MATH3405 Course Notes

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This document was written based on Ramiro's 2022 handwritten notes. The main reference books used (mainly for the handwritten notes, although I also used them sometimes) were *Differential Geometry of Curves and Surfaces* by Do Carmo, *Riemannian Geometry* by Do Carmo and *Introduction to Smooth Manifolds* by Lee.

This is stated within the body of this text as well, but it is so important and disconcerting I felt it was necessary to state here too: in this course, differentiable means "smooth", i.e. infinitely differentiable. When differentiable *actually* means differentiable, I'll add (as Ramiro does) "in the usual sense" in parentheses.

Another thing to note when comparing results/proofs in here to those found in other sources is (it appears to me that) there are no accepted conventions for many definitions in differential geometry, so, for example, some of our curvatures are the negatives of the curvatures in other sources. This may lead to some confusion when trying to compare proofs.

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Part I
Curves in $\mathbb{R}^3$

1 Regular parameterised Curves

1.1 Definitions

Definition 1.1.1. Let $I \subseteq \mathbb{R}$ be an interval. A **parameterised curve** in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$. We say it is **regular** if $\alpha'(t) \neq 0$ for all $t \in I$.

Remark. Some things to note:

1. Differentiable will mean "of class $\mathcal{C}^\infty$" (i.e. what we would usually call smooth) in this course.
2. If $I = [a, b]$, differentiable means $\exists \tilde{I} \supset I$ open such that $\alpha = \tilde{\alpha}|_I$, $\tilde{\alpha} : \tilde{I} \to \mathbb{R}^3$ differentiable.
3. The image $\alpha(I)$ of a curve is called the **trace** of α. If the entire trace of α is contained in a single plane, we call α a **plane curve**.
4. $\alpha'(t) = \big(x'(t), y'(t), z'(t)\big)$ is called the velocity at time t, and $\alpha''(t)$ the acceleration. Thus, for any given $t \in I$, $\alpha(t)$ is a point in $\mathbb{R}^3$ while $\alpha'(t)$ is a vector, which we "think of as starting at $\alpha(t)$".
5. α need not be injective.

Example 1.1.2. Some examples of parameterised curves:

1. A straight line, $\alpha(t) = vt + p$, $v, p \in \mathbb{R}^3$. For instance, $\alpha(t) = (2t + 1, 3t + 2, 0)$.
2. A circle or helix, $\alpha(t) = (a \cos t, a \sin t, bt)$, $t \in \mathbb{R}$, $a \in \mathbb{R}_{>0}$. If $b = 0$, we get a circle; if $b \neq 0$, we get a helix. If we imagine the helix inscribed onto the exterior of a cylinder and that we are looking at some "side" of the cylinder, the helix will appear to be a series of upward sloping lines evenly spaced along the side of the cylinder (the rest of the helix is blocked from view by the cylinder). The spacing of these lines is $2\pi \cdot |b|$.
3. A cusp, $\alpha(t) = \big(t^2, t^3\big)$, $t \in \mathbb{R}$. Note that $\alpha'(0) = (0, 0)$, so this is not a regular curve.
4. An ill-parameterised line, $\alpha(t) = \big(t^3, t^3\big)$, $t \in \mathbb{R}$. This also isn't regular, since $\alpha'(0) = 0$. Note that it has the same trace as $\tilde{\alpha}(t) = (t, t)$.

Definition 1.1.3. Let $\alpha : I \to \mathbb{R}^3$ be a parameterised curve. A **change of parameters** for α is a differentiable bijection $\phi : \tilde{I} \to I$ with $\phi'(u) \neq 0$ for all $u \in \tilde{I}$ ($\tilde{I} \subseteq \mathbb{R}$ is an interval). Then $\beta(u) := \alpha\big(\phi(u)\big)$ is a parameterised curve with the same trace as α; we call it a **re-parametrisation**. We say ϕ is **orientation-preserving** if $\phi'(u) > 0$ for all $u \in \tilde{I}$; otherwise, ϕ is orientation-reversing.

Remark. α is regular if and only if β is regular.

Definition 1.1.4. Let $\alpha : I \to \mathbb{R}^3$ be a parameterised curve. The **length** of α between a and b, $a, b \in I$ with $a < b$, is

$$\mathcal{L}\big(\alpha|_{[a,b]}\big) := \int_a^b \big\|\alpha'(t)\big\| \,\mathrm{d}t.$$

We say α is parameterised by **arc-length** (or "unit-speed") if $\big\|\alpha'(t)\big\| = 1$. In that case, we call the parameter s.

Theorem 1.1.5. *Any regular curve $\alpha : I \to \mathbb{R}^3$ can be re-parameterised by arc-length.*

Proof. We seek $\phi : \tilde{I} \to I$ such that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}s}\Big(\alpha\big(\phi(s)\big)\Big) \right\| = 1, \quad \forall s.$$

Let $I = (a, b)$, $t_0 \in I$ and set $s : I \to \mathbb{R}$ defined by

$$s(t) := \int_{t_0}^{t} \big\|\alpha'(x)\big\| \,\mathrm{d}x.$$

By the Fundamental Theorem of Calculus, s is differentiable with

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} \overset{*}{=} \big\|\alpha'(t)\big\| > 0, \quad \forall t \in I$$

(greater than 0 since α is regular). Set $\tilde{I} := s(I)$. Then $s : I \to \tilde{I}$ is an increasing, surjective function. Then since increasing implies injective, s is bijective, so there exists an inverse $\phi : \tilde{I} \to I$. By the Inverse Function Theorem, ϕ is differentiable and $\phi'(s) = \Big(s'\big(\phi(s)\big)\Big)^{-1}$. Then $\beta(s) := \alpha\big(\phi(s)\big)$ is a re-parametrisation and

$$\frac{\mathrm{d}\beta(s)}{\mathrm{d}s} = \alpha'\big(\phi(s)\big) \cdot \frac{\mathrm{d}\phi}{\mathrm{d}s} = \alpha'\big(\phi(s)\big) \cdot \frac{1}{s'\big(\phi(s)\big)} \overset{*}{=} \frac{\alpha'\big(\phi(s)\big)}{\Big\|\alpha'\big(\phi(s)\big)\Big\|} \implies \left\|\frac{\mathrm{d}\beta(s)}{\mathrm{d}s}\right\| = 1, \quad \forall s.$$

□

Remark. We can pick t_0 to be anything in I because the function $s : I \to \mathbb{R}$ is not *actually* arc-length. The only significant property it has that we care about is that its derivative is $\big\|\alpha'(t)\big\|$ for all $t \in I$. Consequently, the fact that s is negative for some values of t does not matter here and so we do not have to integrate from a, as in Definition 1.1.4.

Example 1.1.6. In practice, finding β is difficult. Here we present an example for which it is possible. Define

$$\alpha : (0, 2\pi) \to \mathbb{R}^2, \quad t \mapsto (r\cos t, r\sin t), \quad r > 0.$$

Then

$$\alpha'(t) = (-r\sin t, r\cos t) \implies \big\|\alpha'(t)\big\| = r,$$

so

$$s(t) = \int_0^t r \,\mathrm{d}x = rt \implies \phi(s) = \frac{s}{r}.$$

Therefore,

$$\beta(s) = \big(r\cos(s/r), r\sin(s/r)\big)$$

is the arc-length re-parametrisation.

1.2 The Frenet Frame

Definition 1.2.1. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, arc-length parameterised (i.e. $\big\|\alpha'(s)\big\| = 1$ for all $s \in I$). The function $\kappa : I \to \mathbb{R}$, $\kappa(s) := \big\|\alpha''(s)\big\|$, is called the **curvature** of α.

Remark. One can think of κ as measuring how fast α pulls away from the tangent at $\alpha(s)$ for any given $s \in I$. We can parametrise the tangent line at $\alpha(s)$ as $\ell(u) = \alpha(s) + u \cdot \alpha'(s)$. Then using Taylor expansion and Big-O notation, near s we can write

$$\alpha(s+t) = \alpha(s) + t \cdot \alpha'(s) + \frac{t^2}{2} \cdot \alpha''(s) + \mathcal{O}\big(t^2\big), \quad \lim_{t \to 0} \frac{\mathcal{O}\big(t^2\big)}{t^2} = 0,$$

where we know that $\kappa(s) = \big\|\alpha''(s)\big\|$.

Example 1.2.2. Recall β from Example 1.1.6:

$$\beta(s) := \big(r\cos(s/r), r\sin(s/r)\big), \quad s \in (0, 2\pi r].$$

Then

$$\beta'(s) = \big(-\sin(s/r), \cos(s/r)\big) \implies \big\|\beta'(s)\big\| = \sqrt{\sin^2(s/r) + \cos^2(s/r)} = 1,$$

so β is indeed parameterised by arc-length. Now

$$\beta''(s) = \left(-\frac{\cos(s/r)}{r}, -\frac{\sin(s/r)}{r}\right) \implies \kappa(s) = \big\|\beta''(s)\big\| = \frac{1}{r}\sqrt{\cos^2(s/r) + \sin^2(s/r)} = \frac{1}{r}.$$

Definition 1.2.3. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parameterised by arc-length and with $\kappa(s) \neq 0$ for all $s \in I$. Then

1. $t(s) := \alpha'(s)$ is the **tangent** vector of α at $\alpha(s)$.
2. $n(s) := \alpha''(s)/\kappa(s)$ is the (unit) **normal** vector of α at $\alpha(s)$.
3. $b(s) := t(s) \times n(s)$ is the **binormal** vector of α at $\alpha(s)$.

The mapping $s \mapsto \big\{t(s), n(s), b(s)\big\}$ is called the **Frenet frame** of α.

Theorem 1.2.4. *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parameterised by arc-length and with $\kappa(s) \neq 0$ for all $s \in I$. Then $\forall s \in I$, $\big\{t(s), n(s), b(s)\big\}$ is an orthonormal basis of $\mathbb{R}^3$.*

Proof. α is parameterised by arc-length (i.e. is unit speed), so $\big\|t(s)\big\| = 1$ for all $s \in I$. By definition of $\kappa(s)$,

$$\big\|n(s)\big\| = \left\|\frac{\alpha''(s)}{\kappa(s)}\right\| = \frac{1}{\big\|\kappa(s)\big\|} \cdot \underbrace{\big\|\alpha''(s)\big\|}_{=\kappa(s)} = 1, \quad \forall s \in I.$$

We now claim that $t(s) \perp n(s)$ for all $s \in I$. Indeed, using that $1 = \big\|t(s)\big\|^2$ for all $s \in I$, we differentiate with respect to s:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}s}\big\langle t(s), t(s)\big\rangle = \big\langle t'(s), t(s)\big\rangle + \big\langle t(s), t'(s)\big\rangle \\ &= 2\big\langle \alpha''(s), \alpha'(s)\big\rangle = 2\kappa(s)\left\langle \frac{\alpha''(s)}{\kappa(s)}, \alpha'(s)\right\rangle = 2\kappa(s)\big\langle n(s), t(s)\big\rangle. \end{aligned}$$

Since $\kappa(s) \neq 0$ for all $s \in I$, it follows that $\big\langle n(s), t(s)\big\rangle = 0$ for all $s \in I$. Finally, $b(s) = t(s) \times n(s)$ is orthogonal to both $t(s)$ and $n(s)$ by definition. Moreover,

$$\big\|b(s)\big\| = \big\|t(s) \times n(s)\big\| = \big\|t(s)\big\| \cdot \big\|n(s)\big\| \cdot \Big|\sin\big(\operatorname{ang}(t, n)\big)\Big| = 1 \cdot 1 \cdot \Big|\sin\big(\pm\pi/2\big)\Big| = 1,$$

where $\operatorname{ang}(t, n)$ denotes the angle between t and n. $\square$

Definition 1.2.5. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parameterised by arc-length and with $\kappa(s) \neq 0$ for all $s \in I$. Then

1. The plane $\operatorname{span}\big\{t(s), n(s)\big\}$ is called the **osculating plane** of $\alpha(s)$.
2. The plane $\operatorname{span}\big\{n(s), b(s)\big\}$ is called the **normal plane** of $\alpha(s)$.
3. The plane $\operatorname{span}\big\{t(s), b(s)\big\}$ is called the **rectifying plane** of $\alpha(s)$.

Lemma 1.2.6. *Let $\alpha : I \to \mathbb{R}^3$ be a regular, unit-speed curve with $\kappa(s) \neq 0$ for all $s \in I$. Then $b'(s)$ is a multiple of $n(s)$.*

Proof. We need to show that $b'(s) \perp t(s)$ and $b'(s) \perp b(s)$. Differentiating $\|b(s)\|^2 \equiv 1$ yields

$$0 = 2\big\langle b'(s), b(s)\big\rangle \implies b'(s) \perp b(s).$$

Differentiating $0 \equiv \big\langle b(s), t(s)\big\rangle$ gives us

$$0 = \big\langle b'(s), t(s)\big\rangle + \big\langle b(s), t'(s)\big\rangle = \big\langle b'(s), t(s)\big\rangle + \big\langle b'(s), \kappa(s)n(s)\big\rangle \implies \big\langle b'(s), t(s)\big\rangle = 0.$$

Now since $\big\{t(s), n(s), b(s)\big\}$ is an orthonormal basis for $\mathbb{R}^3$, we can write $b'(s) = x_1t(s) + x_2n(s) + x_3b(s)$, $x_1, x_2, x_3 \in \mathbb{R}$. But from above, $0 = \big\langle b'(s), t(s)\big\rangle = x_1$ and $0 = \big\langle b'(s), b(s)\big\rangle = x_3$, so $b'(s) = x_2n(s)$. □

Definition 1.2.7. Let $\alpha : I \to \mathbb{R}^3$ be a regular, unit-speed curve with $\kappa(s) \neq 0$ for all $s \in I$. For each $s \in I$, we define the **torsion** of α at $\alpha(s)$ to be the unique scalar $\tau(s)$ such that $b'(s) = \tau(s)n(s)$. In other words, $\tau(s) := \big\langle b'(s), n(s)\big\rangle$.

Remark. Observe that $b(s)$ determines the osculating plane. Thus, the torsion measures how much the curve deviates itself from its osculating plane.

Theorem 1.2.8 (Frenet Formulas)**.** *Let $\alpha : I \to \mathbb{R}^3$ be a regular, unit-speed curve with $\kappa(s) \neq 0$ for all $s \in I$. Then*

$$\begin{aligned} t'(s) &= \kappa(s)n(s) \\ n'(s) &= -\kappa(s)t(s) - \tau(s)b(s) \\ b'(s) &= \tau(s)n(s), \end{aligned}$$

which can be written more concisely as

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}.$$

Proof. The first and third equations are true by definition. For the second, we again use that $\big\{t(s), n(s), b(s)\big\}$ is an orthonormal basis for $\mathbb{R}^3$ to write

$$n'(s) = \big\langle n'(s), t(s)\big\rangle t(s) + \big\langle n'(s), n(s)\big\rangle n(s) + \big\langle n'(s), b(s)\big\rangle b(s).$$

Now, $1 \equiv \|n(s)\|^2$ means that by differentiating,

$$0 = 2\big\langle n'(s), n(s)\big\rangle.$$

Similarly, we know that $0 \equiv \big\langle n(s), t(s)\big\rangle$, so differentiating,

$$0 = \big\langle n'(s), t(s)\big\rangle + \big\langle n(s), t'(s)\big\rangle = \big\langle n'(s), t(s)\big\rangle + \kappa(s)\underbrace{\big\langle n(s), n(s)\big\rangle}_{=1} \implies \big\langle n'(s), t(s)\big\rangle = -\kappa(s).$$

Finally, differentiating $0 \equiv \big\langle n(s), b(s)\big\rangle$, we have

$$0 = \big\langle n'(s), b(s)\big\rangle + \big\langle n(s), b'(s)\big\rangle = \big\langle n'(s), t(s)\big\rangle + \tau(s)\underbrace{\big\langle n(s), n(s)\big\rangle}_{=1} \implies \big\langle n'(s), b(s)\big\rangle = -\tau(s).$$

□

Definition 1.2.9. A regular, unit-speed curve $\alpha : I \to \mathbb{R}^3$ with $\kappa(s) \neq 0$ for all $s \in I$ is called a **helix** if $\exists u \in \mathbb{R}^3$ a unit vector such that $\big\langle \alpha'(s), u\big\rangle \equiv C$ a constant for all $s \in I$.

Theorem 1.2.10. *Let $\alpha : I \to \mathbb{R}^3$ be a regular, unit-speed curve with $\kappa(s) \neq 0$ for all $s \in I$. Then α is a helix if and only if $\tau(s)/\kappa(s) \equiv \tilde{C}$ a constant.*

Proof. ($\Rightarrow$). Let θ denote the angle between $\alpha'(s)$ and u. Then

$$C = \langle \alpha'(s), u \rangle = \underbrace{\|\alpha'(s)\|}_{=1} \cdot \underbrace{\|u\|}_{=1} \cdot \cos\theta = \cos\theta$$

implies that θ is constant, and so that $\langle t(s), u \rangle \equiv \cos\theta$. Differentiating, we get

$$0 = \langle t'(s), u \rangle + \underbrace{\langle t(s), u' \rangle}_{=0} = \kappa(s)\langle n(s), u \rangle \implies \langle n(s), u \rangle = 0, \quad \forall s \in I,$$

since $\kappa(s) \neq 0$ for all $s \in I$. Differentiating again, we get

$$0 = \langle n'(s), u \rangle = \langle -\kappa(s)t(s) - \tau(s)b(s), u \rangle = -\kappa(s)\langle t(s), u \rangle - \tau(s)\langle b(s), u \rangle = -\kappa(s)\cos\theta - \tau(s)\langle b(s), u \rangle,$$

from which it follows that

$$\frac{\tau(s)}{\kappa(s)} \langle b(s), u \rangle = -\cos\theta.$$

It remains to show that $\langle b(s), u \rangle$ is constant:

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle b(s), u \rangle = \langle b'(s), u \rangle = \tau(s) \underbrace{\langle n(s), u \rangle}_{=0} = 0.$$

This concludes the proof.

($\Leftarrow$). Suppose $\tau(s)/\kappa(s) \equiv \tilde{C}$; in particular, let θ be such that $\tau(s)/\kappa(s) \equiv -\cot\theta$ (such a θ exists since cot is surjective). We claim that $u = \cos(\theta)t(s) + \sin(\theta)b(s)$ is constant (it being a unit vector is clear). Indeed, from our hypothesis, we have $\cos(\theta)\kappa(s) \equiv -\sin(\theta)\tau(s)$, so using the Frenet formulas, we have, $\forall s \in I$,

$$u' = \cos(\theta)t'(s) + \sin(\theta)b'(s) = \big(\cos(\theta)\kappa(s) + \sin(\theta)\tau(s)\big)n(s) = \big(-\sin(\theta)\tau(s) + \sin(\theta)\tau(s)\big)n(s) = 0.$$

Thus, u is indeed constant. Finally,

$$\langle \alpha'(s), u \rangle = \langle t(s), \cos(\theta)t(s) + \sin(\theta)b(s) \rangle = \cos\theta, \quad \forall s \in I,$$

a constant, since $\langle t(s), b(s) \rangle \equiv 0$. α is therefore a helix. $\square$

2 Curvature of Planes

2.1 Signed Curvature

Let $\alpha : I \to \mathbb{R}^2$ be a regular, unit-speed curve (we always consider $\mathbb{R}^2 \subseteq \mathbb{R}^3$, as given by the embedding $(x, y) \mapsto (x, y, 0)$, so that everything we have defined and showed applies to curves in $\mathbb{R}^2$). Recall that $\kappa(s) := \|\alpha''(s)\| \geq 0$ for all $s \in I$. We will show that for curves in $\mathbb{R}^2$ one can associate a sign to the curvature.

Given a unit vector $v = (x, y) \in \mathbb{R}^2$, how can we complete it to an orthonormal basis of $\mathbb{R}^2$? $w_+ = (-y, x)$ and $w_- = (y, -x)$ are both valid choices. We prefer w_+, because $\{v, w_+\}$ has the same **orientation** ("positively oriented") as the canonical basis $\{e_1, e_2\}$, i.e. the change of basis matrix has det > 0. Observe that if identify $\mathbb{R}^2 \simeq \mathbb{C}$, then if $w_+ = i$, v corresponds to a rotation of $\pi/2$ counterclockwise.

Given $\alpha : I \to \mathbb{R}^3$, using the above we define a new normal $\underline{n}(s)$, so that $\{\alpha'(s), \underline{n}(s)\}$ is an orthonormal basis for $\mathbb{R}^2$, positively oriented. Then $\alpha''(s)$ is a multiple of $\underline{n}(s)$, but now the coefficient will have a sign. We define $\underline{\kappa}(s)$ via $\alpha''(s) = \underline{\kappa}(s)\underline{n}(s)$ to be the **signed curvature** of α.

Theorem 2.1.1. *Let $\alpha : I \to \mathbb{R}^2 \subseteq \mathbb{R}^3$ be a regular, unit-speed curve. Then $\big\|\underline{\kappa}(s)\big\| = \kappa(s)$ for all $s \in I$.*

Proof. $\underline{n}(s) = \pm n(s)$, so

$$\kappa(s)\underline{n}(s) = \pm\kappa(s)n(s) = \pm\alpha''(s) = \pm\underline{\kappa}(s)\underline{n}(s).$$

Taking norms, we find that $\kappa(s) = \big\|\underline{\kappa}(s)\big\|$. □

2.2 Linear Rigid Motions of Euclidean Space

Definition 2.2.1. A function $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a **rigid motion** if

$$d\big(f(p), f(q)\big) = d(p, q), \quad \forall p, q \in \mathbb{R}^3.$$

Remark. Recall that $d(p, q) := \|p - q\| := \langle p - q, p - q\rangle^{1/2}$. In other words, rigid motions are the isometries of $\mathbb{R}^3$, viewed as a metric space with respect to the metric induced by the Euclidean scalar product.

Recall that linear transformations are not matrices (but are closely related): If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then by using the canonical basis $\beta = \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$, we can define the matrix representation $A := [T]_\beta \in \mathrm{M}_n(\mathbb{R})$ of T (note that in MATH2301 notation, this would be $[T]_\beta^\beta$) by defining its matrix elements A_{ij} to be those unique values for which

$$Te_j = \sum_{i=1}^{n} A_{ij}e_i, \quad j = 1, \dots, n.$$

Conversely, given $A \in \mathrm{M}_n(\mathbb{R})$, the function

$$L_A : \mathbb{R}^n \to \mathbb{R}^n, \quad p \mapsto Ap,$$

$p \in \mathbb{R}^n$ a column vector, is a linear transformation (in MATH2301, these p's would be the coordinate representations of the row vector/n-tuple elements of $\mathbb{R}^n$; in this course, we skip this and assume that the elements of $\mathbb{R}^n$ are column vectors to begin with, or at least I assume so). Finally, observe that if L_A is a rigid motion, then

$$\|p\| = \|p - 0\| = \big\|L_A(p) - L_A(0)\big\| = \big\|L_A(p)\big\|.$$

Definition 2.2.2. A matrix $A \in \mathrm{M}_n(\mathbb{R})$ is called **orthogonal** if it preserves norms, i.e. $\|Ap\| = \|p\|$ for all $p \in \mathbb{R}^n$. The set of all such matrices forms a group under the matrix product, called the **orthogonal group** $\mathrm{O}(n)$.

Remark. We saw above that if L_A is a rigid motion, then $A \in \mathrm{O}(n)$.

Theorem 2.2.3. *Let $A \in \mathrm{M}_n(\mathbb{R})$. The following are equivalent:*

1. *$L_A : \mathbb{R}^n \to \mathbb{R}^n$, $p \mapsto Ap$, is a rigid motion.*
2. *A is orthogonal,*
3. *$\big\langle L_A(p), L_A(q)\big\rangle = \langle p, q\rangle$ for all $p, q \in \mathbb{R}^n$.*
4. *L_A sends orthonormal bases to orthonormal bases.*
5. *The columns of A form an orthonormal basis for $\mathbb{R}^n$.*
6. *$A^TA = \mathrm{Id}_{n\times n}$, where A^T denotes the transpose of A.*

Proof. (1) $\Rightarrow$ (2). Above.

(2) $\Rightarrow$ (3). Suppose A is orthogonal. Then by the polarisation identity,

$$\begin{aligned}
\big\langle L_A(p), L_A(q)\big\rangle &= \langle Ap, Aq\rangle \\
&= \frac{\|Ap+Aq\|^2 - \|Ap-Aq\|^2}{4} \\
&= \frac{\big\|A(p+q)\big\|^2 - \big\|A(p-q)\big\|^2}{4} \\
&= \frac{\|p+q\|^2 - \|p-q\|^2}{4} \\
&= \langle p, q\rangle
\end{aligned}$$

(3) $\Rightarrow$ (4). If $\{v_1, \ldots, v_n\}$ is an orthonormal basis, then

$$\big\langle v_i, v_j\big\rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Therefore, $\big\{L_A(v_1), \ldots, L_A(v_n)\big\}$ satisfies

$$\Big\langle L_A(v_i), L_A\big(v_j\big)\Big\rangle = \big\langle v_i, v_j\big\rangle = \delta_{ij},$$

so $\big\{L_A(v_1), \ldots, L_A(v_n)\big\}$ is an orthonormal basis.

(4) $\Rightarrow$ (5). The columns of A are $L_A(e_1), \ldots, L_A(e_n)$. Since $\{e_1, \ldots, e_n\}$ is an orthonormal basis, so is $\big\{L_A(e_1), \ldots, L_A(e_n)\big\}$.

(5) $\Rightarrow$ (6). We observe that

$$\Big(A^T A\Big)_{ij} = \sum_{k=1}^{n} \Big(A^T\Big)_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj} = \Big\langle (A_{1i}, \ldots, A_{ni}), \big(A_{1j}, \ldots, A_{nj}\big)\Big\rangle = \big\langle \mathrm{col}_i(A), \mathrm{col}_j(A)\big\rangle = \delta_{ij},$$

where $\mathrm{col}_i(A)$ denotes the i^{th} column of A; these are orthonormal by assumption. Thus, $A^T A = \mathrm{Id}$.

