Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 5: differentiation 3

Introductory problems

1. Differentiate the following functions with respect to x, using the stated rules where indicated:

   a) Product rule: $\sqrt{x}\, e^x$

   b) Product rule: $3x^2 \log(x)$

   c) Chain rule: e^{-4x^3}

   d) Chain rule: $\ln \sqrt{\left(\frac{x^{3/2}}{6}\right)}$

   e) Chain rule: 10^{x^2}

   f) Any rules: $\dfrac{\ln x}{5x-7}$

   g) Any rules: $\dfrac{e^x}{2x^3-1}$

   h) Any rules: $\log_2 (x \cos(x))$

2. If $\quad y = e^{-ax} \quad$ show that $\quad 2\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a\dfrac{\mathrm{d}y}{\mathrm{d}x} - a^2 y = 0.$

3. If $\quad y = e^{-x}\sin(x) \quad$ show that $\quad \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$

Main problems

1. The power, W, that a certain machine develops is given by the formula

$$W = EI - RI^2$$

   where I is the current and E and R are positive constants.

   Find the maximum value of W as I varies.

2. Environmental health officers monitoring an outbreak of food poisoning after a wedding banquet were able to model the time course of the recovery of the guests using the equation:

$$r = \frac{100t}{1+t}$$

   where t represents the number of days since infection and r is the percentage of guests who no longer display symptoms. Determine an expression for the rate of recovery.

3. An experiment called 'the reptilian drag race' looks at how agamid lizards accelerate from a standing start. The distance x travelled in time t on a horizontal surface has been modelled as

$$x = v_{\max}\left(t + \frac{e^{-kt}}{k} - \frac{1}{k}\right),$$

   where $v_{\max}$ is the maximum velocity, and k is a rate constant.

   a) Find expressions for the velocity v and acceleration a as functions of time.

b) For $v_{\max} = 3\,\mathrm{m\,s^{-1}}$ and $k = 10\,\mathrm{s^{-1}}$, sketch x, v and $a/10$ on the same axes for $0 \leq t \leq 1\,\mathrm{s}$.

4. The distance x that a particular organism travels over time from its starting location is modelled by the equation

$$x(t) = t^2 e^{k(1-t)},$$

where k is a positive constant and $0 \leq t \leq 1\,\mathrm{s}$.

   a) Sketch x over time for $k = \frac{1}{2}$ and $k = 3$.

   b) Calculate an expression for the organism's velocity as a function of time.

   c) What is the largest value of k such that the organism never starts moving back towards where it started?

5. The function

$$S = S_{max}(1 - e^{\frac{-t}{\tau}})$$

is used to describe sediment thickness accumulating in an extensional basin through time.

   a) What is the sedimentation rate?
   b) At what time is the sediment accumulating fastest? (*Hint: differentiate again*)

Extension problems

1. Let $\quad z = \dfrac{2}{3}x^3 - \dfrac{3}{4}x^2 y + \dfrac{2}{5}y^3.$

   a) Find z_x and z_y

   b) Find z_{xx} and z_{yy}

   c) Show that $z_{xy} = z_{yx}$

2. Show that $f_{xy} = f_{yx}$ for the following functions:

   a) $f(x, y) = x^2 - xy + y^3$

   b) $f(x, y) = e^y \ln(2x - y)$

   c) $f(x, y) = 2\,x\,y\,e^{2xy}$

   d) $f(x, y) = x \sin(y)$

3. The body mass index, B, is used as a parameter to classify people as underweight, normal, overweight and obese. It is defined as their weight in kg, w, divided by the square of their height in meters, h.

   a) Sketch a graph of B against w for a person who is 1.7m tall.

   b) Find the rate of change of B with weight of this person.

   c) Sketch a graph of B against h for a child whose weight is constant at 35 kg.

   d) Find the rate of change of B with height h of this child.

   e) Show that $\left(\dfrac{\partial^2 B}{\partial h \partial w}\right) = \left(\dfrac{\partial^2 B}{\partial w \partial h}\right).$

4. A light wave or a sound wave propagated through time and space can be represented in a simplified form by:

$$y = A \sin \left(2\pi \left(\frac{x}{\lambda} + \omega t \right) \right)$$

where A is the amplitude, λ is the wavelength and ω is the frequency of the wave. x and t are position and time respectively.

An understanding of this function is essential for many problems such as sound, light microscopy, phase microscopy and X-ray diffraction.

a) Draw a graph of y as a function of x assuming $t = 0$.

b) Draw a graph of y as a function of t assuming $x = 0$.

c) At what values of x and t does the function repeat itself?

d) Find the rate at which y changes at an arbitrary fixed position.

e) Show that $y_{xt} = y_{tx}$.