Constrained Convex Generators: A Tool Suitable for Set-Based Estimation With Range and Bearing Measurements

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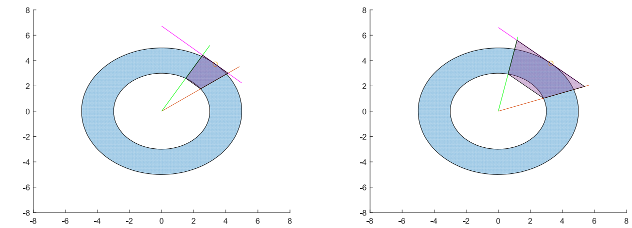
***Abstract*—Autonomous vehicles in GPS-denied areas or when cooperating in missions might have access to bearing and range measurements corrupted by noise, rendering the reachable set to be nonconvex since the measurement set is a segment of an annulus in 2D or a spherical shell in 3D. There are various alternatives that could be used in the literature to over-approximate the set by a convex one. However, given the circular part caused by the range measurement, adopting an exact polytopic description would require an infinite number of hyperplanes. In a similar fashion, using ellipsoids suffers from the same problem due to the hyperplane constraints arising from the bearing part. Moreover, if only bearing measurements are available, the measurement set should be unbounded. Motivated by these observations, we propose a generalization of the definition for constrained zonotopes recently introduced in the literature to also consider the ℓ_2 norm and cones (or any other convex set for that matter) as to represent these sets with less conservatism. Given the exact nature of the propagation, these can serve as a worst-case bounds for the true state which is relevant in some applications such as collision avoidance. In simulations, we also illustrate the performance of the computations to be suitable for relatively small sampling times.**

***Index Terms*—State reachability, range and bearing readings, constrained convex generators.**

I. Introduction

In MISSIONS with autnomous vehicles in GPS-denied locations or where the local sensors are reduced to a minimum, it is beneficial to have a small subset of the agents be equipped with range and bearing sensors to provide estimates for the remaining autonomous vehicles of their relative positioning (see for example [1], [2]). If the available measurements are bearing and range corrupted by noise, ideally, the measurement set is a segment (between minimum and maximum bearing angle) of an annulus where the inside circle has radius equal to the smallest possible value for the range and the outer circle is the maximum value admissible for the range. In three dimensions, it would be the intersection of a cone (defined by the bearing angles) and spherical shell (defined based on the range). A typical solution in the literature is to over-approximate the measurement set by a partition of intervals as done in [3] for range-only measurements or by an ellipsoid as done in [4]. However, these introduce unnecessary conservatism to the solution.



(a) Tight over-approximation for small angles.

(b) Conservative case caused by a larger error in the angle.

Fig. 1. Two examples of different levels of conservativeness on the over-approximation by a polytope of the measurement set corresponding to a bearing and range reading corrupted by noise. The line on top is perpendicular to the outer circle.

A direct approach based on the aforementioned techniques would be to over-approximate by a polytope, be it described in the hyper-plane representation such as [5] or in a constrained zonotope formulation [6]. Figure 1 depicts two cases where the approximation presents different levels of conservatism. In Figure 1(a), for a small angle error and short distances, the approximation does not introduce too much conservatism to be problematic. However, if we increase the angle error, the conservatism added is represented in Figure 1(b), where the added area would increase with the distance. Therefore, the presence of bearing measurements requires a set representation capable of including unbounded sets and, the addition of range measurements requires the possibility to have facets

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like an ellipsoid and others like a polytope, which motivated the introduction of a novel set representation.

There is a vast literature on set representations for state estimation of Linear Time-Varying (LTV) systems in discrete time such as: interval arithmetic [7], zonotopes [8], ellipsoids [9], constrained zonotopes [6] and polytopes [10] for the propagation and update of the set-valued estimates while using an over-approximation for the nonconvex annulus that corresponds to the choice of set representation. If we considered a nonlinear system, these techniques can be applied provided that the dynamics are approximated by a linear function that enables the propagation of the sets, as was done in [11], [12], [13], [14], [15], for each of the respective representation.

In the literature of autonomous vehicles, this problem is often tackled by employing a stochastic estimation resorting to a Kalman filter or an Extended Kalman filter depending on the assumed model dynamics. For instance, the problem with a single beacon range measurement was addressed in [16] by first converting the dynamics to a LTV model and then applying a Kalman Filter. Working directly with the nonlinear model of the vehicle was done in [17]–[19] where an Extended Kalman filter was presented and extensively studied. However, these approaches are not possible if the intended objective is to ensure safe passage of the vehicle with no collision with obstacles modeled as convex bodies.

In this letter, the main contributions can be summarized as follows.

