Hybrid zonotopes: a new set representation for reachability analysis of mixed logical dynamical systems ⋆

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Abstract

This article presents a new set representation named the *hybrid zonotope*. The hybrid zonotope is shown to be equivalent to 2^N constrained zonotopes through the addition of N binary zonotope factors and is well-suited for the analysis of hybrid systems with both continuous and discrete states and inputs. The major contribution of this manuscript is a closed-form solution for exact forward reachable sets of linear mixed logical dynamical systems. This is given by a simple identity and does not require solving any optimization programs or taking set approximations. The proposed approach captures the worst-case exponential growth in the number of convex sets required to represent the nonconvex reachable set of a hybrid system while exhibiting only linear growth in the complexity of the hybrid zonotope set representation. To reduce both set representation complexity and the computational burden of reachability analysis, a binary tree is used to store only the combinations of binary factors of the hybrid zonotope that map to nonempty convex sets. The proposed approach is applied to an established benchmark example where the exact reachable set of a discrete-time hybrid system with six continuous and two discrete states is given by a single hybrid zonotope equivalent to the union of 657 constrained zonotopes, and is represented using only 283 continuous factors, 29 binary factors, and 177 linear equality constraints. Furthermore, the hybrid zonotope is closed under linear mappings, Minkowski sums, generalized intersections, and halfspace intersections.

Key words: Set-based computing, Zonotopes, Hybrid systems, Reachability analysis, Verification and abstraction of hybrid systems, Mixed logical dynamical systems

1 Introduction

Hybrid system theory has found increased use for modeling and control synthesis due to its ability to capture the mixed continuous and discrete dynamics exhibited by many engineered systems [1]. While providing a powerful tool, the analysis and control of hybrid systems is inherently complex. Even in the case of linear hybrid systems, basic properties such as stability and controllability may not be easily determined from the system model [2,3,4]. Thus, hybrid systems under closed-loop control may not not exhibit the intended behavior under certain operating conditions. Set-based methods for reachability analysis and safety verification are often deployed when certain properties of a system, such as safety or performance, must be guaranteed. These methods are well established for linear time invariant systems using convex sets, for which multiple representations exist [5]. However, the application of set-based methods to nonlinear and hybrid systems are non-convex [6]. The reader is directed to the review papers [7,8,9] and the references therein for detailed discussion on the state-of-the-art.

In the case of linear hybrid systems, non-convexity arises in reachable sets due to discrete inputs, switching of dynamic subsystems, and reset maps. The exact reachable set may be determined by partitioning the state space into a set of closed convex sets, often referred to as guard sets. Using a finite number of convex sets and applying techniques developed for linear systems, the reachable set may be found by iteratively propagating the

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appropriate linear dynamics within each partition [7]. However, when an intersection with a guard set occurs or an uncertain discrete input is applied, the reach set branches, resulting in a worst-case exponential growth in the number of convex sets required to represent the reachable space as their implicit union [10]. This approach becomes computationally intractable for large time horizons.

To avoid this exponential growth in set representation complexity, researchers often approximate the true reachable set, given by the implicit union of a finite number of convex sets, by a reduced number of convex sets. One such method propagates the dynamics of the system by branching along each guard set, then uses clustering methods to over-approximate groups of convex sets by fewer convex sets [11,12]. This approach provides computational efficiency at the cost of conservatism in the reachable set itself, although the specific trade-off is application-dependent. Another approach is to search each region of the partitioned state space individually and then over-approximate transitions along the guard sets [10,13]. This approach is computationally efficient as it only deals with one convex set at a time and avoids unnecessary error by only over-approximating non-convex sets along guard set intersections. However, it is not guaranteed to converge when the reach set intersects a guard partially without fully transitioning into another partition.

While useful and efficient, existing approaches rely on over-approximations and are therefore only valid for safety verification and avoiding unsafe regions in robust control. Furthermore, the error associated with such over-approximations may be large and difficult to quantify, thus resulting in conservative results at best, and trivial solutions at worst [14]. In all of these approaches, detecting guard set intersections and avoiding explosion in the number of convex sets required to represent the non-convex reachable set remains the primary challenge [7].

This paper presents a non-convex set representation named the *hybrid zonotope* that contains both continuous and binary zonotope factors. It is shown that the hybrid zonotope is equivalent to the union of 2^N constrained zonotopes through the use of N binary factors. The major contribution of this work is an identity for representing exact reachable sets of discrete-time hybrid automata modeled as Mixed Logical Dynamical (MLD) systems [15] using hybrid zonotopes. This identity contains all guard set intersections implicitly as properties of the MLD model and avoids solving any optimization programs or using any approximation techniques to determine guard crossings, changes in dynamics, or reset maps. The resulting reachable set is represented as a single hybrid zonotope equivalent to an exponential number of convex sets while exhibiting linear growth in set representation complexity. This approach is desirable as it is both computationally efficient and exact. By leveraging binary trees and mixed integer techniques to identify empty subsets of the hybrid zonotope, it is shown how the complexity growth of the reachable set may be further reduced. In addition to representing reachable sets of MLD systems, hybrid zonotopes are closed under linear mappings, Minkowski sums, generalized intersections, and halfspace intersections, which can all be calculated through simple identities that exhibit modest growth in set representation complexity.

The remainder of the manuscript is organized as follows. Notation and preliminary information for zonotopes and constrained zonotopes, as well as a brief overview of MLD systems, is provided in Sec. 2. The hybrid zonotope set representation is presented in Sec. 3. In Sec. 4, a closed-form solution to the forward reachable sets of MLD systems is proven and a redundancy removal technique is described. In Sec. 5 it is shown how each hybrid zonotope has an associated binary tree that can be used to reduce the number of binary factors required to represent the set. Two numerical examples of the forward reachable sets of MLD systems are provided in Sec. 6. Concluding remarks are given in Sec. 7.

2 Notation and preliminaries

Matrices are denoted by uppercase letters, e.g., $G \in \mathbb{R}^{n \times n_g}$, and sets by uppercase calligraphic letters, e.g., $\mathcal{Z} \subset \mathbb{R}^n$. Vectors and scalars are denoted by lowercase letters, e.g., $b \in \mathbb{R}^{n_c}$. Commas in subscripts are used to distinguish between properties that are defined for multiple sets; e.g., $n_{g,z}$ describes the complexity of set $\mathcal{Z}$ while $n_{g,w}$ describes the complexity of $\mathcal{W}$. The n-dimensional unit hypercube is denoted by $\mathcal{B}_\infty^n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. The power set of an n-dimensional vector of binary variables is denoted by $\{-1,1\}^n$, e.g.,

$$\{-1,1\}^2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} .$$

The cardinality of the discrete set $\mathcal{T}$ is denoted by $|\mathcal{T}|$ and the i^{th} element as $\mathcal{T}(i)$; e.g., for $\mathcal{T} = \{-1,1\}^2$ as previously defined, $|\mathcal{T}| = 4$ and $\mathcal{T}(2) = [1\ -1]^T$. The concatenation of two column vectors to a single column vector is denoted by $(\xi_1\, \xi_2) = [\xi_1^T\, \xi_2^T]^T$. Finally, $\mathbf{1}$ and $\mathbf{0}$ denote matrices of all 1 and 0 elements, respectively, of appropriate dimension, and I denotes the identity matrix.

Given the sets $\mathcal{Z}$, $\mathcal{W} \subset \mathbb{R}^n$, $\mathcal{Y} \subset \mathbb{R}^m$, and matrix $R \in \mathbb{R}^{m \times n}$, the linear mapping of $\mathcal{Z}$ by R is $R\mathcal{Z} = \{Rz : z \in \mathcal{Z}\}$, the Minkowski sum of $\mathcal{Z}$ and $\mathcal{W}$ is $\mathcal{Z} \oplus \mathcal{W} = \{z + w : z \in \mathcal{Z},\, w \in \mathcal{W}\}$, the generalized intersection of $\mathcal{Z}$ and $\mathcal{Y}$ under R is $\mathcal{Z} \cap_R \mathcal{Y} = \{z \in \mathcal{Z} : Rz \in \mathcal{Y}\}$, and the union of $\mathcal{Z}$ and $\mathcal{W}$ is $\mathcal{Z} \cup \mathcal{W} = \{x \in \mathbb{R}^n : x \in \mathcal{Z} \vee x \in \mathcal{W}\}$.

