-1)^4 =$$

$$= \frac{1}{e^2}\left[1 - 2(x-1) + 2(x-1)^2 - \frac{4}{3}(x-1)^3\right] + \frac{2}{3e^{2c}}(x-1)^4$$

c) $f:(0,\infty)\to\mathbb{R}$, $f(x) = \ln x$, $n\in\mathbb{N}^*$, $a=1$

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^n +$$
$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$$

$$f(1) = \ln 1 = 0;\ f'(x) = (\ln x)' = \frac{1}{x};\ f'(1) = 1$$

$$f''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2};\ f''(1) = -1$$

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$$f'''(x) = +\frac{2x}{x^4} = \frac{2}{x^3}\ ;\ f'''(1) = 2 = 2!$$

$$f^{(4)}(x) = -\frac{2\cdot 3x^2}{x^6} = -\frac{6}{x^4}\ ;\ f^{(4)}(1) = -6 = -3!$$

$$f^{(5)}(x) = +6\,\frac{4x^3}{x^8} = \frac{24}{x^5}\ ;\ f^{(5)}(1) = 24 = 4!$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}\ ;\ f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

$$f^{(n+1)}(x) = \frac{(-1)^n\, n!}{x^{n+1}}$$

$$f(x) = 0 + \frac{1}{1}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 +$$

$$+ \ldots + \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n + \frac{(-1)^n\cdot n!}{(n+1)!}\cdot\frac{1}{c^{n+1}}\cdot(x-1)^{n+1}$$

d) $f:(0,\infty)\to\mathbb{R}\quad f(x)=\sqrt{x},\ n\in\mathbb{N}^*,\ a=1$

$$f(1)=1$$

$$f'(x) = \frac{1}{2\sqrt{x}}\ ;\ f'(1) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\cdot\left(x^{-\frac{1}{2}}\right)' = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{1}{2}-1} = -\frac{1}{4}\,x^{-\frac{3}{2}} = -\frac{1}{4}\cdot\frac{1}{\sqrt{x^3}} =$$

$$= -\frac{1}{4}\cdot\frac{1}{x\sqrt{x}} = -\frac{1}{2^2}\cdot\frac{1}{x\sqrt{x}}\ ;\ f''(1) = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\cdot\left(x^{-\frac{3}{2}}\right)' = -\frac{1}{4}\cdot\left(-\frac{3}{2}\right)\cdot x^{-\frac{3}{2}-1} = \frac{3}{8}\,x^{-\frac{5}{2}} = \frac{3}{8}\,\frac{1}{\sqrt{x^5}} =$$

$$= \frac{3}{8}\cdot\frac{1}{x^2\sqrt{x}} = \frac{3}{2^3}\cdot\frac{1}{x^2\sqrt{x}}\ ;\ f'''(1) = \frac{3}{2^3}$$

$$f^{(4)}(x) = \frac{3}{8}\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{x^3\sqrt{x}} = -\frac{15}{2^4}\cdot\frac{1}{x^3\sqrt{x}}\ ;\ f^{(4)}(1) = -\frac{15}{2^4}$$

$$f^{(5)}(x) = \frac{1\cdot 3\cdot 5}{2^4}\cdot\frac{7}{2}\cdot\frac{1}{x^4\sqrt{x}} = \frac{1\cdot 3\cdot 5\cdot 7}{2^5}\cdot\frac{1}{x^4\sqrt{x}}\ ;\ f^{(5)}(1) = \frac{1\cdot 3\cdot 5\cdot 7}{2^5}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot 7\cdot\ldots\cdot(2n-3)}{2^n}\cdot\frac{1}{x^{n-1}\sqrt{x}}$$

$$f^{(n)}(1) = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-3)}{2^n}$$

$$f^{(n+1)}(x) = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-3)(2n-1)}{2^{n+1}} \cdot \frac{1}{x^n \sqrt{x}}$$

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{4} \cdot \frac{1}{2!}(x-1)^2 + \frac{3}{2^3} \cdot \frac{1}{3!}(x-1)^3 - \frac{1 \cdot 3 \cdot 5}{2^4} \cdot \frac{1}{4!}(x-1)^4 +$$

$$+ \ldots + (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-3)}{2^n} \cdot \frac{1}{n!}(x-1)^n$$

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2. Dezvoltați în serie Taylor în jurul originii următoarelor funcții:

Seria Taylor:

$f: J \to \mathbb{R}$ pp f admite derivate de orice ordin în pct $a \in J$.