- By noticing that the constrained zonotope formulation for polytopes can be adapted to have the generators belong to a convex set, we propose a novel set representation Constrained Convex Generators (CCGs), which in particular can include the ℓ_2 unit balls to generate smooth surfaces, ℓ_∞ unit balls to generate facets, and cones to generate unbounded sets;
- We then present how to apply these novel sets to address the three variants of the problem with reduced conservatism: i) range-only, ii) bearing-only, and iii) range and bearing measurements.

The remainder of the letter is organized as follows. We formalize the estimation problem in Section II highlighting the limitations arising from the current polytopic descriptions. Based on the insights of Section II, we present a general set representation to include convex sets as generators in Section III. In Section IV, it is provided the details on how to represent the measurement sets arising from range and bearing measurements and simulations showcasing the performance and accuracy are provided in Section V. Conclusions and directions of future work are given in Section VI.

Notation: We let $\mathbf{0}_n$ denote the n-dimensional vector of zeros and I_n the identity matrix of size n. The transpose of a vector v is denoted by $v^\intercal$, while the Euclidean norm for vector x is represented as $\|x\|_2 := \sqrt{x^\intercal x}$. On the other hand, $\|x\|_\infty := \max_i |x_i|$. The cartesian product is denoted by $\times$, the Minkowski sum of two sets by $\oplus$ and the intersection after applying a matrix R to the first set by $\cap_R$.

II. Problem Statement

In this letter, we model the vehicle dynamics by a linear model:

$$x(k+1) = F_k x(k) + B_k u(k) + d(k) \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$ and $d(k) \in \mathbb{R}^{n_d}$ are respectively the state, input and disturbance signals. Following the derivation in [20], one can assume the model for a vehicle like a quadrotor to be a double integrator subject to constraints on the velocity and acceleration, which fits the formulation for a linear dynamical system. The objective is to estimate the state $x(k)$ using a set reachability approach from bearing and range measurements. Let us assume that there are j nodes providing measurements, which will be referred by towers. Under such conditions, we are interested in 3 different problems, namely:

Problem 1 (State Estimation Using Range/Bearing Data): Let us consider a LTV model for a vehicle as in (1) with position evolving in $\mathbb{R}^p$ and j towers with known locations $\text{tower}_1, \dots, \text{tower}_j$.

- If range measurements are available, they will be denoted by $y^r(k)$ defined as follows:

$$y^r(k) = \begin{bmatrix} \|x_{[1,\dots,p]}(k) - \text{tower}_1\|_2 \\ \vdots \\ \|x_{[1,\dots,p]}(k) - \text{tower}_j\|_2 \end{bmatrix} + w(k),$$

- If bearing measurements are available, they will be denoted by $y^b(k)$ defined as follows:

$$y^b(k) = \begin{bmatrix} \text{ang}(x_{[1,\dots,p]}(k) - \text{tower}_1) \\ \vdots \\ \text{ang}(x_{[1,\dots,p]}(k) - \text{tower}_j) \end{bmatrix} + w(k),$$

where we used the notation $x_{[1,\dots,p]}(k)$ to select the entries 1 through p that correspond to the position of the vehicle and $w(k) \in \mathbb{R}^{n_w}$ is the noise signal. The operator $\text{ang}(v)$ returns either the angle of vector v in polar coordinates when $p = 2$ or a vector of 2 angles in spherical coordinates for v. Let us define the vector of all available measurements at time k by $y(k)$.

The problem is defined as computing a set of possible state values $X(k)$ for $k > 0$ such that $x(k) \in X(k), \forall d(k) \in D(k), \forall w(k) \in W(k)$, for some convex sets $D(k)$ and $W(k)$, from range measurements (i.e., $y(k) = y^r(k)$), bearing measurements (i.e., $y(k) = y^b(k)$, or range and bearing measurements (i.e., $y(k) = \begin{bmatrix} y^r(k) \\ y^b(k) \end{bmatrix}$).

III. Generalization of the Constrained Zonotope Representation

In this section, we first review the constrained zonotope representation introduced in [6] to model polytopes, highlighting how the basic set operations can be viewed in terms of convexity-preserving operations of a basic generator set.

Let us first review the definition of constrained zonotope from [6] and the basic set operations: affine map (2), Minkowski sum (3) and intersection after a linear map (4).

