Calcul diferențial în $\mathbb{R}^2$

1. Fie $g:\mathbb{R}\to\mathbb{R}$ o funcție derivabilă. Arătați că funcția $f:(0,\infty)^2\to\mathbb{R}$, definită prin $f(x,y)=g(xy)+\sqrt{xy}\cdot g\left(\frac{y}{x}\right)$ $\forall\, x,y>0$, verifică relația

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y)-y\frac{\partial^2 f}{\partial y^2}(x,y)=0,\ \forall (x,y)\in(0,\infty)^2$$

• Notăm arg. lui $g(xy)$ cu $u=xy$

• Notăm arg. lui $g\left(\frac{y}{x}\right)$ cu $v=\frac{y}{x}$

$$\frac{\partial u}{\partial x}=(xy)'_x=y \qquad \frac{\partial v}{\partial x}=\left(\frac{y}{x}\right)'_x=\frac{-y}{x^2}$$

$$\frac{\partial u}{\partial y}=(xy)'_y=x \qquad \frac{\partial v}{\partial y}=\left(\frac{y}{x}\right)'_y=\frac{1}{x}$$

$\Rightarrow f(x,y)=g(u)+\sqrt{xy}\cdot g(v),\quad x,y>0$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\big(g(u)\big)+\left(\sqrt{xy}\right)'_x\cdot g(v)+\sqrt{xy}\cdot\frac{\partial}{\partial x}\big(g(v)\big)$$

$$=g'(u)\cdot\frac{\partial u}{\partial x}+\frac{\sqrt{y}}{2\sqrt{x}}\cdot g(v)+\sqrt{xy}\cdot g'(v)\cdot\frac{\partial v}{\partial x}$$

$$=g'(u)\cdot y+\frac{\sqrt{y}}{2\sqrt{x}}\cdot g(v)+\sqrt{xy}\cdot\frac{-y}{x^2}\cdot g'(v)$$

$$=y\cdot g'(u)+\frac{\sqrt{y}}{2\sqrt{x}}\cdot g(v)-\frac{y\cdot\sqrt{xy}}{x^2}\cdot g'(v)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\big(g(u)\big)+\left(\sqrt{xy}\right)'\cdot g(v)+\sqrt{xy}\cdot\frac{\partial}{\partial y}\big(g(v)\big)$$

$$=g'(u)\cdot\frac{\partial u}{\partial y}+\frac{\sqrt{x}}{2\sqrt{y}}\cdot g(v)+\sqrt{xy}\cdot g'(v)\cdot\frac{\partial v}{\partial y}$$

$$=g'(u)\cdot x+\frac{\sqrt{x}}{2\sqrt{y}}\cdot g(v)+\sqrt{xy}\cdot g'(v)\cdot\frac{1}{x}$$

$$=x\cdot g'(u)+\frac{\sqrt{x}}{2\sqrt{y}}\cdot g(v)+\frac{\sqrt{xy}}{x}\cdot g'(v)$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\cdot g'(u)+\frac{\sqrt{y}}{2\sqrt{x}}\cdot g(v)-\frac{y\cdot\sqrt{xy}}{x^2}\cdot g'(v)\right)$$

$$=y\cdot\frac{\partial}{\partial x}\big(g'(u)\big)+\frac{\partial}{\partial x}\left(\frac{\sqrt{y}}{2\sqrt{x}}\right)\cdot g(v)+\frac{\partial}{\partial x}\big(g(v)\big)\cdot\frac{\sqrt{y}}{2\sqrt{x}}-\frac{\partial}{\partial x}\left(\frac{\sqrt{xy}\cdot y}{x^2}\right)\cdot$$

$$\cdot g'(v)-\frac{\partial}{\partial x}\big(g'(v)\big)\cdot\frac{\sqrt{xy}\cdot y}{x^2}$$

$$= y \cdot g''(u) \cdot \frac{\partial u}{\partial x} + \frac{-\sqrt{y}}{4\sqrt{x^3}} \cdot g(v) + g'(v) \cdot \frac{\partial v}{\partial x} \cdot \frac{\sqrt{y}}{2\sqrt{x}} - \left(\frac{-2y\sqrt{y}}{2x^2\sqrt{x}}\right) \cdot g'(v) -$$

