Unions and Complements of Hybrid Zonotopes

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***Abstract*—Hybrid zonotopes have been recently introduced as a mixed-integer set representation that is equivalent to the union of up to 2^N constrained zonotopes through the use of N binary factors. This letter further develops this new set representation by deriving identities for union and complement set operations of hybrid zonotopes. The addition of these set operations proves the hybrid zonotope's closure under linear mappings, Minkowski sums, intersections, unions, and complements, thus increasing the usefulness of the set representation to a broad class of set-theoretic controls problems. Beyond set operations, a major contribution of this letter is a demonstration of how linear mixed-integer constraints may be embedded within the hybrid zonotope set definition. A numerical example using hybrid zonotopes as a nonconvex safety constraint in an obstacle avoidance problem is provided.**

***Index Terms*—Hybrid systems, predictive control, set-based computing, zonotopes.**

I. Introduction

THE USE of sets is ubiquitous in modern control theory. While present in the majority of robust and optimal control formulations, set based methods have found further use for evaluation of reachable sets, safety verification, parameter estimation, global optimization, and fault detection [1]. The set representations with the most mature theory and widespread use in solving these problems are ellipsoids, half-space and vertex representation polytopes, and zonotopes. The reader is directed to [1] for a review of common set representations and their use for control design and formal verification. While suitable for many applications, these set representations share a common disadvantage in their convexity. Nonconvexity is inherent in many applications, such as reachability of nonlinear and hybrid systems [2], active fault diagnosis [3], and safety constraints in optimal and robust control [4]. When admissible, nonconvex sets are often represented as an implicit union of a collection of convex sets and merging and approximation algorithms are used to reduce computational burden at the expense of accuracy [2]. However, in the case of safety constraints, e.g., in obstacle avoidance, nonconvexity is inherent and unavoidable [4].

Convex polytopes given by their half-space and vertex representation, denoted by H- and V-rep, have been a primary tool for many set theoretic methods. Their construction is intuitive, and the ability to convert between H- and V-rep leads to closure under all set operations that maintain convexity. When convexity is not maintained, such as with complement and union operations, the resulting nonconvex set may be defined as a collection of convex polytopes [5]. These collections of convex polytopes may then be used in optimization algorithms by enforcing their explicit union as hyperplane arrangements through the introduction of binary variables and mixed-integer constraints [4]. This flexibility has led to algorithms for solving many set-theoretic controls problems [6].

While useful, the computational burden and complexity of H- and V-rep polytopes has a worst-case exponential growth for basic set operations. This growth limits their use to problems with small dimension (generally less than six) and over short time horizons. On the other hand, zonotopes have found increased popularity due to their ability to compactly represent complex and high dimensional sets. Furthermore, the introduction of constrained zonotopes has established a set capable of representing arbitrary convex polytopes and is closed under linear mappings, Minkowski sums, and generalized intersections while maintaining most of the benefits of zonotopes [7]. Constrained zonotopes may be used in place of H- and V-rep polytopes in many applications involving convex sets and boast improvements in computation time and complexity [7], [8]. Moreover, using a mix of continuous and binary variables within the definition of a zonotope leads to a compact representation of nonconvex (and possibly disconnected) sets with an exponential number of features [9].

In the recent work of the authors, the hybrid zonotope has been introduced to represent the nonconvex reachable sets of mixed logical dynamical systems [10]. Furthermore, the authors have shown that the hybrid zonotope is equivalent to the union of a collection of constrained zonotopes [10]. Although a method is provided for converting from a hybrid zonotope to a collection of constrained zonotopes in [10], how to convert a collection of constrained zonotopes to a hybrid zonotope is not addressed. Therefore, the contributions of this letter are the derivations and proofs of identities for both the unions and complements of hybrid zonotopes. This is accomplished by directly embedding linear mixed-integer constraints within the hybrid zonotope set definition. The union operation

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allows for the conversion of a collection of constrained zonotopes to a hybrid zonotope and vice versa, providing a bridge to leverage work previously derived for other set representations. Moreover, complements of hybrid zonotopes may be used in applications such as obstacle avoidance, robust safety constraints, and set containment problems.

The remainder of the manuscript is organized as follows. In Section II necessary notation and preliminary work is described. In Section III the union operation is derived and in Section IV the complement operation is derived. In Section V an example of the application of the set operations is provided in the context of obstacle avoidance. Finally, concluding remarks are provided in Section VI.

II. Notation and Preliminaries

Sets are denoted by uppercase calligraphic letters, e.g., $\mathcal{Z} \subset \mathbb{R}^n$. Commas in subscripts are used to distinguish between properties that are defined for multiple sets; e.g., $n_{g,z}$ describes the complexity of the representation of $\mathcal{Z}$ while $n_{g,w}$ describes the complexity of the representation of $\mathcal{W}$. The n-dimensional unit hypercube is denoted by $\mathcal{B}_\infty^n = \{x \in \mathbb{R}^n : \|x\|_\infty \le 1\}$. The power set of an n-dimensional vector of binary variables is denoted by $\{-1, 1\}^n$. The cardinality of the discrete set $\mathcal{T}$ is denoted by $|\mathcal{T}|$. The concatenation of two column vectors to a single column vector is denoted by $(\xi_1\ \xi_2) = [\xi_1^T\ \xi_2^T]^T$. The bold numbers $\mathbf{1}$ and $\mathbf{0}$ denote matrices of all 1 and 0 elements, respectively, and I denotes the identity matrix; subscripts are used to identify the dimension of these matrices when not easily deduced from the context. The topological boundary of a set is denoted by $\partial\mathcal{Z}$ and the interior by $\mathcal{Z}^\circ$. Given the sets $\mathcal{Z}$, $\mathcal{W} \subset \mathbb{R}^n$, the intersection of $\mathcal{Z}$ and $\mathcal{W}$ is $\mathcal{Z} \cap \mathcal{W} = \{x \in \mathbb{R}^n : x \in \mathcal{Z} \wedge x \in \mathcal{W}\}$, the union of $\mathcal{Z}$ and $\mathcal{W}$ is $\mathcal{Z} \cup \mathcal{W} = \{x \in \mathbb{R}^n : x \in \mathcal{Z} \vee x \in \mathcal{W}\}$, the closure of the complement of $\mathcal{Z}$ is $\overline{\mathcal{Z}^c} = \{x \in \mathbb{R}^n : x \notin \mathcal{Z}^\circ\}$, and the closure of the complement of $\mathcal{Z}$ defined over the set $\mathcal{X}$ is denoted by $\mathcal{C}_\mathcal{X}(\mathcal{Z}) = \overline{\mathcal{Z}^c} \cap \mathcal{X}$.

