Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 12: linear algebra 1

Introductory problems

1. Write the following system of equations:

$$\begin{aligned} x + y + z &= 7 \\ 2x - y + z &= 2 \\ x - 2y + 2z &= 5 \end{aligned}$$

   in the form

   a) $A\mathbf{x} = \mathbf{b}$

   b) $\mathbf{y}B = \mathbf{c}$

   Where A and B are 3×3 matrices, and $\mathbf{x}, \mathbf{x}, \mathbf{x}$ and $\mathbf{x}$ are vectors whose size and shape you should carefully indicate.

   Check that your answers make sense by expanding your expressions to ensure you get back to the original equations.

2. Given

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 4 \\ 7 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}; \quad D = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \quad E = \begin{pmatrix} 2 & -1 \end{pmatrix};$$

   a) write down $a_{21},\ b_{12}$ and c_{22}

   b) calculate:

      i. $A + A$ ii. $A - B$ iii. $4C$

   c) calculate, where possible, or explain why the product is not defined:

      i. AB ii. CD iii. DC iv. EE

   d) Do A and B commute? Do A and C commute?

   e) Does $(AB)C = A(BC)$? Does this either prove or disprove that matrix multiplication is associative?

   f) Does $AC+BC = (A+B)C$? Does this either prove or disprove the distributive property of matrices?

Main problems

1. If $A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find A^2 and A^3 and comment on your results.

2. Given $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & -1 & 1 \\ -5 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix}$,

   Find AB, BA, $A^TB^T, B^TA^T, (AB)^T, (BA)^T$. Comment on your results.

3. Find the determinany, $|A|$, of the following matrices:

   a) $A = \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix}$

   b) $A = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}$

4. Find the inverse, A^{-1}, of the following matrices:

   a) $A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$

   b) $A = \begin{pmatrix} -3 & 2 \\ -1 & 7 \end{pmatrix}$

5. Use Python's `numpy.linalg.solve` to solve the following systems of equations:

$$\text{a)}\quad \begin{aligned} x + 2y - 3z &= 9, \\ 2x - y + z &= 0, \\ 4x - y + z &= 4. \end{aligned} \qquad \text{b)}\quad \begin{aligned} x + 5y + 3z &= 17, \\ 5x + y - 2z &= 4, \\ x + 2y + z &= 7. \end{aligned} \qquad \text{c)}\quad \begin{aligned} 2y + z &= -8, \\ x - 2y - 3z &= 0, \\ -x + y + 2z &= 3. \end{aligned}$$

```python
# hint
import numpy as np

A = np.array([[1,2,-3], [2,-1,1], [4,-1,1]])
b = np.array([9,0,4])

x = np.linalg.solve(A, b)

print(x)
```

6. Given $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$,

   calculate XY, YX, X^{-1}, Y^{-1}, $X^{-1}Y^{-1}$, $(XY)^{-1}$, $(YX)^{-1}$. Comment on your results.

Extension problems

1. Show that

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \lambda_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

   where λ_1 and λ_2 are constants to be determined.