AOSS 605 - Project 3

1 Assignment Questions

1. Show that the forward-in-time central-in-space (FTCS) scheme
$$\psi_j^{n+1} = \psi_j^n - \frac{\nu}{2}(\psi_{j+1}^n - \psi_{j-1}^n), \tag{1}$$
is first-order in time and second-order in space, where $\nu = c\Delta t/\Delta x$ is the CFL number. **Hint:** Rewrite as LHS $= 0$ then use a Taylor series expansion.

2. Using von Neumann analysis, show that the FTCS scheme is unstable for all values of the CFL number.

3. Recall the 1D Shallow-water equations
$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{2}$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \tfrac{1}{2}gh^2\right) = 0. \tag{3}$$
Show that the total energy, defined by
$$E = h\left[\frac{1}{2}u^2 + \frac{1}{2}gh\right], \tag{4}$$
satisfies a conservation law. Conclude that total energy is conserved in a closed system. **Hint:** Download `http://kiwi.atmos.colostate.edu/group/dave/pdf/ShallowWater.pdf` and fill in the gaps of the result in section 5 with $h_S = 0$.

2 Comparison of Advection Schemes

Your goal in this project is to eventually author your own advection scheme. You are free to choose which language you use for your advection scheme, but we have provided some sample code to get you started if you are using MATLAB. For simplicity we will be performing our calculations on an evenly spaced grid of N elements over $[0,1]$. To ensure conservation, we will be working with the flux-form version of the equations. Hence, we define
$$x_j = \left(j - \frac{1}{2}\right)\Delta x, \tag{5}$$
where
$$\Delta x = \frac{1}{N}. \tag{6}$$
Hence, the leftmost edge ($x = 0$) will be at index $j = 1/2$ and the rightmost edge ($x = 1$) will be at $j = N + 1/2$.

Recall that a basic linear reconstruction for ψ in cell j can be written as
$$\psi_j(x) = \overline{\psi}_j + \left(\frac{\delta\psi}{\delta x}\right)_j (x - x_j). \tag{7}$$
We can also define the limited slope via
$$\left(\frac{\delta\psi}{\delta x}\right)_j = \phi(\theta)\frac{(\psi_{j+1} - \psi_j)}{\Delta x}, \tag{8}$$
where ϕ is a slope limiter and θ is the ratio of consecutive slopes
$$\theta = \frac{\psi_j - \psi_{j-1}}{\psi_{j+1} - \psi_j}. \tag{9}$$

2.1 Project A: Second-Order Semi-Lagrangian Schemes

In a semi-Lagrangian context the flux of ψ_j through edge $j+1/2$ can be written as the integral over all material that will flow through edge $j+1/2$ in time Δt. Mathematically, we have

$$F^*_{j+1/2} = \int_{x_j+\Delta x/2-c\Delta t}^{x_j+\Delta x/2} \psi_j dx, \tag{10}$$

or, using (7),

$$F^*_{j+1/2} = \nu \Delta x \left[\overline{\psi}_j + \left(\frac{\delta\psi}{\delta x}\right)_j (\Delta x - c\Delta t) \right]. \tag{11}$$

2.1.1 Project A1: Unlimited Schemes

A variety of unlimited schemes can be defined using the semi-Lagrangian framework.

(a) Lax-Wendroff Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j+1} - \overline{\psi}_j}{\Delta x}. \tag{12}$$

(b) Beam-Warming Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_j - \overline{\psi}_{j-1}}{\Delta x}. \tag{13}$$

(c) Fromm Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j+1} - \overline{\psi}_{j-1}}{2\Delta x}. \tag{14}$$

(d) Wide-stencil centered scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j-2} - 8\overline{\psi}_{j-1} + 8\overline{\psi}_{j+1} - \overline{\psi}_{j+2}}{12\Delta x}. \tag{15}$$

2.1.2 Project A2: With Slope Limiters

Alternatively, we can author schemes which guarantee monotonicity by applying slope limiters to the reconstruction.

