3. URe-Series Cobot Denavit-Hartenberg (DH) Parameters

The modified Denavit-Hartenberg (DH, Denavit and Hartenberg, 1955) parameters are presented in this section for the serial chain of the Universal Robots 6-dof URe-Series cobots. The Denavit-Hartenberg (DH) Parameters (1955) are used to describe the links/joints geometry of a serial-chain robot. DH parameters have been adopted for standard kinematics analysis in serial-chain robots (Craig, 2005). The community has come to call Craig-style DH Parameters as 'modified', with the original DH Parameters interpretation by Paul as 'standard'. The modified DH parameters have certain advantages over the standard (the main one being that a Craig coordinate frame rotates right at it joint, rather than distal from the joint as in Paul).

The Cartesian reference frame definitions for the URe-Series 6-dof serial arm are shown in Figure 6. Table 7 gives the associated DH parameters (Craig convention, known as 'modified DH parameters').

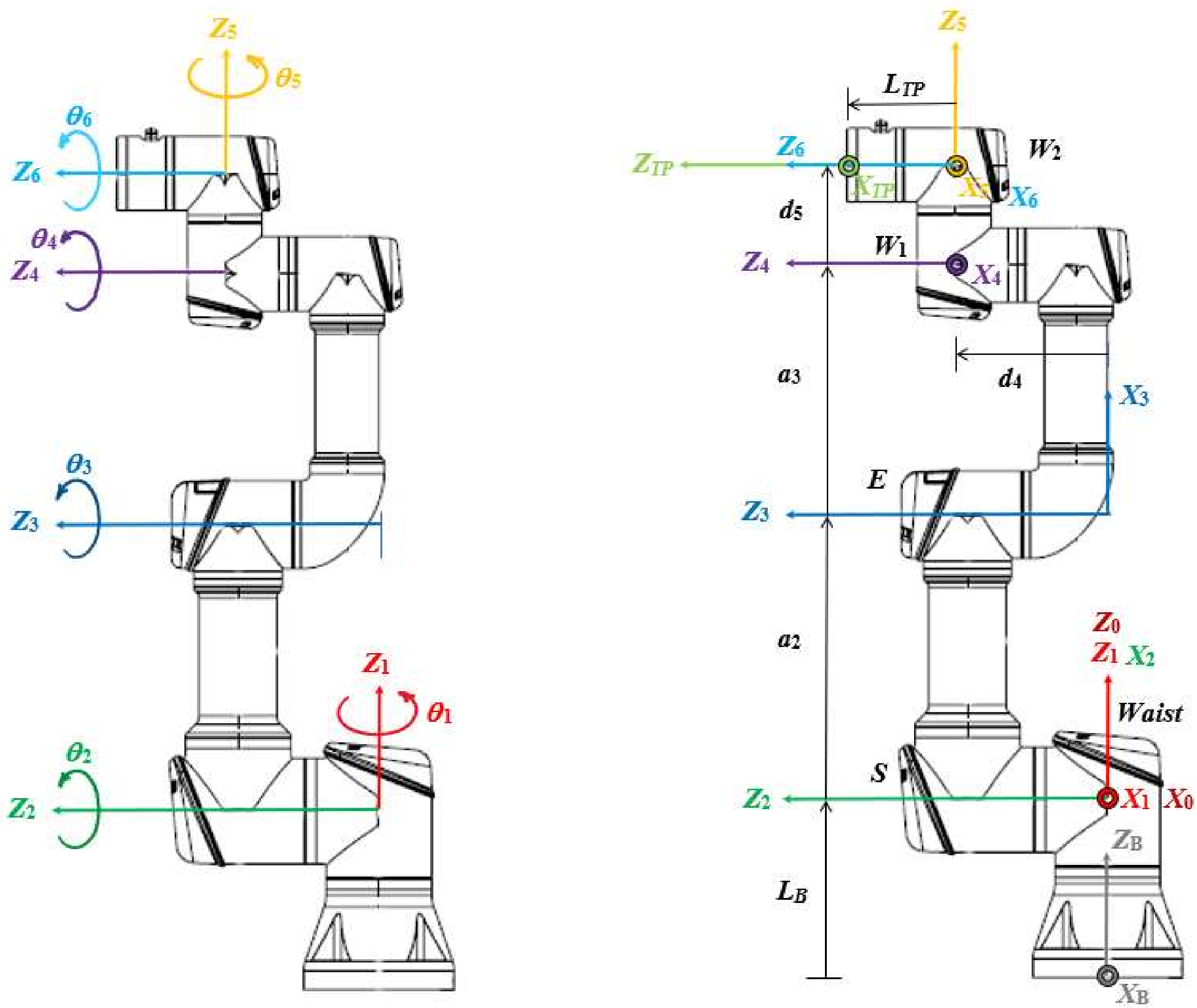

Figure 6. Six-dof Universal URe-Series Kinematic Diagram with Coordinate Frames

Table 7. Six-dof Universal Robot URe-Series DH Parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	0	$\theta_2 + 90^\circ$
3	0	a_2	0	θ_3
4	0	a_3	d_4	$\theta_4 - 90^\circ$
5	-90°	0	d_5	θ_5
6	90°	0	0	θ_6

Kinematic Notation

See the right diagram in Figure 5 above; relative to Universal Robot's notation, this document prefers to refer to joint 1 as the Waist (instead of Base), since the Base commonly refers to a fixed Cartesian coordinate frame $\{B\}$ rather than a joint. The Shoulder joint notation (S), and the Elbow joint notation (E) is straight-forward. Point W_1 above is the origin for the Wrist 1 Cartesian coordinate frame $\{4\}$, and point W_2 above is the origin shared by the Cartesian coordinate frames $\{5\}$ and $\{6\}$ for the Wrist 2 and Wrist 3 joints.

Table 8. Specific DH Lengths in the Six-dof URe-Series (mm)

Parameter	UR3e	UR5e	UR10e	UR16e
L_B	152	163	181	170
a_2	244	425	613	476
a_3	213	392	572	361
d_4	131	133	174	194
d_5	85	100	120	120
L_{TP}	92	100	117	112

4. URe-Series Cobot Forward Pose Kinematics