Definition 1 (Constrained Zonotope): A set Z is a constrained zonotope defined by the tuple $(G, c, A, b) \in \mathbb{R}^{n \times n_g} \times$

$\mathbb{R}^n \times \mathbb{R}^{n_c \times n_g} \times \mathbb{R}^{n_c}$ such that:

$$Z = \{G\xi + c : \|\xi\|_\infty \leq 1, A\xi = b\}.$$

Definition 2 (Set Operations): Consider three constrained zonotopes as in Definition 1:

- $Z = (G_z, c_z, A_z, b_z) \subset \mathbb{R}^n$;
- $W = (G_w, c_w, A_w, b_w) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y) \subset \mathbb{R}^m$;

and a matrix $R \in \mathbb{R}^{m\times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$RZ + t = (RG_z, Rc_z + t, A_z, b_z) \tag{2}$$

$$Z \oplus W = \left(\begin{bmatrix} G_z & G_w \end{bmatrix}, c_z + c_w, \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}\right) \tag{3}$$

$$Z \cap_R Y = \left(\begin{bmatrix} G_z & 0 \end{bmatrix}, c_z, \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}\right). \tag{4}$$

Given the Definition 2 for the three major set operations that will be required, let us write the complete definition for these operations, which will make it easier to introduce the novel definition for the proposed sets. The affine map for a constrained zonotope Z can also be defined as:

$$\begin{aligned} RZ + t &= \{RG_z\xi + Rc_z + t : \|\xi\|_\infty \leq 1, A_z\xi = b_z\} \\ &= \{RG_z\xi + Rc_z + t : \xi \in \mathcal{C}_z, A_z\xi = b_z\} \end{aligned} \tag{5}$$

where the second equation is implicitly assuming that $\mathcal{C}_z$ is the unit ℓ_∞-norm ball. In a similar fashion, we can present the same extended definition for the Minkowski sum and the intersection after linear map, where we use ξ_z, ξ_w and ξ_y as the auxiliary variables for the constrained zonotopes Z, W and Y:

$$\begin{aligned} Z \oplus W = \{&G_z\xi_z + G_w\xi_w + c_z + c_w : A_z\xi_z = b_z, A_w\xi_w = b_w, \\ &\xi_z \in \mathcal{C}_z, \xi_w \in \mathcal{C}_w\} \end{aligned} \tag{6}$$

$$\begin{aligned} Z \cap_R Y = \{&G_z\xi_z + c_z : A_z\xi_z = b_z, A_y\xi_y = b_y, \xi_y \in \mathcal{C}_y, \\ &RG_z\xi_z + Rc_z = G_y\xi_y + c_y, \xi_z \in \mathcal{C}_z\} \end{aligned} \tag{7}$$

From the above definition, it becomes clear that there is nothing forcing the use of the unit ℓ_∞-norm ball as the generator and one could resort to any unit ball following a p-norm but also extend the definition to convex cones (and other convex sets). We are now in condition of presenting the definition for Constrained Convex Generators (CCG) and a proposition establishing the equivalence between the set operations and its definition in the CCG format that will explore the relationship identified in (5), (6) and (7).

Definition 3 (Constrained Convex Generators): A Constrained Convex Generator (CCG) $\mathcal{Z} \subset \mathbb{R}^n$ is defined by the tuple $(G, c, A, b, \mathfrak{C})$ with $G \in \mathbb{R}^{n\times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c\times n_g}$, $b \in \mathbb{R}^{n_c}$, and $\mathfrak{C} := \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n_p}\}$ such that:

$$\mathcal{Z} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}\}.$$

Remark 1: Remark that it is possible to provide a definition for nonlinear operations g on CCGs as long as g is convex and nondecreasing in each argument of the input vector ξ and have $g(\xi)$ instead of $G\xi + c$ in Definition 3. However, that would only be useful if we were dealing with a nonlinear dynamical model such that its dynamics was control separable (meaning that $g(x, u)$ could be written as $f(x) + h(u)$) in addition to being convex and nondecreasing. Given space constraints in this letter and the type of model assumed for the system, we refrain from presenting such trivial extension from the CCG definition.

Given Definition 3, we could present a proposition asserting all set operations for CCGs.

Proposition 1: Consider three Constrained Convex Generators (CCGs) as in Definition 3:

- $Z = (G_z, c_z, A_z, b_z, \mathfrak{C}_z) \subset \mathbb{R}^n$;
- $W = (G_w, c_w, A_w, b_w, \mathfrak{C}_w) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y, \mathfrak{C}_y) \subset \mathbb{R}^m$;

and a matrix $R \in \mathbb{R}^{m\times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$RZ + t = (RG_z, Rc_z + t, A_z, b_z, \mathfrak{C}_z) \tag{8}$$

$$Z \oplus W = \left(\begin{bmatrix} G_z & G_w \end{bmatrix}, c_z + c_w, \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_w\}\right) \tag{9}$$

$$Z \cap_R Y = \left(\begin{bmatrix} G_z & 0 \end{bmatrix}, c_z, \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_y\}\right). \tag{10}$$

Proof: In order to prove (8), let us define the set resulting from the affine map as $\{z' : z' = Rz + t, \forall z \in Z\}$ which can be expanded as:

$$\{z' : z' = Rz + t, z = G_z\xi_z + c_z, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z\}$$

where we used the notation $\xi_z \in \mathfrak{C}_z$ to mean $\xi_z \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}$, $|\mathfrak{C}_z| = n_p$. Replacing the value of z in the expression we get:

$$\{z' : z' = RG_z\xi_z + Rc_z + t, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z\}$$

which is precisely the definition for the CCG on the right-hand side of (8).