A constrained zonotope $\mathcal{Z}_c \subset \mathbb{R}^n$ is defined in Constrained Generator representation (CG-rep) as

$$\mathcal{Z}_c = \{G\xi + c : \|\xi\|_\infty \le 1, A\xi = b\}, \tag{1}$$

where $G \in \mathbb{R}^{n\times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c\times n_g}$, and $b \in \mathbb{R}^{n_c}$ [7]. The shorthand notation of $\mathcal{Z}_c = \langle G, c, A, b\rangle$ is used to denote the set given by (1). A constrained zonotope is the set of points given by all linear combinations of the center c with the weighted generators—the columns of $G = [g^{(1)} \cdots g^{(n_g)}]$—such that their weights $\xi = (\xi_1 \cdots \xi_{n_g})$, called factors, lie within the constrained unit hypercube $\mathcal{B}_\infty^{n_g}(A, b) = \{\xi \in \mathcal{B}_\infty^{n_g} : A\xi = b\}$. By constraining the factors to lie on the intersection of the hyperplanes $A\xi = b$, a constrained zonotope may be constructed to represent any convex polytope [7]. In the remainder of the manuscript, all constrained zonotopes are taken to be full dimensional and nonempty with linearly independent equality constraints.

Hybrid zonotopes have been recently introduced by including an additional vector of factors that are constrained to be binary. A hybrid zonotope $\mathcal{Z}_h \subset \mathbb{R}^n$ is defined in Hybrid Constrained Generator representation (HCG-rep) as

$$\mathcal{Z}_h = \left\{ \begin{bmatrix} G^c\ G^b \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} + c \;\middle|\; \begin{array}{c} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} \in \mathcal{B}_\infty^{n_g} \times \{-1, 1\}^{n_b}, \\ \begin{bmatrix} A^c\ A^b \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} = b \end{array} \right\}, \tag{2}$$

where $G^c \in \mathbb{R}^{n\times n_g}$, $G^b \in \mathbb{R}^{n\times n_b}$, $c \in \mathbb{R}^n$, $A^c \in \mathbb{R}^{n_c\times n_g}$, $A^b \in \mathbb{R}^{n_c\times n_b}$, and $b \in \mathbb{R}^{n_c}$ [10]. The generator and constraint matrices have been partitioned into continuous and binary parts denoted by superscript c and b, respectively, to form the mixed-integer set representation. These superscripts should not be confused with the vectors c and b used for the center and equality constraints, respectively. The shorthand notation of $\mathcal{Z}_h = \langle G^c, G^b, c, A^c, A^b, b\rangle$ is used to denote the set given by (2). Through the introduction of n_b binary factors, the hybrid zonotope is equivalent to the union of $|\mathcal{T}| \le 2^{n_b}$ constrained zonotopes as

$$\mathcal{Z}_h = \bigcup_{\xi_i^b \in \mathcal{T}} \mathcal{Z}_{c,i}, \tag{3a}$$

$$\mathcal{Z}_{c,i} = \left\langle G^c, c + G^b\xi_i^b, A^c, b - A^b\xi_i^b \right\rangle, \tag{3b}$$

where the i^{th} constrained zonotope (3b) has its center and equality constraints shifted by the i^{th} value of the discrete set $\mathcal{T} = \{\xi_i^b \in \{-1, 1\}^{n_b} | \mathcal{Z}_{c,i} \ne \emptyset\}$ mapped by G^b and A^b, respectfully. Hybrid zonotopes are closed under linear mappings, Minkowski sums, generalized intersections, and halfspace intersections that may all be determined through identities [10, Proposition 2].

III. Unions of Hybrid Zonotopes

In this section, the closure of hybrid zonotopes under union operations is proven. This is achieved by including the generators and constraints of both operating sets within the resulting hybrid zonotope. By introducing one additional binary factor, the union switches between which of two sets are active by constraining the factors of the inactive set to a fixed value. The proposition and technical proof will be followed by a discussion of its underlying principles and how the growth of set representation complexity may be reduced.

Proposition 1: For any $\mathcal{Z}_h = \langle G_z^c, G_z^b, c_z, A_z^c, A_z^b, b_z\rangle \subset \mathbb{R}^n$ and $\mathcal{W}_h = \langle G_w^c, G_w^b, c_w, A_w^c, A_w^b, b_w\rangle \subset \mathbb{R}^n$, define the vectors $\hat{G}^b \in \mathbb{R}^n$, $\hat{c} \in \mathbb{R}^n$, $\hat{A}_z^b \in \mathbb{R}^{n_{c,z}}$, $\hat{b}_z \in \mathbb{R}^{n_{c,z}}$, $\hat{A}_w^b \in \mathbb{R}^{n_{c,w}}$, and $\hat{b}_w \in \mathbb{R}^{n_{c,w}}$, such that

$$\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \hat{G}^b \\ \hat{c} \end{bmatrix} = \begin{bmatrix} G_w^b\mathbf{1} + c_z \\ G_z^b\mathbf{1} + c_w \end{bmatrix},$$

$$\begin{bmatrix} -I & I \\ I & I \end{bmatrix} \begin{bmatrix} \hat{A}_z^b \\ \hat{b}_z \end{bmatrix} = \begin{bmatrix} b_z \\ -A_z^b\mathbf{1} \end{bmatrix},$$

$$\begin{bmatrix} -I & I \\ I & I \end{bmatrix} \begin{bmatrix} \hat{A}_w^b \\ \hat{b}_w \end{bmatrix} = \begin{bmatrix} -A_w^b\mathbf{1} \\ b_w \end{bmatrix}.$$

