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There is no asymptotic PTAS for two-dimensional vector packing [1]

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Abstract

The d-dimensional vector packing problem is a well-known generalization of the classical bin packing problem: For a given list of vectors in $[0,1]^d$, the goal is to partition the list into the minimum number of subsets, such that in every subset the sum of all vectors is at most one in every coordinate. For the case $d = 1$, Fernandez de la Vega and Lueker (1981) designed an asymptotic polynomial time approximation scheme. In this note we prove that already for the case $d = 2$, the existence of an asymptotic polynomial time approximation scheme would imply P = NP. The proof is very simple and uses no new ideas. © 1997 Elsevier Science B.V.

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1. Introduction

Problem statement. In the d-dimensional vector packing problem, the input consists of a list L of vectors in $[0,1]^d$. The goal is to partition the vectors in L into the minimum number of subsets, such that in every subset the sum of all vectors is at most one in every coordinate. We say that such a subset of vectors can be packed into a *unit-bin*. The d-dimensional vector packing problem arises as a crucial subproblem in scheduling with resource constraints; see [5] and [8] for more detailed information. The vector packing problem is easily seen to be NP-complete, since it contains classical bin packing as a special case. This suggests to look for fast approximation algorithms that come close to the optimum solution in polynomial time.

Approximation algorithms. For an instance I of a minimization problem, let $\mathrm{OPT}(I)$ denote the value of the optimum solution (in vector packing, $\mathrm{OPT}(I)$ would be the number of unit-bins used in the optimum packing). For an approximation algorithm A, let $A(I)$ denote the value of the solution that A produces on input I. We say that an approximation algorithm A has *absolute performance guarantee* $1+\varepsilon$, if

$$A(I) \leqslant (1+\varepsilon)\mathrm{OPT}(I) \tag{1}$$

holds for all instances I. Similarly, we say that an approximation algorithm A has *asymptotic performance guarantee* $1+\varepsilon$, if there exists a constant c^* (that

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may depend on ε, but must not depend on I) such that

$$A(I) \leqslant (1+\varepsilon)\mathrm{OPT}(I) + c^* \tag{2}$$

holds for all instances I. In the corresponding definitions for *maximization* problems, the "less-or-equal" sign in definitions (1) and (2) has to be replaced by a "greater-or-equal" sign, and the factor "$(1+\varepsilon)$" has to be replaced by "$(1-\varepsilon)$". These performance guarantees are the usual measures for the quality of approximation algorithms: the closer the performance guarantee is to one, the better is the approximation algorithm. A *polynomial time approximation scheme* (PTAS, for short) is a sequence of polynomial time approximation algorithms A_ε with absolute performance guarantees $\leqslant 1+\varepsilon$ where ε tends to zero. An *asymptotic polynomial time approximation scheme* (ASY-PTAS, for short) is a sequence of polynomial time approximation algorithms A_ε with asymptotic performance guarantees $\leqslant 1+\varepsilon$ where ε tends to zero.

It is folklore that no polynomial time approximation algorithm for bin packing problems can have an absolute performance guarantee better than 3/2, unless P = NP (This follows from the NP-hardness of the PARTITION problem which consists in partitioning a set of numbers into two subsets whose elements add up to at most 1. In the packing terminology, either all these numbers can be packed into two bins, or otherwise one needs at least three bins. An approximation algorithm with absolute performance guarantee better than 3/2 would be able to distinguish between these two cases). However, packing instances with only a small number of bins – like two or three or four bins – do not seem to cover the "real" difficulty of packing problems. Therefore, the *asymptotic performance guarantee* has become the standard tool for measuring the quality of packing algorithms.

Currently, the best polynomial time approximation algorithms for d-dimensional packing have asymptotic worst case guarantees $d+\varepsilon$, where $\varepsilon > 0$ is a positive real that can be made arbitrarily small [4]. The only non-trivial non-approximability result for vector packing is due to Yao [9]: He proves that in a very restricted model of computation (linear time algorithms in decision trees), no approximation algorithm for d-dimensional vector packing can have an asymptotic performance guarantee of less than d.

