Control in Robotics
Part 5: Soft and Underactuated Robotics

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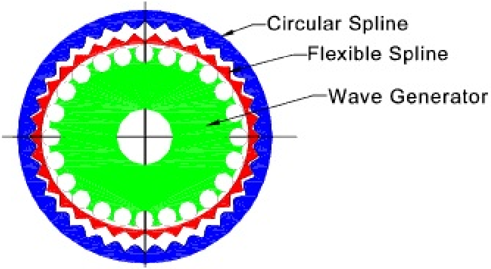
Flexible Joints Robots

- An important effect that limits the achievable performance of a manipulator is flexibility in the motor shaft and/or drive train.
- We refer to this as joint flexibility or joint elasticity.
- For many manipulators, particularly those using so-called harmonic gears, for torque transmission, the joint flexibility is significant.
- Harmonic gears are a type of gear mechanism that are very popular for use in robots due to their low backlash, high torque transmission, and compact size.

Flexible Joints Robots

- A typical harmonic gear, the Harmonic Drive ® gear, is shown here and consists of a rigid circular spline, a flexible flexspline, and an elliptical wave generator.
- The wave generator is attached to the actuator and hence is turned at high speed by the motor.
- The circular spline is attached to the load.



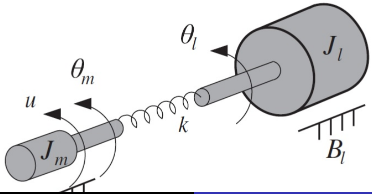
Flexible Joints Robots

- As the wave generator rotates it deforms the flexspline causing a number of teeth of the flexspline to mesh with the teeth of the circular spline.
- However, the principle of the harmonic gear relies on the flexibility of the flexspline.
- This flexibility is the limiting factor to the achievable performance in many cases.

Flexible Joints Robots

- Consider the idealized situation, consisting of an actuator connected to a load through a torsional spring representing the joint exibility. For simplicity we take the motor torque u, rather than the armature voltage, as input. The equations of motion are

$$
\begin{aligned}
J_l\ddot{\theta}_l + B_l\dot{\theta}_l + k(\theta_l - \theta_m) &= 0 \\
J_m\ddot{\theta}_m + B_m\dot{\theta}_l + k(\theta_m - \theta_l) &= 0
\end{aligned}
\tag{1}
$$



Flexible Joints Robots

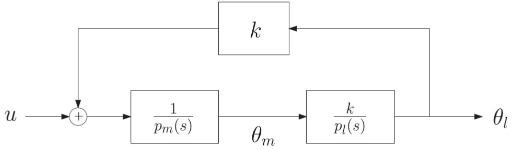
- J_l, J_m are the load and motor inertias, B_l and B_m are the load and motor damping constants, and u is the input torque applied to the motor shaft. The joint stiffness constant k represents the torsional stiffness of the harmonic gear.
- In the Laplace domain we can write the above system as

$$\begin{aligned} p_l(s)\Theta_l(s) &= k\Theta_m(s) \\ p_m(s)\Theta_m(s) &= k\Theta_l(s) + U(s) \end{aligned} \tag{2}$$

where

$$\begin{aligned} p_l(s) &= J_l s^2 + B_l s + k \\ p_m(s) &= J_m s^2 + B_m s + k \end{aligned} \tag{3}$$

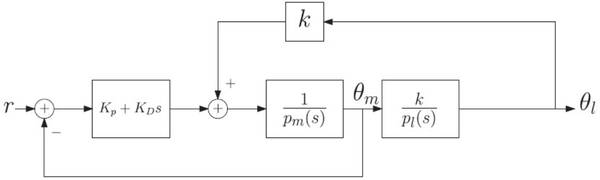
Flexible Joints Robots



- This system is represented by the above block diagram. The output to be controlled is, of course, the load angle θ_l.
- The open-loop characteristic polynomial is $p_l p_m - k^2$
- By neglecting the damping coefficients B_l and B_m, the open-loop characteristic polynomial would be $J_l J_m s^4 + k(J_l + J_m)s^2$ which has a double pole at the origin and a pair of complex conjugate poles on the $j\omega$-axis