(6) $\Rightarrow$ (1). We have

$$\big\|L_A(p) - L_A(q)\big\|^2 = \langle Ap - Aq, Ap - Aq\rangle = \big\langle A(p-q), A(p-q)\big\rangle = \Big\langle A^T A(p-q), p-q\Big\rangle, \quad \forall p, q \in \mathbb{R}^n.$$

The last equality follows from the definition of transpose; see below remark. Now by assumption of (6), $A^T A = \mathrm{Id}$, so the above becomes

$$\big\|L_A(p) - L_A(q)\big\|^2 = \langle p-q, p-q\rangle = \|p-q\|^2, \quad \forall p, q \in \mathbb{R}^n.$$

Therefore, L_A is a rigid motion. □

Remark. Recall that A^T is the unique matrix such that $\big\langle A^T v, w\big\rangle = \langle v, Aw\rangle$ for all $v, w \in \mathbb{R}^n$. Indeed, by bilinearity, it is sufficient to prove it for $v, w \in \beta$ (standard basis):

$$\Big\langle A^T e_i, e_j\Big\rangle = \Big(A^T\Big)_{ji} = A_{ij} = \big\langle e_i, Ae_j\big\rangle.$$

Lemma 2.2.4. *Translations are rigid motions.*

Proof. Let $q \in \mathbb{R}^n$ be fixed. A **translation** is a map $T_q :: \mathbb{R}^n \to \mathbb{R}^n$, $p \mapsto p + q$. Now,

$$\big\|T_q(p_1) - T_q(p_2)\big\| = \big\|(p_1 + q) - (p_2 + q)\big\| = \|p_1 - p_2\|,$$

so T_q is a rigid motion. □

Lemma 2.2.5. *Rigid motions form a group under composition.*

Proof. Let f_1 and f_2 be rigid motions, and let $p, q \in \mathbb{R}^n$. Then

$$d\big((f_1 \circ f_2)(p), (f_1 \circ f_2)(q)\big) = d\Big(f_1\big(f_2(p)\big), f_1\big(f_2(p)\big)\Big) = d\big(f_2(p), f_2(q)\big) = d(p, q),$$

where the second last equality follows from f_1 being a rigid motion and the last equality follows from f_2 being a rigid motion. $\square$

Remark. From the previous two lemmas, it follows that to understand rigid motions it is enough to assume that $f(0) = 0$. Indeed, let f be a rigid motion and set $q := f(0)$. Then $T_{-q} \circ f$ is a rigid motion and

$$\big(T_{-q} \circ f\big)(0) = T_{-q}(q) = 0.$$

Theorem 2.2.6. *Any rigid motion of $\mathbb{R}^n$ is of the form $f = T_q \circ L_A$ for some $q \in \mathbb{R}^n$ and $A \in \mathrm{O}(n)$.*

Proof. By the above considerations, we may assume without loss of generality that $f(0) = 0$. Let us then show that f is linear, and then by Theorem 2.2.3 it would follow that $f = L_A$, $A \in \mathrm{O}(n)$.

1. f preserves norms. Let $p \in \mathbb{R}^n$. Then

$$\big\|f(p)\big\| = \big\|f(p) - 0\big\| = \big\|f(p) - f(0)\big\| = \|p - 0\| = \|p\|.$$

2. f preserves the scalar product. Since f is a rigid motion, we have $\forall p, q \in \mathbb{R}^n$:

$$\begin{aligned}
\big\|f(p) - f(q)\big\|^2 = \|p - q\|^2 &\implies \big\langle f(p) - f(q), f(p) - f(q)\big\rangle = \langle p - q, p - q\rangle \\
&\implies \big\|f(p)\big\|^2 + \big\|f(q)\big\|^2 - 2\big\langle f(p), f(q)\big\rangle = \|p\|^2 + \|q\|^2 - 2\langle p, q\rangle \\
&\implies \big\langle f(p), f(q)\big\rangle = \langle p, q\rangle.
\end{aligned}$$

3. f is linear. Let $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}
\big\|f(cp) - cf(p)\big\|^2 &= \big\langle f(cp) - cf(p), f(cp) - cf(p)\big\rangle \\
&= \big\|f(cp)\big\|^2 + c^2\big\|f(p)\big\|^2 - 2\big\langle f(cp), cf(p)\big\rangle \\
&= \|cp\|^2 + c^2\|p\|^2 - 2c\langle cp, p\rangle \\
&= c^2\|p\|^2 + c^2\|p\|^2 - 2c^2\|p\|^2 \\
&= 0.
\end{aligned}$$

   Therefore, $f(cp) = cf(p)$. One can similarly show that

$$\big\|f(p + q) - f(p) - f(q)\big\|^2 = 0 \implies f(p + q) = f(p) + f(q).$$

$\square$

Remark. Observe that if $a \in \mathrm{O}(n)$, then $I = A^T A$, which implies that

$$1 = \det(I) = \det\Big(A^T A\Big) = \det\big(A^t\big)\det(A) = (\det A)^2 \implies \det A = \pm 1.$$

Definition 2.2.7. Let $f = T_q \circ L_A$, $q \in \mathbb{R}^n$ and $A \in \mathrm{O}(n)$, be a rigid motion. We call f **orientation-preserving** if $\det A = 1$, and **orientation-reversing** if $\det A = -1$. If $f = L_A$, $A \in \mathrm{O}(n)$, and $\det A = 1$, we call f a **rotation**.

Example 2.2.8. Some examples of the above for $n = 1, 2, 3$.

1. $n = 1$. Notice that $\mathrm{O}(1) = \{\pm 1\}$. Thus, rigid motions of $\mathbb{R}$ are either translations, or reflections around a point. For example, let $f = T_q \circ L_{-\mathrm{Id}}$ with $q \in \mathbb{R}$. Then $f(x) = -x + q = -(x - q/2) + q/2$.
2. $n = 2$. Let $A \in \mathrm{O}(2)$. Its columns are an orthonormal basis for $\mathbb{R}^2$. Say

$$\mathrm{col}_1(A) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

We have two choices for the second column:

$$\mathrm{col}_2(A) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad \text{or} \quad (-1) \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Thus,

$$\mathrm{O}(2) = \left\{ \begin{pmatrix} \cos\theta & -\varepsilon\sin\theta \\ \sin\theta & \varepsilon\cos\theta \end{pmatrix} : \theta \in [0, 2\pi),\ \varepsilon \in \{\pm 1\} \right\}.$$

3. $n = 3$. Let $A \in \mathrm{O}(3)$. Since A is orthogonal, its eigenvalues are all ± 1. Let $v \neq 0$ be an eigenvector, with eigenvalue $\lambda = \det A = \pm 1$. Because $A \in \mathrm{O}(3)$, it can be seen that L_A preserves the 2-dimensional subspace

$$v^\perp := \{w \in \mathbb{R}^3 : \langle v, w \rangle = 0\}.$$

Moreover, $L_A|_{v^\perp} : v^\perp \to v^\perp$ is a rigid motion (after identifying $v^\perp \simeq \mathbb{R}^2$ via an orthonormal basis for $v^\perp$). Thus, $\exists \mathcal{B}$ an orthonormal basis of $\mathbb{R}^3$ such that

$$[L_A]_\mathcal{B} = \begin{pmatrix} R_\theta & 0_{2\times 1} \\ 0_{1\times 2} & \det A \end{pmatrix}, \quad R_\theta \text{ rotation by an angle } \theta \in [0, 2\pi).$$

Lemma 2.2.9. *Let $A \in M_3(\mathbb{R})$, $v : I \to \mathbb{R}^3$ a curve. Then $\big(Av(s)\big)' = Av'(s)$.*

Proof. Let $v(s) = \big(v_1(s), v_2(s), v_3(s)\big)$ and $A = \big(a_{ij}\big)_{1 \le i,j \le 3}$. Then

$$Av(s) = \big(a_{11}v_1(s) + a_{12}v_2(s) + a_{13}v_3(s), \ldots, \ldots\big),$$

so

$$\big(Av(s)\big) = \big(a_{11}v_1'(s) + a_{12}v_2'(s) + a_{13}v_3'(s), \ldots, \ldots\big) = Av'(s).$$

□

Lemma 2.2.10. *Let $A \in \mathrm{SO}(3) = \{\det A = 1\} \cap \mathrm{O}(3)$ and $v, w \in \mathbb{R}^3$. Then $(Av) \times (Aw) = A(v \times w)$.*

Remark. This is a special case of a more general result in linear algebra: if $A \in M_3(\mathbb{R})$, we have

$$(Av) \times (Aw) = (\det A)\big(A^{-1}\big)^T (v \times w).$$

Below, we will be using this "version" of the lemma; it reduces to the above special case for $A \in \mathrm{SO}(3)$, because then $A = A^{-1} = A^T$ and $\det A = 1$.

Proof. Assume $\|v\| = \|w\| = 1$ and $\langle v, w \rangle = 0$. Then since $A \in \mathrm{O}(3)$, it maps orthonormal bases to orthonormal bases, so $\{Av, Aw, A(v \times w)\}$ is an orthonormal basis of $\mathbb{R}^3$. On the other hand, $\{Av, Aw, (Av) \times (Aw$ is also an orthonormal basis. Now when completing $\{Av, Aw\}$ to be an orthonormal basis, we have two choices for the third vector: it can be positively or negatively oriented (but still orthogonal to Av and Aw in both cases). Thus, $A(v \times w) = \pm(Av) \times (Aw)$. If $\det A = 1$, we choose the positive version, while if $\det A = -1$, we choose the negative version. □

Theorem 2.2.11. *Let $\alpha : I \to \mathbb{R}^3$ be a regular, unit-speed curve, with $\kappa(s) \neq 0$ for all $s \in I$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Then $\beta(s) := f\big(\alpha(s)\big)$ is also a regular, unit-speed curve, and*

$$\kappa_\alpha(s) = \kappa_\beta(s), \quad \tau_\alpha(s) = \pm\tau_\beta(s), \quad \forall s \in I,$$

where the sign is positive (respectively negative) if f is orientation preserving (respectively reversing).

Proof. Let $f = T_q \circ L_A$, $A \in \mathrm{O}(3)$, $q \in \mathbb{R}^3$ fixed. Then $\beta(s) = A\alpha(s) + q$, so $\beta'(s) = A\alpha'(s)$, which means $t_\beta(s)At_\alpha(s)$. Thus,

$$\big\|\beta'(s)\big\| = \big\|A\alpha'(s)\big\| = \big\|\alpha'(s)\big\| = 1,$$

where the second last equality follows from $A \in \mathrm{O}(3)$. Therefore, $\beta(s)$ is parameterised by arc-length. Now $\beta''(s) = A\alpha''(s)$, and so

$$\kappa_\beta(s) = \big\|\beta''(s)\big\| = \big\|A\alpha''(s)\big\| = \big\|\alpha''(s)\big\| = \kappa_\alpha(s).$$

We also have in particular that $\kappa_\beta(s) \neq 0$ for all $s \in I$, so we can talk about the Frenet frame and the torsion. In particular,

$$n_\beta(s) = \frac{\beta''(s)}{\kappa_\beta(s)} = A\,\frac{\alpha''(s)}{\kappa_\alpha(s)} = An_\alpha(s),$$

so

$$b_\beta(s) = t_\beta(s) \times n_\beta(s) = \big(At_\alpha(s)\big) \times \big(An_\alpha(s)\big) = (\det A) \cdot A\big(t_\alpha(s) \times n_\alpha(s)\big) = (\det A) \cdot Ab_\alpha(s),$$

where the second last equality follows from (the general form of) Lemma 2.2.10. Thus, $b'_\beta(s) = (\det A) \cdot Ab'_\alpha(s)$ and therefore,

$$\tau_\beta(s) = \Big\langle b'_\beta(s), n_\beta(s)\Big\rangle = (\det A) \cdot \big\langle Ab'_\alpha(s), An_\alpha(s)\big\rangle \overset{A\in\mathrm{O}(3)}{=} (\det A) \cdot \big\langle b'_\alpha(s), n_\alpha(s)\big\rangle = (\det A) \cdot \tau_\alpha(s).$$

□

2.2.1 Classification of Curves in $\mathbb{R}^3$

Theorem 2.2.12 (Fundamental Theorem of the Local Theory of Curves)**.** *Let $I \subseteq \mathbb{R}$ be an interval and let $\kappa, \tau : I \to \mathbb{R}$ be arbitrary smooth functions, with $\kappa(s) > 0$ for all $s \in I$. Then*

1. *$\exists \alpha : I \to \mathbb{R}^3$ a regular, unit-speed curve whose curvature and torsion are $\kappa(s)$ and $\tau(s)$, respectively.*
2. *If $\tilde{\alpha} : I \to \mathbb{R}^3$ is another curve as in (1), $\exists f : \mathbb{R}^3 \to \mathbb{R}^3$ an orientation-preserving rigid motion such that $\tilde{\alpha}(s) = f\big(\alpha(s)\big)$ for all $s \in I$.*

Remark. There is an analogous statement for plane curves: τ does not appear, and the assumption $\kappa > 0$ is not needed (one works with the signed curvature $\underline{\kappa}$ instead).

Proof (strategy). After setting it up correctly, the theorem is a consequence of the (well-known) existence and uniqueness of solutions to ODE systems (Theorem 5.2.10). More precisely, assume we have found α such that (1) holds. Then the Frenet formulas are satisfied:

$$(\mathrm{FF}) \begin{cases} t'(s) = \kappa(s)n(s) \\ n'(s) = -\kappa(s)t(s) - \tau(s)b(s) \\ b'(s) = \tau(s)n(s) \end{cases}$$

We now change our perspective and view (FF) as a system of (ordinary differential) equations for the unknowns $t_1(s), t_2(s), t_3(s), n_1(s), \dots, b_1(s), \dots$, where

$$t(s) = \big(t_1(s), t_2(s), t_3(s)\big), \quad n(s) = \big(n_1(s), n_2(s), n_3(s)\big), \quad b(s) = \big(b_1(s), b_2(s), b_3(s)\big)$$

and $t_i, n_i, b_i : I \to \mathbb{R}$ are real-valued functions.

1. *Existence.* Consider the ODE system (FF) consisting of 9 equations for 9 unknowns, all of them functions $I \to \mathbb{R}$. Let $s_0 \in I$. Adding to (FF) the initial conditions

$$t(s_0) = (1,0,0), \quad n(s_0) = (0,1,0), \quad b(s_0) = (0,0,1) \tag{IC}$$

we obtain the initial value problem (IVP) consisting of (FF) and (IC). Since the coefficients of (FF) are smooth (κ and τ), we may apply Picard's theorem on ODEs (Theorem 5.2.10) and conclude that there exists a solution $\big(t_1(s), t_2(s), t_3(s), n_1(s), \dots, b_1(s), \dots\big) : I \to \mathbb{R}^9$ to (IVP). We then have three functions $t, n, b : I \to \mathbb{R}^3$ solving (FF).

   (a) *Theorem 1.2.4.* $\big(t(s), n(s), b(s)\big)$ is an orthonormal basis of $\mathbb{R}^3$ for all $s \in I$.

   Set (integrate each component)

$$\alpha(s) := \int_{s_0}^{s} t(u)\, du.$$

   It is clear that $\alpha'(s) = t(s)$ (by the fundamental theorem of calculus), so $t_\alpha(s) = t(s)$. In particular, α is regular and unit-speed. Moreover, $\alpha''(s) = t'(s) = \kappa(s)n(s)$ by (FF). Thus,

$$n_\alpha(s) = \frac{\alpha''(s)}{\big\|\alpha''(s)\big\|} = \frac{\kappa(s)n(s)}{\kappa(s)} \overset{\kappa>0}{=} n(s)$$

   and $\kappa_\alpha(s) = \big\|\alpha''(s)\big\| = \kappa(s)$. Similarly, one can see that $b_\alpha(s) = b(s)$ and $\tau_\alpha(s) = \tau(s)$. Therefore, we have established existence (note that $\alpha(s_0) = (0,0,0)$).

2. *Uniqueness.* Suppose $\tilde{\alpha} : I \to \mathbb{R}^3$ satisfies (1) as well. Let $T_{-\tilde{\alpha}(s_0)} : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation sending $\tilde{\alpha}(s_0)$ to the origin. Let also $R \in \mathrm{O}(3)$ be the (unique) orthogonal transformation sending the Frenet frame $\left\{\tilde{t}(s_0), \tilde{n}(s_0), \tilde{b}(s_0)\right\}$ of $\tilde{\alpha}$ at s_0 to $\left\{e_1 := t(s_0), e_2 := n(s_0), e_3 := b(s_0)\right\}$.

   If $f := R \circ T_{-\tilde{\alpha}(s_0)}$, this is a rigid motion that sends $\tilde{\alpha}(s)$ to $\beta(s) := f\big(\tilde{\alpha}(s)\big)$. By Theorem 2.2.11, κ and τ remain unchanged under f, and so $\kappa_\beta = \kappa$ and $\tau_\beta = \tau$. In particular, the Frenet frame $\left\{t_\beta(s), n_\beta(s), b_\beta(s)\right\}$ of β solves (FF). But since we chose f so that

$$t_\beta(0) = e_1, \quad n_\beta(0) = e_2, \quad b_\beta(0) = e_3,$$

   (IC) is also satisfied. Therefore, by uniqueness of solutions to the IVP, we have

$$t_\beta = t_\alpha, \quad n_\beta = n_\alpha, \quad b_\beta = b_\alpha.$$

   Integrating (using $\beta(s_0) = 0$) yields $\beta(s) = \alpha(s)$ for all $s \in I$, that is, $\exists f : \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion such that $\alpha = f \circ \tilde{\alpha}$.

□

Example 2.2.13. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with constant curvature and torsion $\kappa(s) \equiv 1/r$, $\tau(s) \equiv 0$. Then by the fundamental theorem, $\exists f : \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion such that $f \circ \alpha$ is the circle of radius r in the xy-plane:

$$\beta(s) := \left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right), \quad s \in [0, 2\pi r).$$

Part II
Surfaces in $\mathbb{R}^3$

3 Regular Surfaces

3.1 Surface Patches and Regular Surfaces

Definition 3.1.1. Let $U \subseteq \mathbb{R}^2$ be open. A map $X : U \to \mathbb{R}^3$ is called a (parameterised, regular) **surface patch** (also coordinate patch, local parametrisation or local chart) if it satisfies

(S1) X is differentiable.

(S3) For each $q = (u, v) \in U$, the differential (or Jacobian) $\mathrm{d}X_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

Definition 3.1.2. Let $U \subseteq \mathbb{R}^2$ be open and $S \subseteq \mathbb{R}^3$ be a set. We say S is a **regular surface** if for each $p \in S$, $\exists X : U \to S$ a surface patch with $p \in X(U)$, such that

(S2) $X : U \to X(U) \subseteq S$ is a **homeomorphism**.

Remark. If X satisfies (S1), (S2) and (S3) from above, we will call it a **parametrisation**.

Remark. Some things to note regarding the above two definitions:

1. In (S1), differentiability means that if $X(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$, then $x, y, z : U \to \mathbb{R}$ are differentiable (recall that in this course, this means they are smooth, i.e. infinitely differentiable).
2. For (S3), recall that
$$\mathrm{d}X_q = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix},$$
which we view as a linear transformation $\mathrm{d}X_q : \mathbb{R}^2 \to \mathbb{R}^3$ by using the canonical bases. The following are equivalent:
   (a) $\mathrm{d}X_q$ is injective.
   (b) $\ker \mathrm{d}X_q = \{0\}$.
   (c) $\operatorname{rank} \mathrm{d}X_q = 2$.
   (d) $X_u(q) \times X_v(q) \neq 0$, where
$$X_u(q) = \left(\frac{\partial x}{\partial u}(q), \frac{\partial y}{\partial u}(q), \frac{\partial z}{\partial u}(q)\right), \quad X_v(q) = \left(\frac{\partial x}{\partial v}(q), \frac{\partial y}{\partial v}(q), \frac{\partial z}{\partial v}(q)\right)$$
   (e) One of the 2×2 minors
$$\frac{\partial(x, y)}{\partial(u, v)}, \quad \frac{\partial(x, z)}{\partial(u, v)}, \quad \frac{\partial(y, z)}{\partial(u, v)}$$
   has det $\neq 0$.
3. In (S2), we consider on $X(U)$ the topology induced from $\mathbb{R}^3$. Equivalently, (S2) could be stated as: (S2') $X : U \to S$ is injective, $\exists V \subseteq \mathbb{R}^3$ open such that $X(U) = V \cap S$, and $X^{-1} : V \cap S \to U$ is the restriction of a continuous function $\phi : V \to U$.

4. (S2") $X : U \to S$ is injective, and for any sequence $\big(X(q_n)\big)_{n\in\mathbb{N}} \subseteq X(U)$, if $X(q_n) \to X(q_\infty)$, then $q_n \to q_\infty$.

Example 3.1.3. Some examples:

1. The graph of a function. Let $U \subseteq \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ differentiable. Then

$$S = \mathrm{Gr}(f) := \Big\{\big(u, v, f(u,v)\big) : (u,v) \in U\Big\}$$

is a regular surface. Indeed, $X(u,v) := \big(u, v, f(u,v)\big)$ is differentiable, and

$$\mathrm{d}X_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}$$

is injective, since

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

Thus, X is a surface patch with $X(U) = S$. To prove (S2), we actually show (S2'): X is clearly injective and $X(U) = S \cap V$, $V = \mathbb{R}^3$. Finally, an expression for $X^{-1} : S \to U$ is $X^{-1}\big(u, v, f(u,v)\big) = (u,v)$. That is, $X^{-1} = \phi|_S$, where $\phi(x,y,z) = (x,y)$ is continuous. For instance,

   (a) $U = \mathbb{R}^2$, $f(u,v) = au + bv$, $a, b \in \mathbb{R}$. This is a plane.
   (b) $U = \mathbb{R}^2$, $f(u,v) = u^2 - v^2$. This is a saddle.
   (c) $U = \big\{(u,v) : u^2 + v^2 < 1\big\} \subseteq \mathbb{R}^2$, $f(u,v) = \sqrt{1 - u^2 - v^2}$. This is the upper half of a sphere of radius 1.
   (d) $U = \mathbb{R}^2$, $f(u,v) = \sqrt{u^2 + v^2}$, a double cone extending above and below the $\mathbb{R}^2$ plane with centre/"point" at $(0,0)$. This is *not* a regular surface, since f is not differentiable at $(0,0)$.

Intuitively, locally a regular surface is a 2-dimensional object which "has a tangent plane" at each point.

2. The sphere $S^2 = \big\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\}$ is a regular surface. Indeed, let $p \in S^2$, $p = (x,y,z)$. Suppose $z > 0$. Then $p \in X_1(U)$, where $U = \big\{u^2 + v^2 < 1\big\}$, and $X_1(u,v) = \Big(u, v, \sqrt{1 - u^2 - v^2}\Big)$, which is a surface patch satisfying (S2') (by the example above). Now if $z < 0$, then $p \in X_2(U)$, where $X_2(u,v) = \Big(u, v, -\sqrt{1 - u^2 - v^2}\Big)$. Now say $z = 0$ and $y > 0$. Then $p \in X_3(U)$. We can similarly apportion the rest of the sphere. The corresponding X_i are

$$\begin{aligned} X_3(u,v) &= \Big(u, \sqrt{1 - u^2 - v^2}, v\Big), \quad X_4(u,v) = \Big(u, -\sqrt{1 - u^2 - v^2}, v\Big) \\ X_5(u,v) &= \Big(\sqrt{1 - u^2 - v^2}, u, v\Big), \quad X_6(u,v) = \Big(-\sqrt{1 - u^2 - v^2}, u, v\Big). \end{aligned}$$

These 6 surface patches satisfy (S2) and cover all of S^2.

3. The helicoid. Let $U = \mathbb{R}^2$, $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by $X(u,v) = (u\cos v, u\sin v, v)$, with $S = X\big(\mathbb{R}^2\big)$. (S1) is clear. For (S3), we compute

$$\mathrm{d}X_{(u,v)} = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 0 & 1 \end{pmatrix}.$$

For each v, either $\cos v \neq 0$ or $\sin v \neq 0$, so take the corresponding minor with the last row; the determinant of this matrix will be non-zero, so $\mathrm{d}X_{(u,v)}$ is injective. Regarding (S2), we first prove injectivity:

$$(u\cos v, u\sin v, v) = (\tilde{u}\cos\tilde{v}, \tilde{u}, \sin\tilde{v}, \tilde{v}) \implies v = \tilde{v}.$$

Now either $\cos v \neq 0$ or $\sin v \neq 0$. In both cases, we obtain $u = \tilde{u}$ from one of the first two equations, so X is injective. For the inverse, suppose $(x, y, z) = X(u, v) = (u\cos v, u\sin v, v)$. Then $v = z$, $x = u\cos z$ and $y = u\sin z$, so $x\cos z + y\sin z = u$. Thus, $X^{-1}(x, y, z) = (x\cos z + y\sin z, z)$, which is continuous. Therefore, (S2) holds.

From the above examples, it is clear that proving S is a regular surface is usually *hard*. The next result will help us do that.

Definition 3.1.4. Let $V \subseteq \mathbb{R}^3$ be open and $F : V \subseteq \mathbb{R}^3 \to \mathbb{R}$ differentiable. We call $p \in V$ a **critical point** if $\boldsymbol{\nabla} F(p) = \mathrm{d}F_p = 0$. An element $a \in F(V) \subseteq \mathbb{R}$ is called a **regular value** for F if the preimage $F^{-1}\big(\{a\}\big) := \big\{p \in V : F(p) = a\big\}$ contains *no* critical points of F.

Theorem 3.1.5. *If $F : V \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $a \in \mathbb{R}$ is a regular value for F, then $F^{-1}\big(\{a\}\big)$ is a regular surface.*

Proof. Since $\boldsymbol{\nabla} F = \big(\partial_x F, \partial_y F, \partial_z F\big) \neq 0$ on $S := F^{-1}\big(\{a\}\big)$, we can cover S with three open sets

$$S \cap \left\{\frac{\partial F}{\partial x} \neq 0\right\}, \quad S \cap \left\{\frac{\partial F}{\partial y} \neq 0\right\}, \quad S \cap \left\{\frac{\partial F}{\partial z} \neq 0\right\}.$$

We now claim that each one of them is locally the graph of a differentiable function. Thanks to Example 3.1.3 (1), this would prove each $p \in S$ belongs to a surface patch (with (S1) and (S3)), satisfying, in addition, (S2).

