I. Vector Analysis

1.1 Vector algebra

1.1.1 Vector Operations

When you want to take the direction & magnitude into account, you need a vector. Quantities that have magnitude but no directions are called scalars

Four operations:

(i) addition of two vectors

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

rules:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

$$\vec{A} - \vec{B} = \vec{A} + \underbrace{(-\vec{B})}_{\vec{B}\text{'s opposite}}$$

(ii) Multiplication by a scalar

$$a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$$

(iii) Dot product of two vectors

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

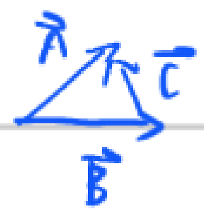
rules:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

$$\vec{A} \cdot \vec{A} = A^2$$

if $\vec{A}$ & $\vec{B}$ are perpendicular then $\vec{A} \cdot \vec{B} = 0$

eg 1.1:

$\vec{C} = \vec{A} - \vec{B}$ i.e. 

$C^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})(\vec{A}-\vec{B}) = A^2 + B^2 - 2\vec{A}\cdot\vec{B} = A^2 + B^2 - 2AB\cos\theta$

(iv) Cross product of two vectors

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

↳ a unit vector s.t. $\vec{A}, \vec{B}, \vec{n}$ make a right-hand rule

rules:

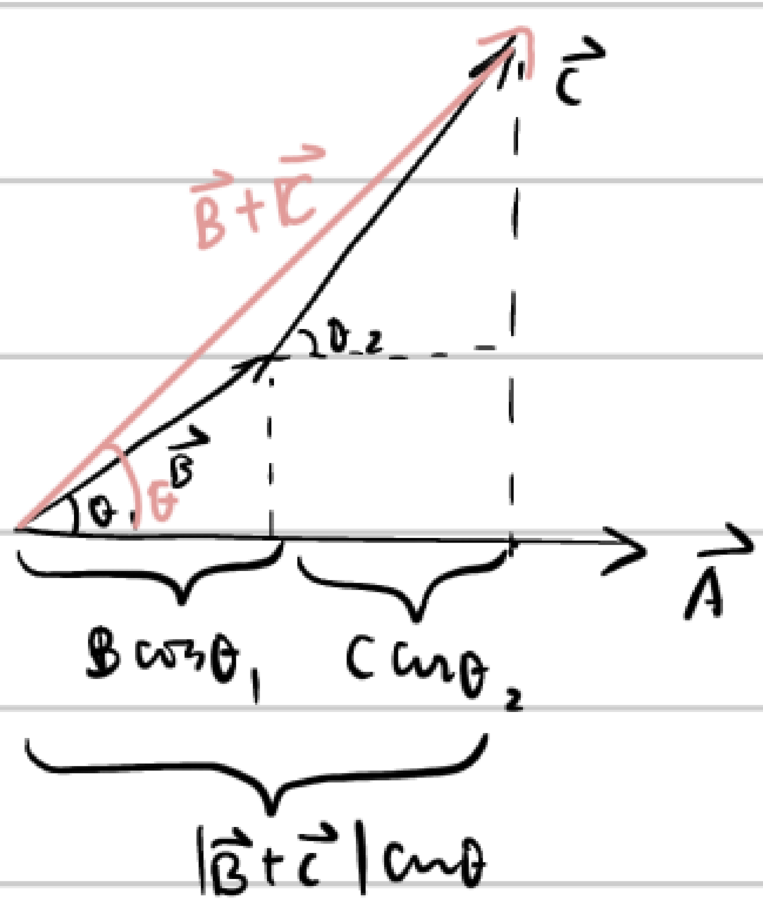
$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

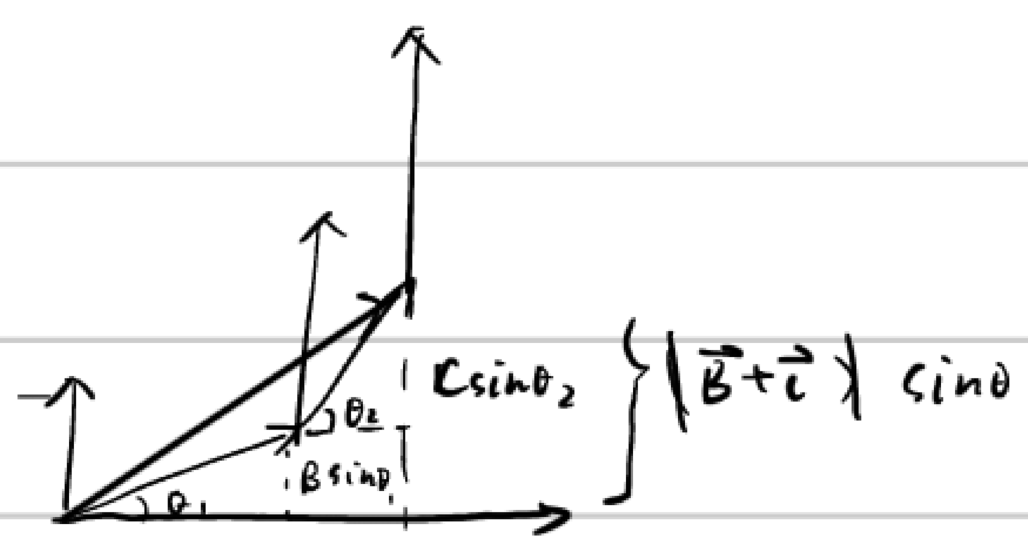
$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\vec{A}\times\vec{A} = \vec{0}$