In general, the Forward Pose Kinematics (FPK) problem for a serial-chain robot is stated: Given the joint values, calculate the pose (position and orientation) of the end-effector frame of interest. For serial-chain robots, the FPK problem set up and solution is straight-forward. It is based on substituting each line of the Denavit-Hartenberg Parameters Table (Table 7) into the equation below (Craig, 2005), giving the pose of frame $\{i\}$ with respect to its nearest neighbor frame $\{i–1\}$ back along the serial chain:

$$\left[{}^{i-1}_{i}T\right]=\begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \left[{}^{i-1}_{i}R\right] & \left\{{}^{i-1}P_i\right\} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$

Where the following abbreviations were used: $c\theta_i = \cos\theta_i$, $s\theta_i = \sin\theta_i$, $c\alpha_i = \cos\alpha_i$, and $s\alpha_i = \sin\alpha_i$.

The equation above represents pose (position and orientation) of frame $\{i\}$ with respect to frame $\{i–1\}$ by using a 4x4 homogeneous transformation matrix. The upper left 3x3 matrix is the rotation matrix $\left[{}^{i-1}_{i}R\right]$ giving the orientation of frame $\{i\}$ with respect to frame $\{i–1\}$, expressed in $\{i–1\}$ coordinates. The upper right 3x1 vector $\left\{{}^{i-1}P_i\right\}$ is the position vector from the origin of $\{i–1\}$ to the origin of $\{i\}$, expressed in $\{i–1\}$ coordinates.

Then homogeneous transformation equations are used to find the pose of the overall end-effector frame of interest with respect to the base reference frame, to complete the FPK solution for each serial chain.

4.1 Ure-series Analytical Six-dof FPK Expressions

The statement of the FPK problem for the 6-dof serial chain of the Universal Cobot URe-Series is:

Given $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$, calculate $\left[{}^{0}_{6}T\right]$ and $\left[{}^{B}_{TP}T\right]$.

where $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$ are the six joint angles, $\{TP\}$ is the tool-plate frame and $\{B\}$ is the fixed robot base reference frame (see Figure 6). Dextral Cartesian coordinate frames are indicated by the curly brackets { }. There are also seven numbered Cartesian coordinate frames {0}, {1}, … {6}. {0} is a fixed frame, while {1} through {6} are the active moving joint frames.