Take $S \cap \{\partial_z F \neq 0\}$; the other two are analogous. Let $V \subseteq \mathbb{R}^3$ be open, $V_z := V \cap \{\partial_z F \neq 0\}$ and $G : V_z \to \mathbb{R}^3$, $G(x, y, z) = \big(x, y, F(x, y, z)\big)$. Notice that

$$\mathrm{d}G_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x & F_y & F_z \end{pmatrix} \implies \det\big(\mathrm{d}G_p\big) = F_z \neq 0, \quad \forall p \in V_z.$$

Thus, by the inverse function theorem, there exist neighbourhoods U of p and W of $G(p)$ such that $G : U \to W$ is invertible, with differentiable inverse $G^{-1} : W \to U$. It follows that

$$G^{-1}(u, v, w) = \big(u, v, g(u, v, w)\big), \quad g : W \to \mathbb{R} \text{ differentiable.}$$

In particular, the map $h(x, y) := g(x, y, a)$ is a differentiable function defined on

$$U_{12} := \big\{(x, y) : \exists z \text{ such that } (x, y, z) \in U\big\}.$$

Notice that

$$\begin{aligned} \mathrm{Gr}(h) &= \Big\{\big(x, y, h(x, y)\big) : x, y \in U_{12}\Big\} \\ &= \big\{G^{-1}(x, y, a) : x, y \in U_{12}\big\} \\ &= \big\{(x, y, z) \in U : F(x, y, z) = a\big\} \\ &= F^{-1}\big(\{a\}\big) \cap U. \end{aligned}$$

Therefore, $F^{-1}\big(\{a\}\big) \cap U = \mathrm{Gr}(h)$ as we wanted.

Alternative. To show $F^{-1}\big(\{a\}\big) \cap \{F_z \neq 0\}$ is covered by surface patches, we use the implicit function theorem, which states that if $F_z(p) \neq 0$, the points (x, y, z) solving $F(x, y, z) = a$ can be described (locally around p) as $z = h(x, y)$, for some differentiable function h. It follows from Example 3.1.3 (1) that $F^{-1}\big(\{a\}\big)$ is a regular surface. ☐

Example 3.1.6. Two examples:

1. The ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
is a regular surface. Indeed, $F(x,y,z)=x^2/a^2+y^2/b^2+z^2/c^2$ has 1 as a regular value, since
$$\nabla F(x,y,z)=2\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)=(0,0,0)\iff(x,y,z)=(0,0,0),$$
but $F(0,0,0)\neq 1$.
2. The hyperboloid of two sheets, $-x^2-y^2+z^2=1$ is a regular surface. Notice it is *disconnected.*

Theorem 3.1.7. *Let $S\subseteq\mathbb{R}^3$ be a regular surface and let $p\in S$. Then $\exists V\subseteq\mathbb{R}^3$ open such that $p\in V$ and $V\cap S$ is the graph of a differentiable function of one of the forms $z=f(x,y)$, $y=g(x,z)$ or $x=h(y,z)$.*

Proof. Let $X(u,v)=\big(x(u,v),y(u,v),z(u,v)\big)$ be a parametrisation for S around $p=X(u_0,v_0)$. By (S3), one of the 2×2 minors in $\mathrm{d}X_{(u_0,v_0)}$ has det $\neq 0$, say
$$\det\frac{\partial(y,z)}{\partial(u,v)}\neq 0.$$
Consider $\pi\circ X:U\subseteq\mathbb{R}^3\to\mathbb{R}^2$, where $\pi(x,y,z)=(y,z)$. Then
$$\mathrm{d}(\pi\circ X)_{(u_0,v_0)}=\frac{\partial(y,z)}{\partial(u,v)},$$
so by the inverse function theorem, $\exists V_1,V_2\subseteq\mathbb{R}^2$ open, $(u_0,v_0)\in V_1$ and $\big(y(u_0,v_0),z(u_0,v_0)\big)\in V_2$, such that $\pi\circ X:V_1\to V_2$ has a smooth inverse $(\pi\circ X)^{-1}:V_2\to V_1$, $(\pi\circ X)^{-1}(y,z)=(u,v)$. Thus, for $(y,z)\in V_2$, we have
$$x=x(u,v)=x\Big((\pi\circ X)^{-1}(y,z)\Big)\quad\text{and we may set}\quad h(y,z):=\Big(x\circ(\pi\circ X)^{-1}\Big)(y,z).$$
Notice that this describes the set $X(V_1)\subseteq S$, which by (S2) is open in S and so of the form $V\cap S$, $V\subseteq\mathbb{R}^3$ open. □

3.2 Maps between Surfaces

The main reason we take (S2) as an axiom is for the following to hold. It expresses a certain compatibility between different parametrisations in a regular surface. This result is of such fundamental importance that, when talking about manifolds at the end of this document, we will take it as an axiom.

Theorem 3.2.1 (Change of Coordinates are Smooth)**.** *Let $S\subseteq\mathbb{R}^3$ be a regular surface and $X_1:U\subseteq\mathbb{R}^3\to S$, $X_2:V\subseteq\mathbb{R}^3\to S$ two parametrisations of S around p with $X_1(U)\cap X_2(V)=:W\neq\emptyset$. Then the* ***change of coordinates*** *$X_2^{-1}\circ X_1:X_1^{-1}(W)\to X_2^{-1}(W)$ is differentiable.*

Proof. Omitted. □

Remark. Applying the above to X_1 and X_2, we see that the change of coordinates is actually a **diffeomorphism**: it is differentiable and has a differentiable inverse.

In Do Carmo, it is claimed that the proof of the above theorem uses (S2) in a fundamental way. Ramiro thinks this is *not* the case, and that differentiability of $X_2^{-1}\circ X_1$ is always true, regardless of (S2).

Definition 3.2.2. Let S be a regular surface.

1. A function $f : S \to \mathbb{R}^n$ is said to be **differentiable** at $p \in S$ if for *some* parametrisation $X : U \subseteq \mathbb{R}^2 \to S$, $p \in X(U)$, $f \circ X : U \to \mathbb{R}^n$ is differentiable (in the usual sense, not smoothness) at $X^{-1}(p)$. f is called differentiable on S if it is differentiable at all $p \in S$.
2. $f : V \subseteq \mathbb{R}^n \to S$, V open, is said to be **differentiable** at $v \in V$ if there exists a parametrisation $X : U \subseteq \mathbb{R}^2 \to S$ around $f(v) \in S$ such that for some neighbourhood $V_1 \subseteq V$ of v, $f(V_1) \subseteq X(U)$, and $X^{-1} \circ f : V_1 \to \mathbb{R}^2$ is differentiable at v (in the usual sense; again, not smoothness). f is called differentiable on V if it is differentiable at all $v \in V$.

Remark. Recall from earlier that "differentiable" (in the usual sense) means that the multivariable derivatives exist (once). Both definitions are independent of the choice of parametrisation thanks to Theorem 3.2.1.

Example 3.2.3. The "height" function $h : S^2 \to \mathbb{R}$ (recall that S^2 is the unit sphere in $\mathbb{R}^3$), $h(x, y, z) = z$, is a differentiable function on S^2. Indeed, around a point $p = (x, y, z)$ with $z > 0$, we have a parametrisation

$$X : \left\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\right\} \to S^2, \quad X(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

and $h \circ X(u, v) = \sqrt{1 - u^2 - v^2}$, which is a differentiable function on $\left\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\right\}$. An analogous argument deals with the case $z < 0$. For $z = 0$, suppose without loss of generality that $y > 0$. Then we have a parametrisation

$$Y(u, v) = \left(u, \sqrt{1 - u^2 - v^2}, v\right),$$

and $h \circ Y(u, v) = v$, differentiable. All remaining cases are analogous.

Theorem 3.2.4. *Let S be a regular surface. If $V \subseteq \mathbb{R}^3$ is open and $F : V \to \mathbb{R}$ is differentiable (usual sense), then $f := F|_{V \cap S} : V \cap S \to \mathbb{R}$ is differentiable.*

Proof. Let $p \in V \cap S$ and $X : U \subseteq \mathbb{R}^2 \to V \cap S$ be a parametrisation around p. By (S1), X is differentiable as a map $X : U \to V$ (usual sense). Hence, by the "chain rule",

$$f \circ X = F \circ X : U \to \mathbb{R}$$

is a differentiable map. Therefore, f is differentiable (as a function on S). $\square$

Definition 3.2.5. A function $f : S_1 \to S_2$ between regular surfaces is said to be **differentiable** at $p \in S_2$ if $\exists X_1 : U_1 \subseteq \mathbb{R}^2 \to S_1$, $X_2 : U_2 \subseteq \mathbb{R}^2 \to S_2$, $p \in X_1(U_1)$, $f\big(X_1(U_1)\big) \subseteq X_2(U_2)$, such that $X_2^{-1} \circ f \circ X_1 : U_1 \to U_2$ is differentiable (usual sense). f is called differentiable if it is so at every $p \in S_2$. A differentiable function $f : S_1 \to S_2$ which has a differentiable inverse $f^{-1} : S_2 \to S_1$ is called a **diffeomorphism**, and in this case S_1 and S_2 are called **diffeomorphic**.

Remark. As before, this definition does not depend on the parametrisations.

Example 3.2.6. The sphere $S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$ and the ellipsoid

$$E^2 = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

are diffeomorphic. Indeed, $F : S^2 \to E^2$, $(x, y, z) \mapsto (ax, by, cz)$ is differentiable ($f : \mathbb{R}^3 \to \mathbb{R}^3$ with the same map is clearly differentiable since it is linear and we already know that S^2 is regular, the restriction $F = f|_{\mathbb{R}^3 \cap S^2}$ is also differentiable; it also maps to E^2) with differentiable inverse $F^{-1}(u, v, w) = (u/a, v/b, w/c)$.

3.3 Tangent Plane and Differentials

Definition 3.3.1. Let S be a regular surface and let $p \in S$. The **tangent plane** to S at p is defined as

$$T_pS := \big\{\alpha'(0) : \alpha : (-\varepsilon, \varepsilon) \to S \text{ is differentiable and } \alpha(0) = p\big\}.$$

Theorem 3.3.2. *Let $X : U \subseteq \mathbb{R}^2 \to S$ be a parametrisation for S around $p = X(q)$, $q \in U$. Then $T_pS = \mathrm{d}X_q\big(\mathbb{R}^2\big)$. In particular, T_pS is a 2-dimensional vector subspace of $\mathbb{R}^3$, with basis given by $\big\{X_u(q), X_v(q)\big\}$.*

Proof. $T_pS \subseteq \mathrm{d}X_q\big(\mathbb{R}^2\big)$. Let $w \in T_pS$. Then $w = \alpha'(0)$ for some $\alpha : (-\varepsilon, \varepsilon) \to S$ differentiable, with $\alpha(0) = X(q)$. Let $\beta := X^{-1} \circ \alpha$. Then $\beta(0) = q$ and β is differentiable. Set $v := \beta'(0)$. It follows by the chain rule that

$$w = \alpha'(0) = (X \circ \beta)'(0) = \mathrm{d}X_{\beta(0)}\big(\beta'(0)\big) = \big(\mathrm{d}X_q\big)(v).$$

$\mathrm{d}X_q\big(\mathbb{R}^2\big) \subseteq T_pS$. Let $w\mathrm{d}X_q(v) \in \mathrm{d}X_q\big(\mathbb{R}^2\big)$, $v \in \mathbb{R}^2$. Define $\beta : (-\varepsilon, \varepsilon) \to U$, $\beta(t) = q + tv$. Then $\beta(0) = q$ and $\beta'(0) = v$. Set $\alpha := X \circ \beta : (-\varepsilon, \varepsilon) \to S$. It is differentiable, because in the parametrisation X, it looks like β. Finally, by the chain rule,

$$\alpha'(0) = \mathrm{d}X_q(v) = w \implies w \in T_pS.$$

□

Definition 3.3.3. Given a differentiable map $f : S_1 \to S_2$ between two regular surfaces and $p \in S_1$, we define its **differential** at p to be the (linear) map

$$\mathrm{d}f_p : T_pS_1 \to T_{f(p)}S_2, \quad \big(\mathrm{d}f_p\big)\big(\alpha'(0)\big) = \beta'(0),$$

where $\beta = f \circ \alpha : (-\varepsilon, \varepsilon) \to S_2$.

Remark. Let $X : U \to S_1$ be a local parameterisation around $p \in S$ such that $X(q) = p$. Then

$$f_u := \mathrm{d}f_p(X_u) = \frac{\partial}{\partial u}(f \circ X)(q), \quad f_v := \mathrm{d}f_p(X_v) = \frac{\partial}{\partial v}(f \circ X)(q).$$

The chain rule also applies for differentials. Let $g : S_2 \to S_3$ be a differentiable map. Then $g \circ f : S_1 \to S_3$ is differentiable and its derivative (differential) is

$$\mathrm{d}(g \circ f)_p = \mathrm{d}_{f(p)}g \circ \mathrm{d}f_p : T_pS_1 \to T_{g(f(p))}S_3.$$

In other words,

$$\mathrm{d}(g \circ f)_p(w) = \mathrm{d}_{f(p)}g\big(\mathrm{d}f_p(w)\big), \quad \forall w \in T_pS_1.$$

Theorem 3.3.4. *$\mathrm{d}f_p$ from the above definition is well-defined, i.e. $\beta'(0)$ depends only on $\alpha'(0)$ and not on the specific curve α. Moreover, $\mathrm{d}f_p$ is indeed a linear map.*

Proof. Let $X : U \subseteq \mathbb{R}^2 \to S_1$ be a parametrisation of S_1 around $p = X(q)$, $q \in U$. Then by Theorem 3.3.2, $\big\{X_u(q), X_v(q)\big\}$ is a basis for T_pS_1. Thus,

$$\tilde{\alpha}(t) := \big(X^{-1} \circ \alpha\big)(t) = \big(u(t), v(t)\big),$$

where $\big(u(t), v(t)\big)$ is the "coordinate representation of α", is a differentiable curve in $\mathbb{R}^2$ and $\big(u(0), v(0)\big) = q$. Therefore, $\alpha(t) = X\big(\tilde{\alpha}(t)\big) = X\big(u(t), v(t)\big)$. Now by the chain rule,

$$\alpha'(0) = \big(\mathrm{d}X_q\big)\big(u'(0), v'(0)\big) = \begin{pmatrix} X_u(q) & X_v(q) \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} = u'(0)X_u(q) + v'(0)X_v(q).$$

(notice that this means $u'(0)$ and $v'(0)$ are the coordinates of $\alpha'(0)$ with respect to $\{X_u(q), X_v(q)\}$). Now given $f : S_1 \to S_2$, we choose a parametrisation $Y : V \subseteq \mathbb{R}^2 \to S_2$ for S_2 around $f(p)$, and assume without loss of generality that $f\big(X(U)\big) \subseteq Y(V)$. Then $\big(Y^{-1} \circ f \circ X\big) : U \to V$ can be written as

$$\big(Y^{-1} \circ f \circ X\big)(u, v) = \big(f_1(u, v), f_2(u, v)\big),$$

for $f_1, f_2 : U \to \mathbb{R}$ differentiable (this is the "coordinate representation of our function f"). Thus, the coordinate representation of β satisfies

$$\begin{aligned}\big(Y^{-1} \circ \beta\big)(t) &= \big(Y^{-1} \circ f \circ \alpha\big)(t)\\ &= \big(Y^{-1} \circ f \circ X\big) \circ \big(X^{-1} \circ \alpha\big)(t)\\ &= \big(Y^{-1} \circ f \circ X\big)\big(u(t), v(t)\big)\\ &= \Big(f_1\big(u(t), v(t)\big), f_2\big(u(t), v(t)\big)\Big)\\ &=: \big(a(t), b(t)\big).\end{aligned}$$

As above, $\beta'(0) = a'(0)Y_u(\tilde{q}) + b'(0)Y_v(\tilde{q})$, where $Y(\tilde{q}) = f(p)$. Finally, using the above, we may compute $a'(0)$ and $b'(0)$:

$$a'(0) = \frac{\partial f_1}{\partial u}(q) \cdot u'(0) + \frac{\partial f_1}{\partial v}(q) \cdot v'(0), \quad b'(0) = \frac{\partial f_2}{\partial u}(q) \cdot u'(0) + \frac{\partial f_2}{\partial v}(q) \cdot v'(0).$$

Therefore,

$$\big(\mathrm{d}f_p\big)\big(\alpha'(0)\big) = \beta'(0) = \underbrace{\begin{pmatrix} a'(0) \\ b'(0) \end{pmatrix}}_{\in T_{f(p)}S_2} = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{pmatrix}\right|_q \underbrace{\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}}_{\in T_pS_1}.$$

This is an expression which only depends on $u'(0)$ and $v'(0)$ (i.e. on $\alpha'(0)$), and *not* $\alpha(t)$. Linearity is clear from the above equality. $\square$

Example 3.3.5. Let $S = F^{-1}\big(\{a\}\big)$ be a regular surface, where $F : V \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $a \in \mathbb{R}$ is a regular value for F. Then $T_pS = \big(\boldsymbol{\nabla} F(p)\big)^\perp$, where $\boldsymbol{\nabla} F(p) \neq 0$ since a is a regular value. Indeed, both are 2-dimensional vector subspaces, so it is sufficient to show that $T_pS \subseteq \big(\boldsymbol{\nabla} F(p)\big)^\perp$. If $\alpha'(0) \in T_pS$ with $\alpha : (-\varepsilon, \varepsilon) \to S$ a differentiable curve and $\alpha(0) = p$, then $F\big(\alpha(t)\big) = a$ for all $t \in (-\varepsilon, \varepsilon)$, by definition of $S = F^{-1}\big(\{a\}\big)$. Therefore, by the chain rule,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} F\big(\alpha(t)\big)\bigg|_{t=p} = \Big\langle \boldsymbol{\nabla} F\big(\alpha(0)\big), \alpha'(0) \Big\rangle \implies \alpha'(0) \perp \boldsymbol{\nabla} F(p) \implies \alpha'(0) \in \big(\boldsymbol{\nabla} F(p)\big)^\perp.$$

Theorem 3.3.6 (Inverse Function Theorem for Surfaces)**.** *Let $f : S_1 \to S_2$ be a differentiable map between two regular surfaces, and assume that at some $p \in S_1$, $\mathrm{d}f_p : T_pS_1 \to T_{f(p)}S_2$ is non-singular. Then f is a* ***local diffeomorphism*** *at p, that is, $\exists V_1, V_2 \subseteq \mathbb{R}^3$ open, $p \in V_1$ and $f(p) \in V_2$ such that $f|_{V_1 \cap S_1} : V_1 \cap S_1 \to V_2 \cap S_2$ is a diffeomorphism.*

Remark. Recall that if S is a regular surface and $V \subseteq \mathbb{R}^3$ is open, then $V \cap S$ is a regular surface, provided $V \cap S \neq \emptyset$. Recall also, from MATH2301, that a linear transformation between finite dimensional vector spaces is invertible if and only if it is an isomorphism. Thus, the above "non-singular" is equivalent to being a linear isomorphism $\Leftrightarrow$ invertible $\Leftrightarrow$ $\det\big(\mathrm{d}f_p\big) \neq 0$.

Proof (idea). By definition of f being differentiable (between surfaces), there exist parametrisations $X_1 : U_1 \to S_1$ and $X_2 : U_2 \to S_2$ such that $f\big(X_1(U_1)\big) \subseteq X_2(U_2)$ and $\tilde{f} := X_2^{-1} \circ f \circ X_1$ is differentiable. Say

$X_1(q) = p$. Since $\mathrm{d}f_p$ is an isomorphism, the same holds for $\mathrm{d}\tilde{f}_q : \mathbb{R}^2 \to \mathbb{R}^2$. We then apply the inverse function theorem in $\mathbb{R}^2$. It follows that $\exists W \subseteq U$ a neighbourhood of q in $\mathbb{R}^2$ such that $\tilde{f}|_W : W \to \tilde{f}(W)$ is bijective and has a differentiable inverse $\tilde{g} := \tilde{f}^{-1} : \tilde{f}(W) \to W$. Then $X_2\Big(\tilde{f}(W)\Big) \subseteq S_2$ and $X_1(W) \subseteq S_1$ are open and $g := X_1 \circ \tilde{g} \circ X_2^{-1} : X_2\Big(\tilde{f}(W)\Big) \to X_1(W)$ satisfies

1. $g = f^{-1}$. We have

$$g \circ f = \left(X_1 \circ \tilde{g} \circ X_2^{-1}\right) \circ f = X_1 \circ \underbrace{\tilde{f}^{-1} \circ \overbrace{\left(X_2^{-1} \circ f \circ X_1\right)}^{=\tilde{f}}}_{=\mathrm{Id}_W} \circ X_1^{-1} = \mathrm{Id}_{X_1(W)}.$$

   Similarly for $f \circ g$.

2. g is differentiable. Indeed, in the parametrisation X_1, X_2, g "looks like" $X_1^{-1} \circ g \circ X_2 = \tilde{g}$, which is differentiable (usual sense).

□

4 Geometry on Surfaces

4.1 Brief Linear Algebra Review

Definition 4.1.1. Let V be an n-dimensional vector space over $\mathbb{R}$. A **bilinear form** on V is a map $b : V \times V \to \mathbb{R}$ which is bilinear (linear in both arguments).

Definition 4.1.2. Given a vector space V with a basis $\mathcal{B}$ of vectors indexed by an index set I, the **dual set** of $\mathcal{B}$ is a set $\mathcal{B}^*$ of vectors in the dual space V^* with the same index set I such that $\mathcal{B}$ and $\mathcal{B}^*$ form a biorthogonal system. The dual set is always linearly independent but does not necessarily span V^*. If it does span V^*, then $\mathcal{B}^*$ is called the **dual basis** for the basis $\mathcal{B}$.

Remark. Denoting the indexed vector sets as $\mathcal{B} = \{v_i\}_{i \in I}$ and $\mathcal{B}^* = \{w_i\}_{i \in I}$, being biorthogonal means that evaluating a dual vector in V^* on a vector in the original space V:

$$v_i\big(w_j\big) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Remark. Here we give a quick recap of coordinate representations. Given a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V, the matrix $\big(B_{ij}\big)_{1 \le i,j \le n}$ given by $B_{ij} = b\big(v_i, v_j\big)$ is called the matrix of b with respect to $\mathcal{B}$. Notice that from $\mathcal{B}$ and B one can recover b: let $x = \sum x_i v_i, y = \sum y_i v_i \in V$. By bilinearity,

$$b(x, y) = \sum_{i,j=1}^{n} x_i y_j b\big(v_i, v_j\big) = \sum_{i,j=1}^{n} x_i y_j B_{ij} = [x]^T B[y],$$

where $[x] = (x_1, \ldots, x_n)^T$ and $[y] = (y_1, \ldots, y_n)^T$ are the column vectors representing x and y in the basis $\mathcal{B}$. If $\mathcal{B}^* = \{w_i, \ldots, w_n\}$ denotes the dual basis, then $x_i = w_i(x)$ and $y_j = w_j(y)$. Let $w_i \otimes w_j : V \times V \to \mathbb{R}$ be the bilinear form given by $\big(w_i \otimes w_j\big)(x, y) = x_i y_j$. Then, b can be written as

$$b = \sum_{i,j} B_{ij}\big(w_i \otimes w_j\big).$$

Definition 4.1.3. Let V be a real vector space. A bilinear form $b : V \times V \to \mathbb{R}$ is called **symmetric** if $b(v, w) = b(w, v)$ for all $v, w \in V$. Equivalently, $B_{ij} = B_{ji}$ for all $1 \leq i, j \leq n$ for any basis $\mathcal{B}$.

Definition 4.1.4. Let V be a real vector space. Given a symmetric bilinear form $b : V \times V \to \mathbb{R}$, there is an associated **quadratic form** $Q_b : V \to \mathbb{R}$, defined by $Q_b(v) := b(v, v)$. We call a symmetric bilinear form b an **inner product** if it is also positive definition, i.e. if $Q_b(v) \geq 0$ with equality if and only if $v = 0$.

4.2 The First Fundamental Form

Let $S \subseteq \mathbb{R}^3$ be a regular surface. The Euclidean inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ induces on each tangent plane $T_pS \subseteq \mathbb{R}^3$ an inner product $\langle \cdot, \cdot \rangle_p$.

Definition 4.2.1. The quadratic form $I_p : T_pS \to \mathbb{R}_{\geq 0}$ associated with $\langle \cdot, \cdot \rangle_p$ is called the **First Fundamental Form** (FFF) (of S at $p \in S$):

$$I_p(v) := \|v\|^2, \quad v \in T_pS.$$

4.2.1 The Coordinate Representation of the First Fundamental Form

Let $X : U \subseteq \mathbb{R}^2 \to S$ be a parametrisation around $p \in S$. Recall that $\{X_u(q), X_v(q)\}$ is a basis for T_pS with $X(q) = p$. The matrix of I_p (i.e. that of $\langle \cdot, \cdot \rangle_p$) with respect to this basis is denoted by

$$\begin{pmatrix} E(q) & F(q) \\ F(q) & G(q) \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

where

$$E(q) = \langle X_u(q), X_u(q) \rangle, \quad F(q) = \langle X_u(q), X_v(q) \rangle, \quad G(q) = \langle X_v(q), X_v(q) \rangle$$

from the discussion in Section 4.1. If $\{\mathrm{d}u|_q, \mathrm{d}v|_q\}$ denotes the basis of $(T_pS)^*$ dual to $\{X_u(q), X_v(q)\}$, then we may write

$$(I_p \circ X)(q) = E(q)(\mathrm{d}u|_q)^2 + F(q)(\mathrm{d}u|_q \otimes \mathrm{d}v|_q + \mathrm{d}v|_q \otimes \mathrm{d}u|_q) + G(q)(\mathrm{d}v|_q)^2,$$

where $(\mathrm{d}u|_q)^2 := \mathrm{d}u|_q \otimes \mathrm{d}u|_q$. Notice that $E, F, G : U \to \mathbb{R}$ are differentiable functions. They are called the **coefficients of the First Fundamental Form** with respect to the parametrisation $X : U \to S$.

Example 4.2.2. Some examples:

1. *Planes.* Let $p_0, w_1, w_2 \in \mathbb{R}^3$. The (affine) plane Π spanned by w_1 and w_2 passing through p_0 may be parameterised by
$$X : \mathbb{R}^2 \to \mathbb{R}^3, \quad X(u, v) = p_0 + uw_1 + vw_2,$$
which means $X_u = w_1$ and $X_v = w_2$ for all $q \in \mathbb{R}^2$. Assuming $\{w_1, w_2\}$ is orthonormal, we have
$$E(u, v) = \langle w_1, w_1 \rangle \equiv 1, \quad F(u, v) = \langle w_1, w_2 \rangle \equiv 0, \quad G(u, v) = \langle w_2, w_2 \rangle \equiv 1.$$
Thus, $I_p(w) = a^2 + b^2$ for all $w = aw_1 + bw_2 \in T_p\Pi$, $p \in \Pi$.
2. *Cylinder.* Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, $U := (0, 2\pi) \times \mathbb{R}$ and
$$X : U \to C, \quad X(u, v) = (\cos u, \sin u, v).$$
Thus, $X_u(u, v) = (-\sin u, \cos u, 0)$ and $X_v(u, v) = (0, 0, 1)$, meaning $E \equiv 1$, $F \equiv 0$ and $G \equiv 1$.