2.1 Zonotopes and constrained zonotopes

A zonotope is a centrally symmetric, polytopic set representation that is defined as the affine image of a unit hypercube.

Definition 1 [16] *The set $\mathcal{Z} \subset \mathbb{R}^n$ is a zonotope if there exists $G \in \mathbb{R}^{n \times n_g}$ and $c \in \mathbb{R}^n$ such that*

$$\mathcal{Z} = \{G\xi + c : \|\xi\|_\infty \leq 1\} \ . \tag{1}$$

The zonotope is given in Generator-representation (G-rep), and the shorthand notation of $\mathcal{Z} = \{G, c\}$ is used to denote the set given by (1). A zonotope is the set of points given by all linear combinations of the center c with the weighted generators—the columns of $G = [g^{(1)} \ldots g^{(n_g)}]$—such that their weights $\xi = (\xi_1 \ldots \xi_{n_g})$, called factors, lie within the closed unit hypercube. The complexity of the set is reflected by the number of generators, n_g, and the order of the zonotope is defined as $o = n_g/n$. The utility of the zonotope may be extended by adding a set of linear equality constraints to the mapped hypercube.

Definition 2 [17] *The set $\mathcal{Z} \subset \mathbb{R}^n$ is a constrained zonotope if there exists $G \in \mathbb{R}^{n \times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c \times n_g}$, and $b \in \mathbb{R}^{n_c}$ such that*

$$\mathcal{Z}_c = \{G\xi + c : \|\xi\|_\infty \leq 1, A\xi = b\} \ . \tag{2}$$

The constrained zonotope is given in Constrained Generator-representation (CG-rep), and the shorthand notation of $\mathcal{Z}_c = \{G, c, A, b\}$ is used to denote the set given by (2). Through the addition of the linear equality constraints $A\xi = b$ to the projected unit hypercube, the affine image of the constrained space of factors is no longer restricted to be symmetric, and the constrained zonotope may be used to represent any convex polytope [17]. The degree of freedom order of a constrained zonotope is defined as $o_d = (n_g - n_c)/n$. Constrained zonotopes are closed under linear mappings, Minkowski sums, and intersections, so that performing these operations on constrained zonotopes results in yet another constrained zonotope.

2.2 Mixed Logical Dynamical (MLD) systems

As first introduced in [15], the Mixed Logical Dynamical (MLD) system modeling framework combines continuous and binary variables with logical relations in mixed integer inequalities to express complex dynamic systems. It has been shown in [15,18,19] that such a framework can be used to model systems that have mixed continuous and discrete states and inputs, piece-wise affine and bilinear dynamics, finite state machines, qualitative outputs, and those with any combination of the former. An MLD system with linear discrete-time dynamics may be expressed as

$$x_+ = Ax + B_u u + B_z z + B_\delta \delta + B_{aff} \ , \tag{3a}$$

$$\text{s.t.} \ \ E_x x + E_u u + E_z z + E_\delta \delta \leq E_{aff} \ , \tag{3b}$$

where $x \in \mathbb{R}^{n_{xc}} \times \{0,1\}^{n_{xl}}$ is the system state, $u \in \mathbb{R}^{n_{uc}} \times \{0,1\}^{n_{ul}}$ is the control input, $z \in \mathbb{R}^{n_{rc}}$ are continuous auxiliary variables, and $\delta \in \{0,1\}^{n_{rl}}$ are binary auxiliary variables. The dimensions of the matrices A, B_u, B_z, B_δ, and B_{aff} may be deduced from the dimensions of the states, inputs, and auxiliary variables, while the number of inequality constraints is denoted by n_e such that $E_{aff} \in \mathbb{R}^{n_e}$. Given a unique system state x and control input u, the propagation of the system dynamics of (3a) through a single discrete time step may be determined using any value of the auxiliary variables z and δ that satisfy (3b) [15].

When formulating an MLD model (3), the so-called "big-M" constants used in the mixed integer inequalities to relate continuous values to logical statements [20] are chosen for a specific bounded state space, $\mathcal{X} \subset \mathbb{R}^{n_{xc}} \times \{0,1\}^{n_{xl}}$, and set of admissible control inputs, $\mathcal{U} \subset \mathbb{R}^{n_{uc}} \times \{0,1\}^{n_{ul}}$ [15]. It follows that for the bounded state-input domain over which the MLD model is defined, the auxiliary continuous variables will belong to the interval set $\mathcal{Z} = [z_l, \ z_u] \subset \mathbb{R}^{n_{rc}}$ and the auxiliary binary variables will belong to the discrete set $\Delta = \{0,1\}^{n_{rl}}$. The set of states reachable from an initial set of $\mathcal{X}_0 \subseteq \mathcal{X}$ by the MLD system (3) in one time step is therefore given by

$$\mathcal{R}_+ = \left\{ \begin{matrix} x \in \mathcal{X} \mid x_0 \in \mathcal{X}_0, \ u \in \mathcal{U}, \ z \in \mathcal{Z}, \ \delta \in \Delta, \\ x = Ax_0 + B_u u + B_z z + B_\delta \delta + B_{aff}, \\ E_x x_0 + E_u u + E_z z + E_\delta \delta \leq E_{aff} \end{matrix} \right\} . \tag{4}$$

The set of states reachable from $\mathcal{X}_0$ in k discrete time steps may be found through iterative applications of (4) and is denoted by $\mathcal{R}_k$.

3 The hybrid zonotope

This section introduces the definition of hybrid zonotopes as an extension of the constrained zonotope through the addition of a vector of binary factors.

Definition 3 *The set $\mathcal{Z}_h \subset \mathbb{R}^n$ is a hybrid zonotope if there exists $G^c \in \mathbb{R}^{n \times n_g}$, $G^b \in \mathbb{R}^{n \times n_b}$, $c \in \mathbb{R}^n$, $A^c \in \mathbb{R}^{n_c \times n_g}$, $A^b \in \mathbb{R}^{n_c \times n_b}$, and $b \in \mathbb{R}^{n_c}$ such that*

$$\mathcal{Z}_h = \left\{ \begin{bmatrix} G^c \ G^b \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} + c \ \middle| \ \begin{matrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} \in \mathcal{B}_\infty^{n_g} \times \{-1,1\}^{n_b}, \\ \begin{bmatrix} A^c \ A^b \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} = b \end{matrix} \right\} . \tag{5}$$

The hybrid zonotope is given in *Hybrid Constrained Generator-representation* (HCG-rep), and the shorthand notation of $\mathcal{Z}_h = \{G^c, G^b, c, A^c, A^b, b\} \subset \mathbb{R}^n$ is used to denote the set given by (5). When $n_b = 0$, the hybrid zonotope set representation is equivalent to the constrained zonotope given by Def. 2. When $n_b \neq 0$, the vector of binary factors may take on values from the discrete set $\{-1, 1\}^{n_b}$ containing 2^{n_b} elements. The hybrid zonotope therefore consists of a continuous portion equivalent to that of the constrained zonotope shifted by contributions from a discrete, finite set.

Given that $\|\xi^b\|_\infty = 1 \; \forall \xi^b \in \{-1, 1\}^{n_b}$, the hybrid zonotope is a subset of the zonotope and constrained zonotope set representations. That is, the hybrid zonotope definition is more strict by adding one additional constraint on the space of factors being projected – namely, that some of them must be binary.

Lemma 1 *Given any hybrid zonotope*

$$\mathcal{Z}_h = \{G^c, G^b, c, A^c, A^b, b\} \subset \mathbb{R}^n \,,$$

the zonotope $\mathcal{Z} = \{[G^c \; G^b], c\} \subset \mathbb{R}^n$ *and constrained zonotope* $\mathcal{Z}_c = \{[G^c \, G^b], c, [A^c \, A^b], b\} \subset \mathbb{R}^n$ *satisfy* $\mathcal{Z}_h \subseteq \mathcal{Z}_c \subseteq \mathcal{Z}$.

PROOF. For $\mathcal{Z}$ and $\mathcal{Z}_c$ it holds that $\mathcal{Z}_c \subseteq \mathcal{Z}$ [7]. For any $z \in \mathcal{Z}_h$ there exists some $\|\xi^c\|_\infty \leq 1$ and $\xi^b \in \{-1, 1\}^{n_b}$ such that $A^c\xi^c + A^b\xi^b = b$ and $z = G^c\xi^c + G^b\xi^b + c$. Letting $\xi = (\xi^c \; \xi^b)$ implies that $\|\xi\|_\infty \leq 1$, $z = [G^c \; G^b]\xi + c$, and $[A^c \; A^b]\xi = b$, thus $z \in \mathcal{Z}_c$, and therefore $\mathcal{Z}_h \subseteq \mathcal{Z}_c \subseteq \mathcal{Z}$. □

Many of the identities derived for the set operations of constrained zonotopes may be modified to hold for hybrid zonotopes by including the additional binary constraint as follows. First, the point containment problem for constrained zonotopes formulated as a linear program [17, Prop. 2] may be modified for hybrid zonotopes as a mixed integer linear program.