$$- g''(v) \cdot \frac{\partial v}{\partial x} \cdot \frac{\sqrt{xy} \cdot y}{x^2}$$

$$= g''(u) \cdot y \cdot y + \frac{-\sqrt{y}}{4\sqrt{x^3}} \cdot g(v) + g'(v) \cdot \frac{-y}{x^2} \cdot \frac{\sqrt{y}}{2\sqrt{x}} + \frac{2y\sqrt{y}}{2x^2\sqrt{x}} \cdot g'(v) -$$

$$- g''(v) \cdot \left(\frac{-y}{x^2}\right)$$

$$= y^2 \cdot g'(u) + \frac{-\sqrt{y}}{4\sqrt{x^3}} \cdot g(v) - \frac{y\sqrt{y}}{2x^2\sqrt{x}} \cdot g'(v) + \frac{2y\sqrt{y}}{2x^2\sqrt{x}} \cdot g'(v) + g''(v) \cdot \frac{y}{x^2}$$

2. Determinați punctele de extrem și valorile extreme ale următoarelor funcții $f: \mathbb{R}^2 \to \mathbb{R}$

a) $f(x,y) = x^3 + y^3 - 3xy$

$$\frac{\partial f}{\partial x} = (x^3 + y^3 - 3xy)'_x = 3x^2 - 3y$$

$$\frac{\partial f}{\partial y} = (x^3 + y^3 - 3xy)'_y = 3y^2 - 3x$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \iff \begin{cases} 3x^2 - 3y = 0 & (1) \\ 3y^2 - 3x = 0 & (2) \end{cases}$$

$(1) \Rightarrow 3x^2 = 3y \Rightarrow x^2 = y$

$(2) \Rightarrow 3x^4 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x^3 - 1 = 0 \Rightarrow x^3 = 1 \Rightarrow x_2 = \sqrt[3]{1} = 1 \end{cases}$

Pentru $x_1 = 0 \Rightarrow y_1 = 0^2 = 0$

$x_2 = 1 \Rightarrow y_2 = 1^2 = 1$

$\Rightarrow (0,0)$ și $(1,1)$ puncte critice

$$A = \frac{\partial^2 f}{\partial x^2} = (3x^2 - 3y)'_x = 6x$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = (3y^2 - 3x)'_x = -3$$

$$C = \frac{\partial^2 f}{\partial y} = (3y^2 - 3x)'_y = 6y$$

Pentru $(0,0)$:

$$\begin{cases} A = 6 \cdot 0 = 0 \\ B = 0 \\ C = 6 \cdot 1 = 6 \end{cases} \Rightarrow \Delta = AC - B^2 = 0 - (-3)^2 = -9$$

$\Delta < 0 \Rightarrow (0,0)$ nu este punct de extrem

Pentru $(1,1)$:

$$\begin{cases} A = 6 \cdot 1 = 6 \\ B = 0 \\ C = 6 \cdot 1 = 6 \end{cases} \Rightarrow \Delta = AC - B^2 = 36 - (-3)^2 = 27$$

$\Delta > 0$ și $A > 0 \Rightarrow (1,1)$ este punct de minim local

b) $f(x,y) = e^{2x+3y}(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \left[e^{2x+3y}(x^2+y^2) \right]'_x$$
$$= (e^{2x+3y})'(x^2+y^2) + e^{2x+3y}(x^2+y^2)'$$
$$= 2e^{2x+3y}(x^2+y^2) + e^{2x+3y} \cdot 2x$$
$$= e^{2x+3y}(2x^2 + 2y^2 + 2x)$$

$$\frac{\partial f}{\partial y} = \left[e^{2x+3y}(x^2+y^2) \right]'_y$$
$$= (e^{2x+3y})'(x^2+y^2) + e^{2x+3y}(x^2+y^2)'$$
$$= 3e^{2x+3y}(x^2+y^2) + e^{2x+3y} \cdot 2y = e^{2x+3y}(3x^2+3y^2+2y)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \iff \begin{array}{l} e^{2x+3y}(2x+2x^2+2y^2)=0 \\ \\ e^{2x+3y}(2y+3x^2+3y^2)=0 \end{array}$$

