Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 4: differentiation 2

Introductory problems

1. Differentiate the following functions, using the stated rules where indicated:

    a) Product rule: $y = (3x + 4x^2)(8 + 3x^2)$

    b) Product rule: $y = (5x^2 - 12)(4x^{-1} + 2)$

    c) Product rule: $y = x \cos x$

    d) Product rule: $y = (3\sqrt{x} + 4x^3) \sin x$

    e) Chain rule: $y = (6x + 2)^4$

    f) Chain rule: $y = (5x^3 + 10)^{1/2}$

    g) Any rules: $y = \sqrt{\dfrac{1}{1 - 3x}}$

    h) Any rules: $y = (3x + 1)\sqrt{5x^2 - x}$

2. Differentiate the following trigonometric functions with respect to x:

    a) $3\sin(x) + 5\cos(x)$

    b) $\cos(3 - 2x)$

    c) $\sqrt{\sin(x)}$

    d) $\dfrac{\cos(x)}{4x}$

Main problems

1. Let $y = x^2$.

    a) Find the **exact** value of y when $x = 2.1$.
    b) Now **estimate** the value of y when $x = 2.1$ by using the linear approximation formula

    $$f(x_1 + h) = f(x_1) + hf'(x_1)$$

    and letting $x_1 = 2.0$ and $h = 0.1$.
    c) Compare your estimate to the true value. Which is bigger? What is there about the shape of the graph of $y = x^2$ that accounts for this?
    d) Repeat parts a) and b) for $x = 2.01$.
    e) Calculate the absolute error in each estimate. How does this error change with the value of h?

2. The energy, E, carried by a photon of wavelength λ is given by

    $$E = \frac{hc}{\lambda}$$

    where h is Planck's constant ($h = 6.63 \times 10^{-34}$ Js) and c is the speed of light ($c = 3 \times 10^8$ ms^{-1}).

a) Calculate the energy carried by a photon of wavelength 500 nm.
b) Sketch a graph to show how E varies with λ.
c) Derive an expression for $\frac{\mathrm{d}E}{\mathrm{d}\lambda}$, and calculate the slope of your graph when $\lambda = 500\,\mathrm{nm}$.
d) Hence, or otherwise, estimate the difference in energy between a photon of wavelength 500 nm and one of wavelength 505 nm.

3. On February 10, 1990, the water level in Boston harbour was given by:

$$y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right)$$

where t is the number of hours since midnight and y is measured in feet.

a) Sketch y.
b) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$. What does it represent, in terms of water level?
c) For $0 \leq t \leq 24$, when is $\frac{\mathrm{d}y}{\mathrm{d}t}$ zero? Explain what it means for $\frac{\mathrm{d}y}{\mathrm{d}t}$ to be zero.

4. In laminar flow of blood through a cylindrical artery, the resistance R is inversely proportional to the fourth power of the radius r. Use a linear approximation to show that if r is decreased by 2%, R will increase by approximately 8%.

5. Imagine an ocean basin with an inlet on one side. We can model the salinity of the water as a function of the distance from the ocean inlet. Let s be the water salinity, and x is the distance from the inlet. Then

$$s = \frac{s_0 \alpha X}{(\alpha X - x)}$$

where s_0 is the initial salinity at the inlet, X is the width of the basin, and α is a constant. Prove that the rate of increase of salinity with distance from the inlet is given by

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{s}{\alpha X - x}$$

6. The sine and cosine functions can be written in the form of the following infinite series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Differentiate these series term by term to verify the standard expressions for $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)$.

Extension problems

1. The focal length of the lens of the eye, $f(t)$, can be controlled so that an object at distance $u(t)$ in front of the eye can be brought to perfect focus on the retina at a constant $v = 1.8\,\mathrm{cm}$ behind the lens.

A fly is moving towards the eye at a speed of $0.7\,\mathrm{ms^{-1}}$.

Assuming that the optics of the eye lens obeys the thin lens formula

$$\frac{1}{f(t)} = \frac{1}{u(t)} + \frac{1}{v},$$

find the rate of change of focal length required to keep the fly in perfect focus at a distance of 3 m.

Note: consider carefully what you are differentiating with respect to, and the physical interpretation of the mathematics.

2. A contaminated lake is treated with a bactericide. The rate of change of harmful bacteria t days after treatment is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{2000t}{1+t^2}$$

where $N(t)$ is the number of bacteria in 1 ml of water.

a) State with a reason whether the count of bacteria increases or decreases during the period $0 \le t \le 10$.

b) Find the minimum value of $\dfrac{\mathrm{d}N}{\mathrm{d}t}$ during this period.

3. Consider a population of lions $L(t)$ and zebra $Z(t)$ interacting over time t. One type of model for this situation is

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = aZ - bZL \qquad \text{and} \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -cL + dZL,$$

where a, b, c and d are positive constants.

a) What values of $\dfrac{\mathrm{d}Z}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}L}{\mathrm{d}t}$ correspond to stable populations?

b) How would the statement ‘zebra go extinct’ be represented mathematically?

c) For parameters $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.00001$, find all population pairs (Z, L) that yield stable populations. Is extinction inevitable?