# Essential Maths for DTC DPhil Students

## Michaelmas Term 2020

## Problem Sheet 6: integration 1

### Introductory problems

1. Integrate the following functions with respect to $x$. Remember that you can check your own answers by differentiating your results:

   a) $x^3 - \dfrac{1}{x^4} + x^2$

   b) $\sqrt[3]{x} + \dfrac{1}{3\sqrt[4]{x}}$

   c) $\dfrac{1}{x^2} + \dfrac{1}{\sqrt[3]{x}} - 7$

2. Evaluate the following definite integrals:

   a) $\displaystyle\int_1^2 x^{1/2}\ \mathrm{d}x$

   b) $\displaystyle\int_2^3 x^{-2/3}\ \mathrm{d}x$

   c) $\displaystyle\int_0^{\ln(2)} e^{3x}\ \mathrm{d}x$

   d) $\displaystyle\int_0^2 (x+1)^{1/5}\ \mathrm{d}x$

3. Find the integrals below by making the substitution suggested:

   a) $\displaystyle\int x^2(2x^3-5)^3\ \mathrm{d}x \qquad \text{using} \qquad u = 2x^3 - 5$

   b) $\displaystyle\int x\sqrt{7-2x^2}\ \mathrm{d}x \qquad \text{using} \qquad u = 7 - 2x^2$

   c) $\displaystyle\int_0^1 e^x(3e^x-10)^4\ \mathrm{d}x \qquad \text{using} \qquad u = 3e^x - 10$

   d) $\displaystyle\int x^3\sqrt{15-3x^4}\ \mathrm{d}x \qquad \text{using} \qquad u = 15 - 3x^4$

   e) $\displaystyle\int_0^4 \sqrt{x^3}\,\sqrt{4+x^{5/2}}\ \mathrm{d}x \qquad \text{using} \qquad u = 4 + x^{5/2}$

   f) $\displaystyle\int_0^1 x^{n-1}(1-x^n)^2\ \mathrm{d}x \qquad \text{using} \qquad u = 1 - x^n$

4. By making suitable substitutions, find the indefinite integrals of:

   a) $2(6x-5)^3$

   b) $\dfrac{7x}{x^2-2}$

   c) $\dfrac{3}{\sqrt{5-x}}$

   d) $\dfrac{1}{x-a}$

**Main problems**

1. Electrostatic work: the force acting between two electric charges $q_1$ and $q_2$ separated by distance $x$ in a vacuum is given by Coulomb's inverse square law:

$$F(x) = \frac{q_1 q_2}{4\pi\epsilon_0 x^2}$$

where $\epsilon_0$ is the permittivity of a vacuum. Like charges (charges with the same sign, such as 2 nuclei or 2 electrons) repel so that the force acting on $q_2$ due to the presence of $q_1$ acts in a positive $x$ direction, away from $q_1$. Unlike charges (of opposite signs such as the proton and electron in the hydrogen atom) attract, and the force on $q_2$ is directed towards $q_1$ (i.e. F is negative).

Consider 2 like charges, initially infinitely far apart. Because the charges repel, work must be done on the system to bring $q_2$ from infinity to the distance $x$ from $q_1$. The force $F$ must be applied to overcome the repulsion.

   a) Explain why the total work done is given by $W = -\int_\infty^x F(x')\,\mathrm{d}x' = -\frac{q_1 q_2}{4\pi\epsilon_0}\int_\infty^x \frac{\mathrm{d}x'}{x'^2}$.

   b) Calculate the work, $W$, in Joules when $x = 5.3 \times 10^{-11}\,\mathrm{m}$, $q_1 = q_2 = 1.6 \times 10^{-19}\,\mathrm{C}$, $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{Fm}^{-1}$.

2. The rate at which the world's oil is being consumed is continuously increasing. Suppose the rate (in billions of barrels per year) is given by the function $r = f(t)$, where $t$ is measured in years and $t = 0$ is the start of 1990.

   a) Write down a definite integral which represents the total quantity of oil used between the start of 1990 and the end of 2020.

   b) Calculate this integral using the function $r(t) = 32e^{0.05t}$.