Result of this note. We prove that already for the case $d = 2$, the existence of an ASY-PTAS for vector packing would imply P = NP. The proof is very simple and uses no new ideas: We show that the existence of an ASY-PTAS for 2-dimensional vector packing would imply the existence of a PTAS for a MAX SNP-hard optimization version of 3-DIMENSIONAL MATCHING (and it is well known that this in turn would imply P = NP). This proof is given in Section 2. Section 3 discusses some consequences of our result and some related problems.

2. The non-approximability proof

Let us start with the following version of 3-DIMENSIONAL MATCHING, that has been shown to be MAX SNP-hard by Kann [7].

MAXIMUM BOUNDED 3-DIMENSIONAL MATCHING (MAX-3-DM)

Input: Three sets $X = \{x_1, \dots, x_q\}$, $Y = \{y_1, \dots, y_q\}$, and $Z = \{z_1, \dots, z_q\}$. A subset $T \subseteq X \times Y \times Z$ such that any element in X, Y, Z occurs in one, two or three triples in T. Note that this implies $q \leqslant |T| \leqslant 3q$.

Goal: Find a maximum cardinality subset T' of T such that no two triples in T' agree in any coordinate.

Measure: The cardinality of T'.

Note that since MAX-3-DM is a MAX SNP-hard problem, the existence of a PTAS for it would imply that P = NP (cf. for example [2]).

Observation 1. *For any instance I of MAX-3-DM, the inequality* $\mathrm{OPT}(I) \geqslant \frac{1}{7}q$ *holds.*

Proof. Select an arbitrary triple t from T, and remove t together with all triples that agree with t in some coordinate from T. Repeat this step until T becomes empty. Since every element occurs in at most 3 triples, every step removes at most 7 triples from T. Hence, there are at least $\frac{1}{7}|T| \geqslant \frac{1}{7}q$ steps and at least $\frac{1}{7}q$ selected triples. In the end, the selected triples form a feasible 3-dimensional matching. □

We start with an instance I of MAX-3-DM and we will construct from it an instance of 2-dimensional vector packing with $3q + |T|$ vectors. Let $r = 32q$. First, let us define $3q$ integers as follows.

$$x_i' = ir + 1, \quad 1 \leqslant i \leqslant q,$$
$$y_j' = jr^2 + 2, \quad 1 \leqslant j \leqslant q,$$
$$z_k' = kr^3 + 4, \quad 1 \leqslant k \leqslant q.$$

Moreover, for every triple $t_l = (x_i, y_j, z_k)$ in T, we define an integer t_l' by

$$t_l' = r^4 - kr^3 - jr^2 - ir + 8. \tag{3}$$

Define $b = r^4 + 15$. Note that $0 < x_i', y_j', z_k', t_l' < b$ holds for all i, j, k and l. Finally, we define a set of $3q + |T|$ vectors that consists of

$$\boldsymbol{x}_i = \left(\frac{2}{10} + \frac{x_i'}{5b}, \frac{3}{10} - \frac{x_i'}{5b}\right), \quad 1 \leqslant i \leqslant q,$$
$$\boldsymbol{y}_j = \left(\frac{2}{10} + \frac{y_j'}{5b}, \frac{3}{10} - \frac{y_j'}{5b}\right), \quad 1 \leqslant j \leqslant q,$$
$$\boldsymbol{z}_k = \left(\frac{2}{10} + \frac{z_k'}{5b}, \frac{3}{10} - \frac{z_k'}{5b}\right), \quad 1 \leqslant k \leqslant q,$$
$$\boldsymbol{t}_l = \left(\frac{2}{10} + \frac{t_l'}{5b}, \frac{3}{10} - \frac{t_l'}{5b}\right), \quad 1 \leqslant l \leqslant |T|.$$

We say that for an element $w \in X \cup Y \cup Z \cup T$, the integer w', the vector $\boldsymbol{w}$, and the element w, respectively, correspond to each other. It is easy to see that one can uniquely reconstruct an element w from its corresponding integer w', or from its corresponding vector $\boldsymbol{w}$. Let U' be the set that contains the $3q+|T|$ integers x_i', y_j', z_k' and t_l', and let $\boldsymbol{U}$ be the set that contains the $3q + |T|$ vectors $\boldsymbol{x}_i$, $\boldsymbol{y}_j$, $\boldsymbol{z}_k$ and $\boldsymbol{t}_l$.

The following observation is a slightly changed and updated version of an argument in [6, p. 98].