For the Minkowski sum, we can do a similar analysis by first defining the set corresponding to the application of this operation by $\{z' : z' = z + w, \forall z \in Z, \forall w \in W\}$ and expand it as:

$$\begin{aligned} \{z' : z' = z + w, z = G_z\xi_z + c_z, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z, \\ w = G_w\xi_w + c_w, A_w\xi_w = b_w, \xi_w \in \mathfrak{C}_w\}. \end{aligned}$$

By replacing the values of z and w we can further simplify the definition as:

$$\begin{aligned} \{z' : z' = G_z\xi_z + c_z + G_w\xi_w + c_w, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z, \\ A_w\xi_w = b_w, \xi_w \in \mathfrak{C}_w\}. \end{aligned}$$

Stacking ξ_z and ξ_w in a single vector, we obtain the following expression:

$$\left\{z' : z' = \begin{bmatrix} G_z & G_w \end{bmatrix}\begin{bmatrix} \xi_z \\ \xi_w \end{bmatrix} + (c_z + c_w), \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix}\begin{bmatrix} \xi_z \\ \xi_w \end{bmatrix} = \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \begin{bmatrix} \xi_z \\ \xi_w \end{bmatrix} \in \{\mathfrak{C}_z, \mathfrak{C}_w\}\right\},$$

which is the right-hand side of (9).

Lastly, the intersection after a linear map can be defined as $\{z' : z' = z, \forall z \in Z, Rz \in Y\}$, which can also be expanded to:

$$\{z' : z' = z, z = G_z\xi_z + c_z, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z,\\ Rz = G_y\xi_y + c_y, A_y\xi_y = b_y, \xi_y \in \mathfrak{C}_y\}.$$

Replacing the value of z in the expression yields:

$$\{z' : z' = G_z\xi_z + c_z, A_z\xi_z = b_z, \xi_z \in \mathfrak{C}_z,\\ RG_z\xi_z + Rc_z = G_y\xi_y + c_y, A_y\xi_y = b_y, \xi_y \in \mathfrak{C}_y\},$$

which by stacking the values of ξ_z and ξ_y into a single vector becomes:

$$\left\{z' : z' = \begin{bmatrix} G_z & 0 \end{bmatrix}\begin{bmatrix} \xi_z \\ \xi_y \end{bmatrix} + c_z, \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}\begin{bmatrix} \xi_z \\ \xi_y \end{bmatrix} = \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix},\right.\\ \left.\begin{bmatrix} \xi_z \\ \xi_y \end{bmatrix} \in \{\mathfrak{C}_z, \mathfrak{C}_y\}\right\},$$

which is the right-hand side of (10), thus concluding the proof. ■

From the operations in Proposition 1 and the fact that by construction, CCG as defined in Definition 3 are convex sets, it means that they are well-suited to be applied to state estimation and fault detection of LTV models. Computationally speaking, it is required to store which type of generator we are using for which entries of the vector of auxiliary variables ξ. In the next section, we illustrate the use of CCGs and will only use unit ℓ_∞-norm balls, unit ℓ_2-norm balls and cones as the convex generators for designing an observer to estimate the state of a vehicle. Also notice that it is always possible to over-approximate CCG sets by the hyper-cubes resulting from interval analysis so the results in [5] regarding the boundedness of the hyper-volume of these sets can be directly applied.

IV. State Estimation Using Constrained Convex Generators (CCGs) With Range/Bearing Data

In this section, the state estimation strategy is presented by exploiting the properties of CCGs introduced in Section III and over-approximations to the exact non-convex measurement sets. The propagation equation of the estimates using the dynamical model in (1) can be accomplished using the set operations in Proposition 1, namely that with previous set-valued estimates $X(k)$ can be propagated to obtain set $X_{\text{prop}}(k+1)$ that contains all points that are consistent with the previous estimate and the dynamics in the following fashion:

$$X_{\text{prop}}(k+1) = F_kX(k) + B_ku(k) \oplus D(k),$$

meaning that $X_{\text{prop}}(k+1)$ is the result of an affine map on $X(k)$ using matrix F_k and vector $B_ku(k)$ and then the Minkowski sum with the disturbance set.