Substitute each row of the DH parameters from Table 7 into the equation for $\left[{}^{i-1}_{i}T\right]$ to obtain the six neighboring homogeneous transformation matrices as a function of the joint angles for the six-dof arm.

$$\left[{}^{0}_{1}T\right]=\begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{1}_{2}T\right]=\begin{bmatrix} -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{2}_{3}T\right]=\begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[{}^{3}_{4}T\right]=\begin{bmatrix} s_4 & c_4 & 0 & a_3 \\ -c_4 & s_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{4}_{5}T\right]=\begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{5}_{6}T\right]=\begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where the following abbreviations were used: $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, for $i = 1, 2, \cdots, 6$.

The basic Universal Cobot FPK solution is found from the following homogeneous transform equation to derive the active-joints FPK result.

$$\left[{}^{0}_{6}T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)\right]=\left[{}^{0}_{1}T(\theta_1)\right]\left[{}^{1}_{2}T(\theta_2)\right]\left[{}^{2}_{3}T(\theta_3)\right]\left[{}^{3}_{4}T(\theta_4)\right]\left[{}^{4}_{5}T(\theta_5)\right]\left[{}^{5}_{6}T(\theta_6)\right]$$

Since Cartesian coordinate frames {2}, {3}, and {4} have parallel Z axes, the basic Universal Cobot FPK solution should be grouped as follows.

$$\left[{}^{0}_{6}T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)\right]=\left[{}^{0}_{1}T(\theta_1)\right]\left[{}^{1}_{4}T(\theta_2,\theta_3,\theta_4)\right]\left[{}^{4}_{6}T(\theta_5,\theta_6)\right]$$

where $\left[{}^{0}_{1}T(\theta_1)\right]$ was given above, and the other two matrices are found by matrix multiplication and simplification:

$$\left[{}^{1}_{4}T(\theta_2,\theta_3,\theta_4)\right]=\begin{bmatrix} c_{234} & -s_{234} & 0 & -a_2 s_2 - a_3 s_{23} \\ 0 & 0 & -1 & -d_4 \\ s_{234} & c_{234} & 0 & a_2 c_2 + a_3 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[{}^{4}_{6}T(\theta_5,\theta_6)\right]=\begin{bmatrix} c_5 c_6 & -c_5 s_6 & s_5 & 0 \\ s_6 & c_6 & 0 & d_5 \\ -s_5 c_6 & s_5 s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where the following abbreviations were used:

$$c_{23} = \cos(\theta_2+\theta_3) \qquad c_{234} = \cos(\theta_2+\theta_3+\theta_4)$$
$$s_{23} = \sin(\theta_2+\theta_3) \qquad s_{234} = \sin(\theta_2+\theta_3+\theta_4)$$

This particular grouping in matrix multiplication was used to significantly simplify the matrix $\left[{}^{1}_{4}T(\theta_2,\theta_3,\theta_4)\right]$ using the standard sum of angles formulae (in two levels of simplification):

$$\cos(a \pm b) = cacb \mp sasb$$
$$\sin(a \pm b) = sacb \pm casb$$

Any time consecutive Z axes are parallel in a serial robot we can expect such sum-of-angles simplifications. Now we can find the basic active-joints FPK solution $\left[{}^{0}_{6}T\right]$ by more matrix multiplications (but no more trigonometric simplifications):

$$\left[{}^{0}_{6}T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{0}x_6 \\ r_{21} & r_{22} & r_{23} & {}^{0}y_6 \\ r_{31} & r_{32} & r_{33} & {}^{0}z_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The orthonormal rotation matrix elements for this FPK result are:

$$\begin{aligned} r_{11} &= -s_1 s_5 c_6 + c_1(-s_{234}s_6 + c_{234}c_5c_6) \\ r_{21} &= c_1 s_5 c_6 + s_1(-s_{234}s_6 + c_{234}c_5c_6) \\ r_{31} &= c_{234}s_6 + s_{234}c_5c_6 \end{aligned}$$

$$\begin{aligned} r_{12} &= s_1 s_5 s_6 - c_1(s_{234}c_6 + c_{234}c_5s_6) \\ r_{22} &= -c_1 s_5 s_6 - s_1(s_{234}c_6 + c_{234}c_5s_6) \\ r_{32} &= c_{234}c_6 - s_{234}c_5s_6 \end{aligned}$$

$$\begin{aligned} r_{13} &= s_1 c_5 + c_1 c_{234} s_5 \\ r_{23} &= -c_1 c_5 + s_1 c_{234} s_5 \\ r_{33} &= s_{234} s_5 \end{aligned}$$

