Homework 4

Professor: PhD. Eunseo Choi
Student: Evelyn Susana Delgado Andino
dlgdndno@memphis.edu

Part 1: Electric field flux

Part a)

We know:

$$Volume\ cube = 1cm^3 = 1x10^{-6}m \qquad sphere\ radius = 1m$$

$$\rho_{charge} = 1C/m^3 \qquad d = 0.125m$$

First, we need to find q:

$$q = \rho_{charge} \times Volume$$

$$q = 1C/m^3 \times 1x10^{-6}m$$

$$q = 1x10^{-6}C$$

For the flux, we use gauss's law:

$$\Phi_E = q_{in}/\varepsilon_0$$

Where $\varepsilon_0 = 8.85 \times 10^{-12}F/m$ is the permittivity in the vacuum.

$$\Phi_E = \frac{1x10^{-6}C}{8.85 \times 10^{-12}F/m}$$

$$\Phi_E = 112994.35Nm^2/C$$

$$\Phi_E \approx 1.13 \times 10^5 Nm^2/C$$

With the gauss's law the flux only depends on the enclosed charge, so it does not matter if the cube is not at the center of the sphere.

Part b)

The flux only depends on the enclosed charge. However, we have the same charge as in part a), for that reason:

$$\Phi_E \approx 1.13 \times 10^5 Nm^2/C$$

Part c)

The vector potential of a dipole is:

$$A = \frac{\mu_0}{4\pi}\frac{m \times \vec{r}}{r^2}$$

Equation 6.10 from Griffiths, (2017). Where μ_0 is the permeability of the free space, m is the magnetic dipole moment, r is the radial distance and $\vec{r}$ is the unit vector in the radial direction.

In cartesian coordinates:

$$\vec{r} = \sin(\theta)\cos(\phi)\,\hat{\imath} + \sin(\theta)\sin(\phi)\,\hat{\jmath} + \cos\,(\theta)\hat{k}$$

$$m = m\hat{k} = m(0,0,1)$$

Then, we calculate $m \times \vec{r}$,

$$m \times \vec{r} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & m \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos\,(\theta) \end{bmatrix}$$

$$m \times \vec{r} = 0\hat{\imath} + m\sin(\theta)\cos(\phi)\hat{\jmath} + 0\hat{k} - m\sin(\theta)\sin(\phi)\,\hat{\imath} - 0\hat{\jmath} - 0\hat{k}$$

$$m \times \vec{r} = m\sin(\theta)\cos(\phi)\hat{\jmath} - m\sin(\theta)\sin(\phi)\,\hat{\imath}$$

Now, we have:

$$A = \frac{\mu_0}{4\pi}\frac{m}{r^2}(\sin(\theta)\cos(\phi)\hat{\jmath} - \sin(\theta)\sin(\phi)\,\hat{\imath})$$

The magnetic field is obtained by:

$$B = \nabla \times A$$

In spherical coordinates, curl operator is:

$$(\nabla \times A)_r = \frac{1}{r sin(\theta)}\left[\frac{\partial A_\phi \sin\,(\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$(\nabla \times A)_\theta = \frac{1}{r}\left[\frac{1}{sin(\theta)}\frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}\right]$$

$$(\nabla \times A)_\theta = \frac{1}{r}\left[\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

For the magnetic potential vector, first we need to transform into spherical coordinates

$$A = \frac{\mu_0}{4\pi}\frac{m}{r^2}(\sin(\theta)\cos(\phi)\hat{\jmath} - \sin(\theta)\sin(\phi)\,\hat{\imath})$$

We have:

$$A_x = -\frac{\mu_0}{4\pi}\frac{m}{r^2}\sin(\theta)\sin(\phi)\,\hat{\imath}$$

$$A_y = \frac{\mu_0}{4\pi}\frac{m}{r^2}\sin(\theta)\cos(\phi)\hat{\jmath}$$

$$A_z = 0$$

To convert, we need to use:

$$\hat{\imath} = \sin(\theta)\cos(\phi)\hat{r} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\phi)\hat{\phi}$$

$$\hat{\jmath} = \sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\phi)\hat{\phi}$$

$$\hat{k} = \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\phi}$$

Now

$$A_x\hat{\imath} = -\frac{\mu_0}{4\pi}\frac{m\sin(\theta)\sin(\phi)}{r^2}(\sin(\theta)\cos(\phi)\hat{r} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\phi)\hat{\phi})$$

$$A_y\hat{\jmath} = \frac{\mu_0}{4\pi}\frac{m\sin(\theta)\cos(\phi)}{r^2}(\sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\phi)\hat{\phi})$$

$$A_z\hat{k} = 0$$

Expanding each term for r component:

$$A_r = -\frac{\mu_0}{4\pi}\frac{m\sin(\theta)\sin(\phi)}{r^2}\sin(\theta)\cos(\phi) + \frac{\mu_0}{4\pi}\frac{m\sin(\theta)\cos(\phi)}{r^2}\sin(\theta)\sin(\phi)$$

$$A_r = 0$$

Expanding each term for $\theta - component$

$$A_\theta = -\frac{\mu_0}{4\pi}\frac{m\sin(\theta)\sin(\phi)}{r^2}\cos(\theta)\cos(\phi) + \frac{\mu_0}{4\pi}\frac{m\sin(\theta)\cos(\phi)}{r^2}\cos(\theta)\sin(\phi)$$

$$A_\theta = 0$$

Expanding each term for $\phi - component$

$$A_\phi = +\frac{\mu_0}{4\pi}\frac{m\sin(\theta)\sin(\phi)}{r^2}\sin(\phi) + \frac{\mu_0}{4\pi}\frac{m\sin(\theta)\cos(\phi)}{r^2}\cos(\phi)$$

$$A_\phi = +\frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r^2}(\sin^2(\phi) + \cos^2(\phi))$$

We know that

$$\sin^2(\phi) + \cos^2(\phi) = 1$$

Now

$$A_\phi = \frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r^2}$$

Calculating B_r

$$B_r = \frac{1}{rsen(\theta)}\left[\frac{\partial A_\phi \sin(\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$B_r = \frac{1}{rsen(\theta)}\left[\frac{\partial A_\phi \sin(\theta)}{\partial\theta} - 0\right]$$

$$B_r = \frac{1}{rsen(\theta)}\left[\frac{\partial}{\partial\theta}\frac{\mu_0}{4\pi}\frac{m\sin^2(\theta)}{r^2} - 0\right]$$

$$\frac{\partial}{\partial\theta}\sin^2(\theta) = 2\sin(\theta)\cos(\theta)$$

Then

$$B_r = \frac{1}{r^3 sin(\theta)}\frac{\mu_0 2\sin(\theta)\cos(\theta)}{4\pi}$$

$$B_r = \frac{2}{r^3}\frac{\mu_0\cos(\theta)}{4\pi}$$

Calculating B_θ

$$B_\theta = \frac{1}{r}\left[\frac{1}{sin(\theta)}\frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}\right]$$

$$B_\theta = \frac{1}{r}\left[0 - \frac{\partial A_\phi}{\partial r}\right]$$

$$B_\theta = \frac{1}{r}\left[-\frac{\partial}{\partial r}\frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r^2}\right]$$

$$\frac{\partial}{\partial r}\frac{1}{r^2} = -\frac{2}{r^3}$$

Now

$$B_\theta = -\frac{1}{r}\frac{\mu_0}{4\pi}\frac{(-2)m\sin(\theta)}{r^3}$$

$$B_\theta = \frac{\mu_0}{2\pi}\frac{m\sin(\theta)}{r^4}$$

Now, we need to calculate the magnetic field flux through a sphere using:

$$\Phi = \oint B \cdot dA$$

In spheric coordinates:

$$dA = r^2 \sin(\theta)\, d\theta d\phi$$

Now, we use B_r, because only the radial component contributes to the flux.

$$\Phi = \oint B_r r^2 \sin(\theta)\, d\theta d\phi$$

$$\Phi = \oint \frac{2}{r^3}\frac{\mu_0 \cos(\theta)}{4\pi} r^2 \sin(\theta)\, d\theta d\phi$$

$$\Phi = \frac{\mu_0}{2\pi r}\oint \cos(\theta)\sin(\theta)\, d\theta d\phi$$

For this case

$$\Phi = \oint_0^{2\pi}\oint_0^{\pi} \cos(\theta)\sin(\theta)\, d\theta d\phi$$

For

$$\oint_0^{\pi} \cos(\theta)\sin(\theta)\, d\theta = \frac{1}{2}\oint_0^{\pi} \sin(2\theta)\, d\theta = \frac{1}{2}\left[-\frac{1}{2}\cos\,(2\theta)\right]_0^{\pi} = 0$$

And

$$\oint_0^{2\pi} d\phi = 2\pi$$

We obtained

$$\Phi = \frac{\mu_0}{2\pi r}(2\pi)(0)$$

$$\Phi = 0$$

The magnetic field flux through the surface of a sphere is zero if the magnetic field is generated by a dipole.