Proposition 1 *For any* $\mathcal{Z}_h = \left\{G^c, G^b, c, A^c, A^b, b\right\} \subset \mathbb{R}^n$,

$$\mathcal{Z}_h \neq \emptyset \iff \min \left\{ \|\xi^c\|_\infty \left| \begin{array}{c} A^c\xi^c + A^b\xi^b = b \,, \\ \xi^b \in \{-1, 1\}^{n_b} \end{array} \right. \right\} \leq 1 \,, \tag{6}$$

$$z \in \mathcal{Z}_h \iff \min \left\{ \|\xi^c\|_\infty \left| \begin{array}{c} \begin{bmatrix} G^c & G^b \\ A^c & A^b \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^b \end{bmatrix} = \begin{bmatrix} z - c \\ b \end{bmatrix} , \\ \xi^b \in \{-1, 1\}^{n_b} \end{array} \right. \right\} \leq 1 \,. \tag{7}$$

PROOF. Following the proof of [17, Prop. 2]. By Def. 3, $z \in \mathcal{Z}_h$ if and only if there exists some ξ^c and ξ^b such that $\|\xi^c\|_\infty \leq 1$, $\xi^b \in \{-1, 1\}^{n_b}$, $A^c\xi^c + A^b\xi^b = b$, and $z = G^c\xi^c + G^b\xi^b + c$. Choosing $(\xi^c \; \xi^b) = \arg\min$ of the right-hand side of (7), $z \in \mathcal{Z}_h$ if and only if (7) holds. When (6) is satisfied, the point $z = G^c\xi^c + G^b\xi^b + c$, where $(\xi^c \; \xi^b) = \arg\min$ of the right-hand side of (6), belongs to $\mathcal{Z}_h$ by Def. 3 and therefore $\mathcal{Z}_h \neq \emptyset$. If no such point exists, (6) will not be satisfied. □

The identities for linear mappings, Minkowski sums, generalized intersections [17, Prop. 1], and halfspace intersections [21, Theorem 1] of constrained zonotopes may be similarly extended to hybrid zonotopes by including the additional binary constraint.

Proposition 2 *For any* $\mathcal{Z}_h = \left\{G^c_z, G^b_z, c_z, A^c_z, A^b_z, b_z\right\}$, $\mathcal{W}_h = \left\{G^c_w, G^b_w, c_w, A^c_w, A^b_w, b_w\right\} \subset \mathbb{R}^n$, $\mathcal{Y}_h = \left\{G^c_y, G^b_y, c_y, A^c_y, A^b_y, b_y\right\} \subset \mathbb{R}^m$, $R \in \mathbb{R}^{m \times n}$, *and* $\mathcal{H}_- = \{x \in \mathbb{R}^n | h^T x \leq f\}$ *the following identities hold:*

$$R\mathcal{Z}_h = \left\{RG^c_z, RG^b_z, Rc_z, A^c_z, A^b_z, b_z\right\} \,, \tag{8}$$

$$\mathcal{Z}_h + \mathcal{W}_h = \left\{ \begin{bmatrix} G^c_z & G^c_w \end{bmatrix}, \begin{bmatrix} G^b_z & G^b_w \end{bmatrix}, c_z + c_w, \begin{bmatrix} A^c_z & \mathbf{0} \\ \mathbf{0} & A^c_w \end{bmatrix}, \begin{bmatrix} A^b_z & \mathbf{0} \\ \mathbf{0} & A^b_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix} \right\} \,, \tag{9}$$

$$\mathcal{Z}_h \cap_R \mathcal{Y}_h = \left\{ \begin{bmatrix} G^c_z & \mathbf{0} \end{bmatrix}, \begin{bmatrix} G^b_z & \mathbf{0} \end{bmatrix}, c_z, \begin{bmatrix} A^c_z & \mathbf{0} \\ \mathbf{0} & A^c_y \\ RG^c_z & -G^c_y \end{bmatrix}, \begin{bmatrix} A^b_z & \mathbf{0} \\ \mathbf{0} & A^b_y \\ RG^b_z & -G^b_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix} \right\} , \tag{10}$$

$$\mathcal{Z}_h \cap \mathcal{H}_- = \left\{ \begin{bmatrix} G^c_z & \mathbf{0} \end{bmatrix}, G^b_z, c_z, \begin{bmatrix} A^c_z & \mathbf{0} \\ h^T G^c_z & \frac{d_m}{2} \end{bmatrix}, \begin{bmatrix} A^b_z \\ h^T G^b_z \end{bmatrix}, \begin{bmatrix} b_z \\ f - h^T c_z - \frac{d_m}{2} \end{bmatrix} \right\} \,, \tag{11}$$

$$d_m = f - h^T c + \sum_{i=1}^{n_{g,z}} |h^T g_z^{(ci)}| + \sum_{i=1}^{n_{b,z}} |h^T g_z^{(bi)}| \,.$$

PROOF. Proof follows a simple extension of the procedures presented in [17, Prop. 1] and [21, Theorem 1] by including the constraint that $\xi^b_i \in \{-1, 1\}^{n_{b,i}}$ for $i = z, \, w, \, y$.

The equivalence of the hybrid zonotope with a complex of constrained zonotopes is established through the following theorem.

Theorem 1 *The set $\mathcal{Z}_h \subset \mathbb{R}^n$ is a hybrid zonotope if and only if it is the union of a finite number of constrained zonotopes.*

PROOF. Let ξ_i^b be the i^{th} entry of the discrete set $\{-1,1\}^{n_b}$ containing 2^{n_b} elements. Define the constrained zonotope

$$\mathcal{Z}_{c,i} = \left\{G^c, c + G^b \xi_i^b, A^c, b - A^b \xi_i^b\right\} \ . \tag{12}$$

For any $z \in \mathcal{Z}_{c,i}$ there exists some $\|\xi^c\|_\infty \leq 1$ such that $z = G^c\xi^c + G^b\xi_i^b + c$ and $A^c\xi^c + A^b\xi_i^b = b$. Thus $z \in \mathcal{Z}_h$. Given that the choice of z is arbitrary and the set $\{-1,1\}^{n_b}$ is finite,

$$\bigcup_{i=1}^{2^{n_b}} \mathcal{Z}_{c,i} \subseteq \mathcal{Z}_h \ . \tag{13}$$

Conversely, for any $z \in \mathcal{Z}_h$, there exists some $\|\xi^c\|_\infty \leq 1$ and $\xi^b \in \{-1,1\}^{n_b}$ such that $z = G^c\xi^c + G^b\xi^b + c$ and $A^c\xi^c + A^b\xi^b = b$. Also, for $\xi^b = \xi_i^b \implies z \in \mathcal{Z}_{c,i}$. Given that the choice of z is arbitrary and the set $\{-1,1\}^{n_b}$ is finite,

$$\mathcal{Z}_h \subseteq \bigcup_{i=1}^{2^{n_b}} \mathcal{Z}_{c,i} \ , \tag{14}$$

and therefore $\mathcal{Z}_h = \bigcup \mathcal{Z}_{c,i}$. □

The hybrid zonotope exhibits the same combinatorial properties as zonotopes, where a symmetric polytope with 2^{n_g} vertices may be represented with n_g continuous factors [16]. Introducing n_b binary factors, the hybrid zonotope may represent 2^{n_b} zonotopes each having potentially 2^{n_g} vertices. This concept is further explored through the following example.