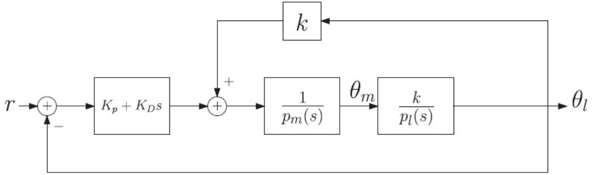
PD Control

- Suppose we implement a PD compensator $C(s) = K_P + K_D s$.
- At this point the analysis depends on whether the position/velocity sensors are placed on the motor shaft or on the load shaft, that is, whether the PD compensator is a function of the motor variables or the load variables.
- If the motor variables are measured then the closed-loop system is given by the below block diagram.

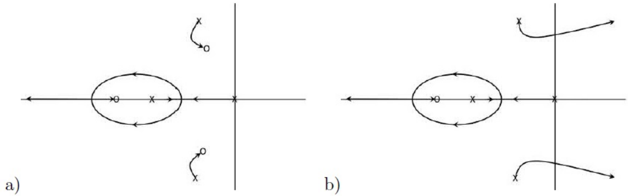


PD Control

- If we measure the load angle instead, the system with PD control is represented by the below block diagram.



- Root loci for the flexible joint systems a) motor rangle feedback and b) link-angle feedback.



PD Control

- In the case that the motor angle is used in the PD control, we see that the closed-loop system is stable for all values of the gain KD but that the presence of the open-loop zeros near the j! axis may result in undesirable oscillations. Also the poor relative stability means that disturbances and other unmodeled dynamics could render the system unstable.
- In the case that the load angle is used in the PD control, we see that the closed-loop system is unstable for large KD. The critical value of KD, that is, the value of KD for which the system becomes unstable, can be found from the Routh-Hurwitz criterion. The best that one can do in this case is to limit the gain KD so that the closed-loop poles remain within the left half plane with a reasonable stability margin.

State Space Design

- Next, we consider the application of state space methods for the control of the flexible joint system above.
- The previous analysis has shown that PD control is inadequate for robot control unless the joint flexibility is negligible or unless one is content with relatively slow response of the manipulator.
- Not only does the joint exibility limit the magnitude of the gain for stability reasons, it also introduces lightly damped poles into the closed-loop system that may result in oscillation in the transient response.
- We can write the flexible joint robot in state space by choosing state variables

$$x_1 = \theta_l, \quad x_2 = \dot{\theta}_l, \quad x_3 = \theta_m, \quad x_4 = \dot{\theta}_m \tag{4}$$

State Space Design

- If we choose an output $y(t) = \theta_l(t)$, then we have

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & \frac{B_m}{J_m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix}, \tag{6}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

- The above system is both controllable andobservable and we can use state feedback together with a state estimator with arbitrary closed loop poles.

State Space Design

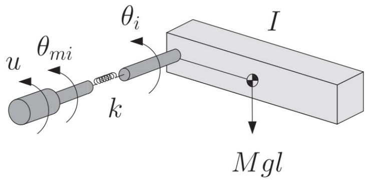
- The result that the closed-loop poles of the system may be placed arbitrarily, under the assumption of controllability and observability, is a powerful theoretical result.
- There are always practical considerations to be taken into account, however.
- The most serious factor to be considered in observer design is noise in the measurement of the output.
- To place the poles of the observer very far to the left of the imaginary axis in the complex plane requires that the observer gains be large.

State Space Design

- Large gains can amplify noise in the output measurement and result in poor overall performance.
- Large gains in the state feedback control law can result in saturation of the input, again resulting in poor performance.
- Also uncertainties in the system parameters, or nonlinearities such as a nonlinear spring characteristic and backlash, will reduce the achievable performance from the above design.
- Therefore, the above ideas are intended only to illustrate what may be possible by using more advanced concepts from control theory.

PD Control for general case

- In previous section, we considered the effect of joint flexibility for a single-link robot. In this section we will discuss the analogous result in the general case of an n-link manipulator.
- We first derive a mode to represent the dynamics of an n-link robot with joint flexibility.
- Let $q_1 = [\theta_1, ..., \theta_n]^T$ be the vector of joint variables and $q_2 = [\frac{1}{r_{m1}}\theta_{m1}, ..., \frac{1}{r_{mn}}\theta_{mn}]^T$ be the vector of motor shaft angles (reflected to the link side of the gears).