3. *Sphere.* Let $U := (0, \pi) \times (0, 2\pi)$ and

$$X : U \to S^2, \quad X(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Then

$$X_u(u, v) = (\cos u \cos v, \cos u \sin v, -\sin u), \quad X_v(u, v) = (-\sin u \sin v, \sin u \cos v, 0),$$

so

$$E(u, v) = \left\|X_u(u, v)\right\|^2 = \cdots \equiv 1, \quad F(u, v) = \langle X_u, X_v\rangle \equiv 0, \quad G(u, v) = \|X_v\|^2 = \sin^2 u.$$

Hence, given $p = X(u_0, v_0)$, $q := (u_0, v_0)$ and $w := aX_u(q) + bX_v(q) \in T_pS^2$, we have $I_p(w) = a^2 + b^2 \sin^2 u$.

Remark. According to Definitions 3.1.1 and 3.1.2, the functions given above in (2) and (3) are not parametrisations of a cylinder and sphere, respectively, since neither is a bijection (though they are still parametrisations of $X(U)$). For example, the "parametrisation" in (3) misses the point $(0, 0, 1) \in S^2$.

Ramiro has confirmed that this is an abuse of notation (or perhaps terminology, in this case) and that technically, these functions are only parametrisations of their images, *not* the whole cylinder/sphere. To construct proper parametrisations of say, S^2, we would need to take the union of multiple parametrisations, as we did in fact do in Example 3.1.3 (2).

The above parametrisations are still useful for various purposes, such as finding area, since, e.g., (3) above, only misses a "line" on the sphere, which is irrelevant when finding area since it has no "thickness" (like how often sets of measure zero are of no concern in analysis) and so we can proceed anyway.

4.3 Area

Let S be a regular surface and let $p \in S$. The FFF allows us to compute the areas of parallelograms on T_pS. For example, let $X : U \to S$ be a parametrisation around p. Then the area A of the parallelogram generated by $X_u(q)$ and $X_v(q)$ (visualised as two vectors lying in T_pS with origins at p and an angle between them of θ) in T_pS satisfies

$$A^2 = \left\|X_u(q) \times X_v(q)\right\|^2 = \left\|X_u(q)\right\|^2 \left\|X_v(q)\right\|^2 \sin^2 \theta = \left\|X_u(q)\right\|^2 \left\|X_v(q)\right\|^2 \left(1 - \cos^2 \theta\right).$$

Noting that $\left|\left\langle X_u(q), X_v(q)\right\rangle\right|^2 = \left\|X_u(q)\right\|^2 \left\|X_v(q)\right\|^2 \cos^2 \theta$, the above becomes

$$A^2 = \left\|X_u(q)\right\|^2 \left\|X_v(q)\right\|^2 - \left|\left\langle X_u(q), X_v(q)\right\rangle\right|^2 = E(q)G(q) - F(q)^2 \implies A = \sqrt{E(q)G(q) - F(q)^2}.$$

Definition 4.3.1. Let S be a regular surface, $X : U \subseteq \mathbb{R}^2 \to S$ are parametrisation of it and $B \subseteq U$ closed and bounded. Then the **area** of the region $R := X(B)$ is defined to be

$$\operatorname{Area}(R) := \iint_B \|X_u \times X_v\| \,\mathrm{d}u \,\mathrm{d}v = \iint_B \sqrt{E(u, v)G(u, v) - F(u, v)^2} \,\mathrm{d}u \,\mathrm{d}v.$$

Theorem 4.3.2. *Area, as defined above, is well-defined, i.e. independent of the parametrisation.*

Proof. Let $\tilde{X} : \tilde{U} \to S$ be another parametrisation with $\tilde{X}\left(\tilde{B}\right) = R = X(B)$, $\tilde{B} \subseteq \tilde{U}$ closed and bounded.

Lemma 4.3.3. *Let $p = X(q) = \tilde{X}(\tilde{q}) \in S$. Then the bases $\left\{X_u(q), X_v(q)\right\}$ and $\left\{\tilde{X}_{\tilde{u}}(\tilde{q}), \tilde{X}_{\tilde{v}}(\tilde{q})\right\}$ are related via*

$$\tilde{X}_{\tilde{u}}(\tilde{q}) = aX_u(q) + cX_v(q), \quad \tilde{X}_{\tilde{v}}(\tilde{q}) = bX_u(q) + dX_v(q),$$

where

$$J := \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the Jacobian of the "change of coordinates" map $X^{-1} \circ \tilde{X}$ *(i.e.* $(u,v) = \left(X^{-1} \circ \tilde{X}\right)(\tilde{u},\tilde{v})$*).*

Proof of lemma. Notice that $X^{-1} \circ \tilde{X}$ is nothing but the expression in coordinates of the identity map $\mathrm{Id}_S : S \to S$. Thus, by the proof of the differential being well-defined (Theorem 3.3.4), the matrix of $\mathrm{d}\,\mathrm{Id}_S|_p = \mathrm{Id}_{T_pS} : T_pS \to T_pS$ with respect to the bases $\tilde{\mathcal{B}} = \left\{\tilde{X}_{\tilde{u}}(\tilde{q}), \tilde{X}_{\tilde{v}}(\tilde{q})\right\}$ and $\mathcal{B} = \left\{X_u(q), X_v(q)\right\}$ is given by

$$\left[\mathrm{Id}_{T_pS}\right]_{\tilde{\mathcal{B}},\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In MATH2301 notation, the above is $\left[\mathrm{Id}_{T_pS}\right]_{\tilde{\mathcal{B}}}^{\mathcal{B}}$. By definition of "matrix of a linear transformation", this immediately yields the claim in the lemma. □

By the lemma, which holds $\forall p \in S$,

$$\begin{aligned}
\left\|\tilde{X}_{\tilde{u}}(\tilde{q}) \times \tilde{X}_{\tilde{v}}(\tilde{q})\right\| &= \left\|\big(aX_u(q) + cX_v(q)\big) \times \big(bX_u(q) + dX_v(q)\big)\right\| \\
&= \left\|ab\big(X_u(q) \times X_v(q)\big) + cd\big(X_v(q) \times X_u(q)\big)\right\| \\
&= |ab - cd| \cdot \left\|X_u(q) \times X_v(q)\right\| = |\det J| \cdot \left\|X_u(q) \times X_v(q)\right\|,
\end{aligned}$$

for all $q = (u,v) \in U$ and $\tilde{q} = (\tilde{u},\tilde{v}) \in \tilde{U}$. Hence, by the theorem of change of variables from multivariable calculus (MATH2001), we have

$$\begin{aligned}
\iint_{\tilde{B}} \sqrt{\tilde{E}(\tilde{u},\tilde{v})\tilde{G}(\tilde{u},\tilde{v}) - \tilde{F}(\tilde{u},\tilde{v})^2}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} &= \iint_{\tilde{B}} \left\|\tilde{X}_{\tilde{u}}(\tilde{u},\tilde{v}) \times \tilde{X}_{\tilde{v}}(\tilde{u},\tilde{v})\right\|\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} \\
&= \iint_{\tilde{B}} \left\|X_u(u,v) \times X_v(u,v)\right\| \cdot |\det J|\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} \\
&= \iint_{B} \left\|X_u(u,v) \times X_v(u,v)\right\|\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{B} \sqrt{E(u,v)G(u,v) - F(u,v)^2}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

□

Example 4.3.4. We calculate in this example the area of the torus of revolution. Let $U := (0,2\pi)\times(0,2\pi)$, $0 < r < a$ (a is the distance from the central ring of the "tube" to the centre of the torus hole and r is the radius of the "tube") and define the parametrisation

$$X : U \to T^2, \quad X(u,v) = \big((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u\big).$$

Again, this technically isn't a parametrisation of the torus since we don't quite get "all the way around"; $X(U)$ is really T^2 *excluding* a ring along the outside of the torus "tube" along the positive x axis and the circle constituting the outer "perimeter" of the torus. As I explained earlier, in terms of the area calculation, which takes the form an integral, these circles are of no consequence since they "have no width". Thus, we continue:

$$X_u(u,v) = (-r\sin u\cos v, -r\sin u\sin v, r\cos u), \quad X_v(u,v) = \big(-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0\big).$$

Therefore,

$$E(u,v) = \|X_u(u,v)\|^2 \equiv r^2, \quad F(u,v) = \langle X_u, X_v \rangle = \cdots \equiv 0, \quad G(u,v) = \|X_v\|^2 = (r\cos u + a)^2,$$

meaning $\sqrt{EG - F^2} = r(r\cos u + a)$. $\forall \varepsilon > 0$, let $\ell_\varepsilon = [\varepsilon, 2\pi - \varepsilon] \times [\varepsilon, 2\pi - \varepsilon] \subseteq U$, and set $R_\varepsilon := X(\ell_\varepsilon)$. Then by definition of area, we have

$$\begin{aligned}
\operatorname{Area}(R_\varepsilon) &= \iint_{R_\varepsilon} \sqrt{E(u,v)G(u,v) - F(u,v)^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_\varepsilon^{2\pi-\varepsilon} \int_\varepsilon^{2\pi-\varepsilon} r(r\cos u + a)\,\mathrm{d}u\,\mathrm{d}v \\
&= (2\pi - 2\varepsilon) \int_\varepsilon^{2\pi-\varepsilon} \left(r^2 \cos u + ar\right) \mathrm{d}u \\
&= (2\pi - 2\varepsilon) \left[r^2 \sin u + aru\right]_\varepsilon^{2\pi-\varepsilon} \\
&= (2\pi - 2\varepsilon)\Big(r^2\big(\sin(2\pi - \varepsilon) - \sin\varepsilon\big) + ar(2\pi - \varepsilon)\Big).
\end{aligned}$$

Thus,

$$\operatorname{Area}\big(T^2\big) = \lim_{\varepsilon \to 0^+} \operatorname{Area}(R_\varepsilon) = 2\pi^2 ar.$$

4.4 Orientation

Let V be a vector space over $\mathbb{R}$ with $\dim V = n$. Given two bases $\{b_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$, $\exists! A : V \to V$ invertible linear transformation such that $c_i = Ab_i$ for all $i \in \{1, \ldots, n\}$. By definition, $A \in \mathrm{GL}(V)$, the **general linear group** (group of all automorphisms of V with composition as the group operation). Fixing a basis yields an isomorphism $V \simeq \mathbb{R}^n$, giving rise to $\mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{R})$, the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, under the operation of matrix multiplication.

The group $\mathrm{GL}_n(\mathbb{R})$ is also a topological space (it is open in $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$), and has *two* connected components, since $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ is a continuous function with image $(-\infty, 0) \cup (0, \infty)$. Two bases of V are said to define the **same orientation** if the corresponding change of basis transformation A satisfies $\det A > 0$ (recall from MATH2301 that we define the determinant of a general linear transformation to be the determinant of its matrix representation). Thus, each vector space has **two different orientations**.

We now specify the above to the case where $V = T_pS$ is a tangent plane of a regular surface S, with $p \in S$. Notice that orientation on T_pS is equivalent to the choice of basis $\{v, w\}$ on T_pS. We also know that

$$(Av) \times (Aw) = \det A \cdot (v \times w), \quad \forall A \in \mathrm{GL}\big(T_pS\big) \simeq \mathrm{GL}_2(\mathbb{R}).$$

Thus, whether or not two bases $\{v, w\}$, $\{\tilde{v}, \tilde{w}\}$ of T_pS define the same orientation can be read off from the **normal vector** they define:

$$\frac{v \times w}{\|v \times w\|} = \frac{\tilde{v} \times \tilde{w}}{\|\tilde{v} \times \tilde{w}\|} \iff \{v, w\} \sim \{\tilde{v}, \tilde{w}\},$$

where $\sim$ means "define the same orientation".

Definition 4.4.1. A regular surface S is called **orientable** if there exists a differentiable field of unit normal vectors:

$$N : S \to \mathbb{R}^3, \quad \|N(p)\| \equiv 1, \quad N(p) \perp T_pS, \quad \forall p \in S.$$

Remark. In the above, "field" refers to "vector field", not the algebraic structure. It is "of unit normal vectors" because the vector it maps each point on S to has unit norm.

Remark. Observe that this says one can choose orientations on each T_pS in a constant (i.e. continuous) way throughout all of S. Orientability is a **global** concept, and whether or not a surface S is orientable is indeed a topological property of S.

Example 4.4.2. Some examples:

1. Regular surfaces defined **implicitly** as
$$S = F^{-1}\big(\{a\}\big), \quad F : V \subseteq \mathbb{R}^3 \to \mathbb{R},$$
with V open and $a \in \mathbb{R}$ a regular value for F, are always orientable. Indeed,
$$N : S \to \mathbb{R}^3, \quad N(x, y, z) := \frac{\boldsymbol{\nabla} F(x, y, z)}{\big\|\boldsymbol{\nabla} F(x, y, z)\big\|}$$
is a differentiable unit normal field (denominator is never zero, since a is a regular value). For instance, this implies that the following surfaces are orientable:
   (a) *Planes.* $\{ax + by + cz + d = 0\} = F^{-1}\big(\{0\}\big)$, $F(x, y, z) = ax + by + cz + d$.
   (b) *Sphere.* $S^2 = F^{-1}\big(\{1\}\big)$, $F(x, y, z) = x^2 + y^2 + z^2$.
   (c) Graphs of differentiable functions $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$:
$$\mathrm{Gr}(f) = \Big\{\big(x, y, f(x, y)\big) : (x, y) \in U\Big\} = F^{-1}\big(\{0\}\big), \quad F(x, y, z) = z - f(x, y).$$
2. Surface patches $X : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ are orientable. Let $S := X(U)$ and take
$$N : S \to \mathbb{R}^3, \quad N\big(X(u, v)\big) := \frac{X_u(u, v) \times X_v(u, v)}{\big\|X_u(u, v) \times X_v(u, v)\big\|}.$$
The denominator is never zero by (S3), and N is clearly a unit normal field. It is differentiable by (S1). In particular, **locally**, every regular surface is orientable (i.e. $\forall p \in S$, $\exists V \subseteq \mathbb{R}^3$ open such that $p \in V$ and $S \cap V$ is orientable).
3. The **Möbius band** $S = \big\{X(u, v) : u \in \mathbb{R}, |v| < r\big\}$, $r > 0$, with
$$X(u, v) = \alpha(u) + v\beta(u), \quad \alpha(u) = (\cos u, \sin u, 0), \quad \beta(u) = \cos\left(\frac{u}{2}\right)\alpha(u) + \sin\left(\frac{u}{2}\right)(0, 0, 1),$$
is *not* orientable.

Theorem 4.4.3. *A regular surface S is orientable if and only if it can be completely covered with a family of parametrisations $\big\{(X_\alpha, U_\alpha)\big\}_{\alpha \in I}$ such that for any pair $\alpha, \beta \in I$ with $X_\alpha(U_\alpha) \cap X_\beta\big(U_\beta\big) \neq \emptyset$, the corresponding change of parameters map $X_\beta^{-1} \circ X_\alpha$ has Jacobian with positive determinant everywhere.*

Proof. Omitted. □

4.5 Curvature

Definition 4.5.1. Let S be an oriented regular surface. The unit normal field
$$N : S \to S^2 \subseteq \mathbb{R}^3, \quad N(p) \perp T_pS, \quad \big\|N(p)\big\| \equiv 1$$
defining the orientation is called the **Gauss map**.

Remark. First, note that the condition $\big\|N(p)\big\| \equiv 1$ is equivalent to just saying $N(p) \in S^2$ for all $p \in S$. Second, notice that $T_pS = \big(N(p)\big)^\perp$, while $T_{N(p)}S^2 = \big(N(p)\big)^\perp$ as well. Therefore, $T_pS = T_{N(p)}S^2$.

Definition 4.5.2. Let S be an oriented regular surface and N its Gauss map. For each $p \in S$, the linear transformation

$$\mathrm{W}_p := -\mathrm{d}N_p : T_pS \to T_{N(p)}S^2 = T_pS$$

is called the **Weingarten map** (or **shape operator**). Also, $\mathrm{K}(p) := \det \mathrm{W}_p$ is the **Gaussian curvature** of S at p and $\mathrm{H}(p) := \operatorname{tr}\big(\mathrm{W}_p\big)/2$ is the **mean curvature** of S at p.

Remark. For a different orientation $\tilde{N}(p) = -N(p)$, H changes sign but K stays the same. Thus, K is well defined on *any* regular surface (because it is locally orientable), and the same is true for $|\mathrm{H}|$. Note also that in many sources (including Wikipedia), they define $\mathrm{H}(p) = -\operatorname{tr}\big(\mathrm{W}_p\big)/2$.

Example 4.5.3. Some examples:

1. A plane $\Pi = \big\{(x,y,z) \in \mathbb{R}^3 : ax + by + cz + d = 0\big\}$ has

$$N(x,y,z) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}},$$

which is constant. Therefore, $\mathrm{W}_p = -\mathrm{d}N_p \equiv 0$ for all $p \in \Pi$. Therefore, $\mathrm{K} \equiv 0$ (Π is **flat**) and $\mathrm{H} \equiv 0$ (Π is a **minimal surface**).

2. The sphere of radius r is defined by $S_r^2 := \big\{(x,y,z) : x^2+y^2+z^2 = r^2\big\}$. Take the Gauss map

$$N : S_r^2 \to S^2 = S_1^2, \quad N(p) = \frac{p}{r}.$$

Since N is the restriction of the linear map $\mathrm{Id}/r : \mathbb{R}^3 \to \mathbb{R}^3$, it follows that

$$\mathrm{W}_p = -\mathrm{d}N_p = -\frac{1}{r}\,\mathrm{Id}_{T_pS} \implies \mathrm{K}(p) = \frac{1}{r^2}, \quad \mathrm{H}(p) = -\frac{1}{r}.$$

Both are constant, and so we say that

- S_r^2 is a **space form** (constant curvature).
- $S_r^2 \subseteq \mathbb{R}^3$ is a **constant mean curvature surface** (CNC surface).

3. $S = \mathrm{Gr}(f) = \Big\{\big(x,y,f(x,y)\big) : (x,y) \in U\Big\}$, the graph of a differentiable function $f : U \to \mathbb{R}$. Take the parametrisation $X : U \to S$, $X(u,v) = \big(u,v,f(u,v)\big)$. Then $\big\{X_u(q), X_u(q)\big\}$ is a basis of T_pS for all $p = X(q) \in S$. We see that $X_u = (1,0,f_u)$ and $X_v = (0,1,f_v)$. Thus, we can take

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \cdots = \frac{(-f_u, -f_v, 1)}{\sqrt{f_u^2 + f_v^2 + 1}}.$$

Suppose that p is a critical point for f, i.e. $f_u = f_v = 0$ at (u_0,v_0), where $X(u_0,v_0) = p$. Then $T_pS = \operatorname{span}\{e_1,e_2\}$, which is just the xy-plane. To compute $\mathrm{W}_p = -\mathrm{d}N_p$, we need to know $\big(\mathrm{W}_p\big)(e_1)$ and $\big(W_p\big)(e_2)$. By definition of $\mathrm{d}N_p$, we take a curve $\alpha_1(t) = (u_0+t, v_0)$, which gives us

$$\gamma_1(t) := X\big(\alpha_1(t)\big) = \big(u_0+t, v_0, f(u_0+t,v_0)\big).$$

Then by definition of the differential, we have

$$\mathrm{W}_p(e_1) = -\mathrm{d}N_p(e_1) = -\frac{\mathrm{d}}{\mathrm{d}t}\,N\big(\gamma_1(t)\big)\Big|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\Big(-f_u\big(\alpha_1(t)\big), -f_v\big(\alpha_1(t)\big), 1\Big)}{\sqrt{f_u\big(\alpha_1(t)\big)^2 + f_v\big(\alpha_1(t)\big)^2 + 1}}\Bigg|_{t=0}.$$

Noticing that the denominator (denote it $D(t)$) satisfies $D(0) = 1$ and $D'(0) = 0$, we obtain

$$\mathrm{W}_p(e_1) = \big(f_{uu}(u_0,v_0), f_{vu}(u_0,v_0), 0\big).$$

Analogously,

$$\mathrm{W}_p(e_2) = \big(f_{uv}(u_0,v_0), f_{vv}(u_0,v_0), 0\big).$$

Defining $\mathcal{B} := \{e_1, e_2\}$, the basis of T_pS, we have

$$\big[\mathrm{W}_p\big]_{\mathcal{B}} = \begin{pmatrix} f_{uu}(u_0,v_0) & f_{uv}(u_0,v_0) \\ f_{vu}(u_0,v_0) & f_{vv}(u_0,v_0) \end{pmatrix} = (\mathrm{Hess}\, f)(u_0,v_0),$$

where $\mathrm{Hess}\, f$ is the Hessian matrix of f. Thus, $\mathrm{K}(p) = \det\big((\mathrm{Hess}\, f)(u_0,v_0)\big)$ and $\mathrm{H}(p) = \boldsymbol{\nabla}^2 f/2$. Observe that if $\mathrm{K}(p) > 0$, then p is a local extremum, while if $\mathrm{K}(p) < 0$, it is a saddle point. In the below table, we list some explicit examples with $p = (0,0,0)$.

$f(x,y)$	$\big[\mathrm{W}_p\big]_{\mathcal{B}}$	$\mathrm{K}(p)$	$\mathrm{H}(p)$
$1 - x^2 - 2y^2$	$\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$	8	-3
$x^2 - y^2$	$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$	-4	0
$1 - x^2$	$\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$	0	-1
$2xy$	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	-4	0

Table 4.1: Some explicit examples of the surfaces formed from the graphs of differentiable functions and the corresponding values of $\big[\mathrm{W}_p\big]_{\mathcal{B}}$, $\mathrm{K}(p)$ and $\mathrm{H}(p)$, where $p = (0,0,0)$.

4.6 The Second Fundamental Form

4.6.1 Preliminaries

Definition 4.6.1. A **Euclidean vector space** is a finite-dimensional inner product space over $\mathbb{R}$.

Lemma 4.6.2. *Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be a Euclidean vector space with $\dim V = n < \infty$, $T : V \to V$ be a linear transformation and $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthonormal basis for $\big(V, \langle\cdot,\cdot\rangle\big)$. Then the matrix $M := [T]_{\mathcal{B}}$ is given by*

$$M_{ij} = \big\langle Tv_j, v_i\big\rangle, \quad i,j = 1,\dots,n.$$

Proof. By definition,

$$Tv_j = \sum_{k=1}^{n} M_{kj} v_k \implies \big\langle Tv_j, v_i\big\rangle = \sum_{k=1}^{n} M_{kj}\langle v_k, v_i\rangle = \sum_{k=1}^{n} M_{kj}\delta_{ij} = M_{ij}.$$

□

Theorem 4.6.3. *Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be a Euclidean vector space and $T : V \to V$ be linear. Then the following are equivalent:*

1. *$\exists\mathcal{B}$ an orthonormal basis such that $M = [T]_{\mathcal{B}}$ is* ***symmetric****, i.e. $M^T = M$.*
2. *$\forall\mathcal{B}$ an orthonormal basis, $M = [T]_{\mathcal{B}}$ is symmetric.*

3. T is ***self-adjoint***, *i.e.* $\langle Tx, y\rangle = \langle x, Ty\rangle$ *for all* $x, y \in V$.

Proof. (2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (3). Since the inner product is bilinear, it is sufficient to prove T is "self-adjoint for $x, y \in \mathcal{B}$", where $\mathcal{B}$ is the orthonormal basis given in (1). In this case, by the above lemma,

$$\big\langle Tv_i, v_j\big\rangle = M_{ji} \overset{(1)}{=} M_{ij} = \big\langle Tv_j, v_i\big\rangle, \quad \forall i, j.$$

Thus, (3) holds.

(3) $\Rightarrow$ (2). Let $\mathcal{B} = \{v_i\}_{i=1}^n$ be an orthonormal basis. Then by the above lemma,

$$M_{ij} = \big\langle Tv_j, v_i\big\rangle \overset{(3)}{=} \big\langle v_j, Tv_i\big\rangle = M_{ji}.$$

$\square$

Theorem 4.6.4 (Spectral Theorem)**.** *Let* $\big(V, \langle\cdot,\cdot\rangle\big)$ *be a Euclidean vector space and* $T : V \to V$ *be self-adjoint. Then* $\exists\mathcal{B}$ *an orthonormal basis of* V *such that* $[T]_{\mathcal{B}}$ *is diagonal.*

Proof. We prove only the 2-dimensional case. Recalling from Section 4.4 that choosing an orthonormal basis yields an isomorphism of inner product spaces $\big(V, \langle\cdot,\cdot\rangle\big) \simeq \big(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\text{Eucl}}\big)$, we will work with $T : \mathbb{R}^2 \to \mathbb{R}^2$ and let $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product. Let $Q : \mathbb{R}^2 \to \mathbb{R}$, $Q(v) := \langle v, Tv\rangle$ be the quadratic from associated with T (i.e. the quadratic form associated with the symmetric bilinear form $b(\cdot,\cdot) = \langle\cdot, T\cdot\rangle$).

Consider $Q|_{S^1} : S^1 = \big\{v \in \mathbb{R}^2 : \|v\| = 1\big\} \to \mathbb{R}_{>0}$. Since Q is continuous and S^1 is compact, $Q|_{S^1}$ attains a global minimum at some $v_1 \in S^1$. Let $v_2 \in S^1$ with $v_2 \perp v_1$, and set $\lambda_1 = Q(v_1)$, $\lambda_2 = Q(v_2)$, $b = \langle Tv_1, v_2\rangle$ and $\mathcal{B} = \{v_1, v_2\}$. Then by Lemma 4.6.2,

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & b \\ b & \lambda_2 \end{pmatrix}.$$

Now, any $v \in S^1$ is of the form $v = \cos\theta v_1 + \sin\theta v_2$, $\theta \in [0, 2\pi]$, so

$$\begin{aligned} Q(v) = \langle v, Tv\rangle &= \cos^2\theta\langle v_1, Tv_1\rangle + 2\cos\theta\sin\theta\langle Tv_1, v_2\rangle + \sin^2\theta\langle v_2, Tv_2\rangle \\ &= \lambda_1\cos^2\theta + 2b\cos\theta\sin\theta + \lambda_2\sin^2\theta. \end{aligned}$$

Since at $\theta = 0$ this expression attains a minimum, we have

$$0 = \left.\frac{\mathrm{d}Q(v)}{\mathrm{d}\theta}\right|_{\theta=0} = \cdots = 2b \implies b = 0 \implies [T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$\square$

Lemma 4.6.5. *Let* S *be an oriented regular surface. Then for each* $p \in S$, $\mathrm{W}_p : \Big(T_pS, \langle\cdot,\cdot\rangle_p\Big) \to \Big(T_pS, \langle\cdot,\cdot\rangle_p\Big)$ *is self-adjoint, i.e.*

$$\big\langle \mathrm{W}_p(x), y\big\rangle_p = \big\langle x, \mathrm{W}_p(y)\big\rangle_p, \quad \forall x, y \in T_pS.$$

Remark. Recall from Section 4.2 that $\langle\cdot,\cdot\rangle_p$ is the inner product induced on $T_pS \subseteq \mathbb{R}^3$ by the Euclidean inner product on $\mathbb{R}^3$.