Observation 2. *Four integers in U' sum up to the value b if and only if* (i) *one of them corresponds to some element $x_i \in X$, one of them to some element $y_j \in Y$, one of them to an element $z_k \in Z$, and one of them to some triple $t_l \in T$, and if* (ii) $t_l = (x_i, y_j, z_k)$ *holds for these four elements.*

Proof. The definition of t_l' in Eq. (3) yields the if-part. The only-if-part follows from working modulo r, modulo r^2, modulo r^3 and modulo r^4. □

Observation 3. *Four vectors in $\boldsymbol{U}$ can be packed into a unit-bin if and only if* (i) *one of them corresponds to some element $x_i \in X$, one of them to some element $y_j \in Y$, one of them to an element $z_k \in Z$, and one of them to some triple $t_l \in T$, and if* (ii) $t_l = (x_i, y_j, z_k)$ *holds for these four elements.*

Proof. Suppose that four vectors $\boldsymbol{w}_i$, $1 \leqslant i \leqslant 4$, can be packed together into a unit bin. Let $s = w_1' + w_2' + w_3' + w_4'$ denote the sum of their corresponding integers. Since the sum of the first coordinates of these vectors must be $\leqslant 1$, we get that $8/10 + s/5b \leqslant 1$, which is equivalent to $s \leqslant b$. Since the sum of the second coordinates of these vectors must be $\leqslant 1$, we get that $12/10 - s/5b \leqslant 1$, which is equivalent to $s \geqslant b$. Hence, $s = b$ must hold and Observation 2 yields the only-if-part. The if-part is straightforward. □

Observation 4. *Any set of three vectors in $\boldsymbol{U}$ can be packed into a unit-bin. No set of five vectors in $\boldsymbol{U}$ can be packed into a unit-bin.*

Lemma 5. *Let $\alpha > 0$ be an integer such that $|T| - \alpha$ is divisible by 3. Then there exists a feasible solution for the instance I of MAX-3-DM that contains at least α triples if and only if there exists a feasible packing for the instance $\boldsymbol{U}$ of the 2-dimensional vector packing instance that uses at most $q + \frac{1}{3}(|T| - \alpha)$ unit-bins.*

Proof. (only if) Let T' with $|T'| = \alpha$ be the feasible solution of MAX-3-DM. For every triple in T', the corresponding four vectors can be packed together into a unit-bin by Observation 3. The remaining $3q+|T|-4\alpha$ vectors can be arbitrarily grouped into 3-sets and every such 3-set can be packed into a unit-bin by Observation 4.

(if) Consider a packing into at most $q+\frac{1}{3}(|T| - \alpha)$ unit-bins. It is easy to check that in this case at least α of the unit-bins must contain exactly four vectors. The claim then follows by Observation 3. □

Theorem 6. *If there exists an asymptotic polynomial time approximation scheme for 2-dimensional vector packing, then* P = NP.

Proof. Suppose that there exists an ASY-PTAS for 2-dimensional vector packing. We show how to get from this a PTAS for the MAX SNP-hard MAXIMUM BOUNDED 3-DIMENSIONAL MATCHING problem. The hardness-of-approximation results from the beginning of this decade (cf. [2]) then imply that P = NP, completing the proof of the theorem.

Indeed, let ε be an arbitrarily small, positive real. Throughout this proof, ε is constant and independent of any input. Define $\delta = \frac{1}{42}\varepsilon$, and consider an approximation algorithm A for 2-dimensional vector packing with asymptotic performance guarantee $1+\delta$ that exists by our assumption. Hence, there exists a constant c^* such that

$$A(L) \leqslant (1+\delta)\mathrm{OPT}(L) + c^* \tag{4}$$

holds for all input lists L.

Now consider an arbitrary instance I of MAX-3-DM with $|X| = |Y| = |Z| = q$ and $q \leqslant |T| \leqslant 3q$. First, check whether $\mathrm{OPT}(I) < \frac{3}{\delta}(1+\delta+c^*)$ holds. This can be done in polynomial time by checking all subsets of T with cardinality at most $\frac{3}{\delta}(1+\delta+c^*)$. In case $\mathrm{OPT}(I) < \frac{3}{\delta}(1+\delta+c^*)$ holds, determine the optimum solution of I in polynomial time and terminate. In the remaining case,