PD Control for general case

- It is straightforward to compute the Euler-Lagrange equations for this system as

$$\begin{aligned} M(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_2 + K(q_2 - q_1) &= u \end{aligned} \tag{7}$$

  where $J = \text{diag}(J_1, ...J_n)$, $K = \text{diag}(K_1, ...K_n)$.
- For the problem of set-point tracking with PD control, consider a control law of the form

$$u = -K_P\tilde{q}_2 - K_D\dot{q}_2 \tag{8}$$

- where $\tilde{q}_2 = q_2 q_d$ and q_d is a vector of constant set points, and suppose again that the gravity vector $g(q_1) = 0$.

PD Control for general case

- To show asymptotic tracking for the closed-loop system consider the Lyapunov function candidate

$$V = \frac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T J\dot{q}_2 + \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2) + \frac{1}{2}\tilde{q}_2^T K_P \tilde{q}_2 \tag{9}$$

- Application of LaSalle's theorem proves global asymptotic stability of the system.
- Consequently, the motor and link angles are equal in the steady state.
- If gravity is present, then it is not apparent how one can achieve gravity compensation in a manner similar to the rigid joint case.
- We will address this question later in the context of feedback linearization

Feedback Linearization

- In this section, we present some basic, but fundamental, ideas from geometric nonlinear control theory.
- We first give some background from differential geometry to set the notation and define basic quantities, such as manifold, vector field, Lie bracket, and so forth that we will need later.
- The main tool that we will use in this section is the Frobenius theorem.
- We then discuss the notion of feedback linearization of nonlinear systems. This approach generalizes the concept of inverse dynamics of rigid manipulators.
- As we shall see, the full power of the feedback linearization technique for manipulator control becomes apparent if one includes in the dynamic description of the manipulator the transmission dynamics, such as gear elasticity.

Preliminary Mathematics

- Vector function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a vector field in $\mathbb{R}^n$.
- Smooth vector field: function $f(x)$ has continuous partial derivatives of any required order.
- Gradient of a smooth scalar function $h(x)$ is denoted by a row vector $\nabla h = \frac{\partial h}{\partial x}$ where $(\nabla h)_j = \frac{\partial h}{\partial x_j}$
- Jacobian of a vector field $f(x)$:an $n \times n$ matrix $\nabla f = \frac{\partial f}{\partial x}$, where $(\nabla f)_{ij} = \frac{\partial f_i}{\partial x_j}$
- Lie derivative of h with respect to f is a scalar function defined by $L_f h = \nabla h f$, where $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth scalar and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field.
- The Lie derivative is the directional derivative of h in along f.
- The derivative of a Lyap. fcn candidate V, is $L_f V$.
- If g is another vector field: $L_g L_f h = \nabla(L_f h) g$
- $L_f^0 h = h$; $L_f^i h = L_f(L_f^{i-1} h) = \nabla(L_f^{i-1} h) f$.

Preliminary Mathematics

- Lie bracket of f and g is a third vector field defined by $[f, g] = \nabla g f - \nabla f g$, where f and g are two vector fields on $\mathbb{R}^n$
- The Lie bracket $[f, g]$ is also written as $ad_f g$ (ad stands for "adjoint")
- $ad_f^0 g = g$; $ad_f^i g = [f, ad_f^{i-1} g]$
- The concept of diffeomorphism can be viewed as a generalization of the familiar concept of coordinate transformation.
- Definition: A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined in a region Ω is called a diffeomorphism if it is smooth, and if its inverse ϕ^{-1} exists and is smooth.
- Lemma: Let $\phi(x)$ be a smooth function defined in a regionΩ in $\mathbb{R}^n$. If the Jacobian matrix $\nabla\phi$ is non-singular at a point $x = x_0$ of Ω, then $\phi(x)$ defines a local diffeomorphism in a subregion of Ω.

