Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 2: indices, logarithms, exponentials

Introductory problems

1. Simplify:

   a) $\dfrac{a^{10}}{a^{-3}}$

   b) $\dfrac{(\sqrt{x})^8}{x^4}$

   c) $\dfrac{y^{\frac{1}{4}}}{y^{-\frac{2}{4}}}$

   d) $\dfrac{10^{-2/3} \times 10^7 \times 10^{-16} \times x^{1/2} \times y^4 \times z^{-1/3}}{10^{-19} \times 10^{43} \times 10^{2/3} \times z^{-1/3} \times y^{1/4} \times x^{5/2}}$

2. Evaluate the following expressions without using a calculator:

   a) 2^{-4}

   b) $36^{\frac{1}{2}} + 64^{\frac{2}{3}}$

   c) $\left(\dfrac{1}{3}\right)^{-2}$

   d) $\left(\dfrac{81}{9}\right)^{\frac{3}{2}}$

3. Express the following in logarithmic form:

   a) $5^3 = 125$

   b) $8^{-\frac{1}{3}} = \dfrac{1}{2}$

   c) $x^y = 4$

4. Evaluate the following expressions without using a calculator:

   a) $\log_4 (64)$

   b) $\log_\pi (1)$

   c) $\log_b (b^a)$

   d) $6^{\log_6 (\pi)}$

5. Solve the following equations for x:

   a) $e^x = 4$

   b) $e^{-x} = 2$

One of your solutions should be exactly -2 times the value of your other solution. Why?

6. Simplify:

   a) $\ln\left(e^3\right)$

   b) $\ln\left(\dfrac{1}{2e}\right)$

   c) $e^{\ln x^4}$

   d) $e^{3+\ln x}$

Main problems

1. In an experiment, the mass, m grams, of a reaction product is measured at various times, t minutes. It is believed that m and t are related by a law of the form:

$$m = At^n$$

   The results are shown in the table below:

time (min)	1	2	3	4	5	6	7	8	12
mass (g)	0.5	0.63	0.72	0.8	0.85	0.9	0.95	0.99	1.0

   a) Confirm this postulate graphically. There is one result that does not conform to this law. Which?
   b) Find appropriate values of A and n, and in this context, explain the meaning of n. What are the units of A?
   c) Explain, with reasons, whether you would use these results to predict the mass when $t = 15$.

2. These problems deal with pH:

   a) What is the pH of 130ml of a solution containing 4.7mg HCl, assuming that HCl is completely ionised in solution, and its molecular mass is 36.46?
   b) What would be the pH if the concentration of HCl were tripled?

3. Express in terms of $\log(a)$, $\log(b)$, $\log(c)$ and $\log(d)$:

   a) $\log\left(\dfrac{b}{ac}\right)$

   b) $\log(a^2bc^3d^4)$

   c) $\log\left(\sqrt{\dfrac{cd}{ab}}\right)$

4. Simplify:

   a) $\log(8) + \log(3) - \log(6)$

   b) $\log\left(x^2 - 1\right) - \log\left(x^2 + 1\right)$

   c) $3\log_a(4) + \log_a(5) - 2\log_a(9)$

   d) $\log\left(x^9\right) - \log\left(x^6\right)$

5. This problem gives you a chance to use the identity $\quad \log_a(b) = \dfrac{1}{\log_b(a)}$.

Consider the equation $\log_3(x) + 4\log_x(3) = 5$:

    a) Verify that $x = 3$ satisfies this equation

    b) There is one other value of x that also satisfies this equation. Find it.

6. Solve the following equations for x:

    a) $2\left(2^{2x}\right) - 5\left(2^x\right) + 2 = 0$

    b) $3^{2x+1} - 28\left(3^x\right) + 9 = 0$

    c) $16 = \log_2(x)$

    d) $\left(2\sqrt{3}\log(x)\right)^2 - 7\log(x^2) + 2 = 0$

7. Write an expression for x or y without using logarithms:

    a) $\log(x) = \log(3r) - 5\log(s) + 3\log(t) - 3$

    b) $3^{2x+1} - 28\left(3^x\right) + 9 = 0$

    c) $\log(2y) = 5 + 5\log\left(4^3\right) - 15\log\left(\dfrac{2}{x}\right) - 6\log(y)$

8. Write x in terms of y for each of the following:

    a) $y = 2e^{4x}$

    b) $\ln y = 3 + 2\ln x$

9. Express as a sum or difference of logarithms:

    a) $\ln\sqrt{\left(\dfrac{x-1}{x+1}\right)}$

10. Express as a single logarithm:

    a) $1 - \ln 4x$

    b) $3\ln x - \dfrac{1}{2}\ln\left(5 - x^2\right)$