$$e^{2x+3y} > 0$$

$$\Rightarrow \begin{cases} 2x+2x^2+2y^2=0 & \mid \cdot(-3) \\ 2y+3x^2+3y^2=0 & \mid \cdot(2) \end{cases} \Rightarrow \begin{cases} -6x-\cancel{6x^2}-\cancel{6y^2}=0 \\ 4y+\cancel{6x^2}+\cancel{6y^2}=0 \end{cases}$$

$$-6x+4y=0$$

$$\Rightarrow -6x=-4y \Rightarrow y=\frac{6x}{4}$$

Avem: $2y+3x^2+3y^2=0$

$$\Rightarrow 2\cdot\frac{3}{2}x+3x^2+3\left(\frac{3}{2}x\right)^2=0$$

$$\Rightarrow 3x+3x^2+\frac{27}{4}x^2=0 \quad \Big| \cdot 4$$

$$\Rightarrow 12x+12x^2+27x^2=0 \quad \Big| : 3$$

$$\Rightarrow 4x+4x^2+9x^2=0$$

$$\Rightarrow 4x+13x^2=0$$

$$\Rightarrow x(4+13x)=0$$

$$\Rightarrow \begin{cases} x_1=0 \\ 4+13x=0 \end{cases} \Rightarrow x_2=-\frac{4}{13}$$

Pentru $x_1=0 \Rightarrow y_1=\frac{3}{2}\cdot 0=0$

$$x_2=-\frac{4}{13} \Rightarrow y_2=\frac{3}{2}\cdot\frac{-4}{13}=-\frac{6}{13}$$

$$\Rightarrow (0,0) \text{ și } \left(-\frac{4}{13},-\frac{6}{13}\right)$$

$$A=\frac{\partial^2 f}{\partial x^2}=\left[e^{2x+3y}(2x+2x^2+2y^2)\right]'_x$$

$$=\left(e^{2x+3y}\right)'(2x+2x^2+2y^2)+e^{2x+3y}(2x+2x^2+2y^2)'$$

$$= 2e^{2x+3y}(2x+2x^2+2y^2) + e^{2x+3y}(2+4x)$$
$$= e^{2x+3y}(4x+4x^2+4y^2+2+4x)$$
$$= e^{2x+3y}(4x^2+4y^2+8x+2)$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = \left[e^{2x+3y}(2x+3x^2+3y^2)\right]'_x$$
$$= (e^{2x+3y})' \cdot (2y+3x^2+3y^2) + e^{2x+3y}(2y+3x^2+3y^2)'$$
$$= 2 \cdot e^{2x+3y}(2y+3x+3y^2) + e^{2x+3y}(6x)$$
$$= e^{2x+3y}(4y+6x^2+6y^2+6x)$$

$$C = \frac{\partial^2 f}{\partial y^2} = \left[e^{2x+3y}(2y+3x^2+3y^2)\right]'_y$$
$$= (e^{2x+3y})'(2y+3x^2+3y^2) + e^{2x+3y}(2y+3x^2+3y^2)'$$
$$= 3 \cdot e^{2x+3y}(2y+3x^2+3y^2) + e^{2x+3y}(2+6y)$$
$$= e^{2x+3y}(6y+9x^2+9y^2+2+6y)$$
$$= e^{2x+3y}(9x^2+9y^2+12y+2)$$

Pentru $(0,0)$:

$$\begin{cases} A = e^0(0+0+0+2) = 2 \\ B = e^0(0+0+0+0) = 0 \\ C = e^0(0+0+0+2) = 2 \end{cases} \Rightarrow \Delta = AC - B^2 = 2 \cdot 2 - 0 = 4$$