Then the union of $\mathcal{Z}_h$ and $\mathcal{W}_h$ is the hybrid zonotope $\mathcal{Z}_h \cup \mathcal{W}_h = \langle G_u^c, G_u^b, c_u, A_u^c, A_u^b, b_u\rangle \subset \mathbb{R}^n$ where

$$G_u^c = \begin{bmatrix} G_z^c & G_w^c & \mathbf{0} \end{bmatrix}, G_u^b = \begin{bmatrix} G_z^b & G_w^b & \hat{G}^b \end{bmatrix}, c_u = \hat{c},$$

$$A_u^c = \begin{bmatrix} A_z^c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_w^c & \mathbf{0} \\ & A_3^c & I \end{bmatrix}, A_u^b = \begin{bmatrix} A_z^b & \mathbf{0} & \hat{A}_z^b \\ \mathbf{0} & A_w^b & \hat{A}_w^b \\ & A_3^b & \end{bmatrix}, b_u = \begin{bmatrix} \hat{b}_z \\ \hat{b}_w \\ b_3 \end{bmatrix},$$

$$A_3^c = \begin{bmatrix} I & \mathbf{0} \\ -I & \mathbf{0} \\ \mathbf{0} & I \\ \mathbf{0} & -I \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, A_3^b = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \\ \frac{1}{2}I & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ -\frac{1}{2}I & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \frac{1}{2}I & -\frac{1}{2}\mathbf{1} \\ \mathbf{0} & -\frac{1}{2}I & -\frac{1}{2}\mathbf{1} \end{bmatrix}, b_3 = \begin{bmatrix} \frac{1}{2}\mathbf{1} \\ \frac{1}{2}\mathbf{1} \\ \frac{1}{2}\mathbf{1} \\ \frac{1}{2}\mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}. \tag{4}$$

Proof: Let $\mathcal{X} = \langle G_u^c, G_u^b, c_u, A_u^c, A_u^b, b_u \rangle$ denote the hybrid zonotope given by (4). For any $x \in \mathcal{X}$ there exists some $\xi_x^c \in \mathcal{B}_\infty^{n_{g,x}}$ and $\xi_x^b \in \{-1, 1\}^{n_{b,x}}$ such that $A_u^c \xi_x^c + A_u^b \xi_x^b = b_u$ and $x = G_u^c \xi_x^c + G_u^b \xi_x^b + c_u$. Let $\xi_x^c = (\xi_z^c\ \xi_w^c\ \xi_u^c)$, where $\xi_z^c \in \mathbb{R}^{n_{g,z}}$, $\xi_w^c \in \mathbb{R}^{n_{g,w}}$, and $\xi_u^c \in \mathbb{R}^{2(n_{g,z}+n_{g,w}+n_{b,z}+n_{b,w})}$, and $\xi_x^b = (\xi_z^b\ \xi_w^b\ \xi_u^b)$, where $\xi_z^b \in \{-1, 1\}^{n_{b,z}}$, $\xi_w^b \in \{-1, 1\}^{n_{b,w}}$, and $\xi_u^b \in \{-1, 1\}$. Expanding the third row of the equality constraints $A_u^c \xi_x^c + A_u^b \xi_x^b = b_u$ gives

$$\xi_z^c = 1/2\mathbf{1} - 1/2\xi_u^b - \xi_{u,1}^c = -1/2\mathbf{1} + 1/2\xi_u^b + \xi_{u,2}^c, \tag{5a}$$
$$\xi_w^c = 1/2\mathbf{1} + 1/2\xi_u^b - \xi_{u,3}^c = -1/2\mathbf{1} - 1/2\xi_u^b + \xi_{u,4}^c, \tag{5b}$$
$$1/2\xi_z^b = -1/2\xi_u^b - \xi_{u,5}^c = -\mathbf{1} + 1/2\xi_u^b + \xi_{u,6}^c, \tag{5c}$$
$$1/2\xi_w^b = 1/2\xi_u^b - \xi_{u,7}^c = -\mathbf{1} - 1/2\xi_u^b + \xi_{u,8}^c, \tag{5d}$$

where $\xi_u^c = (\xi_{u,1}^c\ \cdots\ \xi_{u,8}^c)$. Letting $\xi_u^b = 1$, (5) reduces to

$$\xi_z^c = -\xi_{u,1}^c = \xi_{u,2}^c, \tag{6a}$$
$$\xi_w^c = \mathbf{1} - \xi_{u,3}^c = -\mathbf{1} + \xi_{u,4}^c, \tag{6b}$$
$$\xi_z^b = -\mathbf{1} - 2\xi_{u,5}^c = -\mathbf{1} + 2\xi_{u,6}^c, \tag{6c}$$
$$\xi_w^b = \mathbf{1} - 2\xi_{u,7}^c = -3\mathbf{1} + 2\xi_{u,8}^c. \tag{6d}$$

Given that $\|\xi_u^c\|_\infty \leq 1$, (6b) and (6d) are only satisfied for $\xi_w^c = \mathbf{0}$ and $\xi_w^b = -\mathbf{1}$ respectively, while (6a) and (6c) are satisfied for any $\|\xi_z^c\|_\infty \leq 1$ and $\xi_z^b \in \{-1, 1\}^{n_{b,z}}$. Let $\xi_x^c = (\xi_z^c\ \mathbf{0}\ \xi_u^c)$ and $\xi_x^b = (\xi_z^b\ -\mathbf{1}\ 1)$. Expanding $x = G_u^c \xi_x^c + G_u^b \xi_x^b + c_u$ gives

$$x = G_z^c \xi_z^c + G_w^c \mathbf{0} + \mathbf{0}\xi_u^c + G_z^b \xi_z^b - G_w^b \mathbf{1} + \hat{G}^b + \hat{c}, \tag{7}$$

and, after substituting $-G_w^b \mathbf{1} + \hat{G}^b + \hat{c} = c_z$, reduces to $x = G_z^c \xi_z^c + G_z^b \xi_z^b + c_z$. Expanding the first two rows of the equality constraints $A_u^c \xi_x^c + A_u^b \xi_x^b = b_u$ results in

$$\begin{aligned} A_z^c \xi_z^c + A_z^b \xi_z^b + \hat{A}_z^b &= \hat{b}_z, \\ A_w^c \mathbf{0} - A_w^b \mathbf{1} + \hat{A}_w^b &= \hat{b}_w, \end{aligned} \tag{8}$$

which, after substituting $\hat{b}_z - \hat{A}_z^b = b_z$ and $\hat{b}_w - \hat{A}_w^b = -A_w^b \mathbf{1}$, gives $A_z^c \xi_z^c + A_z^b \xi_z^b = b_z$ and $-A_w^b \mathbf{1} = -A_w^b \mathbf{1}$. Combining (6)-(8) results in $x \in \mathcal{Z}_h$ for $\xi_u^b = 1$.