Preliminary Mathematics

- We present the Frobenius theorem as an important tool.
- Consider PDEs with known f_i, g_i and unknown h

$$\begin{aligned} \frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 = 0 \\ \frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 = 0 \end{aligned} \tag{10}$$

- If the solution $h(x_1, x_2, x_3)$ exists, the set of vector fields $\{f, g\}$ is **completely integrable**.
- Frobenius theorem states that the above PDE has a solution$h(x_1, x_2, x_3)$ iff there exists scalar functions $\alpha_1(x_1, x_2, x_3)$ and $\alpha_2(x_1, x_2, x_3)$ such that $[f, g] = \alpha_1 f + \alpha_2 g$. This condition is called **involutivity** of the vector fields $\{f, g\}$.
- Theorem (Frobenius): Let f_1, f_2, ..., f_m be a set of linearly independent vector fields. The set is completely integrable iff it is involutive.

Input-State Linearization

- Definition (Input-State Linearization): The nonlinear system $\dot{x} = f(x) + g(x)u$ where $f(x)$ and $g(x)$ are smooth vector fields in $\mathbb{R}^n$ is input-state linearizable if there exist region Ω in $\mathbb{R}^n$, a diffeomorphism mapping $\phi : \Omega \to \mathbb{R}^n$, and a control law

$$u = \alpha(x) + \beta(x)v$$

s.t. new state variable $z = \phi(x)$ and new input variable v satisfy an LTI relation:

$$\dot{z} = Az + Bv$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Input-State Linearization

- Theorem: The nonlinear system $\dot{x} = f(x) + g(x)u$ with $f(x)$ and $g(x)$ being smooth vector field is input-state linearizable iff there exists a region Ω s.t. the following conditions hold:
  - The vector fields $\{g, ad_f g, ..., ad_f^{n-1} g\}$ are linearly independent in Ω
  - The set $\{g, ad_f g, ..., ad_f^{n-2} g\}$ is involutive in Ω
- The first condition can be interpreted as a controllability condition.
- The second condition is always satisfied for linear systems since the vector fields are constant, but for nonlinear system is not necessarily satisfied.
- The second condition is necessary according to Ferobenius theorem for existence of $z_1(x)$.

Input-State Linearization

How to perform input-state Linearization?

1. Construct the vector fields $\{g, ad_f g, ..., ad_f^{n-1} g\}$
2. Check the controllability and involutivity conditions
3. If the conditions hold, obtain the first state z_1 from:

$$\nabla z_1 ad_f^i g = 0 \quad i = 0, 1, ..., n-2$$
$$\nabla z_1 ad_f^{n-1} g \neq 0$$

4. Compute the state transformation $z(x) = [z_1, L_f z_1, ..., L_f^{n1} z_1]^T$ and the input transformation $u = \alpha(x) + \beta(x)v$:

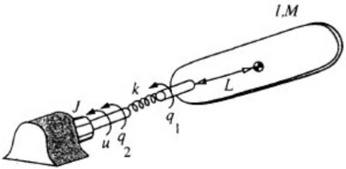
$$\alpha(x) = -\frac{L_f^n z_1}{L_g L_f^{n-1} z_1}$$
$$\beta(x) = \frac{1}{L_g L_f^{n-1} z_1}$$

Feedback linearization of a flexible-joint link

- Equations of motion:

$$
\begin{aligned}
I\ddot{q}_1 + MgL\sin q_1 + K(q_1 - q_2) &= 0 \\
J\ddot{q}_2 - K(q_1 - q_2) &= u
\end{aligned}
$$

- Nonlinearities appear in the first equation and torque is in the second equation



Feedback linearization of a flexible-joint link

- Let:

$$x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}, f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{K}{I}(x_1 - x_3) \\ x_4 \\ \frac{K}{J}(x_1 - x_3) \end{bmatrix}, g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix},$$

- Controllability and involutivity conditions:

$$\begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{K}{IJ} \\ 0 & 0 & \frac{K}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & \frac{K}{J^2} \\ \frac{1}{J} & 0 & -\frac{K}{J^2} & 0 \end{bmatrix}$$

- Its full rank for $K > 0$ and $IJ < \infty$.
- Since vector fields are constant, they are involutive.