$\Delta > 0$ și $A > 0 \Rightarrow (0,0)$ punct de minim local

c) $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

$$\frac{\partial f}{\partial x} = \left[x^3 + 3xy^2 - 15x - 12y\right]'_x = 3x^2 + 3y^2 - 15$$

$$\frac{\partial f}{\partial y} = \left[x^3 + 3xy^2 - 15x - 12y\right]'_y = 6xy - 12$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 3y^2 - 15 = 0 \mid :3 \\ 6xy - 12 = 0 \mid :6 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 - 5 = 0 \\ xy = 2 \Rightarrow y = \frac{2}{x} \end{cases}$$

Avem: $x^2 + y^2 - 5 = 0$

$\Rightarrow x^2 + \frac{4}{x^2} - 5 = 0 \mid \cdot x^2$

$\Rightarrow x^4 + 4 - 5x^2 = 0$

Notăm $t = x^2 \Rightarrow t^2 - 5t + 4 = 0$

$\Delta = 25 - 16 = 9 \Rightarrow t_1 = \frac{5+3}{2} = 4 \Rightarrow x^2 = 4 \Rightarrow x_1 = 2,\ x_1 = -2$

$\Rightarrow t_2 = \frac{5-3}{2} = 1 \Rightarrow x^2 = 1 \Rightarrow x_2 = 1,\ x_2 = -1$

Pentru $\begin{cases} x_1 = 2 \Rightarrow y_1 = 1 \\ x_1 = -2 \Rightarrow y_1 = -1 \\ x_2 = 1 \Rightarrow y_2 = 2 \\ x_2 = -1 \Rightarrow y_2 = -2 \end{cases}$

$\Rightarrow (2,1); (-2,-1); (1,2); (-1,-2)$ puncte critice

$A = \frac{\partial f}{\partial x^2} = [3x^2 + 3y^2 - 15]'_x = 6x$

$B = \frac{\partial f}{\partial x \partial y} = [6xy - 12]'_x = 6y$

$C = \frac{\partial f}{\partial x \partial y} = [6xy - 12]'_y = 6x$

Pentru $(2,1)$:

$$\begin{cases} A = 6 \cdot 2 = 12 \\ B = 6 \cdot 1 = 6 \\ C = 6 \cdot 2 = 12 \end{cases} \Rightarrow \Delta = AC - B^2 = 144 - 36 = 108$$

$\Delta > 0$ și $A > 0 \Rightarrow (2,1)$ punct de minim local

Pentru $(-2,-1)$:

$$\begin{cases} A = 6\cdot(-2) = -12 \\ B = 6\cdot(-1) = -6 \\ C = 6\cdot(-2) = -12 \end{cases} \Rightarrow \Delta = AC - B^2 = 144 - 36 = 108$$

$\Delta > 0$ și $A < 0 \Rightarrow (-2,-1)$ punct de minim local

Pentru $(1,2)$:

$$\begin{cases} A = 6\cdot 1 = 6 \\ B = 6\cdot 2 = 12 \\ C = 6\cdot 1 = 6 \end{cases} \Rightarrow \Delta = AC - B^2 = 36 - 144 = -108$$

$\Delta < 0 \Rightarrow (1,2)$ nu este punct de extrem

Pentru $(-1,-2)$

$$\begin{cases} A = 6\cdot(-1) = -6 \\ B = 6\cdot(-2) = -12 \\ C = 6\cdot(-1) = -6 \end{cases} \Rightarrow \Delta = AC - B^2 = 36 - 144 = -108$$

$\Delta < 0 \Rightarrow (-1,-2)$ nu este punct de extrem

d) $f(x,y) = xy^2 e^{x-y}$

$$\frac{\partial f}{\partial x} = [xy^2 \cdot e^{x-y}]'_x$$
$$= (xy^2)' \cdot e^{x-y} + xy^2 \cdot (e^{x-y})'$$
$$= y^2 \cdot e^{x-y} + xy^2 \cdot e^{x-y}$$
$$= e^{x-y}(y^2 + xy^2)$$

$$\frac{\partial f}{\partial y} = [xy^2 \cdot e^{x-y}]'_y$$
$$= (xy^2)' \cdot e^{x-y} + xy^2 \cdot (e^{x-y})'$$
$$= 2xy \cdot e^{x-y} + xy^2 \cdot (-e^{x-y})$$
$$= e^{x-y}(2xy - xy^2)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \iff \begin{cases} e^{x-y}(y^2+xy^2) = 0 \quad \Big| : e^{x-y} \quad , e^{x-y} \neq 0 \\ e^{x-y}(2xy - xy^2) = 0 \quad \Big| : e^{x-y} \quad , e^{x-y} \neq 0 \end{cases}$$