The update set of the observer requires performing an intersection following the linear map $C = \begin{bmatrix} \mathsf{e}_1 & \cdots & \mathsf{e}_p \end{bmatrix}^\intercal$, which is defined to obtain the first p entries of vector x that store the vehicle position, i.e., $x_{[1,\ldots,p]}(k) = Cx(k)$. This means that the set-valued estimates for state at time $k+1$, $X(k+1)$, can be obtained as an intersection between propagated set $X_{\text{prop}}(k+1)$ with the measurement set $Y(k+1)$ (which will be defined for the three cases of range, bearing and range/bearing data) as follows:

$$X(k+1) = X_{\text{prop}}(k+1) \cap_C Y(k+1).$$

Note that the set $Y(k+1)$ is going to be the intersection of the individual measurement set for each tower. For simplicity of exposition, we will present the various $Y(k+1)$ assuming a single tower (and drop the subscript for that matter) but we can perform the intersection using the identity linear map over all the sets.

A. Bearing-Only Measurements

Let us define the minimum and maximum error on a single bearing measurement as b_l and b_u, meaning that we have in two dimensions that $\text{ang}(x_{[1,2]}(k) - \text{tower}) \in \begin{bmatrix} y^b(k) - b_l & y^b(k) + b_u \end{bmatrix}$.

Thus, the bearing-only measurement set $Y^b(k)$ is a cone, which can be expressed by the CCG:

$$Y^b(k) = \left(\begin{bmatrix} \cos(y^b(k) + b_u) & \cos(y^b(k) - b_l) \\ \sin(y^b(k) + b_u) & \sin(y^b(k) - b_l) \end{bmatrix}, \text{tower}, 0_2^\intercal, 0, \{\mathbb{R}_+^2\}\right), \tag{11}$$

where $\mathbb{R}_+^2$ is the nonnegative orthant in $\mathbb{R}^2$. The definition in (11) did an affine transformation of $\mathbb{R}_+^2$ where G was selected as to change the canonical vectors to the desired ones corresponding to the minimum and maximum angles allowed by the measurement $y^b(k)$. We remark that setting $A = \begin{bmatrix} 0 & 0 \end{bmatrix}$ in $Y^b(k)$ as we have done can be omitted and treated as the empty matrix provided that dimensions are kept consistent when using block diagonal operations. In 3 dimensions, one can address each angle separately and then use the intersection to construct the shape.

B. Bearing and Range Measurements

Prior to presenting the range-only measurements, it is easier to first look at a segment of the annulus and then to resort to the equivalent of Zonotope Bundles [21] with CCGs instead. Let us define the minimum and maximum error on a single range measurement as r_l and r_u, meaning that we have in two dimensions that $\|x_{[1,2]}(k) - \text{tower}\|_2 \in \begin{bmatrix} y^r(k) - r_l & y^r(k) + r_u \end{bmatrix}$ in addition to the constraint from the bearing. Figure 1(b) hints at the need to first compute the four points that result from the intersection of each of the circles and the minimum and maximum angles (getting two outer points in the outer circle and conversely two inner points). The coordinates of each point is simply $\rho\begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$ for $\rho \in \{y^r(k) - r_l, y^r(k) + r_u\}$ and $\alpha \in \{y^b(k) - b_l, y^b(k) + b_u\}$. We also need a fifth point with the maximum range possible and $\alpha = (2y^r(k) - r_l + r_u)/2$. It is then straightforward to find the line equation that is parallel to the outer points and passes through the fifth point as well as the remaining line equations and write the trapezoidal shape as $Mx \leq m$. By applying the formula from [6, Th. 1], we obtain the Constrained Zonotope (G, c, A, b) that is equivalent to the CCG representation $\mathcal{Z}_{\text{trap}} = (G, c, A, b, \{\mathcal{B}_\infty\})$ where $\mathcal{B}_\infty$ is the unit ℓ_∞-ball. The outer circle is given by the

CCG $((y^r(k)+r_u)I_2, \text{tower}, 0_2^\intercal, 0, \{\mathcal{B}_2\})$, where $\mathcal{B}_2$ is the unit ℓ_2-ball. Thus, the measurement set $Y^{br}(k)$ is given by:

$$Y^{br}(k) = \mathcal{Z}_{\text{trap}} \cap_{I_2} \big((y^r(k)+r_u)I_2, \text{tower}, 0_2^\intercal, 0, \{\mathcal{B}_2\}\big).$$