Feedback linearization of a flexible-joint link

- Therefore, the system is input-state linearizable.
- Computing z_1:

$$\frac{\partial z_1}{\partial x_2} = 0, \quad \frac{\partial z_1}{\partial x_3} = 0, \quad \frac{\partial z_1}{\partial x_4} = 0, \quad \frac{\partial z_1}{\partial x_1} \neq 0$$

- Hence, z_1 is the fcn of x_1 only. Let $z_1 = x_1$, then

$$\begin{aligned} z_2 &= \nabla z_1 f = x_2 \\ z_3 &= \nabla z_2 f = -\frac{MgL}{I} \sin x_1 - \frac{K}{I}(x_1 - x_3) \\ z_4 &= \nabla z_3 f = -\frac{MgL}{I} x_2 \cos x_1 - \frac{K}{I}(x_2 - x_4) \end{aligned}$$

Feedback linearization of a flexible-joint link

- The input transformation is given by:

$$u = (v - \nabla z_4 f)/(\nabla z_4 g) = \frac{IJ}{K}(v - a(x))$$

$$a(x) = \frac{MgL}{I}\sin x_1 (x_2^2 + \frac{MgL}{I}\cos x_1 + \frac{K}{I}) + \frac{K}{I}(x_1 - x_3)(\frac{K}{I} + \frac{K}{J} + \frac{MgL}{I}\cos x_1)$$

- As a result, we get the following set of linear equations

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = v$$

Feedback linearization of a flexible-joint link

- The inverse of the state transformation is given by:

$$
\begin{aligned}
x_1 &= z_1 \\
x_2 &= z_2 \\
x_3 &= z_1 + \frac{I}{K}(z_3 + \frac{MgL}{I}\sin z_1) \\
x_4 &= z_2 + \frac{I}{K}(z_4 + \frac{MgL}{I} z_2 \cos z_1)
\end{aligned}
$$

- The transformed state have physical meaning, z_1 : link position, z_2 : link velocity, z_3 : link acceleration, z_4 : link jerk.
- Once, the linearized dynamics is obtained, either a tracking or stabilization problem can be solved.
- The linear dynamics of flexible joint manipulator is expressed as

$$z_1^{(4)} = v$$

Feedback linearization of a flexible-joint link

- Then, a tracking controller can be obtained as

$$v = z_{d1}^{(4)} - a_3 \tilde{z}_1^{(3)} - a_2 \ddot{\tilde{z}}_1 - a_1 \dot{\tilde{z}}_1 - a_0 \tilde{z}_1$$

  where $\tilde{z}_1 = z1 - z_{d1}$.

- The above dynamics is exponentially stable if a_i are selected s.t. the roots of

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

  are in LHP.

Flexible Link Robots

- The problem of modeling and control of flexible-link manipulators has received much attention in the past several years.
- There are a number of potential advantages arising from the use of light-weight flexible-link manipulators.
- For instance, in designing a space manipulator, energy efficiency and microgravity must be considered.
- For this reason, the robot manipulator is normally designed as a light-weight structure which results in smaller actuators for driving the joints and consequently less energy is consumed.
- On the other hand, increased structural flexibility may be desirable in tasks such as cleaning a delicate surface or avoiding damage to the manipulator system due to accidental collisions.

Flexible Link Robots

- The use of light-weight manipulators also results in a high ratio of payload to arm weight. (Traditional manipulators have a poor load-carrying capacity 5 to 10 percent of their own weight)
- For a rigid manipulator, the tip trajectory is completely defined by the trajectory of the joint. Effective control of the joint is equivalent to good control of the tip.
- The situation is not as straightforward for a flexible manipulator and difficulties arise when one tries to track a specified end-effector position trajectory by applying the torque at the joint.
- In this case, the control difficulty is due to the non-colocated nature of the sensor and actuator positions which results in unstable zero dynamics

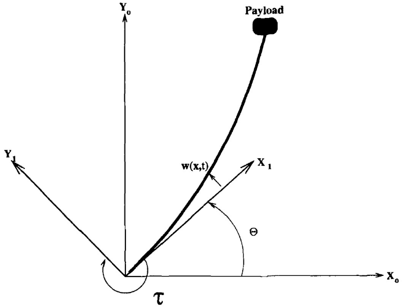
Flexible Link Robots

summary of control difficulties

- Instability of the zero dynamics related to the tip position which yields a non-minimum phase system.
- Highly nonlinear nature of the system
- Requiring a large number of states for accurate modeling
- Unmodeled dynamics due to model truncation and presence of various friction and backlash terms
- Variation of the payload

Modelling of Flexible Link Robots

- We consider a single flexible-link manipulator which is fixed at one end and is driven by a torque τ.
- The other end is free to flex in a horizontal plane, and has a payload mass M_l.