Proof. Let $X : U \to S$ be a parametrisation around $p = X(q)$, giving a basis $\{X_u(q), X_v(q)\}$ for T_pS. It is sufficient to prove the self-adjointness of W_p for x, y basis elements. Notice that if $x = y$, then clearly $\langle \mathrm{W}_p(x), y\rangle_p = \langle x, \mathrm{W}_p(y)\rangle_p$. Thus, let $x = X_u(q)$ and $y = X_v(q)$. Since

$$\Big\langle X_u(u,v), N\big(X(u,v)\big)\Big\rangle \equiv 0, \quad \forall (u,v) \in U,$$

(because $X_u(u,v) \in T_pS$, while $N\big(X(u,v)\big) \perp T_pS$), it follows that differentiating with respect to v at $q = (u_0, v_0)$ and applying the Leibniz rule gives:

$$0 = \big\langle X_{uv}(q), N(p)\big\rangle + \left\langle X_u(q), \frac{\partial}{\partial v}\big(N \circ X(u,v)\big)\bigg|_{v=q}\right\rangle.$$

But

$$\frac{\partial}{\partial v}\big(N \circ X(u,v)\big)\bigg|_{q=(u_0,v_0)} = \frac{\mathrm{d}}{\mathrm{d}t} N\big(X(u_0, v_0 + t)\big)\bigg|_{t=0} = \mathrm{d}N_p\big(X_v(q)\big),$$

by definition of $\mathrm{d}N_p$ and where $X(u_0, v_0 + t) = \alpha(t)$ for $\alpha : (-\varepsilon, \varepsilon) \to S$, $\alpha(0) = p$ and $\alpha'(0) = X_v(q)$. Therefore,

$$\big\langle X_{uv}(q), N(p)\big\rangle = \Big\langle \mathrm{W}_p\big(X_v(q)\big), X_u(q)\Big\rangle.$$

Analogously,

$$\big\langle X_{vu}(q), N(p)\big\rangle = \Big\langle \mathrm{W}_p\big(X_u(q)\big), X_v(q)\Big\rangle.$$

□

4.6.2 Second Fundamental Form

For this section, let S be an oriented regular surface, with unit normal field $N : S \to S^2$ and let $\mathrm{W}_p = -\mathrm{d}N_p$, $p \in S$, be the corresponding Weingarten maps.

Definition 4.6.6. The quadratic form associated with the self-adjoint linear transformation $\mathrm{W}_p : T_pS \to T_pS$ is called the **Second Fundamental Form** (of S at $p \in S$; SFF), and is denoted by

$$\mathbb{I}_p : T_pS \times T_pS \to \mathbb{R}, \quad \mathbb{I}_p(v) := \big\langle v, \mathrm{W}_p(v)\big\rangle = -\big\langle v, \mathrm{d}N_p(v)\big\rangle.$$

Given $p \in S$, let $\mathcal{B} = \{v_1, v_2\}$ be an orthonormal basis for $\Big(T_pS, \langle\cdot,\cdot\rangle_p\Big)$ that diagonalises W_p:

$$\big[\mathrm{W}_p\big]_{\mathcal{B}} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Then

1. $\pm v_1$ and $\pm v_2$ are called **principal directions** of S at p.
2. k_1 and k_2 are called **principal curvatures**.
3. If $k_1 = k_2$, p is called an **umbilical point**.
4. For $v \in T_pS$, $\|v\| = 1$, we call $\mathbb{I}_p(v)$ the **normal curvature** of S at p, in the direction v.

Remark. $\mathrm{K}(p) = k_1 k_2$ and $\mathrm{H}(p) = (k_1 + k_2)/2$.

Remark. If $v = \cos(\theta)v_1 + \sin(\theta)v_2$, then $\mathbb{I}_p(v) = k_1 \cos^2\theta + k_2 \sin^2\theta$, meaning every normal curvature is a convex combination of k_1 and k_2.

Example 4.6.7. Some examples:

1. On the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, all the points are umbilical: we have, respectively,

$$\mathrm{W}_p \equiv 0 \quad \text{and} \quad \mathrm{W}_p = -\operatorname{Id}_{T_pS} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Consider the cylinder $C_r = \{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R}\}$. $X(\theta, z) = (r\cos\theta, r\sin\theta, z)$ is a local parametrisation. We see that $X_\theta(\theta, z) = (-r\sin\theta, r\cos\theta, 0)$ and $X_z(\theta, z) = (0, 0, 1)$. We also know that $T_{X(\theta,z)}C_r = \operatorname{span}\{X_\theta(\theta, z), X_z(\theta, z)\}$ and take

$$N\big(X(\theta, z)\big) = \frac{X_\theta(\theta, z) \times X_z(\theta, z)}{\big\|X_\theta(\theta, z) \times X_z(\theta, z)\big\|} = (\cos\theta, \sin\theta, 0).$$

Then by definition,

$$\begin{aligned}
\mathrm{d}N_p(X_\theta) &= \frac{\mathrm{d}}{\mathrm{d}t} N\big(X(\theta_0 + t, z_0)\big)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(\cos(\theta_0 + t), \sin(\theta_0 + t), 0\big)\Big|_{t=0} \\
&= (-\sin\theta_0, \cos\theta_0, 0) \\
&= \frac{1}{r} \cdot X_\theta + 0 \cdot X_z \\
\mathrm{d}N_p(X_z) &= \frac{\mathrm{d}}{\mathrm{d}t} N\big(X(\theta_0, z_0 + t)\big)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta_0, \sin\theta_0, 0)\Big|_{t=0} \\
&= 0
\end{aligned}$$

Therefore,

$$\big[\mathrm{W}_p\big]_{\{X_\theta, X_z\}} = \begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix} \implies \mathrm{K} \equiv 0, \quad \mathrm{H} \equiv -\frac{1}{2r}.$$

Thus, X_θ and X_z are principal directions, with principal curvatures $-1/r$ and 0.

Definition 4.6.8. $p \in S$ is called

1. **elliptic** if $\mathrm{K}(p) > 0$.
2. **hyperbolic** if $\mathrm{K}(p) < 0$.
3. **parabolic** if $\mathrm{K}(p) = 0$ but $\mathrm{W}_p \neq 0$.
4. **planar** if $\mathrm{W}_p = 0$.

Theorem 4.6.9. *Let $p \in S$, $v \in T_pS$ with $\|v\| = 1$,*

$$\xi^S_{p,v} := \big\{\gamma : I \to S \text{ a unit speed regular curve} : \gamma(0) = p,\ \gamma'(0) = v\big\},$$

and $\kappa_\gamma(0)$ be the curvature of γ at 0. Then

1.

$$\mathbb{I}_p(v) = \big\langle \gamma''(0), N(p)\big\rangle, \quad \forall \gamma \in \xi^S_{p,v}.$$

2.
$$\left\|\mathbb{I}_p(v)\right\| = \min\left\{\kappa_\gamma(0) : \gamma \in \xi^S_{p,v}\right\}.$$

Proof. (1). Since $\gamma(t) \in S$ for all $t \in I$, $\gamma'(t) \in T_{\gamma(t)}S$ for all $t \in I$, which means $\gamma'(t) \perp N(\gamma(t))$ for all $t \in I$. Thus,
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \gamma'(t), N\big(\gamma(t)\big)\right\rangle\Big|_{t=0} = \big\langle \gamma''(0), N(p)\big\rangle + \underbrace{\big\langle v, \mathrm{d}N_p(v)\big\rangle}_{=-\mathbb{I}_p(v)}.$$

(2). Let $\gamma \in \xi^S_{p,v}$. Then
$$\kappa_\gamma(0) = \left\|\gamma''(0)\right\| = \left\|\gamma''(0)\right\| \cdot \left\|N(p)\right\| \geq \left|\big\langle \gamma''(0), N(p)\big\rangle\right| \overset{(1)}{=} \left\|\mathbb{I}_p(v)\right\|.$$

Therefore,
$$\left\|\mathbb{I}_p(v)\right\| \leq \min\left\{\kappa_\gamma(0) : \gamma \in \xi^S_{p,v}\right\}.$$
To prove equality, it is sufficient to exhibit a curve $\gamma \in \xi^S_{p,v}$ which satisfies $\gamma''(0) \parallel N(p)$ (so that we have equality in Cauchy-Schwarz), i.e. that $\gamma''(0)$ and $N(p)$ are parallel. Let P be the plane through p parallel to $N(p)$ and v. On a small neighbourhood of $p \in S$, $P \cap S$ is a regular curve (intuitively, if a curve is not regular, it "ends" somewhere prior to endpoint of its domain; since S is a regular surface, it has no sharp edges, meaning that for any point $p \in S$ there is an open neighbourhood contained in S, along which $P \cap S$ cannot "end"), and we can parametrise it by arc-length: $\gamma : I \to S$, $\gamma(I) \subseteq P \cap S$, $\gamma'(0) = v$. Since $\gamma(I) \subseteq P$, we have
$$\big\langle \gamma(t) - p, N(p) \times v\big\rangle \equiv 0,$$
meaning γ is twice differentiable and $\big\langle \gamma''(0), N(p) \times v\big\rangle = 0$. But $\gamma''(0) \perp v = \gamma'(0)$, so $\gamma''(0) \parallel N(p)$. □

4.6.3 Second Fundamental Form in Coordinates

Let $S \subseteq \mathbb{R}^3$ be a regular surface with $p \in S$ and $X : U \subseteq \mathbb{R}^2 \to S$ a local parametrisation. Recall that the coefficients of the first fundamental form (with respect to X) are
$$E, F, G : U \to \mathbb{R}, \quad E = \|X_u\|^2, \quad F = \langle X_u, X_v\rangle, \quad G = \|X_v\|^2.$$
Abusing notation, let us write $N = N \circ X : U \to S^2$. Then the coefficients of the second fundamental form with respect to X are given by
$$\begin{aligned} e &:= \langle X_{uu}, N\rangle \overset{*}{=} -\big\langle \mathrm{d}N(X_u), X_u\big\rangle = \big\langle \mathrm{W}_p(X_u), X_u\big\rangle, \\ f &:= \langle X_{uv}, N\rangle = \cdots = \big\langle \mathrm{W}_p(X_u), X_v\big\rangle = \big\langle \mathrm{W}_p(X_v), X_u\big\rangle, \\ g &:= \langle X_{vv}, N\rangle = \cdots = \big\langle \mathrm{W}_p(X_v), X_v\big\rangle, \end{aligned}$$
where
$$(*) : \quad \langle X_u, N\rangle \equiv 0 \implies 0 = \langle X_{uu}, N\rangle + \big\langle X_u, \mathrm{d}N(X_u)\big\rangle.$$
Remark. We also have
$$e = -\langle X_u, N_u\rangle, \quad f = -\langle X_v, N_u\rangle = -\langle X_u, N_v\rangle, \quad g = -\langle X_v, N_v\rangle.$$
If
$$\big[\mathrm{W}_p\big]_{\{X_u, X_v\}} =: \begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} \\ \mathrm{W}_{21} & \mathrm{W}_{22} \end{pmatrix}$$
is the matrix representation of W_p, then
$$\mathrm{W}_p(X_u) = \mathrm{W}_{11}X_u + \mathrm{W}_{21}X_v, \quad \mathrm{W}_p(X_v) = \mathrm{W}_{12}X_u + \mathrm{W}_{22}X_v.$$

Thus,

$$\begin{cases} e = \mathrm{W}_{11}E + \mathrm{W}_{21}F \\ f = \mathrm{W}_{11}F + \mathrm{W}_{21}G \\ f = \mathrm{W}_{12}E + \mathrm{W}_{22}F \\ g = \mathrm{W}_{12}F + \mathrm{W}_{22}G \end{cases} \iff \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} \\ \mathrm{W}_{21} & \mathrm{W}_{22} \end{pmatrix}.$$

Then since $EG - F^2 > 0$, the FFF matrix is invertible, so

$$\boxed{[\mathrm{W}_p]_{\{X_u, X_v\}} = \begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} \\ \mathrm{W}_{21} & \mathrm{W}_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.}$$

4.7 Geometric Characterisations of Gaussian Curvature

Lemma 4.7.1. *Let $Q(v) = \langle Tv, v\rangle$ be the quadratic form associated with the self-adjoint linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Then*

1. *if $\det T > 0$, either $\big(Q(v) > 0, \forall v \neq 0\big)$ or $\big(Q(v) < 0, \forall v \neq 0\big)$.*
2. *if $\det T < 0$, $Q(v)$ changes sign as v varies along $S^1 \subseteq \mathbb{R}^2$.*

Proof. By the spectral theorem (Theorem 4.6.4), $\exists\{v_1, v_2\}$ an orthonormal basis of $\mathbb{R}^2$ diagonalising T, with $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$. For any $v = av_1 + bv_2 \in \mathbb{R}^2$, we have $Q(v) = a^2\lambda_1 + b^2\lambda_2$. Thus, if $\det T = \lambda_1\lambda_2 > 0$, then λ_1 and λ_2 have the same sign, which gives us (1). If $\det T = \lambda_1\lambda_2 < 0$, then λ_1 and λ_2 have different signs, which gives us (2). ☐

Remark. Let $\Pi \subseteq \mathbb{R}^3$ be a plane through $p \in \Pi$, with unit normal N. We study the function "distance to the plane Π": $q_0 =$ orthogonal projection of q onto Π, $q = q_0 + d \cdot N$, where d is the (signed) distance from q to Π. Then

$$\langle N, q - p\rangle = \underbrace{\langle N, q_0 - p\rangle}_{=0} + \langle N, dN\rangle = d.$$

Π divides $\mathbb{R}^3$ into two half spaces: $\big\{q \in \mathbb{R}^3 : d > 0\big\}$ and $\big\{q \in \mathbb{R}^3 : d < 0\big\}$.

Theorem 4.7.2. *Let S be a regular surface and $p \in S$.*

1. *If p is elliptic ($\mathrm{K}(p) > 0$), then $\exists V \subseteq S$ a neighbourhood of p such that V is contained in one half space with respect to T_pS.*
2. *If p is hyperbolic ($\mathrm{K}(p) < 0$), then every neighbourhood contains points on both sides with respect to T_pS.*

Remark. Here, Ramiro gives some examples, but they're only instructive as images; I may add these later at some point. The only other significant piece of information is that he defines "Monkey Saddle" as the graph of $f(u, v) = u^3 - 3uv^2$.

Proof. Let $X : U \subseteq \mathbb{R}^2 \to S$ be a local parametrisation such that $p = X(0, 0)$. By Taylor's approximation,

$$X(u, v) = X(0, 0) + uX_u(0, 0) + vX_v(0, 0) + \frac{1}{2}\big(u^2X_{uu}(0, 0) + 2uvX_{uv}(0, 0) + v^2X_{vv}(0, 0)\big) + \tilde{R},$$

where

$$\lim_{(u,v)\to(0,0)} \frac{\tilde{R}}{u^2 + v^2} = 0.$$

Then, the signed distance from $X(u,v)$ to "T_pS" is

$$\begin{aligned}
d &= \big\langle X(u,v) - X(0,0), N\big\rangle \\
&= u \cdot \underbrace{\big\langle X_u(0,0), N\big\rangle}_{=0} + v \cdot \underbrace{\big\langle X_v(0,0), N\big\rangle}_{=0} + \frac{u^2}{2} \cdot \big\langle X_{uu}(0,0), N\big\rangle + uv \cdot \big\langle X_{uv}(0,0), N\big\rangle \\
&+ \frac{v^2}{2} \cdot \big\langle X_{vv}(0,0), N\big\rangle + \underbrace{\Big\langle \tilde{R}, N\Big\rangle}_{=:R}, \qquad \frac{R}{u^2+v^2} \to 0, \\
&= \frac{1}{2}\, \mathbb{I}_p(w) + R_w \\
&= \|w\|^2 \left(\frac{1}{2}\, \mathbb{I}_p\!\left(\frac{w}{\|w\|}\right) + \frac{R_w}{\|w\|^2} \right),
\end{aligned}$$

where $w = uX_u(0,0) + vX_v(0,0) \in T_pS$ and $R_w/\|w\|^2 \to 0$ as $w \to 0$. By Lemma 4.7.1, if $\mathrm{K}(p) > 0$ then the red term has a fixed sign. Hence, for $\|w\| < \varepsilon$ small enough, d will have a sign, and this implies our claim. The case $\mathrm{K}(p) < 0$ is analogous. □

Theorem 4.7.3. *Let S be a regular surface, $X : U \to S$ a local parametrisation, $(u_0, v_0) \in U$ with $p = X(u_0, v_0)$ and let $\delta > 0$ be small enough so that*

$$D_\delta := \Big\{ (u,v) \in \mathbb{R}^2 : (u-u_0)^2 + (v-v_0)^2 \le \delta^2 \Big\}$$

is contained in U. Then

$$\lim_{\delta \to 0} \underbrace{\frac{\mathrm{Area}_{S^2}\Big(N\big(X(D_\delta)\big)\Big)}{\mathrm{Area}_S\big(X(D_\delta)\big)}}_{=:Q_\delta} = \big|\mathrm{K}(p)\big|.$$

Proof (idea). Let $N = N \circ X : U \to S^2$. By definition,

$$\begin{aligned}
\mathrm{Area}_{S^2}\Big(N\big(X(D_\delta)\big)\Big) &= \iint_{D_\delta} \big\|N_u(u,v) \times N_v(u,v)\big\| \,\mathrm{d}u\,\mathrm{d}v \\
\mathrm{Area}_S\big(X(D_\delta)\big) &= \iint_{D_\delta} \big\|X_u(u,v) \times X_v(u,v)\big\| \,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Now, we have seen that $N_u = \mathrm{d}N(X_u)$ and $N_v = \mathrm{d}N(X_v)$, so

$$N_u \times N_v = \det(\mathrm{d}N)(X_u \times X_v) = \mathrm{K} \cdot (X_u \times X_v).$$

Thus,

$$Q_\delta = \frac{\displaystyle\iint_{D_\delta} |\mathrm{K}| \cdot \|X_u \times X_v\| \,\mathrm{d}u\,\mathrm{d}v}{\displaystyle\iint_{D_\delta} \|X_u \times X_v\| \,\mathrm{d}u\,\mathrm{d}v}.$$

For δ small enough, we have

$$\big|\mathrm{K}(p)\big| - \varepsilon \le \Big|\mathrm{K}\big(X(u,v)\big)\Big| \le \big|\mathrm{K}(p)\big| + \varepsilon, \quad \forall (u,v) \in D_\delta,$$

from which

$$Q_\delta \le \frac{\displaystyle\iint_{D_\delta} \Big(\big|\mathrm{K}(p)\big| + \varepsilon\Big) \cdot \|X_u \times X_v\| \,\mathrm{d}u\,\mathrm{d}v}{\displaystyle\iint_{D_\delta} \|X_u \times X_v\| \,\mathrm{d}u\,\mathrm{d}v} = \big|\mathrm{K}(p)\big| + \varepsilon,$$

and analogously, $\big|\mathrm{K}(p)\big| - \varepsilon \le Q_\delta$. Therefore, $Q_\delta \to \big|\mathrm{K}(p)\big|$ as $\delta \to 0$. □

4.8 Minimal Surfaces

Recall that a regular surface S is called **minimal** if $\mathrm{H} \equiv 0$. Observe that this notion also makes sense for any surface patch (i.e. S may have self-intersections), since given $X : U \to S \subseteq \mathbb{R}^3$, one can compute

$$N = N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|},$$

from which $\mathrm{d}N$, K, H, etc. follow. Such surfaces are called minimal because $S = X(U)$ is a critical point for the **area functional**

$$\mathrm{Area} : \{\text{surfaces}\} \to \mathbb{R}_{>0}, \quad S \mapsto \mathrm{Area}(S) = \iint_U \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Indeed, consider a one-parameter family of surfaces $S_t = X_t(U)$,

$$X_t : U \to \mathbb{R}^3, \quad X_t(u,v) = X(u,v) + th(u,v)N(u,v),$$

where h is the distance from the point $X_0(u,v)$ to $X_t(u,v)$, as measured along the normal vector emanating from $X_0(u,v)$ (i.e. this is not the usual, perpendicular distance). Then

$$\mathrm{Area}(t) = \iint_U \sqrt{E(t)G(t) - F(t)^2}\,\mathrm{d}u\,\mathrm{d}v,$$

where $E(t)$, $F(t)$ and $G(t)$ correspond to S_t.

Lemma 4.8.1.

$$\mathrm{Area}'(0) = -2 \cdot \iint_U h(u,v) \cdot \mathrm{H} \cdot \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Proof. We see that

$$(X_t)_u = X_u + th_u N + thN_u,$$

so

$$E(t) = E + 2th\langle X_u, N_u\rangle + \mathcal{O}(t^2) = E - 2the + \mathcal{O}(t^2),$$

where e is the function defined in Section 4.6.3 and E is the FFF coefficient; note that E is a function of u and v. Similarly,

$$F(t) = F - 2thf + \mathcal{O}(t^2), \quad G(t) = G - 2thg + \mathcal{O}(t^2).$$

Therefore,

$$E(t)G(t) - F(t)^2 = EG - F^2 - 2th(Eg - 2Ff + Ge) + \mathcal{O}(t^2) = \big(EG - F^2\big)(1 - 4th\mathrm{H}) + \mathcal{O}(t^2).$$

Thus,

$$\sqrt{E(t)G(t) - F(t)^2} = \sqrt{EG - F^2}(1 - 2th\mathrm{H}) + \mathcal{O}(t^2),$$

meaning

$$\mathrm{Area}(t) = \mathrm{Area}(0) - 2t \iint_U h\mathrm{H}\sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v + \mathcal{O}(t^2).$$

□

Corollary 4.8.1.1. $\mathrm{Area}'(0) = 0$ *for all variations* S_t *of* S *(i.e. for all* $h : U \to \mathbb{R}$*) if and only if* $\mathrm{H} \equiv 0$.

Remark. Note that we haven't said anything about $\mathrm{Area}''(0)$.

Example 4.8.2. Some examples of minimal surfaces include planes, catenoids, helicoids, the Enneper surface and Scherk's minimal surfaces (there are two of them).

5 Intrinsic Geometry of Surfaces

5.1 Isometries and Invariants

Example 5.1.1. Consider three ellipsoids $\left\{(x,y,z) \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\right\}$ with $(a,b,c) = (1,1,1)$, $(a,b,c) = (1,1.5,2)$ and $(a,b,c) = (10,1,1.5)$. All three are diffeomorphic to S^2, which means they have the same topology. However, the lengths of curves on their surfaces, their curvature, etc. are different, meaning they have different geometries.

Remark. Notions associated to a surface are called **intrinsic** when they can be expressed solely in terms of the FFF. Two surfaces are said to have the same geometry if they share the same FFF.

Definition 5.1.2. Let S and $\tilde{S}$ be regular surfaces. A diffeomorphism $f : S \to \tilde{S}$ is called an **isometry** if

$$\left\langle \mathrm{d}f_p(v), \mathrm{d}f_p(w) \right\rangle_{f(p)} = \langle v, w \rangle_p, \quad \forall p \in S,\ \forall v, w \in T_pS,$$

(i.e. if $\mathrm{d}f_p : \left(T_pS, \langle \cdot, \cdot \rangle_p\right) \to \left(T_{f(p)}\tilde{S}, \langle \cdot, \cdot \rangle_{f(p)}\right)$ is a linear isometry). When such an f exists, we say S and $\tilde{S}$ are **isometric**. We say f is a **local isometry** if $\forall p \in S$, $\exists V \subseteq S$ a neighbourhood of p such that $f|_V : V \to f(V) \subseteq \tilde{S}$ is an isometry.

Remark. Observe that "local isometry" is *not* an equivalence relation since it is not symmetric.

Example 5.1.3. Some examples:

1. The plane $\left\{(x,y,z) \in \mathbb{R}^3 : z = 0\right\}$ and the cylinder $\left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\}$ are locally isometric, with such a local isometry given by

   $$f : \left\{(x,y,z) \in \mathbb{R}^3 : z = 0\right\} \to \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\}, \quad (x,y,0) \mapsto (\cos x, \sin x, y).$$

   However, they are not isometric, since they are not homeomorphic (plane is simply connected, while the cylinder is not) and every isometry is also a homeomorphism.

2. There is a 1-parameter deformation of the helicoid onto the catenoid. All surfaces in this family are locally isometric to each other.

Theorem 5.1.4. *Let S and $\tilde{S}$ be regular surfaces. Suppose $X : U \to S$ and $\tilde{X} : U \to \tilde{S}$ are local parametrisations such that $E = \tilde{E}$, $F = \tilde{F}$ and $G = \tilde{G}$. Then $f := \tilde{X} \circ X^{-1} : X(U) \to \tilde{X}(U)$ is an isometry.*

Proof. Let $p = X(q) \in X(U) \subseteq S$ and write $v \in T_pS$ as

$$v = aX_u + bX_v = \mathrm{d}X|_q(a,b), \quad (a,b) \in \mathbb{R}^2.$$

Then by the chain rule,

$$\mathrm{d}f|_p(v) = \mathrm{d}\tilde{X}|_q \, \mathrm{d}X^{-1}|_p \left(\mathrm{d}X|_q(a,b)\right) = \mathrm{d}\tilde{X}|_q(a,b) = a\tilde{X}_u + b\tilde{X}_v.$$

It follows that

$$\left\langle \mathrm{d}f|_p(v), \mathrm{d}f|_p(v) \right\rangle = a^2\tilde{E} + 2ab\tilde{F} + b^2\tilde{G} = a^2E + 2abF + b^2G = \langle v, v \rangle.$$

Therefore, f is an isometry. $\square$

Example 5.1.5. The cone (without its vertex)

$$C_\alpha = \left\{(x,y,z) \in \mathbb{R}^3 : z = \frac{1}{\tan\alpha}\sqrt{x^2+y^2}\right\} \setminus \{(0,0,0)\}, \quad \alpha \in (0,\pi/2),$$

is locally isometric to the plane $\{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Indeed, a local parametrisation of the plane is $X(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 0)$, $\rho > 0$ and $0 < \theta < 2\pi\sin\alpha$. Then $E \equiv 1$, $F \equiv 0$ and $G \equiv 1$. Meanwhile, we may locally parametrise the cone as

$$\tilde{X}(\rho,\theta) = \left(\rho\sin\alpha\cos\left(\frac{\theta}{\sin\alpha}\right), \rho\sin\alpha\sin\left(\frac{\theta}{\sin\alpha}\right), \rho\cos\alpha\right),$$

with the same restrictions on ρ and θ as before. Then $\tilde{E} \equiv 1$, $\tilde{F} \equiv 0$ and $\tilde{G} \equiv 0$.

