6. URe-Series Cobot Inverse Pose Kinematics (IPK)

According to Pieper's principle, if a 6-dof serial robot has 3 consecutive coordinate frames meeting at the same origin, then an analytical solution is guaranteed to exist for the coupled nonlinear inverse pose kinematics problem. This does not occur for the URe Cobots. There is no spherical wrist where three consecutive wrist frame origins share the same point: instead, the offset d_5 disrupts this desired attribute. But Pieper's principle guarantees an analytical solution if his condition is met; it doesn't say that there is no analytical solution in the absence of this condition. So let us pursue an analytical solution for the Inverse Pose Kinematics (IPK) problem of the URe Cobots, despite the lack of a spherical wrist.

In general, the Inverse Pose Kinematics (IPK) problem for a serial-chain robot is stated: Given the pose (position and orientation) of the end frame of interest, calculate the joint values to obtain that pose. For serial-chain robots, the IPK solution starts with the FPK equations. The solution of coupled nonlinear algebraic equations is required and multiple solution sets generally result.

6.1 Ure-series Analytical Six-dof IPK Solution

The specific statement of the IPK problem for the 6-dof URe serial cobots (the entire series shares the identical kinematic design shown in Figure 6) is:

Given: the constant DH Parameters

and the required end-effector pose $\left[{}^{0}_{6}T\right]=\begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{0}x_6 \\ r_{21} & r_{22} & r_{23} & {}^{0}y_6 \\ r_{31} & r_{32} & r_{33} & {}^{0}z_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Calculate: the joint angles $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$ to achieve this pose

Actually, in the real world, a more general pose input $\left[{}^{B}_{TP}T\right]$ must be given. Then the associated required IPK input $\left[{}^{0}_{6}T\right]$ is calculated from known constant homogeneous transformation matrices as follows:

$$\left[{}^{B}_{TP}T\right]=\left[{}^{B}_{0}T\right]\left[{}^{0}_{6}T\right]\left[{}^{6}_{TP}T\right]$$

$$\left[{}^{0}_{6}T\right]=\left[{}^{B}_{0}T\right]^{-1}\left[{}^{B}_{TP}T\right]\left[{}^{6}_{TP}T\right]^{-1}$$

where the constant homogeneous transformation matrices $\left[{}^{B}_{0}T\right],\left[{}^{6}_{TP}T\right]$ were given in Section 4, Forward Pose Kinematics (FPK). Remember these two matrices do not come from the DH Parameters table, but were found by inspection. The IPK equations come from the FPK expressions; but with the six joint angles now unknown, coupled nonlinear (transcendental) equations result, very difficult to solve compared to FPK.

Here are the form of the FPK equations; remember the LHS is a (consistent) given set of 16 numbers representing the desired pose of {6} with respect to {0}:

$$\left[{}^0_6T\right]=\left[{}^0_1T(\theta_1)\right]\left[{}^1_2T(\theta_2)\right]\left[{}^2_3T(\theta_3)\right]\left[{}^3_4T(\theta_4)\right]\left[{}^4_5T(\theta_5)\right]\left[{}^5_6T(\theta_6)\right]$$

So in principle 16 equations may be written to solve the IPK problem. But 4 of these are useless (the last row [0 0 0 1]). All three translational equations are useful and independent. The remaining nine equations come from the rotation matrix terms, only three of which are independent.

A classic IPK solution approach is to invert some of the consecutive $\left[{}^{i}_{i+1}T(\theta_{i+1})\right]$ homogeneous transformation matrices from the RHS and multiply them to the given numbers $\left[{}^0_6T\right]$ on the appropriate sides of the LHS. For the URe Cobots, use the following homogeneous transformation equation to separate two unknowns θ_1,θ_6 from the other four unknown joint angles $\theta_2,\theta_3,\theta_4,\theta_5$:

$$\left[{}^0_1T(\theta_1)\right]^{-1}\left[{}^0_6T\right]\left[{}^5_6T(\theta_6)\right]^{-1}=\left[{}^1_2T(\theta_2)\right]\left[{}^2_3T(\theta_3)\right]\left[{}^3_4T(\theta_4)\right]\left[{}^4_5T(\theta_5)\right]$$

Now we must inspect the resulting equations in order to find equations that may be solved for all 6 unknown joint angles in some do-able order.