And the FPK translational vector components giving the position of the origin of {6} with respect to the origin of {0}, expressed in the basis of {0} are:

$$\begin{aligned} {}^{0}x_6 &= d_4 s_1 - c_1(a_2 s_2 + a_3 s_{23} + d_5 s_{234}) \\ {}^{0}y_6 &= -d_4 c_1 - s_1(a_2 s_2 + a_3 s_{23} + d_5 s_{234}) \\ {}^{0}z_6 &= a_2 c_2 + a_3 c_{23} + d_5 c_{234} \end{aligned}$$

Note that, since the origins of Cartesian coordinate frames {5} and {6} are coincident at the wrist point, the translational terms above are only functions of the first four joint angles $(\theta_1,\theta_2,\theta_3,\theta_4)$; this will become an issue when we consider Inverse Pose Kinematics (IPK):

$$\left\{ {}^{0}P_{6} \right\} = \left\{ {}^{0}P_{6}(\theta_1,\theta_2,\theta_3,\theta_4) \right\} = \begin{Bmatrix} {}^{0}x_{6} \\ {}^{0}y_{6} \\ {}^{0}z_{6} \end{Bmatrix}$$

Additional, Fixed Transforms – Tool-Plate and Base Coordinate Frames

To complete the FPK solution we need to include the Cartesian coordinate frames $\{TP\}$ and $\{B\}$ to find $\left[{}^{B}_{TP}T \right]$:

$$\left[{}^{B}_{TP}T \right] = \left[{}^{B}_{0}T(L_B) \right]\left[{}^{0}_{6}T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6) \right]\left[{}^{6}_{TP}T(L_{TP}) \right]$$

Where L_B and L_{TP} are known constants. Note that these two additional matrices are not evaluated by any row in the DH parameter table (those were all used above), since there is no variable associated with these fixed homogeneous transformation matrices based on constant lengths. Instead, they are determined by inspection, using the rotation matrix and position vector components of the homogeneous transformation matrix definition.

$$\left[{}^{B}_{0}T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_B \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{6}_{TP}T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_{TP} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note the origin of $\{TP\}$ is now a function of $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ but it is still not a function of θ_6 since the last Z_6 R-joint axis passes through the origin of $\{6\}$. Of course, the position of a general tool held in a gripper attached to the tool-plate would be a function of all six joint angles $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$.

The FPK solutions can be evaluated numerically or symbolically, or using a combination of these two methods.

4.2 Ure-series FPK Examples

Now we present three snapshot FPK examples for the Universal Robot UR3e, for all zero joint angles, a general case, and for the initial angles case. Each shows four MATLAB views of the Cobot pose. Then a FPK trajectory example is given. Translational units are m in all examples.

FPK Example 1: Zero Joint Angles

Given Zero Joint Angles: $\{\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$

The FPK results are:

$$\left[{}^{0}_{6}T\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.131 \\ 0 & 1 & 0 & 0.542 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{B}_{TP}T\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.223 \\ 0 & 1 & 0 & 0.694 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