Remark. Note that if $E \neq \tilde{E}$, $F \neq \tilde{F}$ or $G \neq \tilde{G}$ for *some* parametrisation, we *cannot* conclude that S and $\tilde{S}$ are not isometric. Indeed, consider the parametrisations $X(u,v) = (u,v)$ and $\tilde{X}(u,v) = (2u,2v)$ of $S = \tilde{S} = \mathbb{R}^2$. Then S is clearly isometric to $\tilde{S}$, but $E \equiv 1$, while $\tilde{E} \equiv 4$.

5.1.1 Christoffel Symbols and Theorema Egregium

Let $S \subseteq \mathbb{R}^3$ be an oriented regular surface and let $X : U \subseteq \mathbb{R}^2 \to S$ be a parametrisation. For all $p = X(q) \in S$, $\{X_u(q), X_v(q)\}$ is a basis for T_pS, and we can define the Gauss map to be

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

Thus, $\{X_u, X_v, N\}$ is a basis for $\mathbb{R}^3$. We will now express the derivatives of X_u, X_v and N in this basis and call the collection of these expressions (Γ):

$$(\Gamma)\begin{cases} X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + e\cdot N, & (\Gamma)_1 \\ X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + f\cdot N, & (\Gamma)_2 \\ X_{vu} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + f\cdot N & \\ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + g\cdot N, & (\Gamma)_4 \\ N_u = -\mathrm{W}_{11}X_u - \mathrm{W}_{21}X_v & \\ N_v = -\mathrm{W}_{12}X_u - \mathrm{W}_{22}X_v & \end{cases}$$

If the coefficients of N were not known, we would find them by taking the inner product of each expression with N, from which $e = \langle N, X_{uu}\rangle$, $f = \langle N, X_{uv}\rangle = \langle N, X_{vu}\rangle$ and $g = \langle N, X_{vv}\rangle$. In the case of N_u and N_v, we see that since $\|N\|^2 \equiv 1$, $\langle N, N_u\rangle = \langle N, N_v\rangle = 0$. The W_{ij} coefficients are the entries of the matrix

$$[-\mathrm{d}N]_{X_u,X_v} = \begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} \\ \mathrm{W}_{21} & \mathrm{W}_{22} \end{pmatrix},$$

first defined in Section 4.6.3. Γ_{ij}^k are called the **Christoffel symbols** of S in the parametrisation X. Since $X_{uv} = X_{vu}$, we conclude that $\Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$. Now recall from Section 4.6.3 that

$$\begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} \\ \mathrm{W}_{21} & \mathrm{W}_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{pmatrix}.$$

To determine the Christoffel symbols, we take the inner product of $(\Gamma)_1$, $(\Gamma)_2$ and $(\Gamma)_4$ with X_u and X_v:

$$(\Gamma)_1 \begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle X_{uu}, X_u \rangle = \frac{1}{2}\left(\|X_u\|^2\right)_u = \frac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle X_{uu}, X_v \rangle \overset{*}{=} \left(\langle X_u, X_v \rangle\right)_u - \frac{1}{2}\left(\|X_u\|^2\right)_v = F_u - \frac{1}{2} E_v \end{cases}$$
$$(\Gamma)_2 \begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \langle X_{uv}, X_u \rangle = \frac{1}{2} E_v \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \langle X_{uv}, X_v \rangle = \frac{1}{2} G_u \end{cases}$$
$$(\Gamma)_4 \begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = \langle X_{vv}, X_u \rangle = F_v - \frac{1}{2} G_u \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \langle X_{vv}, X_v \rangle = \frac{1}{2} G_v \end{cases}$$

where $(*)$: $\langle X_{uu}, X_v \rangle + \langle X_u, X_{vu} \rangle - \langle X_u, X_{vu} \rangle$ and similar rearrangements were used to get some of the results. For each these 3 systems, the matrix of coefficients is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} > 0.$$

We have thus proven the following result.

Theorem 5.1.6. *The Christoffel symbols can be expressed solely in terms of E, F, G and their partial derivatives.*

Remark. Given $E, F, G, e, f, g : U \to \mathbb{R}$, there does not necessarily exist a regular surface with parametrisation $X : U \to S \subseteq \mathbb{R}^3$ such that these are the coefficients of first and second FF. Indeed, we require, for instance, that $EG - F^2 > 0$. The **Bonnet theorem** details explicitly the requirements on E, F, G, e, f and g for a corresponding regular surface to (locally) exist. This surface is uniquely determined up to rigid motions of $\mathbb{R}^3$ (Definition 2.2.1).

Since Γ_{ij}^k are functions of E, F and G, we can obtain more relations by differentiating (Γ) again and using:

$$\begin{cases} \left(X_{uu}\right)_v - \left(X_{uv}\right)_u = 0, & (*)_1 \\ \left(X_{vv}\right)_u - \left(X_{uv}\right)_v = 0, & (*)_2 \\ N_{uv} - N_{vu} = 0, & (*)_3 \end{cases}$$

By using (Γ) and writing everything in terms of $\{X_u, X_v, N\}$, the above system becomes

$$\begin{cases} A_1 X_u + B_1 X_v + C_1 N = 0 \\ A_2 X_u + B_2 X_v + C_2 N = 0, & (**) \\ A_3 X_u + B_3 X_v + C_3 N = 0 \end{cases}$$

where A_i, B_i and C_i are functions of E, F, G, e, f, g and their derivatives for all $i \in \{1, 2, 3\}$. Since X_u, X_v and N form a basis and so are linearly independent, it follows from $(**)$ that $A_i = B_i = C_i = 0$ for all $i \in \{1, 2, 3\}$. We can use this fact to determine further relations. For example, we know $A_1 = B_1 = C_1 = 0$. From $(*)_1$ we also know that $\left(X_{uu}\right)_v = \left(X_{uv}\right)_u$. Using (Γ), we can write this as

$$\left(\Gamma_{12}^1 X_u + \Gamma_{11}^2 X_v + eN\right)_v = \left(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN\right)_u,$$

which, expanded, becomes

$$\Gamma_{11}^1 X_{uv} + \Gamma_{11}^2 X_{vv} + e N_v + \left(\Gamma_{11}^1\right)_v X_u + \left(\Gamma_{11}^2\right)_v X_v + e_v N = \Gamma_{12}^1 X_{uu} + \Gamma_{12}^2 X_{vu} + f N_u + \left(\Gamma_{12}^1\right)_u X_u + \left(\Gamma_{12}^2\right)_u X_v + f_u N.$$

Using (Γ) again, we can write everything in terms of X_u, X_v and N. After doing this, we can equate the coefficients of, say, X_v, since we know from earlier that the final coefficient of X_v (B_1) must be zero:

$$\Gamma_{11}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{22}^2 - e\mathrm{W}_{22} + \big(\Gamma_{11}^2\big)_v = \Gamma_{12}^1\Gamma_{11}^2 + \big(\Gamma_{12}^2\big)^2 - f\mathrm{W}_{21} + \big(\Gamma_{12}^2\big)_u.$$

Re-arranging and substituting for the W terms, we have

$$\begin{aligned}\Gamma_{11}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{22}^2 + \big(\Gamma_{11}^2\big)_v - \Gamma_{12}^1\Gamma_{11}^2 - \big(\Gamma_{12}^2\big)^2 - \big(\Gamma_{12}^2\big)_u &= e\mathrm{W}_{11} - f\mathrm{W}_{21} \\ &= \big(egE - efF + feF - f^2E\big)\cdot\frac{1}{EG-F^2} \\ &= E\cdot\frac{eg-f^2}{EG-F^2} = E\mathrm{K}.\end{aligned}$$

The calculations performed in this section prove the following extremely significant theorem, which Gauss proved in 1825.

Theorem 5.1.7 (Theorema Egregium)**.** *The Gauss curvature of a regular surface is invariant under local isometries. That is, if $f : S_1 \to S_2$ is a local isometry, then $\mathrm{K}_2\big(f(p)\big) = \mathrm{K}_1(p)$, for all $p \in S_1$.*

Remark. A consequence of this theorem is that the Earth (or any spherical object) cannot be displayed on a flat map without distortion.

5.2 Geodesics

In Euclidean geometry, straight lines play a key role. Their analogue on an arbitrary surface $S \subseteq \mathbb{R}^3$ are geodesics. Recall that given $p, q \in S$, their **intrinsic distance** is defined as

$$d_S(p,q) := \inf\left\{\int_0^1 \big\|\gamma'(t)\big\|\,\mathrm{d}t : \gamma : [0,1] \to S \text{ is a regular curve with } \gamma(0) = p \text{ and } \gamma(1) = q\right\}.$$

Intuitively, a geodesic is a curve on S realising the distance between its points (i.e. γ achieves the inf above). For technical reasons, it is more convenient to use another definition. We will then derive this "intuition" as a consequence.

Definition 5.2.1. Let I be an interval. A regular parameterised curve $\gamma : I \to S$ on a regular surface S is a **geodesic** if $\gamma''(t) \perp T_{\gamma(t)}S$ for all $t \in I$.

Example 5.2.2. Some examples:

1. Geodesics on the plane $\big\{(x,y,z) \in \mathbb{R}^3 : z = 0\big\} \subseteq \mathbb{R}^3$ are straight lines. Indeed, if $\gamma(t) = \big(x(t), y(t), 0\big)$ satisfies $\gamma''(t) = c(0,0,1)$ for all t ($(0,0,1)$ is normal to the plane $z = 0$), then $x''(t) \equiv 0$ and $y''(t) \equiv 0$, so x and y are linear functions.
2. On the unit sphere $S^2 = \big\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\} \subseteq \mathbb{R}^3$, geodesics are **great circles**, i.e. curves whose traces are of the form $S^2 \cap P$, for $P \subseteq \mathbb{R}^3$ a plane through $(0,0,0)$. The proof of this is requires some results only covered later. The idea is that we want to show $\{\text{great circles}\} = \big\{\text{geodesics on } S^2\big\}$. We can show $\subseteq$ with "a computation"; to show $\supseteq$, we use the existence and uniqueness of geodesics.

Definition 5.2.3. Let $\alpha : I \to S$ be a regular curve. A **vector field along** α (tangent to S) is a map $t \in I \mapsto w(t) \in T_{\alpha(t)}S$. It is **differentiable** if, in some local parametrisation $X(u,v)$, $w(t) = a(t)X_u + b(t)X_v$, with $a, b : I \to \mathbb{R}$ differentiable.

Definition 5.2.4. Let w be a vector field along $\alpha : I \to S$, with $\alpha(0) = p$. Then for $t \in I$, the orthogonal projection of $(\mathrm{d}w/\mathrm{d}t)(t)$ onto T_pS is called the **covariant derivative** of w in the direction $\alpha'(t)$, denoted by $(\mathrm{D}w/\mathrm{d}t)(t)$ or $\big(\mathrm{D}_{\alpha'(t)}w\big)\big(\alpha(t)\big)$ (sometimes $\boldsymbol{\nabla}\, w(t)/\mathrm{d}t$ is also used). We say w is **parallel along** α if $\mathrm{D}w/\mathrm{d}t \equiv 0$.

Remark. Since both $(\mathrm{d}w/\mathrm{d}t)(t)$ and the projection operator are independent of parametrisation, so is $(\mathrm{D}w/\mathrm{d}t)(t)$. Indeed, it follows that when two surfaces are tangent along α, the covariant derivative of w along α is the same for both surfaces. The derivative is thus well defined.

Remark. Observe that γ being a geodesic $\Leftrightarrow$ γ' is parallel along γ $\Leftrightarrow$ $\mathrm{D}\gamma'/\mathrm{d}t \equiv 0$.

Lemma 5.2.5. *If v and w are parallel vector fields along $\alpha : I \to S$, then $\big\langle v(t), w(t)\big\rangle$ is constant $\forall t \in I$. (In particular, $\big\|v(t)\big\|$ and the angle between $v(t)$ and $w(t)$ are constant).*

Proof. By definition, $v'(t), w'(t) \perp T_{\alpha(t)}S$ for all $t \in I$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle v(t), w(t)\big\rangle = \big\langle v'(t), w(t)\big\rangle + \big\langle v(t), w'(t)\big\rangle = 0,$$

which means $\big\langle v(t), w(t)\big\rangle$ is constant. Applying this to $v = w$ yields $\big\|v(t)\big\|^2$ being constant. Same for $\big\|w(t)\big\|^2$. For the angle, one uses

$$\langle v, w\rangle = \|v\| \cdot \|w\| \cdot \cos\big(\mathrm{ang}(v, w)\big).$$

□

5.2.1 Covariant Derivatives in Local Coordinates

Given a local parametrisation $X(u, v)$, we may write $\alpha(t) = X\big(u(t), v(t)\big)$ and $w(t) = a(t)X_u + b(t)X_v$. Then by the chain rule,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = a'X_u + a \cdot \big(X_u uu' + X_{uv}v'\big) + b'X_v + b \cdot \big(X_{vu}u' + X_{vv}v'\big).$$

Recall from Section 5.1.1 that we can write X_{uu}, X_{uv} and X_{vv} in terms of X_u, X_v and N. After removing the N terms, we are left with (we call this (CD))

$$\boxed{\frac{\mathrm{D}w}{\mathrm{d}t} = \big(a' + au'\Gamma^1_{11} + av'\Gamma^1_{12} + bu'\Gamma^1_{21} + bv'\Gamma^1_{22}\big)X_u + \big(b' + au'\Gamma^2_{11} + av'\Gamma^2_{12} + bu'\Gamma^2_{21} + bv'\Gamma^2_{22}\big)X_v.}$$

Since the Christoffel symbols can be obtained from the FFF, this shows the following result.

Theorem 5.2.6. *Covariant derivatives are intrinsic: if $f : S \to \tilde{S}$ is an isometry between regular surfaces and w is a vector field along $\alpha : I \to S$, then $\tilde{w} := \mathrm{d}f \circ w$ is a vector field along $\tilde{\alpha} := f \circ \alpha$ on $\tilde{S}$, and we have $\mathrm{D}w/\mathrm{d}t = \mathrm{D}\tilde{w}/\mathrm{d}t$.*

Lemma 5.2.7. *Let I be an interval and S a regular surface. $\gamma : I \to S$ is a geodesic if and only if for every local parametrisation $X(u, v)$, its components $u(t)$ and $v(t)$ (defined via $\gamma(t) = X\big(u(t), v(t)\big)$) satisfy the* ***geodesic equations***

$$(GE)\begin{cases} u'' + (u')^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + (v')^2\Gamma^1_{22} = 0 \\ v'' + (u')^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + (v')^2\Gamma^2_{22} = 0 \end{cases}$$

Proof. Apply (CD) to $w = \gamma'$ (in which case $a = u'$ and $b = v'$). □

Theorem 5.2.8. *If $f : S \to \tilde{S}$ is a local isometry between regular surfaces, then $\gamma : I \to S$ is a geodesic if and only if $f \circ \alpha$ is a geodesic (i.e. "geodesics are intrinsic").*

Proof. Omitted $\square$

Example 5.2.9. We look here at the geodesics of the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subseteq \mathbb{R}^3$. $X : (-\pi, \pi) \times \mathbb{R} \to C$, $X(u, v) := (\cos u, \sin u, v)$ is a local parametrisation and also a local isometry between C and $\mathbb{R}^2$. In $\mathbb{R}^2$, the geodesics through $(0, 0)$ are straight lines: $\alpha(t) = (at, bt)$, $a, b \in \mathbb{R}$. Therefore, by the above theorem, the geodesics on C through $p := X(0, 0) = (1, 0, 0)$ are of the form $X\big(\alpha(t)\big) = \big(\cos(at), \sin(at), bt\big)$, i.e. a helix.

Remark. Observe that even though the cylinder is **locally** isometric to the plane, **globally**, the "geometry" on C is *not* Euclidean: there are pairs of "straight lines" through p which meet again. In fact, there is even a "straight line" which is closed.

5.2.2 Existence and Uniqueness of ODE Solutions

Theorem 5.2.10 (Picard-Lindelöf)**.** *Let $I \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^n$ be open sets and $F_1, \ldots, F_n : U \times I \to \mathbb{R}$ be functions which are Lipschitz continuous (respectively C^k) in the U-variable (with Lipschitz constant independent of the I-variable), and continuous on the I-variable. Then for each $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $\exists \varepsilon > 0$ such that the initial value problem*

$$\begin{cases} y_1'(t) = F_1(y_1, \ldots, y_n, t), & y_1(t_0) = v_1, \\ \vdots & \vdots \\ y_n'(t) = F_n(y_1, \ldots, y_n, t), & y_n(t_0) = v_n, \end{cases}$$

has a unique solution $(y_1^v, \ldots, y_n^v)(t) : (-\varepsilon, \varepsilon) \to U$ (which is respectively of class C^{k+1}). Moreover, the map

$$(v, t) \mapsto \big(y_1^v(t), \ldots, y_n^v(t)\big)$$

is continuous (respectively of class C^k).

Remark. This is the most fundamental result in the theory of ordinary differential equations.

5.2.3 Existence and Uniqueness of Parallel Transport and Geodesics

Theorem 5.2.11. *Let $\alpha : I \to S$ be a regular curve on S, $w_0 \in T_{\alpha(t_0)}S$, $t_0 \in I$. Then there exists a unique parallel vector field w along α with $w(t_0) = w_0$ (we call such a w the* ***parallel transport*** *of w_0 along α).*

Proof. Let $X : U \to S$ be a local parametrisation, and assume that $\alpha(I) \subseteq X(U)$. Then $w(t) = a(t)X_u + b(t)X_v$ satisfies the statement if and only if the components a and b solve the initial value problem

$$\begin{cases} 0 = a' + au'\Gamma_{11}^1 + av'\Gamma_{12}^1 + bu'\Gamma_{21}^1 + bv'\Gamma_{22}^1 \\ 0 = b' + au'\Gamma_{11}^2 + av'\Gamma_{12}^2 + bu'\Gamma_{21}^2 + bv'\Gamma_{22}^2 \end{cases} \quad a(t_0) = a_0, \quad b(t_0) = b_0, \quad w_0 = a_0X_u + b_0X_v.$$

By Picard-Lindelöf (Theorem 5.2.10), this initial value problem has a unique solution. $\square$

Theorem 5.2.12 (Existence and Uniqueness of Geodesics)**.** *Let S be a regular surface, $p \in S$ and $w \in T_pS$. Then $\exists \varepsilon = \varepsilon(p, w) > 0$ and a geodesic $\gamma_{p,w} : (-\varepsilon, \varepsilon) \to S$ with $\gamma_{p,w}(0) = p$, $\gamma_{p,w}'(0) = w$. Moreover, $\gamma_{p,w}$ depends smoothly on p and w.*

Proof. Let $X : U \to S$ be a local parametrisation. In coordinates, $\gamma_{p,w}(t) = X\big(u(t), v(t)\big)$, where u and v must solve the geodesic equations, with initial conditions

$$u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = a_0, \quad v'(0) = b_0.$$

Here $p = X(u_0, v_0)$ and $w = a_0 X_u + b_0 X_v$. (GE) (from Lemma 5.2.7) is a system of two second order ODE's. By introducing two auxiliary new variables $a(t) := u'(t)$ and $b(t) := v'(t)$, it becomes the first order system

$$(\mathrm{GE})' \begin{cases} u' = a, & u(0) = u_0 \\ v' = b, & v(0) = v_0 \\ a' + a^2\Gamma^1_{11} + 2ab\Gamma^1_{12} + b^2\Gamma^1_{22} = 0, & a(0) = a_0 \\ b' + a^2\Gamma^2_{11} + 2ab\Gamma^2_{12} + b^2\Gamma^2_{22} = 0, & b(0) = b_0 \end{cases}$$

Again by Picard-Lindelöf, (GE)′ has a unique solution, which depends smoothly on the individual conditions. □

5.3 Geodesic Curvature

Let $\alpha : I \to S$ be a unit-speed regular curve. Then $\alpha''(t) \perp \alpha'(t)$ for all $t \in I$. Recall also the covariant derivative $\mathrm{D}\alpha'/\mathrm{d}t = \mathrm{pr}_t\big(\alpha''(t)\big)$, where $\mathrm{pr}_t : \mathbb{R}^3 \to T_{\alpha(t)}S$ denotes orthogonal projection onto $T_{\alpha(t)}S$ denotes orthogonal projection onto $T_{\alpha(t)}S$. Since projections are self-adjoint maps (MATH3402), we have

$$\left\langle \frac{\mathrm{D}\alpha'}{\mathrm{d}t}, \alpha' \right\rangle = \Big\langle \mathrm{pr}_t\big(\alpha''\big), \alpha' \Big\rangle = \Big\langle \alpha'', \mathrm{pr}_t\big(\alpha'\big) \Big\rangle = \langle \alpha'', \alpha' \rangle = 0 \implies \frac{\mathrm{D}\alpha'}{\mathrm{d}t} \perp \alpha',$$

where $\mathrm{pr}_t(\alpha') = \alpha'$ since $\alpha' \in T_{\alpha(t)}S$. By definition, we also know that $\mathrm{D}\alpha'/\mathrm{d}t \perp N$. Given that $\big\{\alpha'(t), N \times \alpha'(t), N\big\}$ is an orthonormal frame, we must then have

$$\frac{\mathrm{D}\alpha'}{\mathrm{d}t} = \kappa_g \cdot \big(N \times \alpha'(t)\big) \quad \text{for some } \kappa_g : I \to \mathbb{R}.$$

The coefficient κ_g is called the **geodesic curvature** of α at t.

Remark. Observe that $\kappa_g \equiv 0$ if and only if α is a geodesic.

Remark. The sign of κ_g depends on the particular choice of orientation for S and α, i.e. changing N to $-N$ or t to $-t$ changes the sign of κ_g.

Lemma 5.3.1. *Let $\alpha : I \to S$ be a regular, unit-speed curve. If κ_α is the curvature of α (as a space curve), and κ_n and κ_g denote its normal and geodesic curvature (as a curve in S), then $\alpha'' = \kappa_g \cdot (N \times \alpha') + \kappa_n \cdot N$, and $\kappa_\alpha^2 = \kappa_n^2 + \kappa_g^2$.*

Proof. Omitted. □

Remark. The intuition is that κ_g describes how curved α is from the point of view of an observer in S.

Theorem 5.3.2. *Let V be a parallel vector field along $\alpha : I \to S$, and denote by $\varphi(s)$ the angle that $\alpha'(s)$ makes with $V(s)$. Then*

$$\kappa_g(s) = \frac{\mathrm{d}\varphi}{\mathrm{d}s}.$$

Remark. In the upcoming proof we use the **mixed product** (also known as the **scalar triple product**): given three vectors $v^1, v^2, v^3 \in \mathbb{R}^3$, it is defined as the expression

$$\mathrm{mix}\big(v^1, v^2, v^3\big) := \big\langle v^1, v^2 \times v^3 \big\rangle = \det \begin{pmatrix} v^1_1 & v^2_1 & v^3_1 \\ v^1_2 & v^2_2 & v^3_2 \\ v^1_3 & v^2_3 & v^3_3 \end{pmatrix}.$$

In particular, it satisfies $\mathrm{mix}\big(v^1, v^2, v^3\big) = -\mathrm{mix}\big(v^2, v^1, v^3\big)$.

Proof. The angle $\varphi(s)$ satisfies $V \times \alpha' = \sin(\varphi) \cdot N$, so

$$\underbrace{\sin\varphi = \langle N, V \times \alpha' \rangle}_{(*)_s}, \quad \underbrace{\cos\varphi = \langle V, \alpha' \rangle}_{(*)_c}.$$

Lemma 5.3.3. *$\varphi' \cdot \cos\varphi = \kappa_g \cos\varphi$.*

Proof. Differentiating $(*)_s$ with respect to s, we obtain:

$$\varphi' \cdot \cos\varphi = \langle N', V \times \alpha' \rangle + \langle N, V' \times \alpha' \rangle + \langle N, V \times \alpha'' \rangle.$$

Now $V \times \alpha' = cN$ and $\alpha'' = \kappa_n N + \kappa_g (N \times \alpha')$ (from Lemma 5.3.1). Furthermore, $V' = \mathrm{D}V/\mathrm{d}t + cN$, but V is parallel, so $\mathrm{D}V/\mathrm{d}t \equiv 0$. Thus, we have

$$\varphi' \cdot \cos\varphi = \underbrace{\langle N', cN \rangle}_{=0} + \underbrace{\langle N, (cN) \times \alpha' \rangle}_{=0} + \kappa_n \underbrace{\langle N, V \times N \rangle}_{=0} + \kappa_g \Big\langle N, V \times \big(N \times \alpha''\big) \Big\rangle.$$

The first term is zero since $\|N\|^2 \equiv 1$ implies $\langle N', N \rangle \equiv 0$. Hence, using the mixed product property from the previous remark and the fact that $N \times (N \times \alpha') = -\alpha'$, we get

$$\varphi' \cdot \cos\varphi = -\kappa_g \Big\langle V, N \times \big(N \times \alpha'\big) \Big\rangle = \kappa_g \langle V, \alpha' \rangle = \kappa_g \cos\varphi.$$

□

Lemma 5.3.4. *$\varphi' \cdot \sin\varphi = \kappa_g \sin\varphi$.*

Proof. Differentiating $(*)_c$ and again using $V \times \alpha' = cN$ and $\alpha'' = \kappa_n N + \kappa_g (N \times \alpha')$, we have

$$-\varphi' \cdot \sin\varphi = \langle V', \alpha' \rangle + \langle V, \alpha'' \rangle = \underbrace{\langle cN, \alpha' \rangle}_{=0} + \kappa_n \underbrace{\langle V, N \rangle}_{=0} + \kappa_g \langle V, N \times \alpha' \rangle.$$

Using the mixed product property from the above remark, we get

$$-\varphi' \cdot \sin\varphi = -\kappa_g \langle N, V \times \alpha' \rangle \overset{(*)_s}{=} -\kappa_g \sin\varphi.$$

□

From the above two lemmas, it follows that $\varphi' = \kappa_g$. □

5.4 The Exponential Map

In Theorem 5.2.12 we proved that given S a regular surface, $p \in S$ and $v \in T_pS$ (Ramiro changes from w to v here), there exists a unique geodesic $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to S$ with $\gamma_{p,v}(0) = p$, $\gamma'_{p,v}(0) = v$.