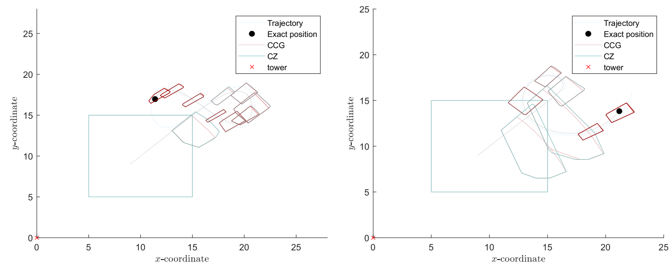
C. Range-Only Measurements

The measurement set $Y^r(k)$ when there are only range measurements available cannot be approximated as done in the previous section. However, one can partition the full circle into segments with $2\pi/\Delta$ angles and use as minimum and maximum angles the values $2\pi q/\Delta$ and $2\pi(q+1)/\Delta$ for $q = 0, 1, \ldots, \Delta-1$ and repeat the computations done in Section IV-B replacing the interval $[y^r(k)-r_l \quad y^r(k)+r_u]$ by $[2\pi q/\Delta \quad 2\pi(q+1)/\Delta]$. These Δ sets $Y_1^r(k), \ldots, Y_\Delta^r(k)$ need to be intersected independently with the propagated set $X_{\text{prop}}(k)$ to obtain $X(k) = \bigcup_{q=0}^{\Delta-1} Y_q^r(k) \cap_{I_2} X_{\text{prop}}(k)$. Naturally, this increases the computational complexity as in the subsequent time step we may end up with Δ times more sets to propagate. Each set can be checked to see if it is empty and discarded if so to reduce the amount of future computation with unnecessary sets. In the envisioned scenario of a tower providing range measurements or a beacon emitting a sound or a signal to help the vehicle in its localization, most of these sets will be empty as the beacon will be far away.

V. Simulations

In this section, simulation results are presented with the objective of characterizing the behavior of the proposed CCG definition with respect to 4 important factors: i) size of the matrices and vectors used in the definitions; ii) computational time to achieve the representation; iii) solver time to check that a point belong to the set and also if it is empty; and, iv) comparison against the size of the estimate set produced by the Constrained Zonotopes. Notice that i) is not of particular interest unless the set representation needs to be sent to other agents (as for example to implement a distributed state estimation algorithm such as in [22]) whereas ii) is crucial when running the observer online as the computation has to be faster than that of the sampling time. Lastly, checking for a collision with another convex obstacle can be achieved by modeling it as a CCG and checking whether the intersection is the empty set, which emphasizes the importance of iii). Moreover, computing the centroid of the set, what is the point maximizing some direction among other questions can all be formulated as an optimization problem which means that it should be efficient to solve a program constrained to a CCG.

In the following simulations, we adopted a double integrator dynamics (which is a fair model provided hard constraints are enforced on the velocity and trajectory [20]) for a vehicle moving in $\mathbb{R}^2$ motivated by applications in maritime and underwater vessels where range and bearing measurements are a relevant type of sensor information. The continuous dynamics are discretized with a sample and hold strategy and a sampling time of $\text{Ts} = 0.1\text{s}$ and it is assumed to exist a state-feedback controller on board corresponding to the solution K of the discrete Linear Quadratic Regulator with parameters



(a) Vehicle performing a 200 time instant trajectory of a figure 8. (b) Vehicle performing a 250 time instant trajectory of a of a spiral.

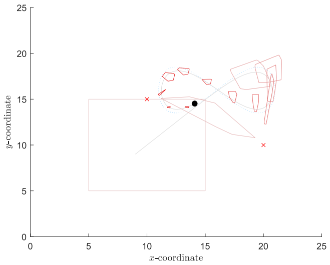
Fig. 2. Trajectory intended (light blue dots) and realized path done by the vehicle (grey) and the set-valued estimates obtained from Range and Bearing measurements drawn at each 15 iterations going from time instant 1 (lighter) to the end of the simulation (darker).

$Q = 10I_4$ and $R = I_2$:

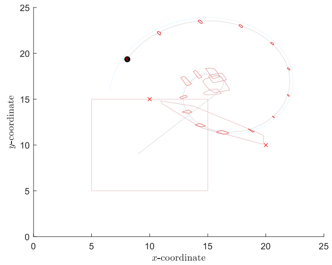
$$K = \begin{bmatrix} 2.5857 & 0 & 3.4434 & 0 \\ 0 & 2.5857 & 0 & 3.4434 \end{bmatrix}. \quad (12)$$

Initial state is $x(0) = [9 \quad 9 \quad 0 \quad 0]^\intercal$ (meaning that the vehicle is stopped at point with coordinates $[9 \quad 9])^\intercal$ and the initial estimate $X(0) = (5I_4, [10 \quad 10 \quad 0 \quad 0]^\intercal, 0_4^\intercal, 0, \{\mathcal{B}_\infty\})$, i.e., both position and velocity have an uncertainty of ± 5 with the center of the estimate representing a stopped vehicle at coordinates $[10 \quad 10]^\intercal$. Lastly, the source of the readings is placed at the origin and the simulations were run in MATLAB R2018a running on a HP machine with a Intel Core i7-8550U CPU @ 1.80GHz and 12 GB of memory resorting to the overloaded plot function by Yalmip to depict the sets and using Mosek as the underlying solver. Videos of the simulations and figures can be found in https://github.com/danielmsilvestre/CCGpaper.