Seria de puteri centrată în a se num. seria Taylor a funcției f în pct a. $\sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n$

a) $f(x) = \cos 2x$

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots +$$

$$+ \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots$$

$$f^{(n)}(x) = (\cos 2x)^{(n)} = 2^n \cos\left(2x + \frac{n\pi}{2}\right)$$

$$f(a) = \cos 2a$$

$$f'(a) = 2\cos\left(2a + \frac{\pi}{2}\right)$$

$$f''(a) = 2^2 \cos\left(2a + \frac{2\pi}{2}\right) = 2^2 \cos(2a + \pi)$$

$$f'''(a) = 2^3 \cos\left(2a + \frac{3\pi}{2}\right)$$

$a=0 \Rightarrow f^{(n)}(0) = 2^n \cos\left(0+\frac{n\pi}{2}\right) = 2^n \cos\frac{n\pi}{2}$

n impar $\Rightarrow \cos\frac{n\pi}{2} = 0$

n par $\Rightarrow \cos\frac{n\pi}{2} \in \{-1;1\}$

$f(0) = \cos 0 = 1$

$f'(0) = 2\cos\frac{\pi}{2} = 0$

$f''(0) = 2^2 \cos(\pi) = -2^2$

$f'''(0) = 2^3 \cos\frac{3\pi}{2} = 0$

$f^{(4)}(0) = 2^4 \cos 2\pi = 2^4$

$a=0$; $f(x) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n\geq 0} \frac{x^n}{n!} f^{(n)}(0)$

$\cos 2x = 1 + \frac{x^2}{2!} f''(0) + \frac{x^4}{4!} f^{(4)}(0) + \frac{x^6}{6!} f^{(6)}(0) + \ldots + \frac{x^{2n}}{(2n)!} f^{(2n)}(0)$

$\cos 2x = 1 + \frac{x^2}{2!} \cdot (-1) \cdot 2^2 + \frac{x^4}{4!} \cdot 2^4 + \frac{x^6}{6!} (-1) 2^6 + \ldots + \frac{x^{2n}}{(2n)!} (-1)^{2n-1} \cdot$

$\cdot 2^{2n} + \ldots$

$\cos 2x = 1 + \sum_{n\geq 1} \frac{(-1)^{2n-1} \cdot 2^{2n}}{(2n)!} \cdot x^{2n}$

b) $f(x) = e^{3x}$; $f'(x) = (e^{3x})' = 3e^{3x}$; $f''(x) = (3\cdot e^{3x})' = 3^2 \cdot e^{3x}$

$f'''(x) = (3^2 \cdot e^{3x})' = 3^3 \cdot e^{3x}$

Fie: $p(n)$: $f^{(n)}(x) = 3^n \cdot e^{3x}$

Pp $p(k)$ adev : $f^{(k)}(x) = 3^k \cdot e^{3x}$

$p(k+1)$: $f^{(k+1)}(x) = 3^{k+1} \cdot e^{3x}$

$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \left(3^k \cdot e^{3x}\right)' = 3^k \cdot 3 \cdot e^{3x} = 3^{k+1} \cdot e^{3x} \Rightarrow$

$\Rightarrow$ Pp făcută este adevărată, $\forall k \in$ a adică și pt $\forall\, n \in \mathbb{N}$.

$f^{(n)}(x) = 3^n \cdot e^{3x}$

$f(x) = \sum_{n\geq 0} \frac{x^n}{n!} f^{(n)}(0)$; $f^{(n)}(0) = 3^n \cdot e^0 = 3^n$

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$$f(x) = \sum_{n\geq 0} \frac{x^n}{n!} \cdot 3^n \; ; \; e^{3x} = \sum_{n\geq 0} \frac{x^n}{n!} 3^n$$

$$e^{3x} = 1 + \frac{3x}{1!} + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + \dots + \frac{3^nx^n}{n!} + \dots$$

c) $f(x) = \sqrt{x+1}$; $f(x) = (x+1)^{\frac{1}{2}}$; $f(0) = 1^{\frac{1}{2}} = 1$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} \; ; \; f'(0) = \frac{1}{2} \cdot 1^{-\frac{1}{2}} = \frac{1}{2}$$