Lemma 5.4.1. *If $\gamma_{p,v}(t)$ is defined $\forall t \in (-\varepsilon, \varepsilon)$, then $\forall \lambda > 0$, $\gamma_{p,\lambda v}(t)$ is defined $\forall t \in (-\varepsilon/\lambda, \varepsilon/\lambda)$, and we have that $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$.*

Proof. Let us denote $\alpha : (-\varepsilon/\lambda, \varepsilon/\lambda) \to S$, $\alpha(t) := \gamma_{p,v}(\lambda t)$. Notice that $\alpha(0) = \gamma_{p,v}(0) = p$ and $\alpha'(t) = \lambda \gamma'_{p,v}(\lambda t)$. Therefore, $\alpha'(0) = \lambda v$ and $\alpha''(t) = \lambda^2 \gamma''_{p,v}(\lambda t)$. It follows that

$$\frac{\mathrm{D}\alpha'}{\mathrm{d}t} = \mathrm{pr}_{T_{\alpha(t)}S}\big(\alpha''(t)\big) = \lambda^2 \, \mathrm{pr}_{T_{\alpha(t)}S}\Big(\gamma''_{p,v}(\lambda t)\Big) = 0,$$

because $\gamma_{p,v}$ is a geodesic. Thus, α is a geodesic. By uniqueness, we must have $\alpha(t) = \gamma_{p,\lambda v}(t)$ for all $t \in (-\varepsilon/\lambda, \varepsilon/\lambda)$. □

Remark. In what follows, 0_p denotes the origin of the tangent plane T_pS to S at p. This is just p, but understood as an element of T_pS.

Let $v \in T_pS \setminus \{0_p\}$ and set $v_1 = v/\|v\|$. Take $\varepsilon_1 > 0$ such that $\gamma_{p,v_1}(t)$ is defined for all $t \in (-\varepsilon_1, \varepsilon_1)$. If $\|v\| < \varepsilon_1$, then $\gamma_{p,v_1}(\|v\|) = \gamma_{p,v/\|v\|}(\|v\|) = \gamma_{p,v}(1)$ is defined. We denote it by

$$\exp_p(v) := \gamma_{p,v}(1),$$

and call it the **exponential map** at the point p. Intuitively, we "follow the geodesic in the direction of $v/\|v\|$, for a total of $\|v\|$ seconds". It is a map $\exp_p : B_\varepsilon(0_p) \subseteq T_pS \to S$. Notice however that since geodesics may *not* be defined for all $t \in \mathbb{R}$, the domain of $\exp_p$ cannot always be taken to be all of T_pS. Nevertheless, we will see that on some open neighbourhood of $0 \in T_pS$, $\exp_p$ is smooth and injective.

5.5 Normal Coordinates

Recall that given $p \in S$, we may define $\exp_p(v) = \gamma_{p,v}(1)$, provided $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to S$ is defined at $t = 1$ (i.e. if $\varepsilon = \varepsilon(p, v) > 1$).

Theorem 5.5.1. *Let S be a regular surface. Given $p \in S$, $\exists \varepsilon = \varepsilon(p) > 0$ such that $\exp_p$ is defined and is differentiable on $B_\varepsilon(0_p) = \{v \in T_pS : \|v\| < \varepsilon\}$.*

Proof. Choose a parametrisation $X : U \to S$ with $X(u_0, v_0) = p$. This also gives a "parametrisation" for $T_pS = \{aX_u(u_0, v_0) + bX_v(u_0, v_0) : a, b \in \mathbb{R}\}$. By Picard-Lindelöf (Theorem 5.2.10), applied to the geodesic equation corresponding to this parametrisation, $\exists \varepsilon_2 > 0$ and a differentiable map

$$\gamma : (-\varepsilon_2, \varepsilon_2)^3 \to S, \quad (t, a, b) \mapsto \gamma_{p,v}(t), \quad v = aX_u + bX_V.$$

By composing with the map $T_pS \to \mathbb{R}^2$, $v \mapsto (a, b)$, it is clear that for some $\varepsilon_1 > 0$, the map

$$\gamma : (-\varepsilon_2, \varepsilon_2) \times B_{\varepsilon_1}(0_p) \to S, \quad (t, v) \mapsto \gamma_{p,v}(t),$$

is differentiable. Take $\lambda = \varepsilon_2/2$, and apply Lemma 5.4.1: $\gamma_{p,\varepsilon_2 v/2}(t)$ is defined $\forall t \in (-\varepsilon_2/\lambda, \varepsilon_2/\lambda) = (-2, 2)$, and for all v such that $\|v\| < \varepsilon_1$. Hence, if $\varepsilon < \varepsilon_1\varepsilon_2/2$, then $\gamma_{p,w}(1) = \exp_p(w)$ is defined and differentiable for all w such that $\|w\| < \varepsilon$. ☐

Theorem 5.5.2. *The differential of $\exp_p : B_\varepsilon(0_p) \subseteq T_pS \to S$ at $0_p \in T_pS$ is Id_{T_pS} (after the natural identification $T_{0_p}(T_pS) \simeq T_pS$). In particular, $\exists U \subseteq B_\varepsilon(0_p)$ a neighbourhood in T_pS such that $\exp_p : U \to S$ is a diffeomorphism onto its image.*

Proof. Let $\alpha : I \to T_pS$, $\alpha(t) = tv$, where $v \in T_pS$. Then $\alpha(0) = 0_p$ and $\alpha'(0) = v$ (this uses the natural identification mentioned above), so we may compute by definition:

$$\left(\mathrm{d}\exp_p\right)_{0_p}(v) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_p(\alpha(t))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma_{p,tv}(1)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma_{p,v}(t)\bigg|_{t=0} = v,$$

where the second last equality follows from Lemma 5.4.1, and the last equality follows from the definition of γ. The last claim in the theorem follows from the inverse function theorem. ☐

Given U as in the above theorem, we call $V := \exp_p(U)$ a **normal neighbourhood** of p in S. If $\{e_1, e_2\}$ is a fixed orthonormal basis for T_pS, yielding the linear isomorphism

$$\mathbb{R}^2 \to T_pS, \quad (u, v) \mapsto ue_1 + ve_2,$$

then on $\tilde{U} := \{(u, v) : ue_1 + ve_2 \in U\}$ there is a map

$$X : \tilde{U} \to S, \quad (u, v) \mapsto \exp_p(ue_1 + ve_2),$$

which is a local parametrisation for S around p. These u and v are called **normal coordinates**.

Remark. Geodesics in S through p are curves of the form $\alpha(t) = X(ta, tb)$, $(a, b) \in \mathbb{R}^2$.

Remark. The coefficient of the FFF at p satisfy $E(p) = 1$, $F(p) = 0$ and $G(p) = 1$.

Using polar coordinates (ρ, θ) on T_pS instead:

$$u = \rho\cos\theta, \quad v = \rho\sin\theta, \quad 0 < \rho, \quad 0 < \theta < 2\pi,$$

yields another natural parametrisation called geodesic polar coordinates (GPC). Let $\ell := \big\{(\rho, \theta) : \theta = 0\big\}$ be a ray in T_pS. Then $\exp_p : U \setminus \ell \to V \setminus \exp_p(\ell)$ is still a diffeomorphism (V normal neighbourhood). The GPC are defined by $(\rho, \theta) \mapsto \exp_p\big(\rho\cos(\theta)e_1 + \rho\sin(\theta)e_2\big)$. The image of circles $\big\{(\rho, \theta) : \rho = \rho_0\big\}$ are called **geodesic circles**, while the image of rays $\big\{(\rho, \theta) : \theta = \theta_0\big\}$ are called **radial geodesics**.

Theorem 5.5.3. *Let $X = X(\rho, \theta) : U \setminus \ell \to V \setminus \ell$ be GPC. Then $E \equiv 1$ and $F \equiv 0$. G need not be constant, but satisfies*

$$\lim_{\rho\to 0^+} G(\rho, \theta) = 0, \quad \lim_{\rho\to 0}\Big(\sqrt{G(\rho, \theta)}\Big)_\rho = 1.$$

Proof. We will only prove the first two assertions. Let $\{e_1, e_2\}$ be a fixed orthonormal basis for T_pS and $v_1 = \cos(\theta)e_1 + \sin(\theta)e_2$. Then

$$X(\rho, \theta) = \exp_p(\rho v_1) = \gamma_{p,\rho v_1}(1) = \gamma_{p,v_1}(\rho) \implies X_\rho(\rho, \theta) = \gamma'_{p,v_1}(\rho), \quad E = \big\|X_\rho\big\|^2 = \Big\|\gamma'_{p,v_1}(\rho)\Big\|^2.$$

But γ_{p,v_1} is a geodesic, so it has constant speed and E is constant. Since

$$\lim_{\rho\to 0} E(\rho, \theta) = \Big\|\gamma'_{p,v_1}(0)\Big\| = \|v_1\| = 1,$$

we have $E \equiv 1$. Regarding F, we look at the second geodesic equation (Lemma 5.2.7):

$$\theta'' + \big(\rho'\big)^2\Gamma^2_{11} + 2\theta'\rho'\Gamma^1_{12} + \big(\theta'\big)^2\Gamma^2_{22} = 0.$$

We know that the curves $\theta \equiv \theta_0$ are solutions, so $\Gamma^2_{11} \equiv 0$. On the other hand, recall from the proof of Theorema Egregium (Theorem 5.1.7) that (everything in red is zero)

$$\begin{cases} \Gamma^1_{11}E + \Gamma^2_{11}F = \frac{1}{2}E_\rho & \implies \Gamma^1_{11} = 0 \\ \Gamma^1_{11}F + \Gamma^2_{11}G = F_\rho - \frac{1}{2}E_\theta & \implies F_\rho = 0. \end{cases}$$

Then it suffices to show that $F \to 0$ as $\rho \to 0$. By definition, $F = \big\langle X_\rho, X_\theta\big\rangle$, and we know that $X_\rho \to v_1$ as $\rho \to 0$. On the other hand, since $X(\rho, \theta) = \exp_p\big(\rho\cos(\theta)e_1 + \rho\sin(\theta)e_2\big)$, by the chain rule we have

$$X_\theta = \Big(\mathrm{d}\exp_p\Big)_{(\rho,\theta)}\big(-\rho\sin(\theta)e_1 + \rho\cos(\theta)e_2\big) = \rho \cdot \underbrace{\Big(\mathrm{d}\exp_p\Big)_{(\rho,\theta)}\overbrace{\big(-\sin(\theta)e_2 + \cos(\theta)e_2\big)}^{=:v_2}}_{\to\,(\mathrm{d}\exp_p)_{(0,0)}v_2 = v_2 \text{ as } \rho\to 0} \to 0, \quad \text{as } \rho \to 0.$$

□

Theorem 5.5.4 (Geodesics Minimise Distance Locally)**.** *Let S be a regular surface and $p \in S$. $\exists W \subseteq S$ a neighbourhood of p such that if $\gamma : I \to W$ is a geodesic with $\gamma(0) = p$, $\gamma(t_1) = q$ and $\alpha : [0, t_1] \to S$ is any regular curve joining p to q, then $\mathcal{L}(\gamma) \le \mathcal{L}(\alpha)$. Moreover, if equality holds, then $\gamma\big([0, t_1]\big) = \alpha\big([0, t_1]\big)$.*

Proof. Let V be a normal neighbourhood of p, and let W be a region in V bounded by a geodesic circle of radius r. Assume first that $\alpha\big([0,t_1]\big) \subseteq W$. Define

$$L := \text{geodesic ray from } p \text{ through } q, \quad \overline{L} := \text{geodesic ray opposite to } L.$$

We know that $\alpha\big([0,t_1]\big) \subseteq L$. Our goal is to estimate $\mathcal{L}(\alpha)$ using GPC. Problem: $\alpha\big([0,t_1]\big)$ may cross $\overline{L}$ (and/or L as well). To address that, we take two systems of GPC: on $W \setminus L$ and on $W \setminus \overline{L}$. Let $A := \alpha^{-1}(W \setminus L)$ and $B := \alpha^{-1}(W \setminus \overline{L})$, two open sets in $[0,t_1]$. They are the union of open intervals in $[0,t_1]$, and $[0,t_1] = A \cup B$. By compactness, there are only finitely many intervals. In other words, we may subdivide $[0,t_1]$ as $0 = a_0 < a_1 < \cdots < a_N = t_1$, such that for each $i = 0,\dots,N-1$, $\alpha\big((a_i,a_{i+1})\big) \subseteq W \setminus L$ or $\alpha\big((a_i,a_{i+1})\big) \subseteq W \setminus \overline{L}$. Write $\alpha(t) = X\big(\rho(t),\theta(t)\big)$ in GPC and use that

$$\big\|\alpha'(t)\big\| = \big\langle \alpha'(t),\alpha'(t)\big\rangle^{1/2} = \sqrt{E\cdot(\rho')^2 + 2\rho'\theta'F + G\cdot(\theta')^2} = \sqrt{(\rho')^2 + G\cdot(\theta')^2},$$

where the last equality follow from the fact that $E \equiv 1$ and $F \equiv 0$. Therefore,

$$\mathcal{L}\Big(\alpha|_{(a_i,a_{i+1})}\Big) = \int_{a_i}^{a_{i+1}} \sqrt{(\rho')^2 + G\cdot(\theta')^2}\,\mathrm{d}t \geq \int_{a_i}^{a_{i+1}} \rho'\,\mathrm{d}t = \rho(a_{i+1}) - \rho(a_i),$$

where the inequality holds because $G\cdot(\theta')^2 \geq 0$. It follows that

$$\mathcal{L}(\alpha) = \sum_{i=0}^{N-1} \mathcal{L}\Big(\alpha|_{(a_i,a_{i+1})}\Big) \geq \sum_{i=0}^{N-1} \rho(a_{i+1}) - \rho(a_i) = \rho(a_N) - \rho(a_0) = t_1 = \mathcal{L}(\gamma).$$

Equality holds if and only if $\theta' \equiv 0$, i.e. θ is constant and $\alpha\big([0,t_1]\big) \subseteq L$. For the general case (i.e. $\alpha\big([0,t_1]\big) \nsubseteq W$), let $t_2 = \inf\big\{t \in [0,t_1] : \alpha(t_1) \notin W\big\}$. Then $\operatorname{im}\big(\alpha|_{[0,t_2]}\big) \subseteq W$, and by the previous case,

$$\mathcal{L}(\alpha) \geq \mathcal{L}\big(\alpha|_{[0,t_2]}\big) \geq r \geq t_1 = \mathcal{L}(\alpha).$$

□

5.6 The Gauss-Bonnet Theorem

Theorem 5.6.1. *Let S be a regular surface and $T \subseteq S$ a geodesic triangle, with interior angles $\varphi_i \in (0,2\pi)$, $i = 1,2,3$. Then*

$$\varphi_1 + \varphi_2 + \varphi_3 - \pi = \iint_T \mathrm{K}\,\mathrm{d}A.$$

Remark. This was the original result that Gauss proved in 1828; several years later, Bonnet generalised Gauss' theorem to regions whose boundary is not necessarily geodesic. One of its immediate consequences is that on surfaces with $\mathrm{K} \neq 0$, the internal angles of a triangle do not sum to $180°$ (i.e. π); for example, on a spheroid like the Earth, with $\mathrm{K} > 0$, the internal angles of a triangle sum to more than π.

Remark. The integral can be defined in analogy with the area: suppose $T = X(D) \subseteq X(U)$, where $X : U \to S$ is a local parametrisation and $D \subseteq U$. Then

$$\iint_T \mathrm{K}\,\mathrm{d}A := \underbrace{\iint_D \mathrm{K}\big(X(u,v)\big)\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v}_{\text{integral in } \mathbb{R}^2}.$$

If T is not contained in a single patch, we write it as a finite union $T = \bigcup_{i=1}^N T_i$, with each $T_i \subseteq X_i(U_i)$, and with $T_i \cap T_j = \emptyset$ or a line segment, if $i \neq j$. Then we set

$$\iint_T \mathrm{K}\,\mathrm{d}A := \sum_{i=1}^N \iint_{T_i} \mathrm{K}\,\mathrm{d}A.$$

It can be shown that this integral is well-defined, that is, independent of the triangulation and of the parametrisations.

Definition 5.6.2. Let S be an oriented regular surface, and let $N : S \to S^2$ be the unit normal (i.e. Gauss map). A continuous parameterised curve $\alpha : [0, \ell] \to S$ is called:

1. **closed**, if $\alpha(0) = \alpha(\ell)$.
2. **simple**, if $\alpha|_{[0,\ell)}$ is injective (i.e. *no* self-intersections).
3. **piecewise regular**, if there exists a subdivision $0 = s_0 < s_1 < \cdots < s_{k+1} = \ell$ such that $\alpha|_{\left(s_i, s_{i+1}\right)}$ is a regular curve $\forall i \in \{0, \ldots, k\}$.

At each **vertex** $\alpha(s_i)$ we set

$$\alpha'_-(s_i) = \lim_{\varepsilon \to 0^+} \alpha'(s_i - \varepsilon), \quad \alpha'_+(s_i) = \lim_{\varepsilon \to 0^+} \alpha'(s_i + \varepsilon).$$

Let $\theta_i \in (-\pi, \pi]$ be the angle between $\alpha'_-(s_i)$ and $\alpha'_+(s_i)$ measured counter-clockwise from $\alpha'_-(s_i)$, so that the ordered basis $\left\{\alpha'_-(s_i), \alpha'_+(s_i), N\big(\alpha(s_i)\big)\right\}$ is positively oriented. We call the θ_i's the **exterior angles**.

Definition 5.6.3. Let S be an oriented regular surface and $N : S \to S^2$ be its unit normal (i.e. Gauss map). In the context of the Gauss-Bonnet theorem, an **admissible region** $R \subseteq S$ (or simply **region** for short) is the union of a connected open set R° (the **interior** of R) and its **boundary** ∂R°. We call R **simple** if ∂R is the trace of a simple, closed, piecewise regular curve $\alpha : I \to S$.

We say a curve $\alpha : I \to S$ with $\alpha(I) \subseteq \partial R$ is **positively oriented** if "when walking along α with head pointing to N, the region R stays to the left".

Theorem 5.6.4 (Gauss-Bonnet, local)**.** *Let $X : U \to S$ be a local parametrisation of a regular surface S, with $U \subseteq \mathbb{R}^2$ homeomorphic to a disk. Let $R \subseteq X(U)$ be a simple region with $\partial R = \alpha(I)$, where $\alpha : I \to S$ is a unit-speed, positively oriented, piecewise regular curve, and let $\alpha(s_0), \ldots, \alpha(s_k)$ and $\theta_0, \ldots, \theta_k$ be, respectively, the vertices and external angles of α. Then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \kappa_g(s)\,\mathrm{d}s + \iint_R \mathrm{K}\,\mathrm{d}A + \sum_{i=0}^{k} \theta_i = 2\pi,$$

where $\kappa_g(s)$ is the geodesic curvature of α and K the Gaussian curvature of S.

Proof (idea). We outline here some of the steps involved in the proof:

1. We may assume without loss of generality that $F \equiv 0$ (i.e. X is an orthogonal parametrisation).
2. Let the elements of U be denoted $(u, v) \in U$. Then we have

$$\sqrt{EG} \cdot \mathrm{K} = -\left(\left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right) \tag{1}$$

   and

$$\kappa_g = \frac{G_u}{2\sqrt{EG}} \cdot \frac{\mathrm{d}v}{\mathrm{d}s} - \frac{E_v}{2\sqrt{EG}} \cdot \frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\mathrm{d}\varphi}{\mathrm{d}s} \tag{2}$$

   where $\varphi = \mathrm{ang}(X_u, \alpha')$ (the angle between X_u and α').

3. Sine U is homeomorphic to a disk, we may use Green's theorem (MATH2001):

$$\iint_U \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v = \int_{\partial U} \left(P\,\frac{\mathrm{d}u}{\mathrm{d}s} + Q\,\frac{\mathrm{d}v}{\mathrm{d}s}\right) \mathrm{d}s \tag{3}$$

Therefore,

$$
\begin{aligned}
\iint_R \mathrm{K}\,\mathrm{d}A &= \iint_{X^{-1}(R)} \mathrm{K}(u,v)\sqrt{EG}\,\mathrm{d}u\,\mathrm{d}v \\
&= -\int_{\partial X^{-1}(R)} \left(\kappa_g(s) - \frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)\mathrm{d}s \\
&= -\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \kappa_g(s)\,\mathrm{d}s + \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{\mathrm{d}\varphi}{\mathrm{d}s}\,\mathrm{d}s \\
&= -\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \kappa_g(s)\,\mathrm{d}s - \sum_{i=0}^{k}\theta_i + 2\pi,
\end{aligned}
$$

where the second equality follows from (1), (2) and (3), while the final equality uses the "theorem of turning tangents". To get from the second line to the third, note that since X is a homeomorphism, it is an open map, so $\partial X^{-1}(R) = X^{-1}(\partial R) = \big(X^{-1}\circ\alpha\big)(I)$. The whole application of Green's theorem takes place in U, so the line integral is taken in uv-space, meaning $\big(X^{-1}\circ\alpha\big)(I)$ is what we want. This set is parameterised by $\big(u(s), v(s)\big)$ with $s \in I$, just as for α, which is how we can turn the line integral into a (sum of) single-variable integral(s) in the third line. We take a sum and integrate piecewise since α is piecewise regular and so this way, our integral "takes into account the corners" and the θ_i's.

□

5.7 The Global Gauss-Bonnet Theorem

Using the same variables as in for the local theorem above, the *global* Gauss-Bonnet theorem differs in that now R does not necessarily have to be contained in a single surface patch, and that R does not necessarily have to be homeomorphic to a disk (i.e. it might have its own topology).

Definition 5.7.1. A region $R \subseteq S$ in a regular surface S is called **regular** if R is compact (closed and bounded, since $S \subseteq \mathbb{R}^3$) and ∂R is a *disjoint* union of finitely many simply, closed, piecewise regular curves.

Remark. Since ∂R must be a disjoint union, it cannot intersect itself. Also, if $R = S$, then since S is regular, $\partial R = \emptyset$.

Definition 5.7.2. A simple region which has only three vertices with external angles $\alpha_i \neq 0$, $i = 1, 2, 3$, is called a **triangle**. A **triangulation** τ of a regular region $R \subseteq S$ is a finite family of triangles $\{T_i\}_{i=1}^F$ such that $\bigcup_{i=1}^F T_i = R$, and if $T_i \cap T_j \neq \emptyset$ $(i \neq j)$ then $T_i \cap T_j$ is either a common edge or a common vertex. Given a triangulation τ, we denote by

$$F := \text{number of triangles ("faces")}, \quad E := \text{number of edges}, \quad V := \text{number of vertices},$$

and we set $\chi := F - E + V$, the **Euler characteristic** (or **Euler-Poincaré characteristic**) of τ.

Remark. Observe that the definition of a "triangle" makes no mention of a requirement for straight edges.

Theorem 5.7.3. *If R is a regular region, then χ does not depend on the triangulation. We thus denote it by $\chi(R)$.*

Proof. Omitted. □

Remark. Smoothness is not essential in the above theorem, so this is really a statement about topology. Moreover, if R_1 and R_2 are **homeomorphic** (i.e. $\exists f : R_1 \to R_2$ continuous, with continuous inverse), then $\chi(R_1) = \chi(R_2)$. Hence, the Euler characteristic is a topological constant.

Example 5.7.4. Some examples:

1. Consider $R \subseteq \mathbb{R}^2$ a disk. If we triangulate it with quarters of the disk, then $F = 4$, $E = 8$ and $V = 5$, so $\chi = 1$. If we triangulate it with thirds, $F = 3$, $E = 6$ and $V = 4$, so $\chi = 1$ again.
2. One can compute that the Euler characteristic of a sphere is 2, that of the torus (homeomorphic to a sphere with one "handle") is zero, and that of the double torus (homeomorphic to a sphere with two handles) is -2. This motivates the following theorem.

Theorem 5.7.5. *If S is a compact oriented surface with g "handles", then $\chi(S) = 2(1-g)$. We call g the **genus** of S and denote it* $\mathrm{genus}(S)$*.*

Proof. Omitted. □

Theorem 5.7.6. *Let $S \subseteq \mathbb{R}^3$ be a compact, oriented surface. Then $\chi(S) \in \{2, 0, -2, \dots, -2k, \dots\}$, and if $\tilde{S}$ is another such surface and $\chi(S) = \chi(\tilde{S})$, then S and $\tilde{S}$ are homeomorphic.*

Proof. Omitted. □

Remark. In other words, every compact connected surface in $\mathbb{R}^3$ is homeomorphic to a sphere with a certain number of handles.

Theorem 5.7.7 (Gauss-Bonnet, global)**.** *Let $R \subseteq S$ be a regular region of an oriented surface S, and write $\partial R = \bigcup_{j=1}^n C_j$, disjoint union of (traces of) simple, closed, positively oriented, regular curves C_j. Let also $\{\theta_i\}_{i=1}^p$ denote the exterior angles at the vertices of ∂R, measured with respect to the orientation of S. Then*

$$\sum_{j=1}^n \int_{C_j} \kappa_g(s)\,\mathrm{d}s + \iint_R \mathrm{K}\,\mathrm{d}A + \sum_{i=1}^p \theta_i = 2\pi \cdot \chi(R),$$

where s denotes the arc length of C_j, and the integral over C_j means the sum of integrals in every regular arc of C_j.