(a) MINMOD limiter

$$\phi_{mm}(\theta) = \max[0, \min(1, \theta)], \tag{16}$$

(b) Superbee limiter

$$\phi_{sb}(\theta) = \max[0, \min(2\theta, 1), \min(\theta, 2)], \tag{17}$$

(c) van Leer Limiter

$$\phi_{vl}(\theta) = (\theta + |\theta|)/(1 + |\theta|). \tag{18}$$

(d) Other limiters

Consult `http://en.wikipedia.org/wiki/Flux_limiter`.

2.2 Project B: Slope-Limited Finite-Volume Schemes

A family of finite-volume methods can be defined similarly to the semi-Lagrangian schemes of the previous section. Again we make use of a slope limiter, but instead of integrating backwards to obtain the total mass that crosses each edge, we simply assume use the reconstructed pointwise value at each edge to compute the flux across each edge. The upwind value is always used, for stability reasons.

For this scheme we use the upwind value obtained from a linear reconstruction,

$$\psi_{j+1/2} = \psi_j(\Delta x/2) = \overline{\psi}_j + \left(\frac{\delta\psi}{\delta x}\right)_j \left(\frac{\Delta x}{2}\right). \tag{19}$$

The flux at each edge is simply given by

$$F^*_{j+1/2} = c\psi_{j+1/2}. \tag{20}$$

To maintain stability, a second-order Runge-Kutta timestep must be used:

$$\mathbf{q}^{(1)} = \mathbf{q}^n - \left(\frac{\Delta t}{2}\right)\left(\frac{F^*_{j+1/2}(\psi^n) - F^*_{j-1/2}(\psi^n)}{\Delta x}\right), \tag{21}$$

$$\mathbf{q}^{n+1} = \mathbf{q}^n - \Delta t\left(\frac{F^*_{j+1/2}(\psi^{(1)}) - F^*_{j-1/2}(\psi^{(1)})}{\Delta x}\right). \tag{22}$$

2.2.1 Project B1: Unlimited Schemes

A variety of unlimited schemes can be defined using the finite-volume framework.

(a) Upwind scheme (first-order-accurate)

$$\left(\frac{\delta\psi}{\delta x}\right)_j = 0. \tag{23}$$

(b) Lax-Wendroff Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j+1} - \overline{\psi}_j}{\Delta x}. \tag{24}$$

(c) Beam-Warming Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_j - \overline{\psi}_{j-1}}{\Delta x}. \tag{25}$$

(d) Fromm Scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j+1} - \overline{\psi}_{j-1}}{2\Delta x}. \tag{26}$$

(e) Wide-stencil centered scheme

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \frac{\overline{\psi}_{j-2} - 8\overline{\psi}_{j-1} + 8\overline{\psi}_{j+1} - \overline{\psi}_{j+2}}{12\Delta x}. \tag{27}$$

2.2.2 Project B2: With Slope Limiters

Alternatively, we can author schemes which guarantee monotonicity by applying slope limiters to the reconstruction.

$$\left(\frac{\delta\psi}{\delta x}\right)_j = \phi(\theta)\frac{(\psi_{j+1} - \psi_j)}{\Delta x}. \tag{28}$$

(a) MINMOD limiter

$$\phi_{mm}(\theta) = \max[0, \min(1, \theta)], \tag{29}$$

(b) Superbee limiter

$$\phi_{sb}(\theta) = \max[0, \min(2\theta, 1), \min(\theta, 2)], \tag{30}$$

(c) van Leer Limiter

$$\phi_{vl}(\theta) = (\theta + |\theta|)/(1 + |\theta|). \tag{31}$$

(d) Other limiters
Consult `http://en.wikipedia.org/wiki/Flux_limiter`.

2.3 Project C: The Piecewise-Parabolic Method

Edge values are defined at the left and right edges of cell j by $\psi_{L,j}$ (left side) and $\psi_{R,j}$ (right side). These are first approximated by

$$\psi_{R,j} = \psi_{L,j+1} = -\frac{1}{12}\left(\overline{\psi}_{j-1} + \overline{\psi}_{j+2}\right) + \frac{7}{12}\left(\overline{\psi}_j + \overline{\psi}_{j+1}\right). \tag{32}$$