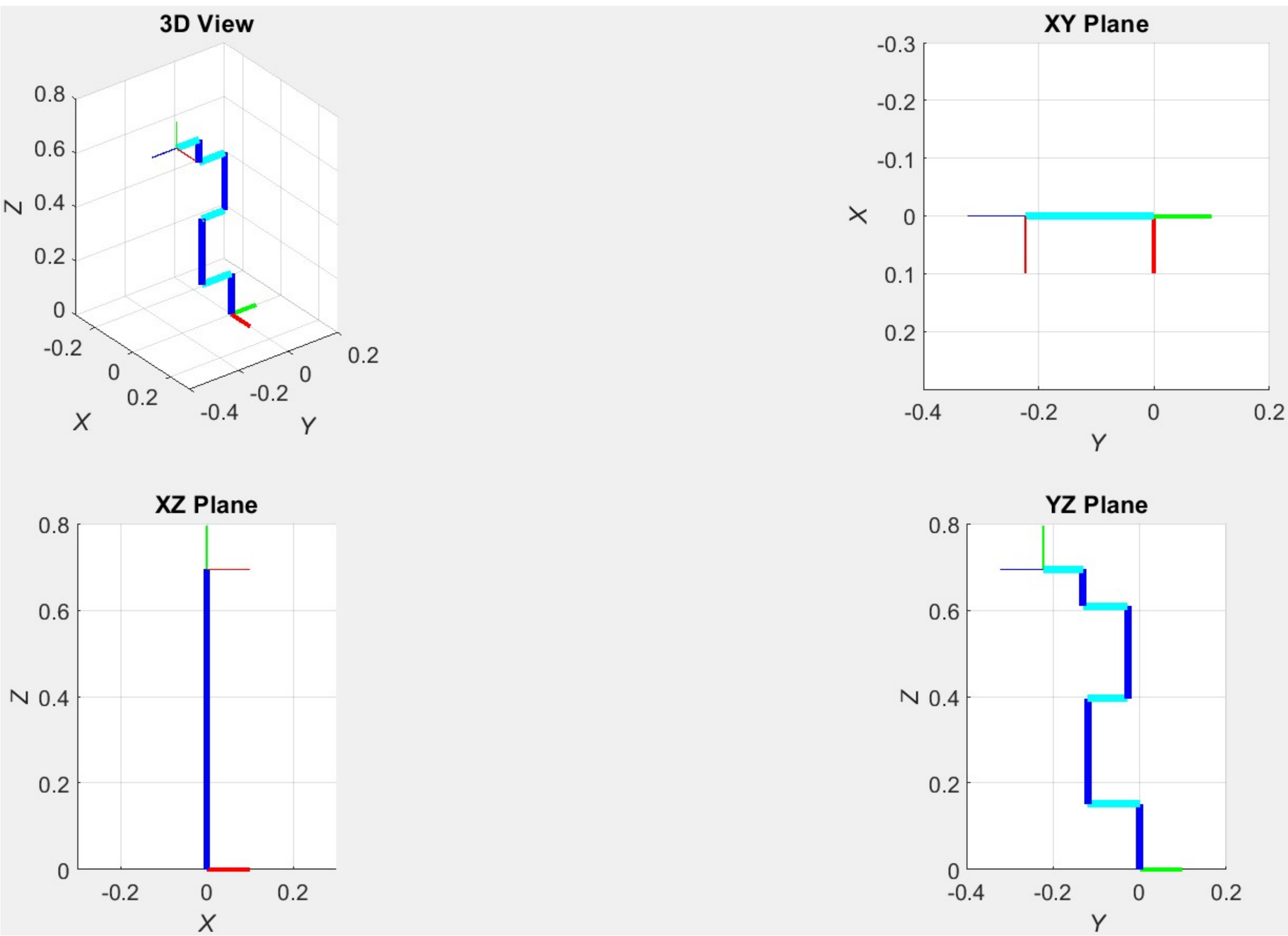

Figure 7. FPK Example 1, Zero Joint Angles

FPK Example 2: General Joint Angles

Given General Joint Angles: $\{\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6\} = \{20^\circ \quad 40^\circ \quad 60^\circ \quad 50^\circ \quad 70^\circ \quad 10^\circ\}$

The FPK results are:

$$\left[{}^{0}_{6}T\right] = \begin{bmatrix} -0.6722 & -0.3586 & -0.6477 & -0.3396 \\ 0.7401 & -0.3042 & -0.5997 & -0.2630 \\ 0.0180 & -0.8826 & 0.4698 & 0.0763 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^{B}_{TP}T\right] = \begin{bmatrix} -0.6722 & -0.3586 & -0.6477 & -0.3992 \\ 0.7401 & -0.3042 & -0.5997 & -0.3182 \\ 0.0180 & -0.8826 & 0.4698 & 0.2715 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