Example 1 *Let the set $\mathcal{Z}_c = \{G_z, c_z, A_z, b_z\} \subset \mathbb{R}^2$ be the example constrained zonotope given in* [17]*, where*

$$\mathcal{Z}_c = \left\{ \begin{bmatrix} 1.5 & -1.5 & 0.5 \\ 1 & 0.5 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, 1 \right\} \ ,$$

and define a hybrid zonotope with continuous generators $G^c = G_z$, binary generators $G^b = 2G_z$, and center c_z giving

$$\mathcal{Z}_{h,1} = \{G_z, 2G_z, c_z, \emptyset, \emptyset, \emptyset\} \ . \tag{15}$$

By adding $n_b = 3$ binary factors, $\mathcal{Z}_{h,1}$ is equivalent to $2^{n_b} = 8$ copies of the zonotope $\mathcal{Z} = \{G_z, c_z\}$ with centers shifted by $2G_z\xi^b \ \forall\ \xi^b \in \{-1,1\}^3$, as depicted in Fig. 1a. Defining another hybrid zonotope that includes linear equality constraints on the continuous factors as $A^c = A_z$, $A^b = \mathbf{0}$, and $b = b_z$ giving

$$\mathcal{Z}_{h,2} = \{G_z, 2G_z, c_z, A_z, \mathbf{0}, b_z\} \ , \tag{16}$$

results in a hybrid zonotope equivalent to eight copies of the constrained zonotope $\mathcal{Z}_c$, again with centers shifted by the contribution of the binary generators as shown in Fig. 1b. Including the binary factors in the equality constraints by defining another hybrid zonotope with $A^b = A_z$ gives

$$\mathcal{Z}_{h,3} = \{G_z, 2G_z, c_z, A_z, A_z, b_z\} \ , \tag{17}$$

as shown in Fig. 1c. In contrast to the previous hybrid zonotopes, $\mathcal{Z}_{h,3}$ does not represent identical copies. Instead, the linear equality constraints on the continuous factors are also shifted by each of the eight discrete values of the binary factors. When doing so, it is possible that these shifted equality constraints may be infeasible and thus map to empty constrained zonotopes, which happens exactly once in the given example.

The result of Theorem 1 provides a method of converting from a hybrid zonotope to a complex of constrained zonotopes, allowing methods developed for the analysis and visualization of other set representations to be applied to hybrid zonotopes. However, the conversion from HCG-rep to a complex of CG-reps, $\mathcal{Z}_{c,i} \forall i \in \{1,\dots,2^{n_b}\}$ given by (12), is an enumeration problem that grows exponentially with respect to the number of binary factors. Use of the hybrid zonotope is therefore most advantageous when these conversions are not necessary and the set may be used directly for the analysis of complex dynamical systems, as discussed in the following sections.

4 Reachable sets of MLD systems

In this section it is shown how the forward reachable sets of MLD systems may be represented as hybrid zonotopes. It is then shown how the representation complexity of the resulting hybrid zonotope may be reduced by removing redundant generators and equality constraints.

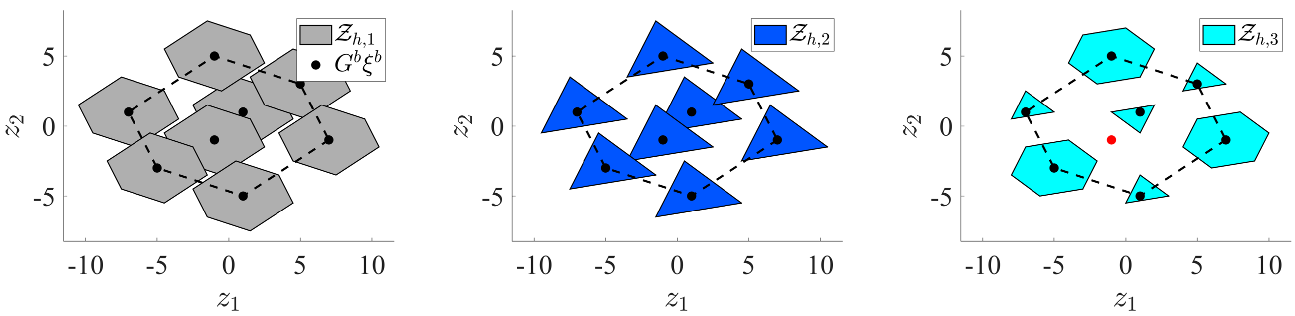
4.1 Forward propagation of MLD dynamics

A closed-form solution to the forward reachable sets of MLD systems as hybrid zonotopes is now presented.

Theorem 2 *Consider the MLD system described by* (3) *with $x \in \mathcal{X}_0 \subseteq \mathcal{X}$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$, where*

$$\begin{aligned} \mathcal{X}_0 &= \{G_x^c, G_x^b, c_x, A_x^c, A_x^b, b_x\} \subset \mathbb{R}^{n_{xc}} \times \{0,1\}^{n_{xl}} \ , \\ \mathcal{U} &= \{G_u^c, G_u^b, c_u, A_u^c, A_u^b, b_u\} \subset \mathbb{R}^{n_{uc}} \times \{0,1\}^{n_{ul}} \ , \\ \mathcal{Z} &= \{G_z^c, \emptyset, c_z, \emptyset, \emptyset, \emptyset\} = [z_l,\ z_u] \subset \mathbb{R}^{n_{rc}}, \\ \Delta &= \{\emptyset, G_\delta^b, c_\delta, \emptyset, \emptyset, \emptyset\} = \{0,1\}^{n_{rl}} \ . \end{aligned}$$

Define the interval set $\mathcal{F} = \{G_{aff}^c, \emptyset, c_{aff}, \emptyset, \emptyset, \emptyset\} = [\alpha,\ E_{aff}]$ such that $\alpha \leq E_x x + E_u u + E_z z + E_\delta \delta$ for all



(a) (b) (c)

Fig. 1. Hybrid zonotopes given in Ex. 1. Note that the convex hull of the eight discrete points given by $G^b\xi^b$ is equivalent to the zonotope $\mathcal{Z} = \{G^b, \mathbf{0}\}$ as depicted by the dashed lines. (a) Without linear equality constraints, the hybrid zonotope $\mathcal{Z}_{h,1}$, given by (15), represents eight copies of the continuous zonotope shifted by the contribution of the binary factors and generators. (b) Including constraints on only the continuous factors results in $\mathcal{Z}_{h,2}$, given by (16), and is equivalent to eight copies of the constrained zonotope $\mathcal{Z}_c$. (c) When the equality constraints include terms for the binary factors in $\mathcal{Z}_{h,3}$, given by (17), the shifted constrained zonotopes are no longer identical. Also note that in this final case, the discrete value of the binary factors depicted by the red $\bullet$ results in an infeasible set of continuous constraints and thus maps to an empty constrained zonotope.

$x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$. Then the set of states reachable in one time step is given by the hybrid zonotope

$$\mathcal{R}_+ = \left\{G_r^c, G_r^b, c_r, A_r^c, A_r^b, b_r\right\} \subset \mathbb{R}^{n_{xc}} \times \{0,1\}^{n_{xl}} ,$$

where

$$G_r^c = \begin{bmatrix} AG_x^c & B_uG_u^c & B_zG_z^c & \mathbf{0} \end{bmatrix} , \tag{18a}$$

$$G_r^b = \begin{bmatrix} AG_x^b & B_uG_u^b & B_\delta G_\delta^b \end{bmatrix} , \tag{18b}$$

$$c_r = Ac_x + B_uc_u + B_zc_z + B_\delta c_\delta + B_{aff} , \tag{18c}$$

$$A_r^c = \begin{bmatrix} A_x^c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_u^c & \mathbf{0} & \mathbf{0} \\ -E_xG_x^c & -E_uG_u^c & -E_zG_z^c & G_{aff}^c \end{bmatrix} , \tag{18d}$$

$$A_r^b = \begin{bmatrix} A_x^b & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_u^b & \mathbf{0} \\ -E_xG_x^b & -E_uG_u^b & -E_\delta G_\delta^b \end{bmatrix} , \tag{18e}$$

$$b_r = \begin{bmatrix} b_x \\ b_u \\ E_xc_x + E_uc_u + E_zc_z + E_\delta c_\delta - c_{aff} \end{bmatrix} . \tag{18f}$$