The values are then scaled so as not to exceed the range $[\overline{\psi}_j, \overline{\psi}_{j+1}]$. We say

$$\psi_{R,j} = \min\left[\psi_{R,j}, \max\left(\overline{\psi}_{j+1}, \overline{\psi}_j\right)\right], \tag{33}$$
$$\psi_{R,j} = \max\left[\psi_{R,j}, \min\left(\overline{\psi}_{j+1}, \overline{\psi}_j\right)\right], \tag{34}$$

and similarly for left edge values $\psi_{L,j}$,

$$\psi_{L,j} = \min\left[\psi_{L,j}, \max\left(\overline{\psi}_{j-1}, \overline{\psi}_j\right)\right], \tag{35}$$
$$\psi_{L,j} = \max\left[\psi_{L,j}, \min\left(\overline{\psi}_{j-1}, \overline{\psi}_j\right)\right], \tag{36}$$

Note that at this point $\psi_{R,j}$ and $\psi_{L,j+1}$ should still be identical. Next the parabolic profile is modified so as not to have maxima within the element. If $(a_{j,R} - \overline{\psi}_j^n)(a_{j,L} - \overline{\psi}_j^n) > 0$, then we set

$$\psi_{j,R} = \psi_{j,L} = \overline{\psi}_j. \tag{37}$$

If $|\psi_{j,R} - \overline{\psi}_j| \geq 2|\psi_{j,L} - \overline{\psi}_j|$ then set

$$\psi_{j,R} = \overline{\psi}_j - 2(\psi_{j,L} - \overline{\psi}_j). \tag{38}$$

Similarly, if $|\psi_{j,L} - \overline{\psi}_j| \geq 2|\psi_{j,R} - \overline{\psi}_j|$ then set

$$\psi_{j,L} = \overline{\psi}_j - 2(\psi_{j,R} - \overline{\psi}_j). \tag{39}$$

Given left and right edge values, we define

$$\Delta\psi_j = \psi_{R,j} - \psi_{L,j}, \qquad \psi_{6,j} = 6\left(\overline{\psi}_j - \tfrac{1}{2}\left(\psi_{L,j} + \psi_{R,j}\right)\right). \tag{40}$$

The interpolation profile is then defined as

$$\psi(x) = \psi_{L,j} + \frac{(x - x_{j-1/2})}{\Delta x}\left(\Delta\psi + \psi_{6,j}\left(1 - \frac{(x - x_{j-1/2})}{\Delta x}\right)\right). \tag{41}$$

Integrating this profile backwards then gives an expression for the flux,

$$F^*_{j+1/2} = \nu\Delta x\left[\psi_{R,j} - \frac{\nu}{2}\left(\Delta\psi_j - \left(1 - \frac{2}{3}\nu\right)\psi_{6,j}\right)\right]. \tag{42}$$

For additional information, see Collela and Woodward (1984)

(a) Unlimited Scheme. First develop the advection scheme by using (32) directly with (42).

(b) Monotone / Limited Scheme. Develop the advection scheme with the limiting procedure described above.

3 Numerical Results

Run your advection schemes using the following initial data:

Sine wave

$$\psi(x) = \frac{1}{2}\left[1 + \sin(10\pi x)\right] \tag{43}$$

Gaussian hill

$$\psi(x) = \exp\left(-\frac{(x - 0.5)^2}{0.01}\right) \tag{44}$$

Window function

$$\psi(x) = \begin{cases} 1, & \text{if } |x - 0.5| \leq 0.1, \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

For simplicity, choose $N = 100$, $c = 1$ and $\nu = 0.9$. Plot the results after 1, 5 and 10 rotations. Comment briefly on each of your results.

For the Gaussian hill test case, calculate error norms

$$L_1 = \frac{1}{N} \sum_j |\overline{\psi}_j^n - \overline{\psi}_j^0|, \tag{46}$$

$$L_2 = \sqrt{\frac{1}{N} \sum_j \left(\overline{\psi}_j^n - \overline{\psi}_j^0\right)^2}, \tag{47}$$

$$L_\infty = \max_j \left|\overline{\psi}_j^n - \overline{\psi}_j^0\right| \tag{48}$$

Run this test case at $N = 50, 100, 200, 400$ and comment on the results. What order-of-accuracy / convergence rate do you observe?