$$\iff \begin{cases} y^2(1+x) = 0 \\ (2-y)\,xy = 0 \end{cases}$$

I $\quad y^2(1+x) = 0 \Rightarrow \boxed{y=0}$

$\qquad\qquad x+1 = 0 \Rightarrow \boxed{x=-1}$

II $\quad (2-y)(xy) = 0 \Rightarrow xy = 0 \Rightarrow \boxed{x=0}$

$\qquad\qquad 2-y = 0 \Rightarrow \boxed{y=2}$

$\Rightarrow (-1,0); (0,2)$ puncte critice

$$A = \frac{\partial^2 f}{\partial x^2} = \left[e^{x-y}(y^2+xy^2)\right]'_x = (e^{x-y})'\cdot(y^2+xy^2) + e^{x-y}(y^2+xy^2)'$$

$$= e^{x-y}(y^2+xy^2) + e^{x-y}(y^2)$$

$$= e^{x-y}(2y^2+xy^2)$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = \cancel{\left[e^{x-y}(y^2+xy^2)\right]'_x = (e^{x-y})'(y^2+xy^2) + (e^{x-y})(y^2+xy^2)'}$$

$$\cancel{= (e^{x-y})(y^2+xy^2) + (e^{x-y})}$$

$$= \left[e^{x-y}(2xy - xy^2)\right]'_x = (e^{x-y})'(2xy-xy^2) + (e^{x-y})(2xy-xy^2)'$$

$$= (e^{x-y})(2xy - xy^2) + (e^{x-y})(2y - y^2)$$

$$= (e^{x-y})(2xy - xy^2 + 2y - y^2)$$

$$C = \frac{\partial^2 f}{\partial y^2} = \left[e^{x-y}(2xy - xy^2)\right]'_y = (e^{x-y})'(2xy-xy^2) + (e^{x-y})(2xy-xy^2)'$$

$$= -(e^{x-y})(2xy - xy^2) + (e^{x-y})(2x - 2xy)$$

$$= e^{x-y}(-2xy + xy^2 + 2x - 2xy) = e^{x-y}(xy^2 + 2x - 4xy)$$

Pentru $(-1,0)$

$$\begin{cases} A = e^{-1}(0+0) = 0 \\ B = e^{-1}(0-0+0-0) = 0 \\ C = e^{-1}(0+(-2)-0) = e^{-1}\cdot(-2) = -\frac{2}{e} \end{cases} \Rightarrow \Delta = AC - B^2 = 0 - 0 = 0$$

Pentru $(0,2)$

$$A = e^{-2}(2\cdot 2^2 + 0) = e^{-2}\cdot 8 = \frac{8}{e^2}$$

$$B = e^{-2}(0-0+4-4) = 0 \quad \Rightarrow \Delta = AC - B^2 = 0 - 0$$

$$C = e^{-2}(0+0-0) = 0$$

3. Determinați punctele de extrem și valorile extreme ale funcției cu legătura $x^2 - y^2 = 1$, $f: \mathbb{R} \to \mathbb{R}$, $f(x,y) = (x-1)^2 + y^2$

$$f(x,y) = (x-1)^2 + y^2 = x^2 - 2x + 1 + y^2$$

Fie $F(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$

$$f(x,y) = x^2 + y^2 - 2x + 1$$

$$x^2 - y^2 = 1$$

$$x^2 - y^2 - 1 = 0$$

$$g(x,y) = x^2 - y^2 - 1$$

$$\Rightarrow F(x,y,\lambda) = x^2 + y^2 - 2x + 1 + \lambda(x^2 - y^2 - 1)$$

$$\frac{\partial F}{\partial x} = [x^2 + y^2 - 2x + 1 + \lambda(x^2 - y^2 - 1)]'_x = 2x - 2 + 2\lambda x$$