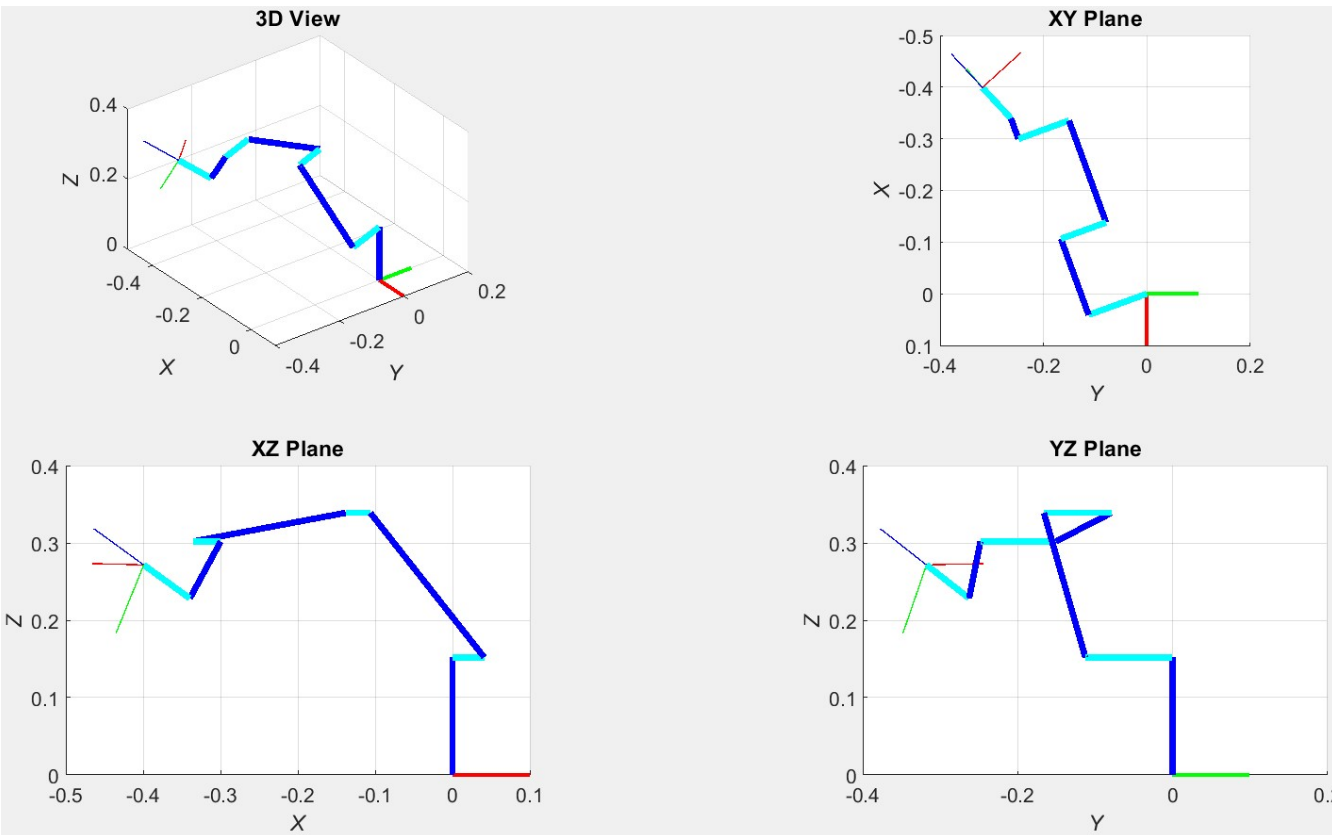

Figure 8. FPK Example 2, General Joint Angles

The cobot arm pose shown in Figures 6 and 7 are for all zero joint angles. The six joint angles for a suggested initial pose are given in Table 9. A third snapshot FPK solution and MATLAB graphic for this initial pose is then presented.

Table 9. Universal URe Initial Joint Angles

Joint Number	Joint Name	Joint Variable	Initial Angle (deg)
1	Base	θ_1	0
2	Shoulder	θ_2	45
3	Elbow	θ_3	90
4	Wrist 1	θ_4	45
5	Wrist 2	θ_5	90
6	Wrist 3	θ_6	0

FPK Example 3: Initial Joint Angles

Given Initial Joint Angles: $\{\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6\} = \{0 \quad 45^\circ \quad 90^\circ \quad 45^\circ \quad 90^\circ \quad 0\}$

The FPK results are:

$$\left[{}^0_6T\right] = \begin{bmatrix} 0 & 0 & -1 & -0.3231 \\ 1 & 0 & 0 & -0.1310 \\ 0 & -1 & 0 & -0.0631 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[{}^B_{TP}T\right] = \begin{bmatrix} 0 & 0 & -1 & -0.4151 \\ 1 & 0 & 0 & -0.1310 \\ 0 & -1 & 0 & 0.0889 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