$$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}} \; ; \; f''(0) = -1 \cdot \frac{1}{2^2} = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-\frac{5}{2}} \; ; \; f'''(0) = \frac{3}{2^3}$$

$$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(x+1)^{-\frac{7}{2}} \; ; \; f^{(4)}(0) = -1 \cdot \frac{3 \cdot 5}{2^4}$$

$$f^{(5)}(x) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5}(x+1)^{-\frac{9}{2}} \; ; \; f^{(5)}(0) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5}$$

$$f^{(n)}(x) = (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n}(x+1)^{-\frac{2n-1}{2}}$$

$$f^{(n)}(0) = (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-3)}{2^n}(0+1)^{-\frac{2n-1}{2}}$$

$$f^{(n)}(0) = (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n}$$

$$\sqrt{x+1} = 1 + \sum_{n\geq 1} \frac{x^n}{n!} \cdot (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n}$$

$$\sqrt{x+1} = 1 + \frac{1}{2} \cdot \frac{x}{1!} + (-1)\frac{1}{2^2} \cdot \frac{x^2}{2!} + \frac{1 \cdot 3}{2^3} \cdot \frac{x^3}{3!} + \ldots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-3)}{2^n}$$

$$\cdot \frac{x^n}{n!} + \ldots$$

d) $f(x) = \ln(x+2)$; $f(x) = \sum_{n \geq 0} \frac{x^n}{n!} f^{(n)}(0)$; $f(0) = \ln 2$

$f'(x) = \frac{1}{x+2}$; $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{(x+2)^2}$; $f''(0) = -\frac{1}{2^2}$

$f'''(x) = \frac{2(x+2)}{(x+2)^4} = \frac{2}{(x+2)^3}$; $f'''(0) = \frac{2}{2^3}$

$f^{(4)}(x) = -\frac{2 \cdot 3(x+2)^2}{(x+2)^6} = -\frac{2 \cdot 3}{(x+2)^4}$; $f^{(4)}(0) = -\frac{2 \cdot 3}{2^4}$

$f^{(5)}(x) = \frac{2 \cdot 3 \cdot 4\,(x+2)^3}{(x+2)^8} = \frac{2 \cdot 3 \cdot 4}{(x+2)^5}$; $f^{(5)}(0) = \frac{2 \cdot 3 \cdot 4}{2^5}$

$f^{(n)}(x) = (-1)^{n-1} \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{(x+2)^n}$; $f^{(n)}(0) = (-1)^{n-1} \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{2^n}$

$f^{(n)}(0) = (-1)^{n-1} \cdot \frac{(n-1)!}{2^n}$

$$\ln(x+2) = \ln 2 + \frac{1}{2} \cdot \frac{x}{1!} - \frac{1}{2^2} \cdot \frac{x^2}{2!} + \frac{\cancel{1 \cdot 2}}{2^3} \cdot \frac{x^3}{\cancel{3!}\,3} - \frac{\cancel{1 \cdot 2 \cdot 3}}{2^4} \cdot \frac{x^4}{\cancel{4!}\,4} + \cancel{\ldots +}$$

$$+ (-1)^{n-1} \frac{\cancel{(n-1)!}}{2^n} \cdot \frac{x^n}{\cancel{n!}\,n} + \ldots$$

$$\ln(x+2) = \ln 2 + \frac{x}{2} - \frac{x^2}{2^3} + \frac{x^3}{3 \cdot 2^3} - \frac{x^4}{4 \cdot 2^4} + \frac{x^5}{5 \cdot 2^5} + \ldots + \frac{(-1)^{n-1} x^n}{n + 2^n} + \ldots$$

INTEGRALE IMPROPRII

Demonstrați convergența următoarelor integrale improprii și calculați valoarea acestora:

a) $\int_1^{\infty} \frac{1}{x^3} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^3} dx = \lim_{t \to \infty} \left. \frac{x^{-3+1}}{-3+1} \right|_1^t = \lim_{t \to \infty} \left. \frac{x^{-2}}{-2} \right|_1^t =$

$= \lim_{t \to \infty} -\frac{1}{2}\left(\overset{0}{\frac{1}{t^2}} - \frac{1}{1}\right) = \frac{1}{2} \in \mathbb{R} \Rightarrow \int_1^{\infty} \frac{1}{x^3} dx$ converge