$$\left[{}^0_1T(\theta_1)\right]^{-1}\left[{}^0_6T\right]\left[{}^5_6T(\theta_6)\right]^{-1}=$$

$$\begin{bmatrix} (r_{11}c_1+r_{21}s_1)c_6-(r_{12}c_1+r_{22}s_1)s_6 & -r_{13}c_1-r_{23}s_1 & (r_{12}c_1+r_{22}s_1)c_6+(r_{11}c_1+r_{21}s_1)s_6 & {}^0x_6c_1+{}^0y_6s_1 \\ (r_{21}c_1-r_{11}s_1)c_6-(r_{22}c_1-r_{12}s_1)s_6 & -r_{23}c_1+r_{13}s_1 & (r_{22}c_1-r_{12}s_1)c_6+(r_{21}c_1-r_{11}s_1)s_6 & {}^0y_6c_1-{}^0x_6s_1 \\ r_{31}c_6-r_{32}s_6 & -r_{33} & r_{32}c_6+r_{31}s_6 & {}^0z_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[{}^1_2T(\theta_2)\right]\left[{}^2_3T(\theta_3)\right]\left[{}^3_4T(\theta_4)\right]\left[{}^4_5T(\theta_5)\right]=\begin{bmatrix} c_{234}c_5 & -c_{234}s_5 & -s_{234} & -a_2s_2-a_3s_{23}-d_5s_{234} \\ s_5 & c_5 & 0 & -d_4 \\ s_{234}c_5 & -s_{234}s_5 & c_{234} & a_2c_2+a_3c_{23}+d_5c_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>First</u>, the (2,4) (y-translational) terms involve only one unknown, θ_1:

$${}^0y_6c_1-{}^0x_6s_1=-d_4$$

Re-writing into classical form yields a well-known equation:

$$E_1\cos\theta_1+F_1\sin\theta_1+G_1=0$$

where:

$$\begin{aligned} E_1 &= {}^0y_6 \\ F_1 &= -{}^0x_6 \\ G_1 &= d_4 \end{aligned}$$

This equation may be solved for the unknown θ_1 by applying the well-known and perennial-favourite tangent half-angle substitution.

If we define $t = \tan\left(\frac{\theta_1}{2}\right)$ then $\cos\theta_1 = \frac{1-t^2}{1+t^2}$ and $\sin\theta_1 = \frac{2t}{1+t^2}$

Substitute this **Tangent Half-Angle Substitution** into the *EFG* equation:

$$E_1\left(\frac{1-t^2}{1+t^2}\right) + F_1\left(\frac{2t}{1+t^2}\right) + G_1 = 0$$

$$E_1(1-t^2) + F_1(2t) + G_1(1+t^2) = 0$$

$$(G_1 - E_1)t^2 + (2F_1)t + (G_1 + E_1) = 0$$

So we see this converts the original first-order trigonometric equation into a quadratic polynomial. Using the quadratic formula, we can solve for the intermediate parameter *t*:

$$t_{1,2} = \frac{-F_1 \pm \sqrt{E_1^2 + F_1^2 - G_1^2}}{G_1 - E_1}$$

Then solve for θ_1 by inverting the original Tangent Half-Angle Substitution definition:

$$\theta_{1_{1,2}} = 2\tan^{-1}(t_{1,2})$$

Note that we do not need to use the quadrant-specific **atan2** function in the above solution, since the multiplier 2 takes care of possible the trigonometric uncertainty (dual values) of inverse trigonometric functions. There are two valid solutions for θ_1, from the $\pm$ in the quadratic formula.

Second, the (2,3) (rotational) terms now comprise one unknown θ_6, since θ_1 has been found:

$$(r_{22}c_1 - r_{12}s_1)c_6 + (r_{21}c_1 - r_{11}s_1)s_6 = 0$$

The solution for θ_6 does not require the Tangent Half-Angle Substitution since the new *G* is zero.

$$\frac{s_6}{c_6} = \tan\theta_6 = \frac{r_{12}s_1 - r_{22}c_1}{r_{21}c_1 - r_{11}s_1}$$

$$\theta_6 = \mathrm{atan2}(r_{12}s_1 - r_{22}c_1, r_{21}c_1 - r_{11}s_1)$$

Note we must use the quadrant-specific inverse tangent function **`atan2`** for the θ_6 solution above. This yields a unique θ_6 for each of the two θ_1 results.

Third, since θ_1 and θ_6 are now known, a ratio of the (2,1) to (2,2) rotational equations can be used to solve for θ_5:

$$\begin{aligned}(r_{21}c_1 - r_{11}s_1)c_6 - (r_{22}c_1 - r_{12}s_1)s_6 &= s_5 \\ -r_{23}c_1 + r_{13}s_1 &= c_5\end{aligned}$$

$$\theta_5 = \mathrm{atan2}((r_{21}c_1 - r_{11}s_1)c_6 + (r_{12}s_1 - r_{22}c_1)s_6, r_{13}s_1 - r_{23}c_1)$$

Note again we must use the quadrant-specific inverse tangent function **`atan2`** for the θ_5 solution above. This yields a unique θ_5 for each of the two θ_1, θ_6 results.