PROOF. For ease of readability, let $\xi_i = (\xi_i^c\ \xi_i^b)$, $G_i = [G_i^c\ G_i^b]$, and $A_i = [A_i^c\ A_i^b]$ for $i = r,\ x,\ u,\ z,\ \delta$. Let $\tilde{\mathcal{R}} = \left\{G_r^c, G_r^b, c_r, A_r^c, A_r^b, b_r\right\}$ denote the set given by (18) and R_+ denote the set given by (4). For any $\tilde{r} \in \tilde{\mathcal{R}}$ there exists some $\xi_r \in \mathcal{B}_\infty^{n_{g,r}} \times \{-1,1\}^{n_{b,r}}$ such that $A_r\xi_r = b_r$ and $\tilde{r} = G_r\xi_r + c_r$. Let $\xi_r^c = (\xi_x^c\ \xi_u^c\ \xi_z\ \xi_{aff})$ and $\xi_r^b = (\xi_x^b\ \xi_u^b\ \xi_\delta)$. Expanding $\tilde{r} = G_r\xi_r + c_r$ then gives $\tilde{r} = A(G_x\xi_x + c_x) + B_u(G_u\xi_u + c_u) + B_z(G_z\xi_z + c_z) + B_\delta(G_\delta\xi_\delta + c_\delta) + B_{aff}$. Expanding the first two rows of $A_r\xi_r = b_r$ gives $A_x\xi_x = b_x$ and $A_u\xi_u = b_u$. Thus given the hybrid zonotope's closure under linear mappings and Minkowski sums, $\tilde{r} = Ax + B_uu + B_zz + B_\delta\delta + B_{aff}$ for some $x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$. Now expanding the third row of $A_r\xi_r = b_r$ gives $E_x(G_x\xi_x + c_x) + E_u(G_u\xi_u + c_u) + E_z(G_z\xi_z + c_z) + E_\delta(G_\delta\xi_\delta + c_\delta) = G_{aff}^c\xi_{aff} + c_{aff}$. Leveraging the hybrid zonotope's closure under linear mappings, Minkowski sums, and intersections, the final row results in $E_xx + E_uu + E_zz + E_\delta\delta \in [\alpha,\ E_{aff}]$ for all $x = G_x\xi_x + c_x$, $u = G_u\xi_u + c_u$, $z = G_z\xi_z + c_z$, $\delta = G_\delta\xi_\delta + c_\delta$, and $A_r\xi_r = b_r$. Given that α was chosen such that $\alpha \leq E_xx + E_uu + E_zz + E_\delta\delta$ for all $x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$, this constraint is equivalent to $E_xx + E_uu + E_zz + E_\delta\delta \leq E_{aff}$. Combining these results gives $\tilde{r} \in \mathcal{R}_+$ and $\tilde{\mathcal{R}} \subseteq \mathcal{R}_+$.

Conversely, for any $r \in \mathcal{R}_+$, $r = Ax + B_uu + B_zz + B_\delta\delta + B_{aff}$ for some $x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$ such that $E_xx + E_uu + E_zz + E_\delta\delta \leq E_{aff}$. Let $\mathcal{X}_+ = A\mathcal{X}_0 \oplus B_u\mathcal{U} \oplus B_z\mathcal{Z} \oplus B_\delta\Delta \oplus B_{aff}$. Given the closure of hybrid zonotopes under linear mappings and Minkowski sums, for all $r \in \mathcal{R}_+ \subseteq \mathcal{X}_+$ there exists some ξ_x, ξ_u, ξ_z, and ξ_δ such that

$$\begin{aligned} &\xi_x \in \mathcal{B}_\infty^{n_{g,x}} \times \{-1,1\}^{n_{b,x}} , \quad A_x\xi_x = b_x , \\ &\xi_u \in \mathcal{B}_\infty^{n_{g,u}} \times \{-1,1\}^{n_{b,u}} , \quad A_u\xi_u = b_u , \\ &\xi_z \in \mathcal{B}_\infty^{n_{rc}} , \ \xi_\delta \in \{-1,1\}^{n_{rl}} , \end{aligned} \tag{19}$$

and $r = A(G_x\xi_x + c_x) + B_u(G_u\xi_u + c_u) + B_z(G_z\xi_z + c_z) + B_\delta(G_\delta\xi_\delta + c_\delta) + B_{aff}$. Let $\mathcal{H} = E_x\mathcal{X}_0 \oplus E_u\mathcal{U} \oplus E_z\mathcal{Z} \oplus E_\delta\Delta$ and $\mathcal{F} = \{G_{aff}^c\xi_{aff} + c_{aff} \mid \xi_{aff} \in \mathcal{B}_\infty^{n_e}\}$. Constraining the factors such that $h \in \mathcal{H} \cap \mathcal{F}$ for all $h = E_x(G_x\xi_x + c_x) + E_u(G_u\xi_u + c_u) + E_z(G_z\xi_z + c_z) + E_\delta(G_\delta\xi_\delta + c_\delta)$ results in $h \leq E_{aff}$. Letting $\xi_r^c = (\xi_x^c\ \xi_u^c\ \xi_z\ \xi_{aff})$ and

$\xi_r^b = (\xi_x^b\ \xi_u^b\ \xi_\delta)$, the above implies that

$$\begin{aligned}\xi_r \in \mathcal{B}_\infty^{n_{g,r}} \times \{-1,1\}^{n_{b,r}}\ ,\ A_r \xi_r = b_r\ ,\\ r = G_r \xi_r + c_r\ .\end{aligned} \tag{20}$$

Therefore, $r \in \tilde{\mathcal{R}}$, $\mathcal{R}_+ \subseteq \tilde{\mathcal{R}}$, and $\mathcal{R}_+ = \tilde{\mathcal{R}}$. □

Remark 1 *Given that the MLD system* (3) *is only defined over the bounded state space $\mathcal{X}$, the set of states reachable from $\mathcal{X}_0$ in one time step is given by Theorem 2 only when $\mathcal{X}_0 \subseteq \mathcal{X}$. This condition may be ensured in two ways,*

(1) assign $\mathcal{X}_0 = \mathcal{X}_0 \cap \mathcal{X}$ prior to applying Theorem 2,
(2) when $\mathcal{X}$ is a polytope, the constraints of $x \in \mathcal{X}\, \forall\, x \in \mathcal{X}_0$ may be defined implicitly within the MLD system inequalities (3b).

Note that when $\mathcal{X}$ is a polytope, these two approaches are mathematically equivalent. This condition is particularly important when applying Theorem 2 iteratively to compute $\mathcal{R}_k$.

Through the introduction of "slack" factors, ξ_{aff}, to enforce linear inequality constraints as intervals within the space of factors, Theorem 2 provides a method of determining the exact set of states reachable by MLD systems defined by (3). This approach is desirable as the propagation of the system dynamics is given by an identity and is computed algebraically. In contrast with existing approaches [7,10,11,12,13], the intersections with guard sets are handled implicitly as properties of the MLD system and require no iterative approximations or optimization programs. Furthermore, the growth in complexity of the set is a linear function of the number of iterative applications of Theorem 2. Specifically, given an initial set of states $\mathcal{X}_0 \subset \mathbb{R}^{n_{xc}} \times \{0,1\}^{n_{xl}}$ and set of admissible control inputs $\mathcal{U} \subset \mathbb{R}^{n_{uc}} \times \{0,1\}^{n_{ul}}$ in HCG-rep, the set of states reachable by the MLD system (3) in k time steps is a hybrid zonotope $\mathcal{R}_k$ with representation complexity given by

$$n_{g,r}(k) = (n_{g,u} + n_{rc} + n_e)k + n_{g,x}\ , \tag{21a}$$
$$n_{b,r}(k) = (n_{b,u} + n_{rl})k + n_{b,x}\ , \tag{21b}$$
$$n_{c,r}(k) = (n_{c,u} + n_e)k + n_{c,x}\ . \tag{21c}$$

4.2 Redundant inequality constraints

It is possible that some of the inequality constraints of the MLD system (3b) are always satisfied by the elements of $\mathcal{X}_0$ and $\mathcal{U}$ and therefore do not need to be enforced within the hybrid zonotope $\mathcal{R}_+$. That is, $e_x^i x + e_u^i u + e_z^i z + e_\delta^i \delta < e_{aff}^i\, \forall x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, and $\delta \in \Delta$, where e^i is the i^{th} row of the matrix E. In this case, including the i^{th} inequality constraint in $\mathcal{R}_+$ is unnecessary and its removal reduces both $n_{c,r}$ and $n_{g,r}$ because the slack factor enforcing the inequality constraint is also unnecessary.