Now let $\xi_u^b = -1$ and (5) reduce to

$$\xi_z^c = \mathbf{1} - \xi_{u,1}^c = -\mathbf{1} + \xi_{u,2}^c, \tag{9a}$$
$$\xi_w^c = -\xi_{u,3}^c = \xi_{u,4}^c, \tag{9b}$$
$$\xi_z^b = \mathbf{1} - 2\xi_{u,5}^c = -3\mathbf{1} + 2\xi_{u,6}^c, \tag{9c}$$
$$\xi_w^b = -\mathbf{1} - 2\xi_{u,7}^c = -\mathbf{1} + 2\xi_{u,8}^c. \tag{9d}$$

Given that $\|\xi_u^c\|_\infty \leq 1$, (9a) and (9c) are only satisfied for $\xi_z^c = \mathbf{0}$ and $\xi_z^b = -\mathbf{1}$ respectively, while (9b) and (9d) are satisfied for any $\|\xi_w^c\|_\infty \leq 1$ and $\xi_w^b \in \{-1, 1\}^{n_{b,w}}$. Let $\xi_x^c = (\mathbf{0}\ \xi_w^c\ \xi_u^c)$ and $\xi_x^b = (-\mathbf{1}\ \xi_w^b\ -1)$. Expanding $x = G_u^c \xi_x^c + G_u^b \xi_x^b + c_u$ gives

$$x = G_z^c \mathbf{0} + G_w^c \xi_w^c + \mathbf{0}\xi_u^c - G_z^b \mathbf{1} + G_w^b \xi_w^b - \hat{G}^b + \hat{c}, \tag{10}$$

and, after substituting $-G_z^b \mathbf{1} - \hat{G}^b + \hat{c} = c_w$, reduces to $x = G_w^c \xi_w^c + G_w^b \xi_w^b + c_w$. Expanding the first two rows of the equality constraints $A_u^c \xi_x^c + A_u^b \xi_x^b = b_u$ results in

$$\begin{aligned} A_z^c \mathbf{0} - A_z^b \mathbf{1} - \hat{A}_z^b &= \hat{b}_z, \\ A_w^c \xi_z^c + A_w^b \xi_w^b - \hat{A}_w^b &= \hat{b}_w, \end{aligned} \tag{11}$$

which, after substituting $\hat{b}_z + \hat{A}_z^b = -A_z^b \mathbf{1}$ and $\hat{b}_w + \hat{A}_w^b = b_w$, gives $-A_z^b \mathbf{1} = -A_z^b \mathbf{1}$ and $A_w^c \xi_w^c + A_w^b \xi_w^b = b_w$. Combining (9)-(11) results in $x \in \mathcal{W}_h$ for $\xi_u^b = -1$. Given that $\xi_u^b \in \{-1, 1\}$ and that the choice of $x \in \mathcal{X}$ is arbitrary, $\mathcal{X} \subseteq \mathcal{Z}_h \cup \mathcal{W}_h$.

Conversely, for any $z \in \mathcal{Z}_h$ there exists some $\xi_z^c \in \mathcal{B}_\infty^{n_{g,z}}$ and $\xi_z^b \in \{-1, 1\}^{n_{b,z}}$ such that $A_z^c \xi_z^c + A_z^b \xi_z^b = b_z$ and $z = G_z^c \xi_z^c + G_z^b \xi_z^b + c_z$. Letting $\xi_x^c = (\xi_z^c\ \mathbf{0}\ \xi_u^c)$ and $\xi_x^b = (\xi_z^b\ -\mathbf{1}\ 1)$, (8) is satisfied and (6) implies that $\|\xi_u^c\|_\infty \leq 1$. Applying (7) then gives $z \in \mathcal{X}$. For any $w \in \mathcal{W}_h$ there exists some $\xi_w^c \in \mathcal{B}_\infty^{n_{g,w}}$ and $\xi_w^b \in \{-1, 1\}^{n_{b,w}}$ such that $A_w^c \xi_w^c + A_w^b \xi_w^b = b_w$ and $w = G_w^c \xi_w^c + G_w^b \xi_w^b + c_w$. Letting $\xi_x^c = (\mathbf{0}\ \xi_w^c\ \xi_u^c)$ and $\xi_x^b = (-\mathbf{1}\ \xi_w^b\ -1)$, (11) is satisfied and (9) implies that $\|\xi_u^c\|_\infty \leq 1$. Applying (10) then gives $w \in \mathcal{X}$. Given that the choice of $z \in \mathcal{Z}_h$ and $w \in \mathcal{W}_h$ is arbitrary, $\mathcal{Z}_h \cup \mathcal{W}_h \subseteq \mathcal{X}$ and therefore $\mathcal{Z}_h \cup \mathcal{W}_h = \mathcal{X}$. ■

The union set operation given by Proposition 1 introduces $2(n_{g,z} + n_{g,w} + n_{b,z} + n_{b,w})$ "slack" continuous factors, ξ_u^c, and one switching binary factor, ξ_u^b. The additional linear equality constraints, $A_3^c(\xi_z^c\ \xi_w^c) + I\xi_u^c + A_3^b(\xi_z^b\ \xi_w^b\ \xi_u^b) = b_3$, implement the switch between which of the two sets in the union is active, according to

$$\begin{aligned} \xi_u^b = 1 \implies & \begin{matrix} (\xi_z^c\ \xi_z^b) \in \mathcal{B}_\infty^{n_{g,z}} \times \{-1, 1\}^{n_{b,z}} \\ (\xi_w^c\ \xi_w^b) = (\mathbf{0}\ -\mathbf{1}) \end{matrix}, \\ \xi_u^b = -1 \implies & \begin{matrix} (\xi_z^c\ \xi_z^b) = (\mathbf{0}\ -\mathbf{1}) \\ (\xi_w^c\ \xi_w^b) \in \mathcal{B}_\infty^{n_{g,w}} \times \{-1, 1\}^{n_{b,w}} \end{matrix}. \end{aligned} \tag{12}$$

The hatted constants, $\hat{G}^b$, $\hat{c}$, $\hat{A}_z^b$, $\hat{b}_z$, $\hat{A}_w^b$, and $\hat{b}_w$, multiplied by the binary switch, ξ_u^b, then account for the binary factors being constrained to $-\mathbf{1}$ instead of $\mathbf{0}$, the change of centers, and the feasibility of the constraints of the inactive set.