$$\frac{\partial F}{\partial y} = [x^2 + y^2 - 2x + 1 + \lambda(x^2 - y^2 - 1)]'_y = 2y - 2\lambda y$$

$$\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2\lambda x - 2 = 0 \\ 2y - 2\lambda y = 0 \\ x^2 - y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} 2x(1+\lambda) = 2 \\ 2y(1-\lambda) = 0 \\ x^2 - y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} 2x = \frac{2}{1+\lambda} \\ 2y = 0 \\ 1 - \lambda = 0 \\ x^2 - y^2 - 1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \dfrac{1}{1+\lambda} \\ y = 0 \\ \lambda = 1 \\ x^2 - y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \dfrac{1}{2} \\ y = 0 \\ \lambda = 1 \end{cases} \Rightarrow \left(\frac{1}{2}, 0, 1\right) \text{ punct critic}$$

Avem $x^2+y^2-1=0 \quad y=0$

$$\Rightarrow \left(\frac{1}{1+\lambda}\right)^2 - 0^2 = 1$$

$$\Rightarrow \frac{1}{(1+\lambda)^2} = 1$$

$$\Rightarrow 1+\lambda^2 = 1$$

$$\Rightarrow 1+2\lambda+\lambda^2 = 1 \Rightarrow 2\lambda+\lambda^2 = 0 \Rightarrow \lambda(\lambda+2) = 0 \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = -2 \end{cases}$$

$$\Rightarrow x = \frac{1}{1+\lambda} \Rightarrow \begin{matrix} x = 1 & \lambda = 0 \\ x = -1 & \lambda = -2 \end{matrix}$$

$\Rightarrow (1,0,0); (-1,0,-2)$ puncte critice

$$A = \frac{\partial^2 f}{\partial x^2} = (2x+2\lambda x-2)'_x = 2+2\lambda$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = (2y-2\lambda y)'_x = 0$$

$$C = \frac{\partial^2 f}{\partial y^2} = (2y-2\lambda y)'_y = 2-2\lambda$$

Pentru $\left(\frac{1}{2}, 0, 1\right)$

$$\begin{cases} A = 2+2\cdot 1 = 4 \\ B = 0 \\ C = 2-2\cdot 1 = 0 \end{cases} \Rightarrow \Delta = AC - B^2 = 0 - 0 = 0 \Rightarrow \left(\frac{1}{2}, 0\right) \text{ nu este punct de extrem}$$

Pentru $(1,0,0)$

$$\begin{cases} A = 2+0 = 2 \\ B = 0 \\ C = 2-0 = 2 \end{cases} \Rightarrow \Delta = AC - B^2 = 4 - 0 = 4 \Rightarrow (1,0) \text{ punct de minim local cu legătură}$$

Pentru $(-1,0,-2)$

$$\begin{cases} A = 2 + (2\cdot(-2)) = -2 \\ B = 0 \\ C = 2 - (2\cdot(-2)) = 6 \end{cases} \quad \Delta = AC - B^2 = -\cancel{12} - 0 = -\cancel{12} \Rightarrow (-1,0) \text{ nu este punct de extrem}$$

Calcul integral în $\mathbb{R}^2$

1. Calculați următoarele integrale cu parametru utilizând formula de derivare

$$I(a) = \int_0^{\frac{\pi}{2}} \ln\left(\cos^2 x + a^2 \sin^2 x\right) dx, \quad a > 0$$

$$I'(a) = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x + a^2 \sin^2 x} \left(\cos^2 x + a^2 \sin^2 x\right)' dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\cos^2 x)' + (a^2 \cdot \sin^2 x)'}{\cos^2 x + a^2 \sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{2\sin x \cos x} + 2a \sin^2 x + \cancel{2 \sin x \cos x}}{\cos^2 x + a^2 \sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2a \sin^2 x}{\cos^2 x + a^2 \sin^2 x} dx$$

$$= 2a \cdot \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos^2 x + a^2 \sin^2 x}$$

Notăm $tg\, x = t \Rightarrow 1 + t^2\, dx = dt \Rightarrow dx = \frac{dt}{1+t^2}$