This redundancy in the i^{th} inequality constraint enforced in $\mathcal{R}_+$ may be detected by evaluating the feasibility of the Mixed Integer Linear Program (MILP) with constraints

$$\begin{aligned}\begin{bmatrix}A_r^c\ A_r^b\end{bmatrix}\begin{bmatrix}\xi_r^c\\ \xi_r^b\end{bmatrix} = b\ ,\ \xi_r^b \in \{-1,1\}^{n_{b,r}}\\ \|(\xi_x^c\ \xi_u^c\ \xi_z\ \xi_{aff,j\neq i})\|_\infty \le 1\ ,\ 1 \le \xi_{aff,i}\ ,\end{aligned} \tag{22}$$

where $\xi_r^c = (\xi_x^c\ \xi_u^c\ \xi_z\ \xi_{aff})$, and the i^{th} slack factor, $\xi_{aff,i}$, is removed from the infinity norm constraint and instead constrained to be greater than or equal to 1. If the MILP is infeasible, then $\nexists\, x \in \mathcal{X}_0$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$, $\delta \in \Delta$ such that $e_x^i x + e_u^i u + e_z^i z + e_\delta^i \delta \ge e_{aff}^i$, and the i^{th} inequality constraint may be removed, thus reducing the number of constraints and continuous generators in $\mathcal{R}_+$ by one without altering the set.

5 Binary trees

As proven in Theorem 1, a hybrid zonotope with n_b binary factors is equivalent to the union of the 2^{n_b} constrained zonotopes $\mathcal{Z}_{c,i}$ given by (12). When it is necessary to decompose a hybrid zonotope into a complex of constrained zonotopes, enumeration of the set $\{-1,1\}^{n_b}$ may become intractable for large n_b. However, it is possible that some of the elements $\xi_i^b \in \{-1,1\}^{n_b}$ map to empty constrained zonotopes and therefore do not contribute to the volume of the hybrid zonotope. In this section it is shown how the enumeration problem of decomposing hybrid zonotopes may be reduced by iteratively growing binary trees in parallel with set operations. It is then shown how the number of binary variables needed to define the hybrid zonotope may be reduced by identifying the nonempty leaves of the binary tree.

For a hybrid zonotope $\mathcal{Z}_h$, let $\mathcal{T} \subseteq \{-1,1\}^{n_b}$ be the set of discrete elements that map to nonempty constrained zonotopes, that is $\mathcal{T} = \{\xi_i^b \in \{-1,1\}^{n_b} | \mathcal{Z}_{c,i} \neq \emptyset\ \}$. Leveraging Theorem 1 and $\mathcal{Z}_h \cup \emptyset = \mathcal{Z}_h$, it follows that

$$\mathcal{Z}_h = \bigcup_{\xi_i^b \in \mathcal{T}} \mathcal{Z}_{c,i}\ . \tag{23}$$

The enumeration problem in decomposing hybrid zonotopes may therefore be reduced by only considering the values of the binary factors belonging to $\mathcal{T}$. The discrete set $\mathcal{T}$ also gives a measure of how efficient the set is—ideally a hybrid zonotope representing 2^N constrained zonotopes would only have N binary factors. The order of the hybrid zonotope is defined by the two values

$$o_d = \frac{n_g - n_c}{n}\ ,\ o_b = \frac{n_b}{\lceil \log_2(|\mathcal{T}|) \rceil}\ , \tag{24}$$

where $\lceil \log_2(|\mathcal{T}|) \rceil$ is the smallest integer greater than $\log_2(|\mathcal{T}|)$. The degree of freedom order of the continuous portion, o_d, is identical to that of the constrained zonotope [17], while the sparsity order of the binary portion, o_b, is given by the ratio of the number of binary factors used with the lower bound of the number required. A full-dimensional hybrid zonotope in $\mathbb{R}^n$ will therefore have $1 \leq o_d$ and $1 \leq o_b$.

The hybrid zonotope is a mixed integer set representation [22] and may be described by a rooted binary tree [23]. The root of the binary tree is the hybrid zonotope $\mathcal{Z}_h$ and the nonempty leaves are the constrained zonotopes $\mathcal{Z}_{c,i} \, \forall \xi_i^b \in \mathcal{T}$. The binary tree consists of n_b layers, where the j^{th} layer branches on the value of the j^{th} binary factor. Each layer of the tree between the root and leaves consists of branch nodes given by hybrid zonotopes

$$\mathcal{Z}_{h,i}^j = \left\{ G^c, G_d^b, c + G_a^b \xi_i^b, A^c, A_d^b, b - A_a^b \xi_i^b \right\} \;, \quad (25)$$

where the binary generator and constraint matrices are partitioned such that $G^b = [G_a^b \; G_d^b]$, where G_a^b are the j columns for the ancestor nodes multiplied by $\xi_i^b \in \{-1,1\}^j$ for the i^{th} branch node of the layer, and G_d^b the remaining columns for the binary factors that are branched on by the descendants. The binary tree and relation between each node for a hybrid zonotope with $o_b = 1.5$ is depicted in Fig. 2.

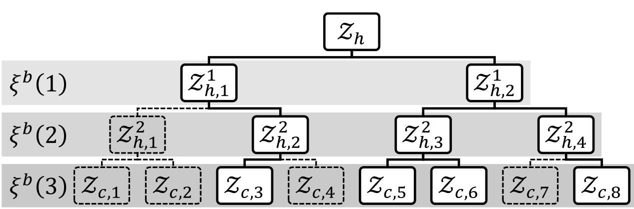


Fig. 2. Example of the binary tree for a hybrid zonotope $\mathcal{Z}_h$ with three binary factors. The set $\mathcal{T}$ is depicted by the bold black lines and empty nodes are grey with dashed borders. In this example, the relations between layers of the binary tree are given by $\mathcal{Z}_h = \mathcal{Z}_{h,1}^1 \cup \mathcal{Z}_{h,2}^1$, $\mathcal{Z}_{h,1}^1 = \mathcal{Z}_{h,2}^2 = \mathcal{Z}_{c,3}$, $\mathcal{Z}_{h,2}^1 = \mathcal{Z}_{h,3}^2 \cup \mathcal{Z}_{h,4}^2$, $\mathcal{Z}_{h,3}^2 = \mathcal{Z}_{c,5} \cup \mathcal{Z}_{c,6}$, and $\mathcal{Z}_{h,4}^2 = \mathcal{Z}_{c,8}$.

5.1 Growing binary trees in parallel with complex sets

The set $\mathcal{T}$ may be found with any MILP algorithm that explores the constrained space of factors, e.g. branch and cut [20], with constraints given by

$$A^c \xi^c + A^b \xi^b = b \;, \; \|\xi^c\|_\infty \leq 1 \;, \quad (26a)$$

$$\xi^b \in \{-1,1\}^{n_b} \;. \quad (26b)$$

The set $\mathcal{T} \subseteq \{-1,1\}^{n_b}$ represents the set of all values of the binary factors that simultaneously satisfy the linear constraints (26a) and integrality constraints (26b) of the hybrid zonotope, referred to as the integer feasible set of the MILP [22]. Although many algorithms exist that may be used to find $\mathcal{T}$ [20], the computational burden grows as the number of variables increases.

Complex sets are rarely given arbitrarily in engineering applications. Instead they are often generated through iterative applications of set operations. Through all set operations of hybrid zonotopes, the constraints on the factors of the operating sets are imposed directly in the resulting hybrid zonotope (see Prop. 2 and Theorem 2). Thus the hybrid zonotope generated through set operations with additional binary factors may only branch from the nonempty leaves of the operating sets.

Given a hybrid zonotope $\mathcal{Z}_{h,1}$ with integer feasible set $\mathcal{T}_1 \subseteq \{-1,1\}^{n_{b,1}}$, let $\mathcal{Z}_{h,2}$ be a hybrid zonotope found through set operations applied to $\mathcal{Z}_{h,1}$ introducing N additional binary factors. Rather than finding $\mathcal{T}_2 \subseteq \{-1,1\}^{n_{b,1}+N}$ by solving the MILP (26) for $\mathcal{Z}_{h,2}$ directly, it is possible to leverage the fact that the leaves of $\mathcal{Z}_{h,2}$ are the descendants of $\mathcal{Z}_{h,1}$, where $\mathcal{T}_1$ is already known. Thus an alternative approach is to solve the MILP (26) for the $|\mathcal{T}_1|$ branch nodes given by (25) at layer $n_{b,1}$, each having only N binary factors. The new integer feasible set $\mathcal{T}_2$ is then given by the union of the results from these $|\mathcal{T}_1|$ MILPs appended to the values of $\mathcal{T}_1$. This approach is described in Alg. 1.