Figure 2(a) depicts the vessel doing a figure-8 with both range and bearing measurements during a simulation of 200 time instants corresponding to 20s. The sets are shown every 15 iterations to avoid cluttering the image. Over the 200 iterations, the construction of the sets took on average 8.1×10^{-3} seconds with a variance of 3.87×10^{-5}, which is much faster than the 10^{-1} for the sampling time. In terms of the sizes of the data structures, the CCG at time k required storing $4+8k$ generator variables, $1+9k$ linear equalities and an additional vector with $4+8k$ entries (one for each generator variable) storing a numeric value starting by an identifier number for the type of generator followed by a unique identifier so that we can group generator variables that are defined within the same generator. This means a linear growth of $2n_x$ and $2n_x+1$ for this type of measurement. At each iteration, we solved a linear objective function constrained on the point belonging to $X(k)$. Over the 200 iterations, it took on average 0.0151 seconds with the maximum being 0.0516 seconds. For instance, a problem at iteration 200 took 0.024 seconds and a collision check took around 0.0197 seconds. In addition, we compared the hypervolume of the sets produced by Constrained Zonotopes against CCGs and on average throughout the whole simulation they are larger 6.6% with a maximum 34.35%. This percentage increases with the difference $b_u - b_l$, i.e., with the error on the bearing measurement as seen in Figure 1(b).



(a) Vehicle performing a 200 time instant trajectory of a figure 8.



(b) Vehicle performing a 300 time instant trajectory of a spiral.

Fig. 3. Trajectory intended (light blue dots) with the actual trajectory done by the vehicle (grey) and the set-valued estimates drawn at each 15 iterations going from time instant 1 (lighter) to time instant 200 (darker) obtained using bearing-only measurements from the two towers (red crosses).

In Figure 2(b), we present a simulation with a different trajectory that promotes a sharper turn that offers the possibility for a better estimate since the control law is exciting the system from an earlier point in the simulation. This is confirmed by the sizes of the sets that decrease after the turn. With respect to the performance of constructing the sets and conservatism, the results are similar to those of Figure 2(a).

We have also simulated a case where the observer has access to bearing-only measurements corrupted by noise from two beacons assuming that the vessel is within 10 distance units. This is a case that cannot be represented by a Constrained Zonotope unless we assume some arbitrarily large constant to eventually bound the set as it is not possible to represent unbounded sets. The beacons are placed at positions $\begin{bmatrix}10 & 15\end{bmatrix}^\intercal$ and $\begin{bmatrix}20 & 10\end{bmatrix}^\intercal$ as to make segments of the trajectory be served by a different beacon. The estimation task has a better performance given that the measurement sets can be described using fewer generator variables. The mean time to compute the sets was 0.0016s with the worst-case taking 0.007s, which reinforces the usability of CCGs in real time. The number of generator variables increases as $4+2k$ given that the set $Y(k)$ can be described with 2 variables. The reduced description of the sets also decreases the time it took to check whether the set is empty or if a point belongs to the set, achieving a mean solver time of 0.0035s with the worst-case being 0.0073s.

The last simulation used a spiral trajectory in order get a richer set of measurements. The results are depicted in Figure 3(b) and show that the set shrinks rapidly and allows for a good estimation performance. No order reductions were performed and the computational times are very similar to the case of the figure 8 with bearing measurements.

VI. Conclusion and Future Work

In this letter, we considered the problem of estimating the state of a dynamical system described by a linear model with nonlinear measurements, namely a combination of bearing and range measurements. Given the nonconvexity of the sets arising when we introduce range measurements, a novel set representation was introduced named Constrained Convex Generators (CCGs) which can be viewed as a generalization of the constrained zonotope representation of polytopes by allowing an arbitrary set of convex sets to encase the generator variables. In doing so, the approximation of the nonconvex measurement set can be made sufficiently tight, while retaining efficiency and performance to be used in real-time scenarios. The three main set operations are given in closed form and the results are illustrated through simulations, validating that the computational times can be accommodated in real-time applications.