$t = 0 \Rightarrow tg\, 0 = 0$

$t = \frac{\pi}{2} \Rightarrow tg \frac{\pi}{2} = \infty$

$$\Rightarrow I'(a) = 2a \cdot \int_0^\infty \frac{t^2}{1+a^2t^2} \frac{dt}{1+t^2}$$

Descompunem fracția în fracții simple

$$\frac{t^2}{(1+a^2t^2)(1+t^2)} = \frac{t^2}{(1+t^2)(1+a^2t^2)} = \frac{At+B}{1+t^2} + \frac{Ct+D}{1+a^2t^2}$$

$$\Rightarrow A = C = 0 \ ; \ B = \frac{1}{a^2-1} \ ; \ D = \frac{-1}{a^2-1}$$

$$\Rightarrow I'(a) = \frac{2a}{a^2-1} \cdot \int_0^\infty \frac{dt}{1+t^2} - \frac{2a}{a^2-1} \cdot \int_0^\infty \frac{1}{1+a^2t^2}\, dt$$

I $$\frac{2\cdot a}{a^2-1} \cdot \int_0^\infty \frac{dt}{1+t^2} = \frac{2a}{a^2-1} \cdot \operatorname{arctg} \Big|_0^\infty = \frac{2a}{a^2-1}(\operatorname{arctg}\infty - \operatorname{arctg} 0) =$$

$$= \frac{2a}{a^2-1} \cdot \frac{\pi}{2} = \frac{a}{a^2-1} \cdot \pi$$

II $$\frac{2a}{a^2-1} \cdot \int_0^\infty \frac{1}{1+a^2t^2}\, dt = \frac{2a}{a^2-1} \cdot \frac{1}{a^2} \cdot \int_0^\infty \frac{1}{t^2+\frac{1}{a^2}}\, dt =$$

$$= \frac{2}{a^2-1} \cdot \frac{1}{a} \cdot \frac{1}{\frac{1}{a}} \cdot \operatorname{arctg}\left(\frac{t}{\frac{1}{a}}\right)\Big|_0^\infty$$

$$= \frac{2}{a^2-1} \cdot \frac{1}{a} \cdot a \cdot \operatorname{arctg}(t\cdot a)\Big|_0^\infty$$

$$= \frac{2}{a^2-1} \cdot \operatorname{arctg}(t\cdot a)\Big|_0^\infty$$

$$= \frac{2}{a^2-1} \cdot (\operatorname{arctg}(\infty \cdot a) - \operatorname{arctg}(0\cdot a))$$

$$= \frac{2}{a^2-1} \cdot \operatorname{arctg}(\infty)$$

$$= \frac{2}{a^2-1} \cdot \frac{\pi}{2}$$

$$\text{I} + \text{II} \Rightarrow I'(a) = \frac{a}{a^2-1} \cdot \pi - \frac{2}{a^2-1} \cdot \frac{\pi}{2} = \frac{2a\pi - 2\pi}{2(a^2-1)} = \frac{\cancel{2}\pi\cancel{(a-1)}}{\cancel{2}\cancel{(a-1)}(a+1)} =$$

$$= \frac{\pi}{a+1}$$

$$\Rightarrow I(a) = \pi \cdot \int \frac{da}{a+1} = \pi \cdot \ln(a+1) + C$$

a) $I(a) = \int_0^a \frac{\ln(1+ax)}{1+x^2}\, dx \,, \; a \geq 0$

$$I'(a) = \int_0^a \frac{[\ln(1+ax)]' \cdot (1+x^2) - \ln(1+ax)(1+x^2)'}{(1+x^2)^2}$$

$$= \int_0^a \frac{\frac{1}{1+ax}(1+ax)' \cdot (1+x^2) - \ln(1+ax) \cdot 2x}{(1+x^2)^2}$$

$$= \int_0^a \frac{\frac{a(1+x^2) - 2x + 2ax^2(\ln(1+ax))}{1+ax}}{(1+x^2)^2}\, dx$$

$$= \int_0^a \frac{a(ax^2 - 2x + 2ax^2(\ln(1+ax))}{(1+ax)(1+x^2)^2}\, dx$$