Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 14: systems of differential equations 1

Introductory problems

1. Find the general solution to the following system of ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y.$$

   Sketch the form of the solution in the x, y plane, using arrows to indicate where the solution moves over time.

2. Take the general decoupled linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = by.$$

   a) Integrate the two equations separately to solve for x and y in terms of t.

   b) If you start at $t = 0$, $x(0) = 0$, $y(0) = 0$ what happens to the solution over time?

   c) If you start at a general position $x(0) = x_0$, $y(0) = y_0$ what happens to the solution as $t \to \infty$? What if a and b are both negative? What if only one of a or b is negative? What if either x_0 or y_0 is negative?

   d) Either by eliminating t from the original equations or by eliminating t from your solutions to part a) find a general solution of the system. (Why not try both methods?) Sketch this solution on the phase plane for
   i. $a > 0,\ b > 0,\ a = b$ ii. $a > 0,\ b < 0,\ a = -b$.

Main problems

1. By reformulating the following system as one first order equation (i.e eliminating t), find the general solution to:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x.$$

   Sketch the form of the solutions in the x, y plane.

2. Again by eliminating t and reformulating the system as one first order equation, find the general solution to the following system of ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x.$$

   Sketch the form of the solutions in the x, y plane.

3. Find the eigenvalues and two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

   a) Put the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as the columns of a 2×2 matrix P. Find P^{-1} and show (by calculation) that $P^{-1}AP$ is diagonal. What are the entries of this matrix? What do they correspond to?

   b) Find the general solution of the system $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x + 4y, \quad \text{and} \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + y.$

   c) Find the particular solution subject to $x(0) = 0$ and $y(0) = 2$.

d) Sketch the trajectory (the $x(t)$, $y(t)$ coordinates over time) in the x, y plane. Draw the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ on the same figure. What happens as $t \to \infty$? What about $t \to -\infty$? What is $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a general point on the y-axis?

Extension problems

1. The force on a damped harmonic oscillator is $f = -kx - m\nu\frac{\mathrm{d}x}{\mathrm{d}t}$, where x is a displacement, $k > 0$ is a spring force constant, $m > 0$ is the mass and $\nu > 0$ is the strength of the damping.

   a) Use Newton's 2nd law of motion to write down an equation for the acceleration $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

   b) Make the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ $\left(\text{and hence } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)$ to obtain a system of two first-order linear ODEs.