For ease of understanding, Proposition 1 applies these constraints to all continuous and binary factors; however, in practice it is only necessary, and beneficial, to apply these constraints to factors that map through non-zero generators. This fact stems from the observation that the factors and generator matrices may be parsed, for the example of $x \in \mathcal{Z}$ in the proof of Proposition 1, such that $\xi_w = (\xi_{w,\neq 0}\ \xi_{w,=0})$ and $G_w = [G_{w,\neq 0}\ \mathbf{0}]$. Using this partition, the equation for x then reduces to $x = G_z \xi_z + G_{w,\neq 0} \xi_{w,\neq 0} + \hat{G}^b + \hat{c}$, and only the factors $\xi_{w,\neq 0}$ must be constrained to cancel their contribution to x. Feasibility is maintained in the remaining equality constraints since the feasible values of $\xi_{w,=0}^c = \mathbf{0}$ and $\xi_{w,=0}^b = -\mathbf{1}$ still exist, although not strictly enforced. This modification is accomplished by replacing the identity matrices in A_3^c and A_3^b with staircase matrices. These staircase matrices are constructed with a single one in each row located in the i^{th} column corresponding to the index of each non-zero generator. For example, the union of hybrid zonotopes $\mathcal{Z}$ and $\mathcal{W}$ with generator matrices given by

$$G_z^c = \begin{bmatrix} g_z^{(c,1)} & \mathbf{0} & \mathbf{0} & g_z^{(c,4)} \end{bmatrix}, \qquad G_z^b = \begin{bmatrix} g_z^{(b,1)} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$G_w^c = \begin{bmatrix} g_w^{(c,1)} & g_w^{(c,2)} \end{bmatrix}, \qquad G_w^b = \begin{bmatrix} g_w^{(b,1)} & \mathbf{0} & g_w^{(b,3)} & \mathbf{0} \end{bmatrix},$$

may be represented using the staircase matrices

$$S_z^c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad S_z^b = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$S_w^c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S_w^b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Making this substitution reduces the growth in the set representation complexity by introducing fewer "slack" factors and equality constraints. This is especially useful when factors appear in the constraints and not the generator matrices—for example, when applying Proposition 1 multiple times. Given two hybrid zonotopes, the set representation complexity growth of the union operation, $\mathcal{Z}_h = \mathcal{Z}_{h,1} \cup \mathcal{Z}_{h,2}$, is given by

$$\begin{aligned} n_g &= n_{g,1} + n_{g,2} + 2(n_{g,1}^r + n_{b,1}^r + n_{g,2}^r + n_{b,2}^r), \\ n_b &= n_{b,1} + n_{b,2} + 1\,, \\ n_c &= n_{c,1} + n_{c,2} + 2(n_{g,1}^r + n_{b,1}^r + n_{g,2}^r + n_{b,2}^r), \end{aligned} \tag{13}$$

where superscript r denotes the number of nonzero generators, i.e., $n^r \leq n$.

IV. Complements of Hybrid Zonotopes

This section provides an identity for the representation of the complements of constrained zonotopes as hybrid zonotopes over a bounded region of interest. It is then shown how this identity implies the closure of hybrid zonotopes under complement set operations.

The point containment problem for the constrained zonotope $\mathcal{Z}_c = \langle G, c, A, b \rangle \subset \mathbb{R}^n$ may be determined by solving the Linear Program (LP) [7, Proposition 2]

$$z \in \mathcal{Z}_c \iff \min\left\{ \|\xi\|_\infty : \begin{bmatrix} G \\ A \end{bmatrix} \xi = \begin{bmatrix} z - c \\ b \end{bmatrix} \right\} \leq 1. \tag{14}$$

The complement of a constrained zonotope may then be defined by modifying the result in [3] using the constrained zonotope's lifted zonotope representation [7].

Lemma 1 [3, Lemma 4]: Given any full dimensional constrained zonotope $\mathcal{Z}_c = \langle G, c, A, b \rangle \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, let

$$\begin{aligned} \delta^*(x) = \min_{\delta,\xi} \quad & \delta \\ \text{s.t.} \quad & \begin{bmatrix} x \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} G \\ A \end{bmatrix} \xi + \begin{bmatrix} c \\ -b \end{bmatrix}, \\ & \|\xi\|_\infty \leq 1 + \delta. \end{aligned} \tag{15}$$

Then $x \notin \mathcal{Z}_c \iff \delta^*(x) > 0$.

The condition given in Lemma 1 may be relaxed to give the closure of the complement by using non-strict inequalities, i.e., $x \in \overline{\mathcal{Z}_c^c} \iff \delta^*(x) \geq 0$, noting that $\delta^*(x) = 0$ occurs when $x \in \partial \mathcal{Z}_c$ [11]. Inspired by the use of complements of zonotopes in [3], the closure of the complement of a constrained zonotope is now defined as a hybrid zonotope.

Proposition 2: Given any full dimensional, nonempty constrained zonotope $\mathcal{Z}_c = \langle G, c, A, b \rangle \subset \mathbb{R}^n$ and a convex, bounded region of interest $\mathcal{X} \supseteq \mathcal{Z}_c$, define positive scalars δ_m and λ_m such that $\mathcal{X} \subseteq \{G\xi + c : \|\xi\|_\infty \leq 1 + \delta_m,\ A\xi = b\}$ and $\lambda_m \geq \max\{\|\lambda\|_\infty : |[G^T\ A^T]\lambda| \leq 1\}$, and let $m = \delta_m + 1$. Define the interval sets

$$\begin{aligned} &\{G_{f,1}\xi_{f,1} + c_{f,1} : \|\xi_{f,1}\|_\infty \leq 1\} \\ &\quad = \left[-\left(m + \tfrac{\delta_m}{2}\right)\mathbf{1}_{2n_{g,z}},\quad \left(1 + \tfrac{\delta_m}{2}\right)\mathbf{1}_{2n_{g,z}} \right], \\ &\{G_{f,2}\xi_{f,2} + c_{f,2} : \|\xi_{f,2}\|_\infty \leq 1\} \\ &\quad = \begin{bmatrix} -\left(m + \frac{3\delta_m}{2} + 1\right)\mathbf{1}_{2n_{g,z}}, & \frac{\delta_m}{2}\mathbf{1}_{2n_{g,z}} \\ -2\mathbf{1}_{2n_{g,z}}, & \mathbf{0}_{2n_{g,z}} \end{bmatrix}. \end{aligned}$$