Part 2

Part a)

A single charge ion, accelerated from rest by an electric potential difference, kinetic energy is given by:

$$\Delta K = q\Delta V$$

We know that

$$\Delta K = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

For this case $v_0 = 0$

$$K = \frac{1}{2}m_1 v_1^2$$

Then

$$\frac{1}{2}mv_1^2 = q\Delta V$$

$$v_1^2 = \frac{2q\Delta V}{m}$$

$$v_1 = \sqrt{\frac{2q\Delta V}{m}}$$

We have $\Delta V = 10{,}000V$, $q = 1.602x10^{-19}C$, correspond to the electron charge, $m = 1.454998x10^{-25}kg$, mass of $^{87}S_r$ ion.

$$v_1 = \sqrt{\frac{2(1.602x10^{-19}C)(10{,}000V)}{1.454998x10^{-25}kg}}$$

$$v_1 = 148393.5606m/s$$

$$v_1 = 1.5x10^5 m/s$$

Part b)

Once the ion enters a vacuum chamber with a uniform magnetic field $B = 0.1T$, it follows a circular path due to the Lorentz force, which equals the centripetal force.

$$qvB = \frac{mv^2}{r}$$

Solving for the radius

$$r = \frac{mv}{qB}$$

We have $v = 148393.5606 m/s$, $q = 1.602x10^{-19}C$, correspond to the electron charge, $m = 1.454998x10^{-25}kg$, mass of $^{87}S_r$ ion, and $B = 0.1T$.

$$r = \frac{(1.454998x10^{-25}kg)(148393.5606 m/s)}{(1.602x10^{-19}C)(0.1T)}$$

$$r = 1.347767378m$$

$$r \approx 1.35m$$

Part 3: inclination and latitude

Part a)

The equation for the inclination I of the magnetic field in an axial geocentric dipole is given by:

$$\tan(I) = 2\tan(\lambda)$$

where λ is the latitude

$$\tan(\lambda) = \frac{\tan(I)}{2}$$

$$\lambda = \tan^{-1}\left(\frac{\tan(I)}{2}\right)$$

For this case $I = 45°$

$$\lambda = \tan^{-1}\left(\frac{\tan(45)}{2}\right)$$

$$\lambda = 26.565°$$

The latitude where the field inclination is $45°$ is $26.565°$

Part b)

Using the same equation, but for the geocentric axial dipole field at latitude $45°$.

$$\tan(I) = 2\tan(\lambda)$$

$$I = \tan^{-1}(2\tan(\lambda))$$

$$I = \tan^{-1}(2\tan(45°))$$

$$I = 63.43494°$$

The inclination where the latitude is $45°$ is $63.43494°$

Part 4

Part a)

We have measurements of the three cartesian components of the Erath's magnetic field at a geomagnetic observatory.

$$Nort\ component: X = +27000nT$$

$$East\ component: Y = -1800nT$$

$$Vertical\ component: Z = -40000nT$$

Because Z is negative, the field in pointing upward rather than downward. In the northern earth's hemisphere, the vertical component is downward ($Z > 0$). In the southern hemisphere Z is typically negative. For that reason, the observatory is in the southern hemisphere.

Part b)

The total field intensity is given by:

$$F = \sqrt{X^2 + Y^2 + Z^2}$$

$$F = \sqrt{(27000nT)^2 + (-1800nT)^2 + (-40000nT)^2}$$

$$F = 48293.27nT$$

Part c)

For this case, the inclination is given by:

$$I = \tan^{-1}\left(\frac{Z}{\sqrt{X^2 + Y^2}}\right)$$

Substituting

$$I = \tan^{-1}\left(\frac{-40000nT}{\sqrt{(27000nT)^2 + (-1800nT)^2}}\right)$$

$$I = -55.9217°$$

The negative indicates the field points 55.9217° upward from the horizontal plane, typically of the southern hemisphere.

For the declination, we must use:

$$D = \tan^{1}\left(\frac{Y}{X}\right)$$

Substituting

$$D = \tan^{1}\left(\frac{-1800nT}{27000nT}\right)$$

$$D = -3.814°$$

The declination means the horizontal field vector is about 3.814° west of true north

Part 5

Part a) In Jupiter Notebooks

Part b) In Jupiter Notebooks

Part c) In Jupiter Notebooks

Part d) In Jupiter Notebooks

Part e) Calculation in In Jupiter Notebooks

We have the scalar potential in spheric coordinates:

$$V_{(r,\theta,\phi)} = R_E \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_E}{r}\right)^{n+1} (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\cos\theta)$$

Where R_E is the Earth's radius, r is the radial distance, θ is the colatitude, $P_n^m(\cos\theta)$ is the Legendre Polynomials, g_n^m and h_n^m are the gauss's coefficients. The magnetic field is given by:

$$B = -\nabla \cdot V$$

Equation 5.67 from Griffiths, (2017).