Sau se demonstrează cu criteriul de comparație cu $\frac{1}{x^\alpha}$

Fie $a \in \mathbb{R}$ și $f : [a, \infty) \to (0, \infty)$ local integrabilă a.î să existe

$$\ell = \lim_{x \to \infty} x^\alpha \cdot f(x)$$

Dacă $\alpha > 1$ și $0 \leq \ell < \infty$, atunci $\int_a^\infty f(x)\,dx$ converge

$\alpha \leq 1$ și $0 < \ell \leq \infty$, atunci $\int_a^\infty f(x)\,dx$ diverge

Fie $f(x) = \frac{1}{x^3}$; $f : [1, \infty) \to \mathbb{R}$ și $\alpha = 3$

$$\ell = \lim_{n \to \infty} x^3 \cdot \frac{1}{x^3} = 1$$

Avem $\alpha = 3 > 1$ și $0 \leq 1 < \infty$ atunci $\int_1^\infty \frac{1}{x^3}\,dx$ converge

b) $\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \Big|_{-1}^{0}$

$= \arcsin 0 - \arcsin(-1) = 0 - \left(\frac{-\pi}{2}\right) = \frac{\pi}{2} \in \mathbb{R}$

$\Rightarrow \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx$ converge

Sau, prin criteriul de comparație cu $\frac{1}{(b-x)^2}$

Fie $f : [a, b) \to [0, \infty)$ local integrabilă a.î să existe

$$\ell = \lim_{x \to b} (b-x)^\alpha \cdot f(x)$$

Dacă $\alpha < 1$ și $0 \leq \ell < \infty$, atunci $\int_a^b f(x)\,dx$ converge

$\alpha \geq 1$ și $0 < \ell \leq \infty$, atunci $\int_a^b f(x)\,dx$ diverge

Fie $f(x) = \frac{1}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$; $f : [0, 1) \to [0, \infty)$; $\alpha = \frac{1}{2}$

$(b-x)^\alpha = (1-x)^{\frac{1}{2}}$; $b = 1$

$$\ell = \lim_{x \to 1} (1-x)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} < \infty$$

$\alpha = \frac{1}{2}$ $\quad \ell < \infty \Rightarrow \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ converge

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c) $\displaystyle\int_0^{\infty} \frac{1}{x^2+9}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{1}{x^2+9}\,dx = \lim_{t\to\infty} \frac{1}{3}\operatorname{arctg}\frac{x}{3}\Big|_0^{t} =$

$\displaystyle= \lim_{t\to\infty} \frac{1}{3}\Big(\underbrace{\operatorname{arctg}\frac{t}{3}}_{\frac{\pi}{2}} - \underbrace{\operatorname{arctg}\frac{0}{3}}_{0}\Big) = \frac{\pi}{6} \in \mathbb{R} \Rightarrow \int_0^{\infty} \frac{1}{x^2+9}\,dx$ converge

sau

$\alpha = 2 > 1$; $f(x) = \dfrac{1}{x^2+9}$; $f : [0,\infty) \to (0,\infty)$

$\displaystyle l = \lim_{n\to\infty} x^2 \cdot \frac{1}{x^2+9} = 1 < \infty$, deci $\displaystyle\int_0^{\infty} \frac{1}{x^2+9}\,dx$ converge

d) $\displaystyle\int_0^{1} \frac{1}{\sqrt[3]{1-x}}\,dx = \int_0^{1} -\frac{-dx}{\sqrt[3]{1-x}}$; $u = 1-x$, $du = -dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$\displaystyle\int -\frac{-dx}{\sqrt[3]{1-x}} \rightarrow J_u = \int -\frac{du}{\sqrt[3]{u}} = -\int u^{\frac{1}{3}}\,du$

$\displaystyle J_u = -\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = -\frac{3}{2}u^{\frac{2}{3}} + C = -\frac{3}{2}\sqrt[3]{u^2} + C$

$\displaystyle\Rightarrow \int_0^{1} \frac{1}{\sqrt[3]{1-x}}\,dx = -\frac{3}{2}\sqrt[3]{(1-x)^2}\Big|_0^{1} = -\frac{3}{2}\Big[\sqrt[3]{(1-1)^2} - \sqrt[3]{(1-0)^2}\Big]$

$\displaystyle= -\frac{3}{2}(-1) = \frac{3}{2} \in \mathbb{R} \Rightarrow \int_0^{1} \frac{1}{\sqrt[3]{1-x}}\,dx$ converge