Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 7: integration 2

Introductory problems

1. By using suitable substitutions, evaluate the following integrals:

   a) $\displaystyle\int x^2(x^3+4)^2 \;\, \mathrm{d}x$

   b) $\displaystyle\int e^{-x}(5-4e^{-x})\, \mathrm{d}x$

   c) $\displaystyle\int (1+x)\sqrt{(4x^2+8x+3)}\, \mathrm{d}x$

   d) $\displaystyle\int 3xe^{(x^2+1)}\, \mathrm{d}x$

2. Find the indefinite integrals, with respect to x, of the following functions:

   a) $x\,e^{3bx}$

   b) $x^3\,e^{-3x}$

   c) $x\cos(x)$

   d) $e^{bx}\sin(x)$

3. Sketch the curve $y=(x-2)(x-5)$ and calculate by integration the area under the curve bounded by $x=2$ and $x=5$.

Main problems

1. Evaluate the following indefinite and definite integrals:

   a) $\displaystyle\int \frac{6}{(7-x)^3}\, \mathrm{d}x$

   b) $\displaystyle\int 13x^3(9-x^4)^5\, \mathrm{d}x$

   c) $\displaystyle\int_2^5 5\log(x)\, \mathrm{d}x$

   d) $\displaystyle\int x^x\,(1+\log(x))\, \mathrm{d}x$

2. Suppose the area $A(t)$ (in cm^2) of a healing wound changes at a rate

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -4t^{-3},$$

   where t, measured in days, lies between 1 and 10, and the area is $2\,\mathrm{cm}^2$ after 1 day. What will the area of the wound be after 10 days?

3. A rocket burns fuel, so its mass decreases over time. If it burns fuel at a constant rate $\rho\,$kg/s, and if the exhaust velocity relative to the rocket is a constant v_e m/s, then there will be a constant force of magnitude ρv_e propelling it. The rocket starts burning fuel at $t=0\,$s with total mass of $m_0\,$kg, and runs out of fuel at a later time $t=t_f\,$s, with a final mass of $m_f\,$kg.

a) Newton's second law tells us that the instantaneous acceleration a of the rocket at time t is equal to the force propelling it at that time, divided by its mass at that time. Write down an expression for a as a function of t.

b) By integrating this expression, show that the rocket's total change in velocity is given by $v_e \ln\left(\dfrac{m_0}{m_f}\right)$.

4. The flow of water pumped upwards through the xylem of a tree, F, is given by:

$$F = M_0(p + qt)^{3/4},$$

where t is the tree's age in days, p and q are positive constants, and $M_0 p^{3/4}$ is the mass of the tree when planted (i.e. at $t = 0$). Determine the total volume of water pumped up the tree in its tenth year (ignoring leap years) if $p = 10$, $q = 0.01\,\mathrm{day}^{-1}$, and $M_0 = 0.92\,\mathrm{l}\,\mathrm{day}^{-1}$.

Extension problems

1. Express $\dfrac{1}{x(x^2-16)}$ in the form $\dfrac{A}{x} + \dfrac{B}{(x+4)} + \dfrac{C}{(x-4)}$. Hence calculate $\displaystyle\int \frac{1}{x(x^2-16)}\,\mathrm{d}x$.

2. The probability that a molecule of mass m in a gas at temperature T has speed v is given by the Maxwell-Boltzmann distribution:

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

where k is Boltzmann's constant. Find the average speed:

$$\overline{v} = \int_0^\infty v\, f(v)\,\mathrm{d}v.$$

3. Baranov developed expressions for commercial yields of fish in terms of lengths, L, of the fish. His formula gave the total number of fish of length L as $k\,e^{-cL}$, where c and k are constants (k is positive).

a) Give a sketch of the graph $f(L) = k\,e^{-cL}$. (Something decreasing, concave upward and asymptotic to horizontal axis will do.) On your sketch, introduce marks on the horizontal axis that represent lengths $L = 1, L = 2, L = 3, L = 4$ and $L = 5$. Now draw a rectangle on your sketch that represents the number of fish whose lengths are between $L = 3$ and $L = 4$.

b) Explain how we can represent the total number of fish N as an area. Show that this number equals k/c.

c) Only fish longer than L_0 count as commercial. Hence, assuming that the fish are all similar in shape (i.e. their width and breadth scales with their length) and of equal density ρ, show that the weight, W, of the commercial fish population is

$$W = \int_{L_0}^{+\infty} a\,k\rho\,L^3 e^{-cL}\,\mathrm{d}L,$$

and hence that

$$W = \frac{N\,a\,\rho\,e^{-cL_0}}{c^3}\left((cL_0)^3 + 3(cL_0)^2 + 6cL_0 + 6\right),$$

where a is a constant.