3. Since 1850, global carbon emissions have been rising exponentially. Let $C(t)$ represent the rate that carbon is emitted into the atmosphere, measured in Gigatonnes per year, where $t$ measures the number of years since the start of 1850. A model for the rate of emission over time is given by

$$C(t) = ke^{pt},$$

where $k$ and $p$ are positive constants.

   a) What are the units of $k$ and $p$?

   b) Given that, at the start of 1850, the global emission rate was 0.2 Gigatonnes per year, and that at the start of 2010 the global emission rate was 32 Gigatonnes per year, calculate $k$ and $p$.

   c) Calculate the total quantity of carbon emitted since the start of 1850.

4. The velocity $v$ of blood in a cylindrical vessel of radius $R$ and length $l$ is given by

$$v(r) = \frac{P(R^2 - r^2)}{4\eta l}$$

where $\eta$ and $P$ are constants, and $r$ is the radial distance from the cylinder's axis.

Find the average velocity of blood along the radius of the cylinder (i.e. for $0 \le r \le R$), and compare this with the maximum velocity.

5. Consider the function $y = 4x^3 + 2x^2 - 8x + 2$.

a) Draw an accurate graph of this function for values of $x$ between $-4$ and 4.

b) Calculate the turning points of the curve and mark these on the graph.

c) On your graph, shade in the region under the curve between $x = -2$ and $x = 2$ and *estimate* its area.

d) Integrate the function between $x = -2$ and $x = 2$. Is this an accurate calculation of the area of the shaded region in c)?

e) Identify and explain any differences you find between your estimate in part (c) and your calculation in part d).

**Extension problems**

1. Evaluate the following definite integrals:

   a) $\displaystyle\int_{-\pi/2}^{\pi/2} 3\cos(x)\,\mathrm{d}x$

   b) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos(x)\,\sin(x)\,\mathrm{d}x$

   c) $\displaystyle\int_{0}^{\pi} (\cos^2(x) + \sin^2(x))\,\mathrm{d}x$

2. Let $u$ and $v$ be functions of $x$.

   a) Given that $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x},$ show that $\displaystyle\int v\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = uv - \int u\,\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x.$

   b) By using this 'integration by parts' formula, and substituting $z = x^2$ or otherwise, show that

   $$\int_0^\infty x^n\,e^{-x^2}\,\mathrm{d}x = \frac{1}{2}(n-1)\int_0^\infty x^{n-2}e^{-x^2}\,dx \qquad \text{for } n > 1.$$

   c) Hence evaluate $\displaystyle\int_0^\infty x^5\,e^{-x^2}\,\mathrm{d}x.$

3. A country wishes to achieve net-zero $CO_2$ emissions in 50 years. At the start of the program, their emissions ($E$) are $800\mathrm{MtCO_2year^{-1}}$. They decide that they will be able to reduce their emissions at a stable rate, so that each year they emit $12\mathrm{MtCO_2year^{-1}}$ less than the previous year.

   a) Write down the rate of change of the countries emissions ($E$), each year ($t$), $\frac{\mathrm{d}E}{\mathrm{d}t}$. Use this to calculate the total emissions that the country had produced over the 50 years.

   After 10 years of these emissions, the country starts a $CO_2$ removal program, whereby a certain amount of $CO_2$ is captured from the atmosphere and sequestered underground each year. This $CO_2$ follows the curve

   $$R = 0.1t^2 - t$$

   where $R$ is the amount of $CO_2$ removal in $\mathrm{MtCO_2year^{-1}}$.

   b) Determine whether the country achieves their 50 year net-zero emissions goal by finding the year in which the emissions produced are equal to the emissions removed.

   After the 50 year program, the countries emission rate stabilises, and they emit the same amount of $CO_2$ each year after that. The $CO_2$ absorption rate per year follows the same trend as before. The country wishes to have not contributed to global warming at all since the start of the program. This means their net total $CO_2$ emissions over the entire program would have to be zero.

c) Show that is takes approximately 109 years for the country to have a nnet-zero effect on global warming since the start of the program.