Proof. Let $\tau = \{T_k\}_{k=1}^F$ be a triangulation of R such that each T_k is contained in an orthogonal coordinate system, compatible with the orientation. Then, the local Gauss-Bonnet theorem applied to each T_k yields

$$\sum_{k=1}^F \left(\int_{\partial T_k} \kappa_g(s)\,\mathrm{d}s + \iint_{T_k} \mathrm{K}\,\mathrm{d}A + \sum_{i=1}^3 \theta_i \right) = 2\pi F. \tag{0}$$

Now, since two adjacent edges must have opposite orientations, the first summand only contributes on boundary edges. That is,

$$\sum_{k=1}^F \int_{\partial T_k} \kappa_g(s)\,\mathrm{d}s = \sum_{j=1}^n \int_{C_j} \kappa_g(s)\,\mathrm{d}s.$$

Also, using that $R = \bigcup_{i=1}^F T_k$, we get

$$\sum_{k=1}^F \iint_{T_k} \mathrm{K}\,\mathrm{d}A = \iint_R \mathrm{K}\,\mathrm{d}A.$$

We are thus left with only the third summand $\sum_{1\leq k\leq F}\sum_{1\leq i\leq 3}\theta_{k,i}$. Adopt the following notation:

$$\begin{aligned}
E_e &:= \text{number of external edges of } \tau = \text{number of edges that lie on } \partial R\\
V_e &:= \text{number of external vertices of } \tau = \text{number of vertices that lie on } \partial R = V_{ec} + V_{et}\\
E_i &:= E - E_e = \text{number of internal edges of } \tau\\
V_i &:= V - V_e = \text{number of internal vertices of } \tau
\end{aligned}$$

where c stands for "corners" and t stands for "T-junctions". Each face has 3 edges, so

$$3F = 2E_i + E_e = 2E - E_e, \tag{1}$$

where we have $2E_i$ because internal edges get counted twice. Now let $\varphi_{k,i} := \pi - \theta_{k,i}$ be the interior angles. Then

$$\sum_{k,i}\theta_{k,i} = 3F\pi - \sum_{k,i}\varphi_{k,i} \overset{(1)}{=} 2\pi E - \pi E_e - \sum_{k,i}\varphi_{k,i} \tag{2}$$

and

$$\begin{aligned}
\sum_{k,i}\varphi_{k,i} &= \sum\{\text{angles at an external vertex}\} + \overbrace{\sum\{\text{angles at an internal vertex}\}}^{=2\pi V_i}\\
&= \sum\{\text{angles at an external corner vertex}\} + \sum\{\text{angles at an external "t"-vertex}\} + 2\pi V_i\\
&= \left(\pi V_{ec} - \sum_{i=1}^{p}\theta_i\right) + \pi V_{et} + 2\pi V_i\\
&= \pi V_e + 2\pi V_i - \sum_{i=1}^{p}\theta_i\\
&= 2\pi V - \pi V_e - \sum_{i=1}^{p}\theta_i. \qquad (3)
\end{aligned}$$

Using (2) and (3), we deduce:

$$\begin{aligned}
\sum_{k,i}\theta_{k,i} &= 2\pi E - \pi E_e - 2\pi V + \pi V_e + \sum_{i=1}^{p}\theta_i\\
&= 2\pi(-V + E - F + F) + \sum_{i=1}^{p}\theta_i\\
&= -2\pi\chi(R) + 2\pi F + \sum_{i=1}^{p}\theta_i, \qquad (4)
\end{aligned}$$

where $E_e = V_e$ since each C_j is a simple closed curve. Hence,

$$\sum_{k=1}^{F}\left(\int_{\partial T_k}\kappa_g(s)\,\mathrm{d}s + \iint_{T_k}\mathrm{K}\,\mathrm{d}A + \sum_{i=1}^{3}\theta_{k,i}\right) = \sum_{j=1}^{n}\int_{C_j}\kappa_g(s)\,\mathrm{d}s + \iint_R \mathrm{K}\,\mathrm{d}A + \sum_{i=1}^{p}\theta_i - 2\pi\chi(R) + 2\pi F. \tag{5}$$

The theorem follows from combining (0) and (5). □

Corollary 5.7.7.1. *If R is a simple region (i.e. R is homeomorphic to a disk), then*

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\kappa_g(s)\,\mathrm{d}s + \iint_R \mathrm{K}\,\mathrm{d}A + \sum_{i=0}^{k}\theta_i = 2\pi.$$

Proof. $\chi(R) = \chi(\text{disk}) = 1$, so this follows from the global Gauss-Bonnet theorem. $\square$

Corollary 5.7.7.2. *Let S be a compact, oriented surface. Then*

$$\iint_S \mathrm{K}\,\mathrm{d}A = 2\pi\chi(S).$$

Proof. In this case, $R = S$ has $\partial R = \emptyset$. $\square$

Remark. Observe that the left hand side depends on the intrinsic geometry of S, whereas the right hand side is purely topological.

Corollary 5.7.7.3. *Let S be a compact, oriented surface with* $\mathrm{K} > 0$*. Then S is homeomorphic to S^2.*

Proof. By Corollary 5.7.7.2 and $\mathrm{K} > 0$, we get

$$\chi(S) = \frac{1}{2\pi}\iint_S \mathrm{K}\,\mathrm{d}A > 0.$$

From the topological classification of compact oriented surfaces (Theorem 5.7.6), it follows that $\chi(S) = 2$ and S is homeomorphic to S^2. $\square$

Example 5.7.8. All oblate and prolate spheroids are homeomorphic to $\chi = 2$. Thus, even though K varies greatly between such surfaces, its integral must always be 4π by Corollary 5.7.7.2.

Part III
Manifolds

6 Differentiable Manifolds

6.1 Definition and Examples of Manifolds

Manifolds generalise (regular) curves and surfaces to higher dimensions. Recall that to define regular surfaces (Definitions 3.1.1 and 3.1.2), we used local parametrisations $X : U \subseteq \mathbb{R}^2 \to S$ which were

1. differentiable
2. a homeomorphism
3. regular: $\mathrm{d}X_q$ is injective

The following theorem follows from these axioms.

Theorem 6.1.1. *On a regular surface, the "change of parameters" maps are differentiable. More precisely, if S is a regular surface, $U_\alpha, U_\beta \subseteq \mathbb{R}^2$ are open and $X_\alpha : U_\alpha \to S$ and $X_\beta : U_\beta \to S$ are two parametrisations of S such that $X_\alpha(U_\alpha) \cap X_\beta\big(U_\beta\big) = W \neq \emptyset$, then the mappings $X_\beta^{-1} \circ X_\alpha : X_\alpha^{-1}(W) \to \mathbb{R}^2$ and $X_\alpha^{-1} \circ X_\beta : X_\beta^{-1}(W) \to \mathbb{R}^2$ are differentiable.*

Proof. Omitted. □

Definition 6.1.2. A **differentiable** (or **smooth**) **manifold** of dimension n is a set M and a family $\big\{(U_\alpha, X_\alpha)\big\}$ of injective maps $X_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M$ such that

1. $\bigcup_\alpha X_\alpha(U_\alpha) = M$.
2. For α and β with $X_\alpha(U_\alpha) \cap X_\beta\big(U_\beta\big) =: W \neq \emptyset$, the sets $X_\alpha^{-1}(W)$ and $X_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and $X_\beta^{-1} \circ X_\alpha : X_\alpha^{-1}(W) \to X_\beta^{-1}(W)$ is differentiable.
3. The family $\big\{(U_\alpha, X_\alpha)\big\}$ is maximal among those satisfying (1) and (2).

Given $p \in X_\alpha(U_\alpha)$, (U_α, X_α) is called a **local parametrisation** or **coordinate system** at p; $X_\alpha(U_\alpha)$ is then called a **coordinate neighbourhood** at p. A family $\big\{(U_\alpha, X_\alpha)\big\}$ satisfying (1) and (2) is called a **differentiable structure** (or **smooth structure**) on M.

Remark. Many authors prefer to work with the pairs $(V_\alpha, \varphi_\alpha)$ instead, where $V_\alpha := X_\alpha(U_\alpha) \subseteq M$ and $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$. These are called **charts**, and the collection of all charts $\big\{(V_\alpha, \varphi_\alpha)\big\}$ is an **atlas**.

Remark. Condition (3) is purely technical and does not contribute to our understanding. In general, any family satisfying (1) and (2) can be uniquely extended to a maximal one (take the union of all the parametrisations that, together with any of the parametrisations of the given structure, satisfy condition (2)).

Remark. We do not require that $M \subseteq \mathbb{R}^N$ for some $N \in \mathbb{N}$. These allows for greater flexibility, but at the same time it makes some concepts more abstract (such as tangent vectors).

Remark. A manifold M carries a natural **topology**: we define $A \subseteq M$ to be open if and only if $X_\alpha^{-1}\big(A \cap X_\alpha(U_\alpha)\big)$ is open in $\mathbb{R}^n$, for all α. With this, the parametrisations $X_\alpha : U_\alpha \to X_\alpha U_\alpha$ are homeomorphisms. Most authors begin the definition of smooth manifolds by assuming that M is not just a set but a topological space.

Example 6.1.3. Some examples:

1. Regular surfaces $S \subseteq \mathbb{R}^3$, and in general abstract regular surfaces (e.g. Klein bottle, $\mathbb{RP}^2$ - the real projective plane, etc.) are two dimensional manifolds.
2. Euclidean space $\mathbb{R}^n$.
3. The n dimensional sphere
$$S^n = \big\{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : a_0^2 + \cdots + a_n^2 = 1\big\}.$$
For each $i \in \{0, \ldots, n\}$, we set $U_i^{\pm} := B_1(0) = \big\{a \in \mathbb{R}^n : \|a\| < 1\big\}$ and
$$X_i^{\pm} : U_i \to S, \quad (a_1, \ldots, a_n) \mapsto \left(a_1, \ldots, a_{i-1}, \pm\sqrt{1 - a_1^2 - \cdots - a_n^2}, a_{i+1}, \ldots, a_n\right).$$
These $2n+2$ coordinate systems cover all of S^n. The change of coordinates are given by
$$\begin{aligned}\left(X_j^+\right)^{-1} \circ \left(X_i^+\right)(a_1, \ldots, a_n) &= \left(X_j^+\right)^{-1}\left(a_1, \ldots, \sqrt{1 - \|a\|^2}, \ldots, a_n\right) \\ &= \left(a_1, \ldots, \hat{a}_j, \ldots, \sqrt{1 - \|a\|^2}, \ldots, a_n\right),\end{aligned}$$
where we have assumed that $j < i$, the square root is in the i^{th} position and the hat indicates that a_j is omitted. This is clearly a differentiable function for $\|a\| < 1$, so S^n is a smooth manifold.
4. Real projective space, defined to be
$$\mathbb{RP}^n := \text{set of lines in } \mathbb{R}^{n+1} \text{ through the origin} = \big(\mathbb{R}^{n+1} \setminus \{0\}\big)/\sim,$$
where $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$. Points in $\mathbb{RP}^n$ are equivalence classes, denoted by $(a_0 : \cdots : a_n)$. Notice that if $a_i \neq 0$, then
$$(a_0 : \cdots : a_i : \cdots a_n) = \left(\frac{a_0}{a_i} : \cdots : 1 : \cdots : \frac{a_n}{a_i}\right).$$
5. Any finite dimensional vector space V has a natural structure to make it a smooth manifold: choose a basis; then there exists a linear isomorphism $L : V \to \mathbb{R}^n$, $n = \dim V$. Now we can use L to construct a single chart covering all of V. In fact, V and $\mathbb{R}^n$ are "diffeomorpic".
6. $\mathrm{M}_{m\times n}(\mathbb{R}) = \{m \times n \text{ matrices with real entires}\}$ is a real vector space of dimension mn. Thus, it has a natural structure to make it a smooth manifold.
7. Let M be any n dimensional manifold, and let $U \subseteq M$ be an open set. Then, given a coordinate system (U_α, X_α), $X_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M$, with $X_\alpha(U_\alpha) \cap U \neq \emptyset$, we set $V_\alpha := X_\alpha^{-1}\big(X_\alpha(U_\alpha) \cap U\big) \subseteq \mathbb{R}^n$ and we can consider the restricted coordinate system $\tilde{X}_\alpha := X_\alpha|_{V_\alpha} : V_\alpha \to U$. It is not difficult to check that $\left(V_\alpha, \tilde{X}_\alpha\right)$ defines a smooth structure on U. Thus, U is a smooth manifold.
8. The general linear group $\mathrm{GL}_n(\mathbb{R}) := \big\{A \in \mathrm{M}_{n\times n}(\mathbb{R}) : \det A \neq 0\big\}$. Because $\det : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a continuous function, $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}\big(\mathbb{R} \setminus \{0\}\big)$ is an open subset of $\mathrm{M}_{n\times n}(\mathbb{R})$. Hence, by (6) and (7) above, it is a smooth manifold.

   Note that it is also a group, and in fact the group operations are smooth, thus $\mathrm{GL}_n(\mathbb{R})$ is also a **Lie group**. If $G \leq \mathrm{GL}_n(\mathbb{R})$ (G is a subgroup of $\mathrm{GL}_n(\mathbb{R})$) is closed and connected, then G is a Lie group.
9. Graphs of smooth functions. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. Its graph is the set
$$\mathrm{Gr}(f) := \Big\{\big(a, f(a)\big) : a \in U\Big\} \subseteq U \times \mathbb{R}^k.$$
It can be covered with a single chart (U, X), where $X(a) = \big(a, f(a)\big)$. Note that at this point, Ramiro switches to his MATH3402 notation of Γ to denote the graph of a function (instead of Gr, which he has used thus far); as far as I can tell, this has no mathematical significance, so I've opted to stay with Gr.

10. Level sets. Let $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. For $c \in \mathbb{R}$, $f^{-1}(\{c\})$ is called a level set of f. Assume c is a regular value, i.e. $\boldsymbol{\nabla} f(x) \neq 0$ for all $x \in f^{-1}(\{c\})$. We claim that $M = f^{-1}(\{c\})$ is a smooth manifold. Indeed, around each $x \in M$ we know there exists $x \in \{1, \dots, n\}$ such that $\partial_{x_i} f(x) \neq 0$. By the implicit function theorem, around x there is a neighbourhood U so that in $U \cap f^{-1}(\{c\})$ we have $x_i = \phi(x_1, \dots, \hat{x}_i, \dots, x_n)$, for some $\phi : V \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ smooth and where the hat again indicates that x_i is omitted. Therefore, after rearranging coordinates, we see that $U \cap f^{-1}(\{c\}) = \mathrm{Gr}(\phi)$. It is not difficult to see, arguing as in the case of the sphere S^n, that the "change of coordinates" maps are smooth.
11. Product manifolds. If M (dimension m) and N (dimension n) are smooth manifolds with coordinate systems $\{(U_\alpha, X_\alpha)\}$ and $\Big\{(V_\beta, X_\beta)\Big\}$, then $M \times N$ is a smooth manifold of dimension $m + n$, with coordinates

$$\Big(U_\alpha \times V_\beta, \big(X_\alpha, Y_\beta\big)\Big), \quad \big(X_\alpha, Y_\beta\big) : U_\alpha \times V_\beta \to M \times N, \quad (a, b) \mapsto \big(X_\alpha(a), Y_\beta(b)\big).$$

Remark. From now on, we shall denote a differentiable manifold M of dimension m by M^m.

Definition 6.1.4. Let M^m and N^n be smooth manifolds. We call a function $f : M \to N$ **differentiable** (or **smooth**) at $p \in M$ if there exist parametrisations (U, X) and (V, Y) at p and $f(p)$ respectively, with $p \in X(U)$ and $f\big(X(U)\big) \subseteq Y(V)$ so that $Y^{-1} \circ f \circ X : U \to V$ is differentiable. It is called **differentiable** if it is differentiable at all $p \in M$. We say $f : M \to N$ is a **diffeomorphism** if it is differentiable and it has a differentiable inverse (which implies $m = n$).

6.2 Tangent Vectors

Let $p \in M^n$ be a point in a smooth manifold, and set

$$C_p^\infty(M) = \{f : U \subseteq M \to \mathbb{R} \text{ differentiable}, \, p \in U\}.$$

Definition 6.2.1. A linear map $v : C_p^\infty \to \mathbb{R}$ is called a **derivation** at p if

$$v(f \cdot g) = v(f)g(p) + f(p)v(g), \quad \forall f, g \in C_p^\infty(M).$$

We call the set of all derivations at p the **tangent space** to M at p, and denote it by T_pM. Its elements are also called **tangent vectors**.

Remark. A more explicit way to look at T_pM is as follows: Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\alpha(0) = p$. Then we define

$$\alpha'(0) : C_p^\infty(M) \to \mathbb{R}, \quad \alpha'(0)(f) := \left.\frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}t}\right|_{t=0}.$$

It is easy to see that $\alpha'(0)$ is a derivation, and hence $\alpha'(0) \in T_pM$. Moreover, *all* $v \in T_pM$ arise in this way.

6.2.1 Tangent Vectors in Coordinates

Given a coordinate system $X : U \subseteq \mathbb{R}^n \to M$, the canonical basis $\{e_i\}_{i=1}^n$ in $\mathbb{R}^n$ gives rise to an associated basis of T_pM: say $X(0) = p$ and set

$$\left.\frac{\partial}{\partial x_i}\right|_p := \left.\frac{\mathrm{d}}{\mathrm{d}t} X(0, \dots, t, \dots, 0)\right|_{t=0} \in T_pM,$$

where t is in the i^{th} position. Then $\left\{\partial_{x_i}\big|_p\right\}_{i=1}^n$ forms a basis for T_pM. Given $f \in C_p^\infty(M)$, we have

$$\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(X(0,\dots,t,\dots,0)\big)\bigg|_{t=0} = \frac{\partial(f\circ X)}{\partial x_i}(0),$$

where the partial derivative on the right hand side is the usual partial derivative of $f \circ X : \mathbb{R}^n \to \mathbb{R}$. It follows that T_pM is an n dimensional real vector space.

6.3 Differential of a Function

T_pM is the "linear approximation" of our manifold M at p. We now wish to define the linear approximation of a differentiable function $f : M^m \to N^n$.

Definition 6.3.1. Let M^m and N^n be smooth manifolds. The **differential** of $f : M \to N$ at $p \in M$ is the linear map

$$\mathrm{d}f_p : T_pM \to T_{f(p)}N, \quad v \in T_pM \mapsto \mathrm{d}f_p(v) \in T_{f(p)}N,$$

where

$$\big(\mathrm{d}f_p(v)\big)(g) := v(g \circ f), \quad \forall g \in C^\infty_{f(p)}(N).$$

Remark. Observe that if $\alpha : (-\varepsilon, \varepsilon) \to M$ is a curve with $\alpha(0) = p$ and $\alpha'(0) = v$, then $\mathrm{d}f_p(v) = \beta'(0)$, where $\beta(t) = f\big(\alpha(t)\big)$.

Remark. We also have the **chain rule**: if $f : M \to N$ and $g : N \to P$ are differentiable functions between smooth manifolds, then $g \circ f$ is differentiable and

$$\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p, \quad \forall p \in M.$$

In particular, if f is a (local) diffeomorphism, then $\mathrm{d}f_p$ is an invertible linear map $\forall p \in M$. We also have the inverse function theorem for manifolds, as stated below.

Theorem 6.3.2 (Inverse Function Theorem for Manifolds)**.** *If $f : M \to N$ is a differentiable function between smooth manifolds and $p \in M$ such that $\mathrm{d}f_p : T_pM \to T_{f(p)}M$ is a linear isomorphism, then $\exists U \subseteq M, V \subseteq N$ neighbourhoods of p and $f(p)$ respectively, such that $f|_U : U \to V$ is a diffeomorphism.*

6.4 Immersions and Submanifolds

Let M^m and N^n be smooth manifolds with $m < n$.

Definition 6.4.1. A smooth map $f : M \to N$ is called an **immersion** if $\mathrm{d}f_p$ is injective $\forall p \in M$. If $M \subseteq N$ is a subset and the inclusion $\iota : M \to N$ is an immersion, we call M a **submanifold** of N. When M is a closed subset of N (under the induced topology), we say it is an **embedded submanifold**.

Example 6.4.2. Some examples:

1. A regular curve $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is an immersion. If $\alpha(I)$ has no self-intersections (i.e. α is injective/simple), then it is a one dimensional submanifold of $\mathbb{R}^3$. If α is also closed, $\alpha(I)$ is an embedded submanifold.
2. Regular surfaces in $\mathbb{R}^3$ are two dimensional embedded submanifolds in $\mathbb{R}^3$.
3. $f^{-1}\big(\{c\}\big) \subseteq M$ for $c \in \mathbb{R}$ a regular value of a smooth function $f : M \to \mathbb{R}$ is always an embedded hypersurface (a submanifold of codimension 1).

6.5 Vector Fields

Definition 6.5.1. A **vector field** F on a smooth manifold M is a map

$$F : M \to \bigcup_{p \in M} T_pM =: TM, \quad p \in M \mapsto F_p \in T_pM,$$

where TM is called a **tangent bundle**. We say F is **smooth** at $p \in M$ if there exists a coordinate neighbourhood (U, X), $p \in X(U)$, such that when we write

$$F_q = \sum_{i=1}^{n} a_i(q) \cdot \left. \frac{\partial}{\partial x_i} \right|_q, \quad q \in X(U),$$

the functions $a_i : X(U) \to \mathbb{R}$ are smooth (well-defined due to smoothness of change of coordinates). The space of all smooth vector fields on M is denoted by $\mathscr{X}(M)$.

Remark. Observe that given $f \in C^\infty(M)$, F a vector field on M, we have a function

$$F(f) : M \to \mathbb{R}, \quad F(f)(p) := F_p(f).$$

If F is smooth, then $F(f)$ will also be smooth, and should be interpreted as the "derivative of f in the direction F".

Definition 6.5.2. Given $F, G \in \mathscr{X}(M)$, their **Lie bracket** is $[F, G] \in \mathscr{X}(M)$ defined by

$$[F, G]_p(f) := F_p\big(G(f)\big) - G_p\big(F(f)\big), \quad \forall f \in C_p^\infty(M).$$

This is skew-symmetric: $[F, G] = -[G, F]$. It also satisfies the **Jacobi identity**:

$$\big[F, [G, H]\big] + \big[G, [H, F]\big] + \big[H, [F, G]\big] = 0,$$

for all smooth vector fields F, G and H on M. Thus, $\mathscr{X}(M)$ is a **Lie algebra**.

7 Riemannian Geometry

7.1 Riemannian Metrics and Manifolds

Definition 7.1.1. Let M^n be a smooth manifold. A **Riemannian metric** on M is a map g that associates to each $p \in M$ an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ on T_pM. We say g is smooth if for any two $F, G \in \mathscr{X}(M)$ (smooth), the function $M \to \mathbb{R}$, $p \mapsto g_p\big(F_p, G_p\big)$, is smooth. The pair (M^n, g) is called a **Riemannian manifold**.

Example 7.1.2. Some examples:

1. On a regular surface $S \subseteq \mathbb{R}^3$, the first fundamental form is a Riemannian metric.
2. The Euclidean metric on $\mathbb{R}^n$. Denote by $\langle \cdot, \cdot \rangle$ the standard scalar product. For any $p \in \mathbb{R}^n$, we have a natural identification $T_p\mathbb{R}^n \simeq \mathbb{R}^n$, so we may define $g_p(\cdot, \cdot) := \langle \cdot, \cdot \rangle$. In this way, $(\mathbb{R}^n, g)$ becomes a Riemannian manifold, and g is called the **Euclidean metric** on $\mathbb{R}^n$.
3. The "**round metric**" on the sphere S^n. $S^n \subseteq \mathbb{R}^{n+1}$, so for any $p \in S^n$, T_pS^n is naturally a linear subspace of $T_p\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$. Hence, we may define $g_p := \langle \cdot, \cdot \rangle|_{T_pS^n}$, the restriction of the Euclidean scalar product. In this way, (S^n, g) is a Riemannian manifold which we call the unit sphere (or round sphere).

4. The hyperbolic space $\mathbb{H}^n$. Let $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. As a smooth manifold, $\mathbb{H}^n$ is diffeomorphic to $\mathbb{R}^n$. We define the hyperbolic metric on it as follows:

$$g_p := \frac{1}{x_n^2} \langle \cdot, \cdot \rangle, \quad p = (x_1, \ldots, x_n), \quad \langle \cdot, \cdot \rangle \text{ Euclidean.}$$

   Then $(\mathbb{H}^n, g)$ is called the **Hyperbolic space**.

5. Example (3) above may be generalised as follows: let (N, g^N) be a Riemannian manifold, and let $M \subseteq N$ be a submanifold. Then, one can naturally see $T_pM \subseteq T_pN$ as a linear subspace $\forall p \in M$ (given a curve $\alpha : (-\varepsilon, \varepsilon) \to M$, we may view it as a curve in N). We define the induced metric on M as $g^M := g^N|_{T_pM \times T_pM}$. Thus, (M, g^M) is a Riemannian manifold.
   - The main object of study in this course are 2 dimensional submanifolds of $\mathbb{R}^3$ with the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^3$.

Remark. The examples $\mathbb{R}^n$, S^n and $\mathbb{H}^n$ are the **model spaces** in Riemannian geometry: they have *constant curvature* 0, 1 and -1, respectively.

7.1.1 Geodesics on Riemannian Manifolds

As in surfaces, a Riemannian metric g on M^n allows us to measure the lengths of curves in M: we get a distance function

$$d_g(p, q) = \inf\{\mathcal{L}(\gamma) : \gamma \text{ a curve on } M \text{ joining } p \text{ to } q\}, \quad p, q \in M.$$

Thus, (M, d_g) is a metric space. The curves on M that locally minimise d_g are called **geodesics**. It is possible to define them as "curves without acceleration", for some notion of derivative intrinsic to M called the **covariant derivative**.

7.2 Curvature on Riemannian Manifolds

As in the case of surfaces, we would like some invariant to differentiate between a Riemannian manifold and the flat Euclidean space. This gives rise to the notion of **curvature**, which is a generalisation of Gauss' curvature to higher dimensions. To do this properly involves some technicalities with tensor calculus and covariant derivatives. Here we will only discuss a geometric interpretation.

For a surface Σ^2, Gaussian curvature is a function $\mathrm{K}^G : \Sigma^2 \to \mathbb{R}$. On an arbitrary Riemannian manifold (M^n, g), its curvature is a mapping $p \in M \mapsto \mathrm{K}_p$, where K_p is defined as follows: Given $\Pi \subseteq T_pM$ a two dimensional subspace (a "plane"), we consider $\Sigma_p \subseteq M$ to be the union of all geodesics emanating from p with initial velocity contained in Π. We then set $\mathrm{K}_p(\Pi) := \mathrm{K}^G(\Sigma_p)$, the Gaussian curvature of the surface Σ_p at p. Thus, $\mathrm{K}_p : \mathbf{Gr}_2(T_pM) \to \mathbb{R}$, where $\mathbf{Gr}_2(T_pM)$ is the Grassmannian of 2-planes in T_pM. Constant curvature means $\mathrm{K}_p(\Pi)$ does not depend on Π or on p.