Now we are halfway home! Before step 4 we need to gather two intermediate equations, from the (3,1) and (3,3) rotational terms:

$$r_{31}c_6 - r_{32}s_6 = s_{234}c_5 \qquad \text{so} \qquad s_{234} = \frac{r_{31}c_6 - r_{32}s_6}{c_5} = A$$

$$r_{32}c_6 + r_{31}s_6 = c_{234} \qquad\qquad c_{234} = r_{32}c_6 + r_{31}s_6 = B$$

Fourth, substitute these intermediate terms c_{234} and s_{234} into the *x* and *z* translational equations (1,4) and (3,4), which have not yet been used. This replaces the sum of three unknowns $(\theta_2 + \theta_3 + \theta_4)$ in these translational equations with two angles θ_5, θ_6 that are now known.

$$\begin{aligned}{}^0x_6c_1 + {}^0y_6s_1 &= -a_2s_2 - a_3s_{23} - d_5s_{234} = -a_2s_2 - a_3s_{23} - d_5A \\ {}^0z_6 &= \quad a_2c_2 + a_3c_{23} + d_5c_{234} = \quad a_2c_2 + a_3c_{23} + d_5B\end{aligned}$$

Rearrange these equations to isolate the $(\theta_2 + \theta_3)$ terms:

$$\begin{aligned}a_3s_{23} &= -a_2s_2 - {}^0x_6c_1 - {}^0y_6s_1 - d_5A \qquad & a_3s_{23} &= -a_2s_2 + a \\ a_3c_{23} &= -a_2c_2 + {}^0z_6 - d_5B \qquad & a_3c_{23} &= -a_2c_2 + b\end{aligned}$$

Where, for convenience, define:

$$a = -{}^0x_6c_1 - {}^0y_6s_1 - d_5A$$
$$b = {}^0z_6 - d_5B$$

Square and add the two equations to eliminate the $(\theta_2+\theta_3)$ terms; this yields the following, our second *EFG*-type equation:

$$E_2\cos\theta_2 + F_2\sin\theta_2 + G_2 = 0$$

where:

$$E_2 = -2a_2b$$
$$F_2 = -2a_2a$$
$$G_2 = a_2^2 + a^2 + b^2 - a_3^2$$

This equation may be solved for the unknown θ_2 by again applying the trusty tangent half-angle substitution, the same method used earlier for θ_1.

$$t_{2_{1,2}} = \frac{-F_2 \pm \sqrt{E_2^2+F_2^2-G_2^2}}{G_2-E_2} \qquad \theta_{2_{1,2}} = 2\tan^{-1}(t_{2_{1,2}})$$

Again, there are two valid solutions for θ_2, due to the $\pm$ in the quadratic formula; there are two θ_2 solutions for each of the two valid θ_1 solutions, for a total of 4 valid θ_1,θ_2 solutions so far.

Fifth, return to the *x* and *z* (1,4) and (3,4) translational equations (repeated below).

$$a_3s_{23} = -a_2s_2 + a$$
$$a_3c_{23} = -a_2c_2 + b$$

Since squaring-and-adding used one independence, we are free to use them again in a different way. Using a ratio of the equations:

$$\theta_{3_{1,2}} = \text{atan2}(a - a_2s_{2_{1,2}}, b - a_2c_{2_{1,2}}) - \theta_{2_{1,2}}$$

Using the quadrant-specific inverse tangent function **`atan2`** for the θ_3 solution above, there is a unique θ_3 for each of the two valid θ_2 solutions (so there is no additional ballooning of number of solution sets here).

Sixth, and last, return to the two intermediate equations above, from the (3,1) and (3,3) rotational terms. Since θ_2,θ_3 are now known (not to mention θ_6 is also known), the solution to θ_4 is now straight-forward:

$$\theta_{4_{1,2}} = \text{atan2}(A, B) - \theta_{2_{1,2}} - \theta_{3_{1,2}}$$

Again, using the quadrant-specific inverse tangent function **`atan2`** for the θ_4 solution above, there is a unique θ_4 for each of the two valid θ_2, θ_3 solutions (so there is no additional ballooning of number of solution sets here).

<u>In Summary</u>, there are 4 overall IPK joint angles solution sets $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ to achieve the commanded pose, as detailed in Table 10.

Table 10. The Four Universal Ure-series IPK Solution Sets

Solution Set	t_1 / t_2 sign	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	elbow
1	+ / –	θ_{1a}	θ_{2a_1}	θ_{3a}	θ_{4a_1}	θ_{5a}	θ_{6a}	up
2	+ / +	θ_{1a}	θ_{2a_2}	$-\theta_{3a}$	θ_{4a_2}	θ_{5a}	θ_{6a}	down
3	– / –	θ_{1b}	θ_{2b_1}	θ_{3b}	θ_{4b_1}	θ_{5b}	θ_{6b}	down
4	– / +	θ_{1b}	θ_{2b_2}	$-\theta_{3b}$	θ_{4b_2}	θ_{5b}	θ_{6b}	up

Here are the patterns seen in the four IPK solution sets: 1. For a given θ_1, the elbow joint angle θ_3 for elbow-up is negative of that for elbow-down, as expected. 2. For a given θ_1, the last two wrist joint angles θ_5, θ_6 are the same for both elbow-up and elbow-down, also as expected.

At first eight distinct solution sets were expected, due to the similarity to PUMA-type 6-dof 6R serial robots. However, that is for a spherical wrist, i.e. one in which the last three active (wrist) joint frames share a common origin. The wrist offset d_5 in the URe-series prevents this.