Algorithm 1. Branching the binary tree of $\mathcal{Z}_{h,2}$ on the descendants of $\mathcal{Z}_{h,1}$.
Input: $\mathcal{Z}_{h,2} = \{G^c, G^b, c, A^c, A^b, b\}$, $\mathcal{T}_1 \subseteq \{-1,1\}^{n_{b,1}}$
Output: $\mathcal{T}_2 \subseteq \{-1,1\}^{n_{b,2}}$

for $i = 1, \ldots, |\mathcal{T}_1|$ **do**
  $\mathcal{Z}_{h,i}^{n_{b,1}} \leftarrow$ (25) for $\xi_i^b \leftarrow \mathcal{T}_1(i)$
  Solve MILP to find integer feasible set $\mathcal{T}$ of $\mathcal{Z}_{h,i}^{n_{b,1}}$
  Append entries of $\mathcal{T}$ to $\mathcal{T}_1(i)$ and store in $\mathcal{T}_2$
end for

Since finding $\mathcal{T}$ amounts to an exhaustive search of the integer feasible space of the MILP (26), Alg. 1 aims to reduce the number of branches that must be searched at each iteration by solving more, smaller MILPs. Each of these smaller MILPs search the subtrees branching on the binary factors added since the last search has been performed. Thus leveraging information stored in the set $\mathcal{T}_1$ prevents searching nodes that have already been determined as infeasible during previous iterations. This approach may allow more complex sets to be considered when many set operations are applied iteratively.

5.2 Reducing the number of binary factors

When the sparsity order of a hybrid zonotope is greater than one, $o_b > 1$, it is possible that the set may be represented with a reduced number of binary factors. One case where the number of binary factors may be reduced is when their values in the integer feasible set $\mathcal{T}$ are not linearly independent. In this case, some of the binary factors may be equivalently represented as a linear combination of the other factors. Another possibility is when all feasible values of a binary factor are the same. In this case, the binary factor can be removed after shifting the hybrid zonotope's center and the right hand side of the linear equality constraints by a constant.

Once $\mathcal{T}$ is known, linearly dependent binary factors may be detected and removed as follows. First, let $T \in \mathbb{R}^{n_b \times |\mathcal{T}|}$ be a matrix with each column an element of $\mathcal{T}$, thus $T(i,j) = \pm 1 \ \forall \ i,j$. Let $n_\phi = \text{rank}(T)$, if $n_\phi < n_b$ then there exists a linear mapping

$$M_1 T(\Phi, \cdot) = T \ , \tag{27}$$

where $\Phi \in \mathbb{N}_+^{n_\phi}$ are the indices of the linearly independent rows of T. Thus the hybrid zonotope $\mathcal{Z}_h = \{G^c, G^b, c, A^c, A^b, b\}$ is equivalent to $\mathcal{Z}_h^* = \{G^c, G^b M_1, c, A^c, A^b M_1, b\}$ [23] where $\mathcal{Z}_h^*$ has $n_b^* = n_\phi < n_b$ binary factors. The integer feasible set of $\mathcal{Z}_h^*$ is then given by

$$\mathcal{T}^* = \bigcup_{i=1}^{|\mathcal{T}|} T(\Phi, i) \ . \tag{28}$$

If all feasible values of a binary factor are the same, it can be removed as follows. Let $T(\Phi)$ be sorted such that the constant linearly independent row occurs first, i.e. $T(\Phi(1), \cdot) = 1$ or -1, and let

$$M_2 = M_1 \begin{bmatrix} \mathbf{0}_{1 \times n_\phi - 1} \\ I_{n_\phi - 1} \end{bmatrix} \ , \ m_2 = M_1 \begin{bmatrix} T(\Phi(1), 1) \\ \mathbf{0}_{n_\phi - 1 \times 1} \end{bmatrix} \ . \tag{29}$$

Then $\mathcal{Z}_h = \{G^c, G^b, c, A^c, A^b, b\}$ is equivalent to $\mathcal{Z}_h^* = \{G^c, G^b M_2, c + G^b m_2, A^c, A^b M_2, b - A^b m_2\}$ [23] where $\mathcal{Z}_h^*$ has $n_b^* = n_\phi - 1 < n_b$ binary factors. The integer feasible set of $\mathcal{Z}_h^*$ is then given by

$$\mathcal{T}^* = \bigcup_{i=1}^{|\mathcal{T}|} T(\Phi_2, i) \ , \tag{30}$$

where $\Phi_2 = \Phi(j)$ for $j = 2, \ldots, n_\phi$.

Although the number of binary factors, and equivalently the number of layers in the binary tree, are reduced, the nonempty leaves of the binary tree are not changed. Thus detecting and removing the redundancy in the binary factors through the described approach reduces the complexity of the hybrid zonotope set representation without altering the set.

6 Numerical examples

This section presents the forward reachable sets of two linear MLD systems in the form of (3). In both examples, the reachable set is found through iterative application of Theorem 2. For comparison, the dimensions and computation times are given for the hybrid zonotope found directly from Theorem 2 as well as that resulting from removal of redundant constraints and binary factors as discussed in Sec. 4.2 and 5.2, respectively. The integer feasible sets of the binary factors are found using a branch and cut algorithm [24] for the final set prior to performing redundancy removal.

MLD representations of the presented hybrid systems are obtained using HYSDEL 3.0 [25]. Optimization problems are solved using GUROBI [24]. Numerical results are generated with MATLAB on a desktop computer using one core of an 3.0 GHz Intel i7 processor and 32 GB of RAM.

6.1 *Piece-wise affine system with two equilibrium points*

Consider the discrete-time Piece-Wise Affine (PWA) system given by

$$x[k+1] = \begin{cases} \begin{bmatrix} 0.75 & 0.25 \\ -0.25 & 0.75 \end{bmatrix} x[k] + \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix} , & \text{if } x_1 \leq 0 \, , \\ \begin{bmatrix} 0.75 & -0.25 \\ 0.25 & 0.75 \end{bmatrix} x[k] + \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} , & \text{otherwise} \, . \end{cases} \tag{31}$$

This hybrid system consists of two stable, autonomous subsystems, each having an equilibrium point at $x = \pm[1 \ 0]^T$. This PWA system can be represented as an MLD system by introducing two continuous auxiliary variables, $n_{rc} = 2$, one binary auxiliary variable, $n_{rl} = 1$, and ten inequality constraints, $n_e = 10$. The states reachable by the PWA system in $N = 15$ time steps are shown in Fig. 3. The set representation dimensions and computation times are given in Table 1 with and without redundancy removal.

Table 1
Results of reachability analysis for (31) with redundancy removal, $\mathcal{R}_{15}^*$, and without, $\mathcal{R}_{15}$.

Set	$n_{g,r}$	$n_{c,r}$	$n_{b,r}$	o_b	Time (sec)
$\mathcal{R}_{15}$	182	150	15	15	0.02
$\mathcal{R}_{15}^*$	142	110	1	1	0.36

In this example, fifteen binary factors are introduced, resulting in $2^{15} = 32,768$ leaves in the full binary tree for $\mathcal{R}_{15}$. However, only one guard crossing occurs, resulting in two nonempty leaves. While performing redundancy removal, these empty leaves are successfully detected and replaced by a tree with *two leaves from a single binary factor*, thus reducing the sparsity order of the hybrid zonotope from $o_b = 15$ to $o_b^* = 1$. In this example, 40 of the inequality constraints are identified as redundant and removed.

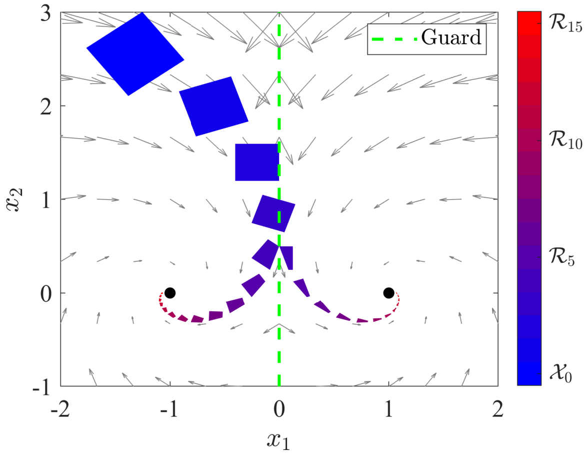


Fig. 3. Reachable set of PWA system (31) with two subsystems, each having an equilibrium point depicted by • and autonomous dynamics with vector fields depicted by →.