Then the closure of the complement of $\mathcal{Z}_c$ within the region of interest $\mathcal{X}$ is given by the hybrid zonotope $\mathcal{C}_{\mathcal{X}}(\mathcal{Z}_c) = \langle G_c^c, G_c^b, c_c, A_c^c, A_c^b, b_c \rangle \subset \mathbb{R}^n$ where,

$$\begin{aligned} &G_c^c = \begin{bmatrix} mG & \mathbf{0} \end{bmatrix},\ G_c^b = \mathbf{0}\,,\ c_c = c, \\ &A_c^c = \begin{bmatrix} mA & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{PF}^c & G_{f,1} & \mathbf{0} \\ A_{DF}^c & \mathbf{0} & \mathbf{0} \\ A_{CS}^c & \mathbf{0} & G_{f,2} \end{bmatrix}, A_c^b = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ A_{CS}^b \end{bmatrix}, b_c = \begin{bmatrix} b \\ c_{f,1} \\ b_{DF} \\ c_{f,2} \end{bmatrix}, \\ &A_{PF}^c = \begin{bmatrix} mI & -\frac{\delta_m}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -mI & -\frac{\delta_m}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ &A_{DF}^c = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \lambda_m[G^T\ A^T] & \frac{1}{2}I & -\frac{1}{2}I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \frac{1}{2}\mathbf{1} \end{bmatrix}, b_{DF} = \begin{bmatrix} \mathbf{0} \\ 1 - n_g \end{bmatrix}, \\ &A_{CS}^c = \begin{bmatrix} -mI & \frac{\delta_m}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ mI & \frac{\delta_m}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix}, A_{CS}^b = \begin{bmatrix} mI & \mathbf{0} \\ \mathbf{0} & mI \\ -I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix}. \end{aligned} \tag{16}$$

Proof: Let $\mathcal{W}_h = \langle G_c^c, G_c^b, c_c, A_c^c, A_c^b, b_c \rangle \subset \mathbb{R}^n$ denote the hybrid zonotope given by (16). For any $w \in \mathcal{W}_h$ there exists some $\xi_w^c \in \mathcal{B}_\infty^{n_{g,w}}$ and $\xi_w^b \in \{-1, 1\}^{n_{b,w}}$ such that $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$ and $w = G_c^c\xi_w^c + G_c^b\xi_w^b + c_c$. Let $\xi_w^c = (\xi_c^c\ \xi_\delta^c\ \xi_\lambda^c\ \xi_{\mu,1}^c\ \xi_{\mu,2}^c\ \xi_{f,1}^c\ \xi_{f,2}^c)$ and $\xi_w^b = (\xi_1^b\ \xi_2^b)$, where $\xi_c^c \in \mathbb{R}^{n_{g,z}}$, $\xi_\delta^c \in \mathbb{R}$, $\xi_\lambda^c \in \mathbb{R}^{n_z + n_{c,z}}$, $\xi_{\mu,1,2}^c \in \mathbb{R}^{n_{g,z}}$, $\xi_{f,1}^c \in \mathbb{R}^{2n_{g,z}}$, $\xi_{f,2}^c \in \mathbb{R}^{4n_{g,z}}$, and $\xi_{1,2}^b \in \{-1, 1\}^{n_{g,z}}$. Then $w = mG\xi_c^c + c$ and the first row of $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$ gives $mA\xi_c^c = b$. Expanding the second row of $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$ gives

$$\begin{bmatrix} m\xi_c^c \\ -m\xi_c^c \end{bmatrix} + \begin{bmatrix} -\delta_m/2\,\xi_\delta^c \\ -\delta_m/2\,\xi_\delta^c \end{bmatrix} = -G_{f,1}\xi_{f,1}^c + c_{f,1}, \tag{17}$$

which implies that $m\xi_c^c - \frac{\delta_m}{2}\xi_\delta^c \leq (1 + \frac{\delta_m}{2})\mathbf{1}$ and $-m\xi_c^c - \frac{\delta_m}{2}\xi_\delta^c \leq (1 + \frac{\delta_m}{2})\mathbf{1}$. Expanding the third row of $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$ gives

$$\begin{aligned} \lambda_m[G^T\ A^T]\xi_\lambda^c + {}^1\!/\!_2\,\xi_{\mu,1}^c - {}^1\!/\!_2\,\xi_{\mu,2}^c &= \mathbf{0}, \\ {}^1\!/\!_2\left(\xi_{\mu,1}^c + \xi_{\mu,2}^c\right)^T\mathbf{1} &= 1 - n_{g,z}. \end{aligned} \tag{18}$$

Expanding the fourth row of $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$ gives

$$\begin{bmatrix} -m\xi_c^c + \delta_m/2\,\xi_\delta^c + (1 + \delta_m)\xi_1^b \\ m\xi_c^c + \delta_m/2\,\xi_\delta^c + (1 + \delta_m)\xi_2^b \\ \xi_{\mu,1}^c - \xi_1^b \\ \xi_{\mu,2}^c - \xi_2^b \end{bmatrix} = -G_{f,2}\xi_{f,2}^c + c_{f,2}, \tag{19}$$

which implies that $-m\xi_c^c + \frac{\delta_m}{2}\xi_\delta^c + (1+\delta_m)\xi_1^b \leq \frac{\delta_m}{2}\mathbf{1}$, $m\xi_c^c + \frac{\delta_m}{2}\xi_\delta^c + (1+\delta_m)\xi_2^b \leq \frac{\delta_m}{2}\mathbf{1}$, $\xi_{\mu,1}^c \leq \xi_1^b$, and $\xi_{\mu,2}^c \leq \xi_2^b$. Define the change of variables

$$\xi = m\xi_c^c\ ,\ \delta = {}^{\delta_m}\!/_2\xi_\delta^c + {}^{\delta_m}\!/_2\ ,\ \lambda = \lambda_m\xi_\lambda^c$$
$$\mu_{1,2} = {}^1\!/_2\xi_{\mu,1,2}^c + {}^1\!/_2\ ,\ p_{1,2} = {}^1\!/_2\xi_{1,2}^b + {}^1\!/_2. \tag{20}$$