Modelling of Flexible Link Robots

- Neglecting the shear deformation and rotary inertia, the deflection of any point on the beam $W(x,t)$ is given by the Euler-Bernoulli beam equation

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + \rho I A^4 \frac{\partial^2 W(x,t)}{\partial t^4} = 0$$

  where E is Young's modulus of the material, A is the link cross-sectional area, I is its inertia and ρ is its uniform density.
- It can be shown that a single solution according to separation of variables $(W(x,t) = \phi(x)\delta(t))$ do not usually satisfy the initial conditions for position and velocity.
- The following infinite summation provides exact solution.

$$W(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\delta_i(t)$$

Modelling of Flexible Link Robots

- Therefore, an exact solution to the Euler-Bernoulli PDE requires an infinite number of modes.
- **The Assumed Modes Method:** The exact solution, can be approximated by the assumed modes method using a finite number of modes

$$W(x,t) = \sum_{i=1}^{n} \phi_i(x)\delta_i(t)$$

- By considering a finite number n of modal terms, the dynamic equations for the one-link flexible arm can be derived by using a Recursive Lagrangian approach.

Control of Flexible Link Robots

Properties of Flexible-Link Manipulators

- A Flexible-Link manipulator presents a non-minimum phase system
- A Flexible-Link manipulator is not input to state linearizable
- It is input-output linearizable, but the zero dynamics associated to the end-point (tip) is not stable
- It is non-minimum phase and has an unstable inverse
- The most common approach for controlling non-minimum phase system is output redefinition

Control of Flexible Link Robots

Output Redefinition

- Suppose, the original input-output pair, is (u, y)
- In output redefinition approach, a new output y_1 is defined such that the I-O pair $(u; y_1)$ is minimum phase, then either
  - Find y_{1d} such that if $y_1 \to y_{1d} \implies y \to y_d$. It is an extremely difficult task especially for nonlinear systems
  - Select y_1 close to y in some mathematical sense such that $y_1 \to y_d \implies y \rightsquigarrow y_d$. Hence, approximate tracking is achieved.
- Consider an LTI system with an Hurwitz polynomial $b(s)$:

$$y(s) = \frac{b(s)(1 - s/b_0)}{a(s)} u(s)$$

- A minimum phase approximation of $G(s)$ is obtained by ignoring the RHP zeros of $G(s)$, i.e. $y_1(s) = \frac{b(s)}{a(s)} u(s)$

Control of Flexible Link Robots

- Since y_1 mapping is minimum phase, an inverse dynamic method can be applied to get a perfect tracking, i.e. $\frac{y_1(s)}{y_d(s)} = 1$, then

$$\frac{y(s)}{y_d(s)} = \frac{y(s)}{y_1(s)} \frac{y_1(s)}{y_d(s)} = 1 - s/b_0$$

- Hence, the tracking is obtained for frequency well below the RHP zero b. In other words, using inverse Laplace transform, one can get:

$$y(t) = y_d(t) - \frac{1}{b}\dot{y}_d(t)$$

- The tracking is obtained if $|\dot{y}_d(t)| << b$

Control of Flexible Link Robots

Output Redefinition for Flexible-Link Manipulator

- The actual output is defined as the end point (tip) of the manipulator:

$$y = \theta + \frac{W(l,t)}{l}$$

- Define the joint angle as the output, i.e. $y_1 = \theta$
- The I-O map corresponding to this output is minimum phase.
- Hence, a stable inverse dynamics can be designed
- It result significant amount of vibration at the tip due to the flexible mode which left uncontrolled especially if the link flexibility is fairly considerable

Control of Flexible Link Robots

- Define a point close to the tip as the output:

$$y_1 = \theta + \alpha \frac{W(l,t)}{l}$$

  It has been shown that here exists an α such that the zero dynamics is Stable for $\alpha < \alpha^*$ (Madhavan and Singh, 1991).
- Unstable for $alpha > \alpha^*$
- For α close to 1, y_1 is close to y and a good approximate tracking is possible.
- A value between 05 to 0.9 has been reported for α.