$$\frac{3}{\delta}(1+\delta+c^*) \leqslant \mathrm{OPT}(I) \tag{5}$$

holds. Proceed as follows: Compute the vector packing instance $\boldsymbol{U}$ from I as described above. Clearly, this can be done in polynomial time. Then apply the approximation algorithm A to $\boldsymbol{U}$ and get a packing with at most $(1+\delta)\mathrm{OPT}(\boldsymbol{U}) + c^*$ unit-bins. Finally, determine an integer M such that M fulfills

$$A(\boldsymbol{U}) = q + \frac{1}{3}(|T| - M). \tag{6}$$

We claim that M is an excellent approximation of $\mathrm{OPT}(I)$. Indeed, by Lemma 5,

$$\mathrm{OPT}(\boldsymbol{U}) \leqslant q + \left\lceil \frac{1}{3}(|T| - \mathrm{OPT}(I)) \right\rceil \tag{7}$$

holds. Combining (4), (6) and (7) yields that

$$q + \frac{1}{3}(|T| - M) \leqslant (1+\delta)\left(q + \left\lceil \frac{1}{3}(|T| - \mathrm{OPT}(I)) \right\rceil\right) + c^* \tag{8}$$

which by applying the inequality $\lceil x \rceil \leqslant x+1$ gives

$$\begin{aligned}(1+\delta)\mathrm{OPT}(I) &\leqslant M + 3\delta q + \delta|T| + 3(1+\delta+c^*)\\ &\leqslant M + 6\delta q + 3(1+\delta+c^*),\end{aligned}$$

where the last inequality follows from $|T| \leqslant 3q$. By Observation 1, $q \leqslant 7\mathrm{OPT}(I)$ holds. Applying this inequality together with (5), we arrive at

$$(1+\delta)\mathrm{OPT}(I) \leqslant M + 42\delta\mathrm{OPT}(I) + \delta\mathrm{OPT}(I), \tag{9}$$

which is equivalent to

$$M \geqslant (1-42\delta)\mathrm{OPT}(I) = (1-\varepsilon)\mathrm{OPT}(I). \tag{10}$$

Summarizing, we have found a bound M that is at most a factor of $(1-\varepsilon)$ away from the optimum of I. Since this can be done in polynomial time for every fixed $\varepsilon > 0$, the proof of the theorem is complete. □

3. Concluding remarks

We conclude this short note with several remarks.

(1) Let us take a look at the usual partial order on 2-dimensional vectors where $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. The vector set $\boldsymbol{U}$ constructed in Section 2 forms an extremal configuration with respect to this order, since the vectors in $\boldsymbol{U}$ are pairwise incomparable. The opposite extreme case, where the vector set is linearly ordered, is easy to approximate. In fact, a slight modification of the method of Fernandez de la Vega and Lueker [4] yields asymptotic polynomial time approximation schemes for the subproblems of d-dimensional vector packing with constant Dilworth number.

(2) Let us denote by $\eta(d)$ the asymptotic approximability threshold of d-dimensional vector packing (i.e. for all numbers $h > \eta(d)$ there exists a polynomial time approximation algorithm with asymptotic performance guarantee h, and for all $h < \eta(d)$ there does not exist such an algorithm). In this terminology, the approximation result of Fernandez de la Vega and Lueker [4] reads $\eta(d) \leqslant d$ for all d, and the main result of this note becomes $\eta(2) > 1$. We strongly conjecture that $\eta(d)$ tends to infinity when d tends to infinity. A natural conjecture would be that $\eta(d) = d$ holds for all d.

(3) A dual problem to vector packing is *vector covering*: Here the goal is to partition the set of vectors into a *maximum* number of subsets, such

that in every subset the sum of all vectors is at *least* one in every coordinate. Alon et al. [1] have derived several approximability results on vector covering. Our proof in Section 2 can be modified in a straightforward way to provide an analogous non-approximability result for 2-dimensional vector covering.

(4) Another 2-dimensional generalization of classical bin packing is *2-dimensional rectangle packing* where the input consists of a list of geometric rectangles. The goal is to pack these rectangles without overlap into the minimum number of unit-squares. In 1982, Chung, Garey, and Johnson [3] designed a polynomial time approximation algorithm with asymptotic performance guarantee $\leqslant 2.125$, and since then, no further progress has been made on this problem. Especially, we do not know how to exclude the existence of an ASY-PTAS for rectangle packing.

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