Carrying these change of variables through the above constraints results in

$$w = G\xi + c,\ A\xi = b,\ \|\xi\|_\infty \leq 1 + \delta, \tag{21a}$$
$$\begin{bmatrix} G^T & A^T \end{bmatrix}\lambda + \mu_1 - \mu_2 = \mathbf{0},\ (\mu_1 + \mu_2)^T\mathbf{1} = 1, \tag{21b}$$
$$\begin{aligned} -2(1+\delta_m)(1-p_1) &\leq \xi - \delta - \mathbf{1},\ \mu_1 \leq p_1, \\ -2(1+\delta_m)(1-p_2) &\leq -\xi - \delta - \mathbf{1},\ \mu_2 \leq p_2, \end{aligned} \tag{21c}$$
$$\delta \in [0, \delta_m],\ \mu_{1,2} \in [0,1]^{n_{g,z}},\ p_{1,2} \in \{0,1\}^{n_{g,z}}, \tag{21d}$$

where (21) are the Karush Kuhn Tucker (KKT) conditions of the LP (15) [3]: (21a) is the primal feasibility, resulting from (17), (21b) is the dual feasibility, resulting from (18), and (21c) is the complementary slackness, resulting from (19). Given that the LP is convex, the KKT conditions are necessary and sufficient; thus $\delta = \delta^*(w)$. Recalling Lemma 1, the constraint $\delta \in [0, \delta_m]$ in (21d) results in $w \in \overline{\mathcal{Z}_c^c}$. Given that the choice of $w \in \mathcal{W}_h$ is arbitrary, $\mathcal{W}_h \subseteq \overline{\mathcal{Z}_c^c}$.

Conversely, for any $z \in \mathcal{C}_\mathcal{X}(\mathcal{Z}_c)$, there exists some ξ such that $z = G\xi + c$, $A\xi = b$, and $\delta^*(z) \in [0, \delta_m]$. Since $\delta^*(z)$ is the minimum of the convex LP (15), there exists some λ, $\mu_{1,2} \geq \mathbf{0}$, and $p_{1,2} \in \{-1,1\}^{n_{g,z}}$ such that (21) holds. Letting $\xi_w^c = (\xi_c^c\ \xi_\delta^c\ \xi_\lambda^c\ \xi_{\mu,1}^c\ \xi_{\mu,2}^c\ \xi_{f,1}^c\ \xi_{f,2}^c)$, $\xi_w^b = (\xi_1^b\ \xi_2^b)$, and applying the change of variables (20), the above implies that $\xi_w^c \in \mathcal{B}_\infty^{n_{g,w}}$, $\xi_w^b \in \{-1,1\}^{n_{b,w}}$, $A_c^c\xi_w^c + A_c^b\xi_w^b = b_c$, and $z = G_c^c\xi_w^c + G_c^b\xi_w^b + c_c$; thus $z \in \mathcal{W}_h$. Given that the choice of $z \in \mathcal{C}_\mathcal{X}(\mathcal{Z}_c)$ is arbitrary, $\mathcal{C}_\mathcal{X}(\mathcal{Z}_c) \subseteq \mathcal{W}_h$ and $\mathcal{W}_h \cap \mathcal{X} = \mathcal{C}_\mathcal{X}(\mathcal{Z}_c)$. ■

The complement set operation given by Proposition 2 embeds the mixed-integer formulation of the KKT conditions of the LP (15) directly within the equality constraints of the hybrid zonotope set definition. The limitation that the proposed identity is only valid over the bounded region $\mathcal{X}$ is due to the so called "big-M" constant, δ_m, that appears in the complementary slackness condition (21c) [12]. The representation complexity of the hybrid zonotope $\mathcal{C}_\mathcal{X}(\mathcal{Z}_c) \subset \mathbb{R}^n$ defined by Proposition 2 is given by

$$\begin{aligned} n_{g,c} &= 9n_{g,z} + n + n_{c,z} + 1, \\ n_{b,c} &= 2n_{g,z}, \\ n_{c,c} &= 7n_{g,z} + n_{c,z} + 1. \end{aligned} \tag{22}$$

Theorem 1: Hybrid zonotopes are closed under complement set operations.

Proof: A set is a hybrid zonotope if and only if it is the union of the finite collection of constrained zonotopes $\mathcal{Z}_{c,i}$ given by (3) [10]. Thus given any hybrid zonotope $\mathcal{Z}_h$, by De Morgan's law, the closure of the complement of $\mathcal{Z}_h$ is given by $\overline{\mathcal{Z}_h^c} = \bigcap_{\xi_i^b \in \mathcal{T}} \overline{\mathcal{Z}_{c,i}^c}$. The representation of $\overline{\mathcal{Z}_{c,i}^c}$ by Proposition 2 as a hybrid zonotope, and the closure of hybrid zonotopes under intersections, concludes the proof. ■

Remark 1: The KKT conditions are necessary but no longer sufficient for the nonconvex point containment problem of the hybrid zonotope [10, Proposition 5]. To overcome this limitation, the result of Theorem 1 ensures that all $|\mathcal{T}|$ local minima satisfying (15) are enforced over the region of interest $\mathcal{X}$. Although the growth in the set representation complexity given by (22) is increased proportionally to the number of nonempty constrained zonotopes $|\mathcal{T}|$, this trend is similar to that encountered when representing the complements of nonconvex sets as hyperplane arrangements [4].

Remark 2: Note that the representations of constrained and hybrid zonotopes are not unique. For example, given the hybrid zonotope $\mathcal{Z}_h = \mathcal{C}_\mathcal{X}(\mathcal{Z}_c)$, it holds that $\mathcal{C}_\mathcal{X}(\mathcal{Z}_h) = \mathcal{Z}_c$ for any $\mathcal{X} \supseteq \mathcal{Z}_c$. While the points represented by these two sets are equivalent, their representations are not.