Underactuated Robots

- Here, we consider the control of an important class of robots known as underactuated robots.
- By an underactuated robot, or more generally an underactuated mechanical system, we mean one in which the number of independent control inputs is fewer than the number of generalized coordinates.
- The class of underactuated robots is large and complex and the control problems are more difficult than for fully-actuated robots.
- Fully-actuated manipulator arms are globally feedback linearizable. This is generally not true for most underactuated systems, flexible-joint robots being an exception.
- As a result, the control problems for underactuated systems often require the development of new tools for controller design.

Underactuated Robots

- The below figure shows two underactuated serial-link robots.
- The shaded joints represent actuated degrees of freedom and the unshaded joints represent unactuated degrees of freedom.

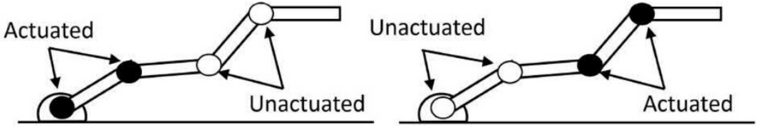


Figure : Upper-actuated (left) and lower-actuated (right) robots

Modelling of Underactuated Robots

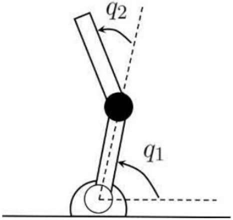
- The dynamic equations of motion of a general n-DOF underactuated system can be expressed as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Bu \tag{11}$$

  where $q \in \mathbb{R}^{n\times 1}$ is the vector of joint positions, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the centrifugal and Coriolis matrix, and $g(q) \in \mathbb{R}^{n\times 1}$ contains the gravitational forces.
- The matrix B is an $n \times m$ matrix of rank m reflecting the fact that there are m independent actuators.

Modelling of Underactuated Robots

- Example: The **Acrobot**, short for Acrobatic Robot, is a two-link RR robot with actuation at the second link.
- The Acrobot is representative of a gymnast on a high bar where $q2$ and u_2 represent a hip angle and hip torque, respectively.
- There is no actuator at the first joint where the hands grasp the bar.



Modelling of Underactuated Robots

- Therefore, the dynamic equations of the Acrobot are identical to the two-link RR robot with the torque at the first joint set to zero

$$
\begin{aligned}
m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + c_1 + g_1 &= 0 \\
m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + c_2 + g_2 &= u
\end{aligned}
$$

- The required terms are defined in Part 1.

Control of Underactuated Robots

- **Partial Feedback Linearization:**
  - Collocated partial feedback linearization: using nonlinear feedback to create a linear relationship between the accelerations of the active joints and their respective inputs
  - Noncollocated partial feedback linearization: establishing a linear relationship between the accelerations of the passive joints and the inputs to the active joints
- In both cases, we obtain systems of double integrator equations

$$\ddot{q}_i = a_i \quad i = 1, 2$$

  where a_i is an outer-loop control, as in the case of the inverse dynamics
- Both the collocated and noncollocated partial feedback linearization approaches lead to normal forms.

Control of Underactuated Robots

Collocated Partial Feedback Linearization

- Consider the system

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + c_1 + g_1 = 0$$
$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + c_2 + g_2 = u$$

- We may solve the first equation for $\ddot{q}_1$ as

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + c_1 + g_1)$$

and substitute it into the second equation to obtain

$$\bar{M}_{22}\ddot{q}_2 + \bar{c}_2 + \bar{g}_2 = u$$

where

$$\bar{M}_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}$$
$$\bar{c}_2 = c_2 - M_{21}M_{11}^{-1}c_1$$
$$\bar{g}_2 = g_2 - M_{21}M_{11}^{-1}g_1$$

Control of Underactuated Robots

- It can be proved that the $m \times m$ matrix $\bar{M}_{22}$ is symmetric and positive definite at each $q \in \mathbb{R}^m$
- We can see that the control law

$$u = \bar{M}_{22} a_2 + \bar{c}_2 + \bar{g}_2$$

where $a_{22} \in \mathbb{R}^m$ is an additional outer-loop control term, results in

$$\ddot{q}_2 = a_2$$

- The complete system up to this point may be written as

$$M_{11}\ddot{q}_1 + c_1 + g_1 = -M_{12} a_2 \tag{12}$$

$$\ddot{q}_2 = a_2 \tag{13}$$

- Equation (12) is called the internal dynamics.