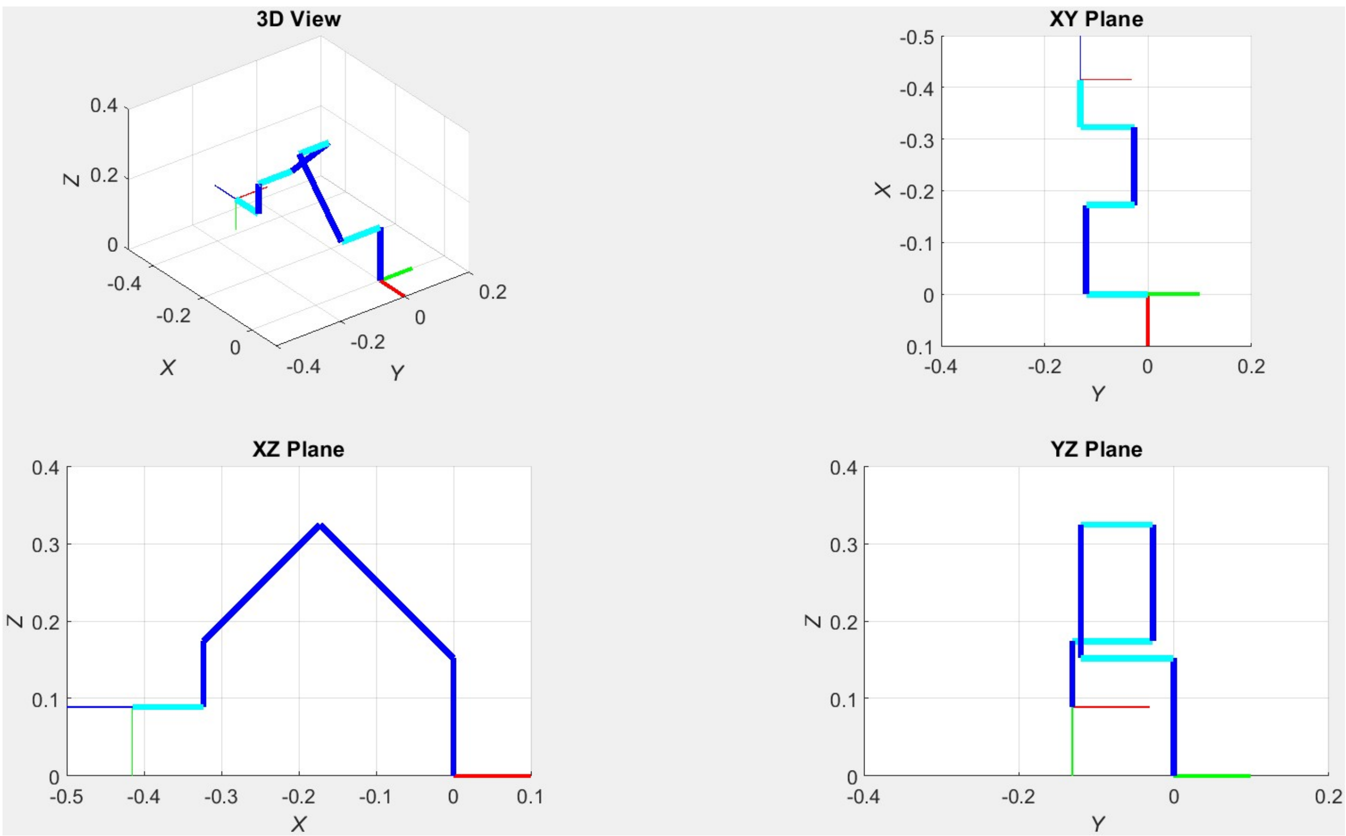

Figure 9. FPK Example 3, initial joint angles

The following FPK solution validations were performed successfully in all 3 snapshot examples:

- The numerical and symbolic solutions for $\left[{}^0_6T\right]$ and $\left[{}^B_{TP}T\right]$ agree.

- The FPK solutions were performed by inspection (verifying the $\{TP\}$ Cartesian position and the XYZ_{TP} pointing directions with respect to $\{B\}$) for Examples 1 and 3, which agreed with the above solutions.