V. Numerical Example: Obstacle Avoidance

This example considers the problem of formulating a model predictive controller (MPC) for an agent moving from an initial condition to a target point while avoiding collision with multiple obstacles. In [13] it is shown that over-approximating polytopic obstacles using zonotopes sharing a common structure leads to considerable improvements in the computation time of the MPC. However, once the over-approximation is found, the zonotope must be converted back to an H-rep polytope [13] to represent its complement as hyperplane arrangements [4]. Here it is shown how the same safety constraint may be formulated directly as a hybrid zonotope representing the complements of the obstacles.

This example considers a single agent in $2D$ space with continuous dynamics given by

$$\dot{x} = \begin{bmatrix} \mathbf{0} & I_2 \\ \mathbf{0} & -\frac{\mu}{M}I_2 \end{bmatrix} x + \begin{bmatrix} \mathbf{0} \\ \frac{1}{M}I_2 \end{bmatrix} u, \tag{23}$$

where $x_{1,2}$ is the position and $x_{3,4}$ the velocity of the agent, the control actions are the acceleration in the $1, 2$ coordinates, and the model parameters are $\mu = 3$ and $M = 60$ [13]. The obstacles are over-approximated by a zonotope denoted by $\mathcal{Z}_i$, and their union is given by the hybrid zonotope $\mathcal{Z}_{h,O} = \bigcup \mathcal{Z}_i$. The optimal action of the agent at each time step under the proposed MPC is given by the solution to the mixed-integer quadratic program

$$\begin{aligned} \min_u\ & \|x_N\|_P^2 + \sum_{k=0}^{N-1} \|x_k\|_Q^2 + \|u_k\|_R^2 \\ \text{s.t.}\ & \forall\, k \in [0, N-1]\ ,\ x_{k+1} = Ax_k + Bu_k, \\ & u_k \in \mathcal{U}\ ,\ x_{k+1} \in \mathcal{C}_\mathcal{X}(\mathcal{Z}_{h,O}) \times \mathbb{R}^2, \end{aligned} \tag{24}$$

where x_0 is fixed to the sampled state of the system, $\mathcal{U} = \{u \in \mathbb{R}^2 : \|u\|_\infty \leq 1\}$ is the set of all admissible control inputs, and the states are constrained to the nonconvex safe set $\mathcal{C}_\mathcal{X}(\mathcal{Z}_{h,O}) \times \mathbb{R}^2$ for $\mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 2\}$. The A and B matrices used in the MPC formulation are given by the zero-order hold transform of (23) with a discrete time step of $T_s = 0.5$. The MPC parameters are set to $Q = I_4$, $P = 10I_4$, $R = I_2$, and $N = 10$ [13]. The safe set, $\mathcal{C}_\mathcal{X}(\mathcal{Z}_{h,O}) \times \mathbb{R}^2$ generated through Theorem 1 and Proposition 2 with two example trajectories of the simulated closed-loop plant are shown in Fig. 1.

The MPC problem (24) is formulated using YALMIP [14] and solved using Gurobi [15] with MATLAB on a desktop

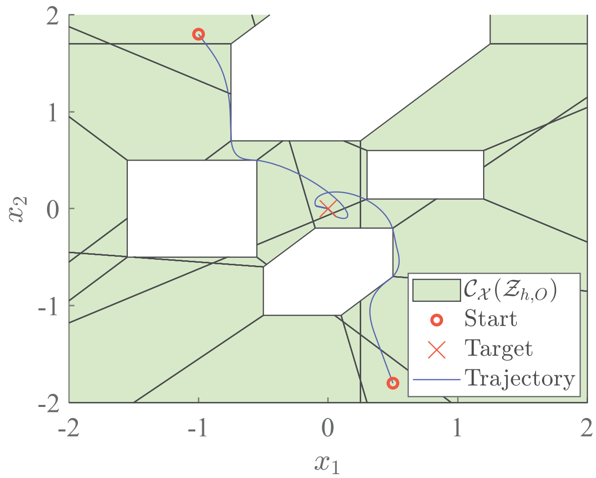

Fig. 1. Hybrid zonotope defined as the complement of the union of the obstacles, $\mathcal{C}_{\mathcal{X}}(\mathcal{Z}_{h,O})$, and trajectories of the simulated closed-loop system. Black lines depict the decomposition of the hybrid zonotope $\mathcal{C}_{\mathcal{X}}(\mathcal{Z}_{h,O})$ into constrained zonotopes by (3).

TABLE I
COMPUTATION TIMES TO FORMULATE AND SOLVE THE MPC PROBLEM (24) ANALYZED OVER 100 TRIALS

$\mathcal{C}_{\mathcal{X}}(\mathcal{Z}_{h,O})$	Average (s)	Maximum (s)
Hybrid Zonotope	0.10	0.24
Hyperplane Arrangement	0.19	0.25

computer using one core of a 3.0 GHz Intel i7 processor with 32 GB of RAM. The computation time to formulate and solve the MPC (24), analyzed over 100 trials with randomly sampled initial conditions, is given in Table I. The computation time is compared to the equivalent problem formulated using hyperplane arrangements to define the safety constraint $\mathcal{C}_{\mathcal{X}}(\mathcal{Z}_{h,O})$ [4, Sec. 2.1]. It is noted that additional methods exist to further reduce the complexity of hyperplane arrangements, such as logarithmic formulations and merging adjacent cells [16], each introducing additional overhead in the formulation of the problem. It is also noted that hyperplane arrangements have a mature theory in obstacle avoidance, including methods to guarantee the constraint satisfaction of the plant's continuous dynamics [17]. Nevertheless, the use of hybrid zonotopes shows a considerable improvement in the average solution time in this case, namely a reduction by $\sim 47\%$.

VI. CONCLUSION

In this letter, the closure of hybrid zonotopes under union and complement set operations has been proven. Both identities for these operations are derived by directly embedding linear mixed-integer constraints within the set definition. The addition of these set operations establishes the hybrid zonotope as a nonconvex set representation closed under linear mappings, Minkowski sums, intersections, unions, and complements, thus increasing its utility to a broad class of set-theoretic controls problems. A numerical example was used to compare the derived set operations to a common approach for representing nonconvex sets in obstacle avoidance problems and showed an improvement in the average computation time. Future work will be focused on developing order reduction techniques for hybrid zonotopes to further reduce computational complexity.

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