6.2 *Thermostat-controlled heated rooms*

This example considers the heated room scenario given in [13], adapted from a benchmark example proposed in [26], where the heat exchange among six adjacent rooms is modeled as a hybrid system. Two of the rooms have heaters controlled by thermostats that turn on when the temperature in the room drops below 22°C and turn off when it rises above 24°C. The continuous dynamics of the hybrid system are modeled as

$$\dot{x}_i = c \cdot h_i + b_i(u - x_i) + \sum_{i \neq j} a(x_j - x_i)\,, \tag{32}$$

where x_i is the temperature of the i^{th} room, the heat transfer coefficient between adjacent rooms is $a = 1$, the heat transfer coefficient between the rooms inside the building and the outside environment is $b_i = 0.16$ for $i = 1, 3, 4, 6$ and $b_i = 0.08$ for $i = 2, 5$, the heating power is $c = 15$ multiplied by $h_i \in \{0, 1\}$ for $i = 1, 6$ and $h_i = 0$ for $i = 2, \ldots, 5$, and the outside temperature may take on any value in the interval $u \in [0, 0.1]$ [13]. The discrete dynamics of the thermostats can be modeled as a finite automaton. The closed-loop temperature dynamics are then modeled as an MLD system with six continuous states $n_{xc} = 6$, one for the lumped temperature of each room, two discrete states $n_{xl} = 2$, one for each thermostat, six binary auxiliary variables $n_{rl} = 6$, and eighteen inequality constraints $n_e = 18$. The MLD system is depicted in Fig. 4. Using a discrete time step of $T_s = 0.01$ and a zero-order-hold discrete transform of the continuous dynamics (32), the reachable set of the MLD system for a time interval of $t = [0,\ 1]$ with both heaters initially turned on is given in Fig. 5. The set representation dimensions and computation times are given in Table 2.

In this example, $\mathcal{R}_{100}$ has 657 nonempty leaves in its binary tree, which is 4×10^{-177} % of the 2^{602} leaves in the full binary tree. Therefore, the reduction achieved using binary trees simplifies a computationally intractable enumeration problem into one for which visualization and analysis are easily possible. While performing redundancy removal, the number of binary factors is reduced from 602 to 29 to represent the exact reachable set. Furthermore, 1623 of the linear inequality constraints are identified as redundant and removed. The hybrid zonotope $\mathcal{R}^*_{100}$ therefore represents the exact reachable set of the MLD system, equivalent to the union of 657 constrained zonotopes, using 283 continuous generators, 29 binary generators, and 177 linear equality constraints.

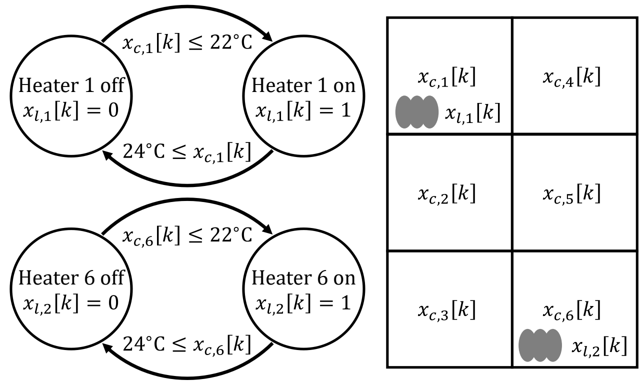


Fig. 4. (Left) Thermostats modeled as finite automata where $h_1 = x_{l,1}[k]$ and $h_6 = x_{l,2}[k]$. (Right) Six adjacent rooms with thermostat-controlled heaters in rooms one and six.

Table 2
Results of reachability analysis for (32) with redundancy removal, $\mathcal{R}^*_{100}$, and without, $\mathcal{R}_{100}$.

Set	$n_{g,r}$	$n_{c,r}$	$n_{b,r}$	o_b	Time (sec)
$\mathcal{R}_{100}$	1906	1802	602	60.2	5.84
$\mathcal{R}^*_{100}$	283	177	29	2.9	18.07

7 Conclusions

Hybrid zonotopes extend zonotopes and constrained zonotopes to represent the non-convex union of an exponential number of convex sets using a linear number of continuous and discrete variables. This is well-suited for reachability analysis of hybrid systems, in which discrete variables can cause branching of sets. The hybrid zonotope is closed under linear mappings, Minkowski sums, intersections, and halfspace intersections. Furthermore, exact reachable sets of linear mixed logical dynamical systems can be calculated as a hybrid zonotope using a simple identity exhibiting linear growth in set representation complexity. Methods for the removal of redundant continuous factors, binary factors, and linear equality constraints of such reachable sets substantially reduced the set representation complexity in two numerical examples. In an established benchmark example of a hybrid system with six continuous and two discrete states, 657 nonempty leaves of the

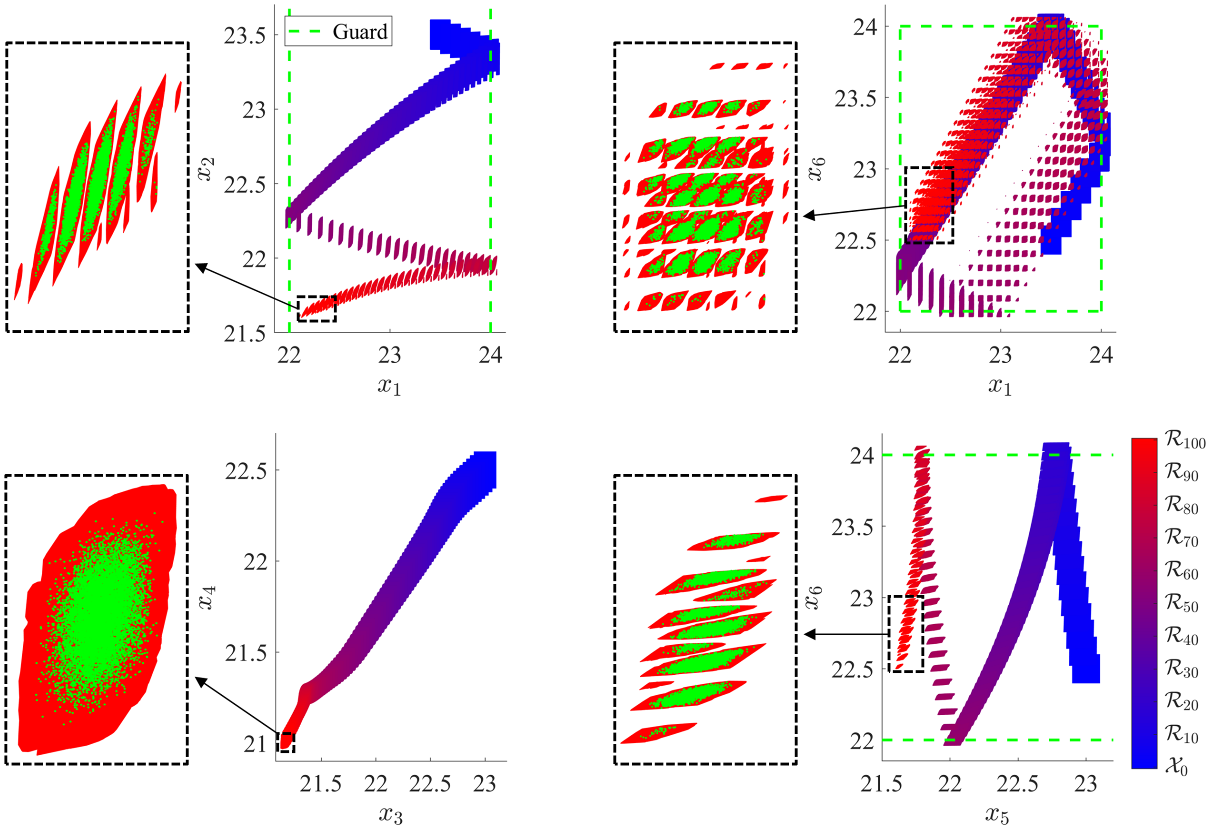


Fig. 5. Reachable set of heated room example modeled as an MLD system. Two dimensional projections for the six continuous states of the hybrid zonotope are provided with zoomed in figures of the final reachable set with randomly sampled, simulated trajectories given by green dots.

hybrid zonotope's binary tree were identified from 2^{602} possible, thus making analysis and visualization computationally tractable. The resulting hybrid zonotope for this example is equivalent to the nonconvex union of 657 constrained zonotopes and is represented using only 283 continuous factors, 29 binary factors, and 177 linear equality constraints.

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