Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 11: differential equations 3

Introductory problems

1. TODO fill in some introductory problems:
   a) TODO fill in some introductory problems

Main problems

1. Not all chemical systems relax exponentially to steady state. Consider the bimolecular decay reaction

$$A + A \xrightarrow{k}$$

   Assuming k is a mass action constant, form and solve a differential equation representing the change in concentration of A.

   If $a(0) = a_0$ you should get $\quad a(t) = \dfrac{1}{2kt + \frac{1}{a_0}}$.

2. The SIS model is an appropriate model for diseases that mutate quickly and can therefore infect people multiple times, such as the common cold or sexually transmitted infections like gonorrhea and chlamydia.

   In the model, individuals are 'susceptible' until they are 'infected', and then they return to being 'susceptible' again. Infection requires the interaction of susceptible individuals with infected individuals and therefore follows the law of mass action, whereas the rate at which an individual becomes susceptible again after infection is constant.

   a) Let S and I be the proportions of the population that are susceptible and infected. If infection happens at rate β and recovery happens at rate γ, write down differential equations for S and I.

   b) Noting that S and I are proportions of the population, which is assumed constant, reduce the system to a single differential equation in terms of I. In other words, write down a single equation, involving just I and its derivative.

   c) Find both steady states of I. Under what conditions on β and γ are each attainable?

   d) Without solving the differential equation, sketch the behaviour of S and I over time, starting with a small quantity of infected individuals. Illustrate how both steady states may be achieved.

3. Consider a closed reaction system consisting of a single reversible reaction:

$$A \underset{k_b}{\overset{k_f}{\rightleftharpoons}} B$$

   where k_f and k_b are mass action coefficients.

   a) Formulate a pair of coupled differential equations for the change in concentration of A and B.

   b) Noting that the total concentration T of reactants is constant ($T = [A] + [B]$), reduce the system of equations to a single differential equation. In other words, write down a single equation, involving either just A and its derivative, or just B and its derivative.

c) Find the steady-state concentrations of A and B.

d) Solve the single differential equation to reveal the transient behaviour. Sketch the behaviour for different illustrative initial conditions.

4. Consider the simple model

$$\frac{\mathrm{d}s}{\mathrm{d}t} = k - \frac{V_{\max} s}{K_M + s}$$

in which species s is produced at a fixed rate and consumed via Michaelis-Menten kinetics. Find the steady state of s, and verify that it is stable for any non-negative parameter values, provided $V_{\max} > k$.

5. Recall the simple model of the production and degradation of a protein from the lecture, shown by the reaction chain

$$\xrightarrow{v_1} S \xrightarrow{v_2}$$

where v_1 and v_2 are reaction rates rather than mass action coefficients.

a) Suppose $v_0 = k_0$ and $v_1 = k_1$. Write down a differential equation describing the rate of change of A, and find the steady state concentration of A in terms of the two parameters k_0 and k_1 (i.e. the concentration at which the rate of change is zero). At what rate is A being produced in steady state?

b) Now suppose that

$$v_1 = k_0 + \frac{k_1 s^2}{k_2 + s^2} \quad \text{and} \quad v_2 = k_3 s$$

and take the parameter values to be $k_0 = 6/11,\ k_1 = 60/11,\ k_2 = 11,\ k_3 = 1$. Determine the number of steady states and the type of each.

Extension problems

1. Various mathematical models have been proposed for the initial growth of solid tumours, and some are summarised in DOI:10.1158/0008-5472.CAN-12-4355. They are differential equations describing the rate of change of tumour volume V as a function of time t, for example:

$$\text{a)}\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t) \qquad \text{b)}\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t)^b \qquad \text{c)}\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = r_0 e^{-\rho t} V(t)$$

Solve each equation both analytically and numerically, using Python. As was done in Figure 1A in the paper, compare the behaviours of the different growth laws over a suitable time interval for an initially small tumour, again using Python.

2. Find the solution to the following differential equations subject to the specified boundary conditions, using integrating factors:

a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x} = 0 \quad \text{with} \quad y(2) = 2 \quad \text{for} \quad x > 0$

b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + (2x - 1)y = 0 \quad \text{with} \quad y(1) = 2$

c) $x^3 \dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{1/x^2} \quad \text{with} \quad y(1) = e$

d) $\sec(x) \dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 1 \quad \text{with} \quad y(0) = 1$

e) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y \tan(x) = \cos(x) \quad \text{with} \quad y(0) = 1 \quad \text{for} \quad 0 \le x < \dfrac{\pi}{2}$

3. Consider the second order differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$$

Show that $y = Ae^{z_1 x} + Be^{z_2 x}$ is a solution to this equation for some complex numbers z_1, z_2 and real constants A, B.

a) Recalling that any complex number z can be written as $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, what does this tell you about the nature of the solution?

b) If $y(0) = 1$ and $y(\pi/2) = 2$ what is the particular solution of the differential equation?