Control of Underactuated Robots

Output Feedback Linearization

- Consider again the previous systemand suppose that we have a p-dimensional output $y = h(q_1, q_2) : \mathbb{R}^n \to \mathbb{R}^p$ defined as a smooth function of the configuration $q = (q_1, q_2)$

$$\begin{aligned} M_{11}\ddot{q}_1 + c_1 + g_1 &= -M_{12}a_2 \\ \ddot{q}_2 &= a_2 \\ y &= h(q_1, q_2) \end{aligned} \tag{14}$$

- Differentiating the output y yields

$$\dot{y} = \frac{\partial h}{\partial q_1}\dot{q}_1 + \frac{\partial h}{\partial q_2}\dot{q}_2 = J_1(q)\dot{q}_1 + J_2(q)\dot{q}_2$$

where $J_i := \frac{\partial h}{\partial q_i}$

Control of Underactuated Robots

- Computing $\ddot{y}$:

$$\begin{aligned}\ddot{y} &= J_1(q)\ddot{q}_1 + J_2(q)\ddot{q}_2 + \dot{J}_1(q)\dot{q}_1 + \dot{J}_2(q)\dot{q}_2 \\ &= (J_2 - J_1 M_{11}^{-1} M_{12})a_2 + \eta(q,\dot{q}) \\ &= \bar{J}a_2 + \eta(q,\dot{q})\end{aligned}$$

- Under the assumption that $\bar{J}$ has full rank, we can then define the control input a_2, using the right pseudo-inverse $\bar{J}^\dagger = \bar{J}^T(\bar{J}\bar{J}^T)^{-1}$ as

$$a_2 = \bar{J}^\dagger(\bar{a}_2 - \eta)$$

to obtain the linearized and decoupled output equation

$$\ddot{y} = \bar{a}_2 \tag{15}$$

- We note that an outer-loop control $\bar{a}_2$ can easily be designed to stabilize y or to track an arbitrary reference trajectory $y_d(t)$

Control of Underactuated Robots

- Definition: With output $y = h(q_1, q_2)$, let $\Gamma = \{(q; \dot{q}) : h(q) = 0, \dot{h}(q) = \frac{\partial h}{\partial q}\dot{q} = 0$. Γ is called the zero-dynamics manifold. An outer-loop control a_2 that asymptotically stabilizes the equilibrium $y = 0$ in (15) makes Γ an invariant manifold for the system (54). The reduced-order dynamics on Γ are called the zero dynamics
- For a general output function $h(q_1; q_2)$, it is not easy to characterize the reduced-order dynamics on the zero-dynamics manifold.
- In the special cases $y = q_i, i = 1$ or 2, the zero dynamics can be easily characterized.

Control of Underactuated Robots

- Consider

$$
\begin{aligned}
M_{11}\ddot{q}_1 + c_1 + g_1 &= -M_{12}a_2 \\
\ddot{q}_2 &= a_2 \\
y &= q_2
\end{aligned}
$$

- The zero dynamics are found by setting the output y identically zero, which implies that $q_2 = 0$, $\dot{q}_2 = 0$, and $a_2 = 0$. Then, the zero dynamics are given by

$$
\tilde{M}_{11}\ddot{q}_1 + \tilde{c}_1 + \tilde{g}_1 = 0
$$

- $\tilde{(\cdot)}$ is obtained by setting $q_2 = 0$, $\dot{q}_2 = 0$ in $(\cdot)$
- The reduced-order model represents the dynamics of a robot with passive joints where the m active joints are fixed, at $q_2 = 0$, and is therefore a (reduced-order) Lagrangian mechanical system.

Robots with Actuator Fault

- What is fault?
- Signal-based fault detection
    - Pattern recognition
    - Classification
- Model-based fault detection
    - Fault detectability
    - Unknown-input observer
    - Parrity
- Monitoring, supervision, and maintenance
- Fault-tolerant control