Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 13: linear algebra 2

Introductory problems

1. Given

$$A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}; \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

verify by hand, and using the `numpy.linalg` module, that

a) $AA^{-1} = A^{-1}A = I$;

b) $BB^{-1} = B^{-1}B = I$;

c) $CC^{-1} = C^{-1}C = I$;

```
# hint
import numpy as np

# In Python the imaginary unit is "1j"
A = np.array([[1, 0], [0, 1j]])

print(A * np.linalg.inv(A))
```

2. Let A be an $n \times n$ invertible matrix. Let I be the $n \times n$ identity matrix and let B be a $n \times n$ matrix. Suppose that $ABA^{-1} = I$. Determine the matrix B in terms of the matrix A.

Main problems

1. Let A be the coefficient matrix of the system of linear equations

$$\begin{aligned} -x_1 - 2x_2 &= 1, \\ 2x_1 + 3x_2 &= -1. \end{aligned}$$

a) Solve the system by finding the inverse matrix A^{-1}.

b) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the solution of the system obtained in part a). Calculate and simplify $A^{2017}\mathbf{x}$.

2. For each of the following matrices

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 4 & 2 \\ 6 & 8 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}; \quad D = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

compute the determinant, eigenvalues and eigenvectors by hand. Check your results by verifying that $Q\mathbf{x} = \lambda_i\mathbf{x}$, where $Q = A$, B, C or D, and by using the `numpy.linalg` module.

```
# hint
import numpy as np

A = np.array([[2, 3], [1, 4]])

e_vals, e_vecs = np.linalg.eig(A)
```

```
print(e_vals)
print(e_vecs)
```

3. Orthogonal vectors. Two vectors $\mathbf{x_1}$ and $\mathbf{x_2}$ are said to be perpendicular or *orthogonal* if their dot/scalar product is zero:

$$\mathbf{x_1}.\mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = (2\times2) + (-1\times7) + (3\times1) = 4 - 7 + 3 = 0,$$

   thus $\mathbf{x_1}$ and $\mathbf{x_2}$ are orthogonal.

   Find vectors that are orthogonal to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. How many such vectors are there?

4. Diagonalize the 2×2 matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ by finding a non-singular matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

5. Find a 2×2 matrix A such that $A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $A \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Extension problems

1. If there exists a matrix M whose columns are those of normalised (unit length) and orthogonal vectors, prove that $M^T M = I$ which implies that $M^T = M^{-1}$.