References

[1] R. Ribeiro, D. Silvestre, and C. Silvestre, "A rendezvous algorithm for multi-agent systems in disconnected network topologies," in *Proc. 28th Mediterr. Conf. Control Autom. (MED)*, 2020, pp. 592–597.
[2] R. Ribeiro, D. Silvestre, and C. Silvestre, "Decentralized control for multi-agent missions based on flocking rules," in *CONTROLO*, vol. 695, J. A. Gonçalves, M. Braz-César, and J. P. Coelho, Eds. Cham, Switzerland: Springer Int., 2021, pp. 445–454. [Online]. Available: https://link.springer.com/chapter/10.1007/978-3-030-58653-9_43#citeas
[3] L. Jaulin, "Range-only SLAM with occupancy maps: A set-membership approach," *IEEE Trans. Robot.*, vol. 27, no. 5, pp. 1004–1010, Oct. 2011.
[4] J. Marcal, J. Jouffroy, and T. I. Fossen, "An extended set-value observer for position estimation using single range measurements," in *Proc. 14th Int. Symp. Unmanned Untethered Submersible Technol. (UUST'05)*, Durham, NH, USA, 2008. [Online]. Available: https://www.ucviden.dk/en/publications/an-extended-set-value-observer-for-position-estimation-using-sing
[5] D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Stochastic and deterministic fault detection for randomized gossip algorithms," *Automatica*, vol. 78, pp. 46–60, Apr. 2017.
[6] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz, "Constrained zonotopes: A new tool for set-based estimation and fault detection," *Automatica*, vol. 69, pp. 126–136, Jul. 2016.
[7] R. E. H. Thabet, T. Raïssi, C. Combastel, D. Efimov, and A. Zolghadri, "An effective method to interval observer design for time-varying systems," *Automatica*, vol. 50, no. 10, pp. 2677–2684, 2014.
[8] C. Combastel, "A state bounding observer based on zonotopes," in *Proc. Eur. Control Conf. (ECC)*, Cambridge, U.K., 2003, pp. 2589–2594.
[9] F. Chernousko, "Ellipsoidal state estimation for dynamical systems," *Nonlinear Anal. Theory Methods Appl.*, vol. 63, no. 5, pp. 872–879, 2005.
[10] D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Set-based fault detection and isolation for detectable linear parameter-varying systems," *Int. J. Robust Nonlinear Control*, vol. 27, no. 18, pp. 4381–4397, 2017.
[11] F. Abdallah, A. Gning, and P. Bonnifait, "Box particle filtering for nonlinear state estimation using interval analysis," *Automatica*, vol. 44, no. 3, pp. 807–815, 2008.
[12] T. Alamo, J. M. Bravo, and E. F. Camacho, "Guaranteed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.
[13] A. A. Julius and G. J. Pappas, "Trajectory based verification using local finite-time invariance," in *Hybrid Systems: Computation and Control*, R. Majumdar and P. Tabuada, Eds, Elsevier, 2009, pp. 223–236.
[14] B. S. Rego, D. M. Raimondo, and G. V. Raffo, "Set-based state estimation of nonlinear systems using constrained zonotopes and interval arithmetic," in *Proc. Eur. Control Conf. (ECC)*, Limassol, Cyprus, 2018, pp. 1584–1589.
[15] J. Wan, S. Sharma, and R. Sutton, "Guaranteed state estimation for nonlinear discrete-time systems via indirectly implemented polytopic set computation," *IEEE Trans. Autom. Control*, vol. 63, no. 12, pp. 4317–4322, Dec. 2018.
[16] P. Batista, C. Silvestre, and P. Oliveira, "Single range aided navigation and source localization: Observability and filter design," *Syst. Control Lett.*, vol. 60, no. 8, pp. 665–673, 2011.
[17] A. S. Gadre and D. J. Stilwell, "A complete solution to underwater navigation in the presence of unknown currents based on range measurements from a single location," in *Proc. IEEE/RSJ Int. Conf. Intell. Robots Syst.*, Edmonton, AB, Canada, 2005, pp. 1420–1425.
[18] T. Casey, B. Guimond, and J. Hu, "Underwater vehicle positioning based on time of arrival measurements from a single beacon," in *Proc. OCEANS*, Vancouver, BC, Canada, 2007, pp. 1–8.
[19] P.-M. Lee, B.-H. Jun, K. Kim, J. Lee, T. Aoki, and T. Hyakudome, "Simulation of an inertial acoustic navigation system with range aiding for an autonomous underwater vehicle," *IEEE J. Ocean. Eng.*, vol. 32, no. 2, pp. 327–345, Apr. 2007.
[20] P. Mali, K. Harikumar, A. Singh, K. Krishna, and P. Sujit, "Incorporating prediction in control barrier function based distributive multi-robot collision avoidance," in *Proc. Eur. Control Conf. (ECC)*, 2021, pp. 2386–2391.
[21] M. Althoff and B. H. Krogh, "Zonotope bundles for the efficient computation of reachable sets," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, Orlando, FL, USA, 2011, pp. 6814–6821.
[22] D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Finite-time convergence policies in state-dependent social networks," in *Proc. Amer. Control Conf. (ACC)*, Chicago, IL, USA, 2015, pp. 1041–1046.