In spherical coordinates

$$B_r = -\frac{\partial V}{\partial r}$$

$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$B_\phi = -\frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}$$

The derivate of V respect to r

$$B_r = -\frac{\partial}{\partial r} R_E \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^{n+1}(g_n^m \cos m\phi + h_n^m \sin m\phi)P_n^m(\cos\theta)$$

Since

$$\frac{\partial}{\partial r}\left(\frac{R_E}{r}\right)^{n+1} = (n+1)\left(\frac{R_E}{r}\right)^{n+2}$$

Then

$$B_r = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(n+1)\left(\frac{R_E}{r}\right)^{n+2}(g_n^m \cos m\phi + h_n^m \sin m\phi)P_n^m(\cos\theta)$$

For $n = 1$

$$P_1^0(\cos\theta) = \cos\theta$$

$$P_1^1(\cos\theta) = \sin\theta$$

Now, we have

$$B_r = -2\left(g_{1,0}P_1^0(\cos\theta) + g_{1,1}\cos m\phi\, P_1^1(\cos\theta) + h_{1,1}\sin m\phi\, P_1^1(\cos\theta)\right)$$

Also, we have $m = 1$

$$B_r = -2\big(g_{1,0}\cos\theta + g_{1,1}\cos\phi\sin\theta + h_{1,1}\sin\phi\sin\theta\big)$$

Calculating B_θ

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta} R_E \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^{n+1}(g_n^m \cos m\phi + h_n^m \sin m\phi)P_n^m(\cos\theta)$$

We have

$$\frac{\partial}{\partial\theta} P_n^m(\cos\theta)$$

$$\frac{\partial}{\partial\theta} P_1^0(\cos\theta) = \frac{\partial}{\partial\theta}\cos\theta = -\sin\theta$$

$$\frac{\partial}{\partial\theta} P_1^1(\cos\theta) = \frac{\partial}{\partial\theta}\sin\theta = \cos\theta$$

Substituting and remember, $n = 1$ and $m = 1$

$$B_\theta = -\frac{1}{r}\left(\frac{R_E}{r}\right)^{1+1} \Big(g_{1,0}(-\sin\theta) + g_{1,1}\cos\phi\,(\cos\theta) + h_{1,1}\sin\phi\,(\cos\theta)\Big)$$

$$B_\theta = -\Big(g_{1,0}(-\sin\theta) + g_{1,1}\cos\phi\,(\cos\theta) + h_{1,1}\sin\phi\,(\cos\theta)\Big)$$

$$B_\theta = g_{1,0}(\sin\theta) - g_{1,1}\cos\phi\,(\cos\theta) - h_{1,1}\sin\phi\,(\cos\theta)$$

For B_ϕ

$$B_\phi = -\frac{1}{rsin\theta}\frac{\partial}{\partial\phi} R_E \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^{n+1} (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\cos\theta)$$

Since

$$\frac{\partial}{\partial\phi}\cos m\phi = -m\sin m\phi$$

$$\frac{\partial}{\partial\phi}\sin m\phi = m\cos m\phi$$

Now, we have $m = 1$

$$\frac{\partial}{\partial\phi} g_{1,1}\cos\phi\sin\theta + h_{1,1}\sin\phi\sin\theta = -g_{1,1}\sin\phi + h_{1,1}\cos\phi$$

Substituting

$$B_\phi = -\frac{1}{rsin\theta}\left(\frac{R_E}{r}\right)^2 \big(-g_{1,1}\sin\phi + h_{1,1}\cos\phi\big) P_1^1(\cos\theta)$$

$$B_\phi = -\frac{1}{sin\theta}\big(-g_{1,1}\sin\phi + h_{1,1}\cos\phi\big) sin\theta$$

$$B_\phi = -\big(-g_{1,1}\sin\phi + h_{1,1}\cos\phi\big)$$

$$B_\phi = g_{1,1}\sin\phi - h_{1,1}\cos\phi$$

Jupiter Notebooks: https://github.com/ESDelgado/ceri8211_S2025/tree/main/Hw4

References

Arfken, G. B., Weber, H. J., & Harris, F. E. (2013). Mathematical Methods for Physicists: A Comprehensive Guide (7th ed.). Academic Press.

Griffiths, D. J. (2017). Introduction to Electrodynamics (4th ed.). Cambridge University Press.

International Geomagnetic Reference Field: the 13th generation, Alken, P., Thébault, E., Beggan, C.D. et al. International Geomagnetic Reference Field: the thirteenth generation. Earth Planets Space 73, 49 (2021). doi: 10.1186/s40623-020-01288-x

Lowrie, W. (2007). Fundamentals of geophysics (2nd ed.). Cambridge University Press.