Problems:

1.1 a)

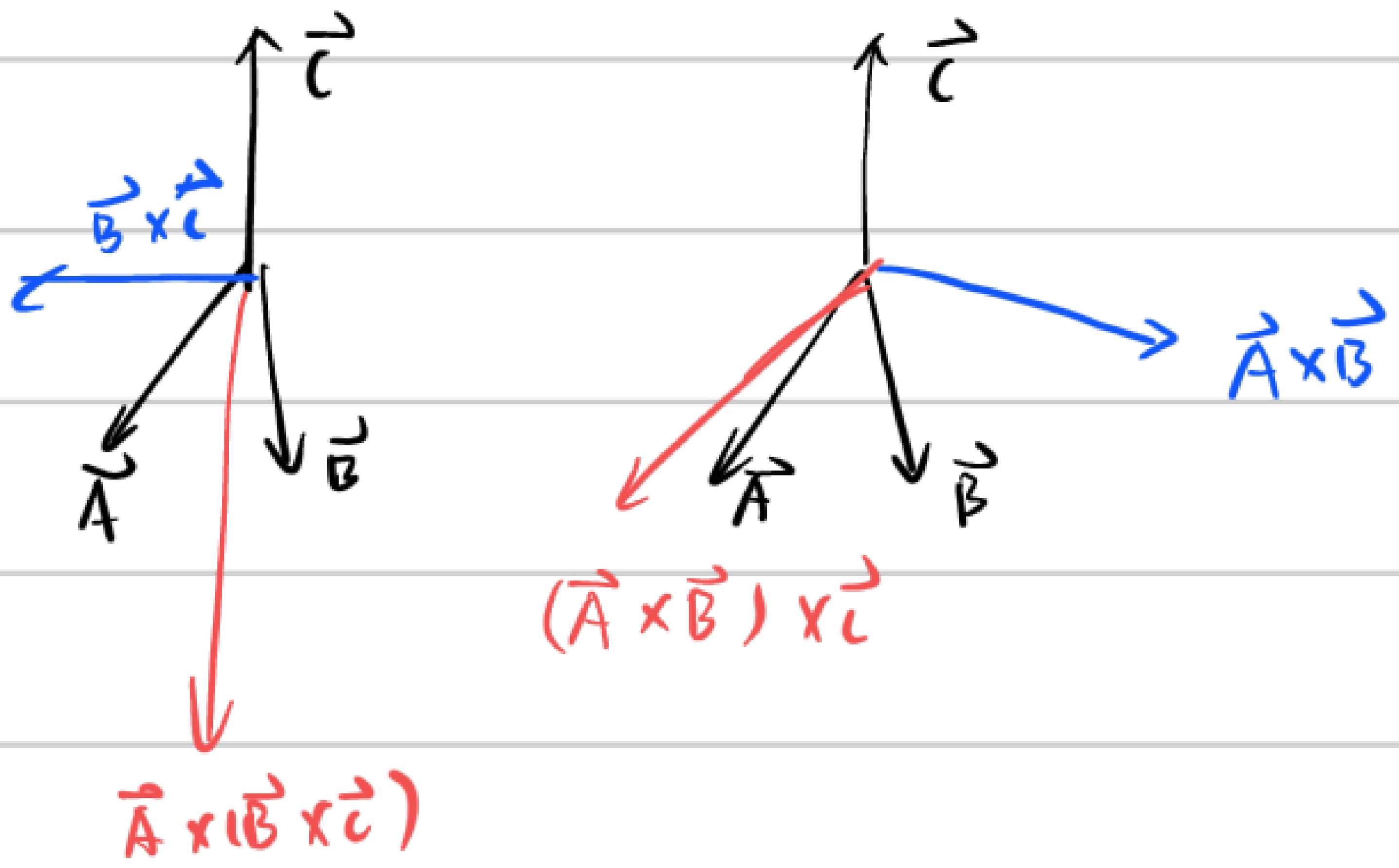




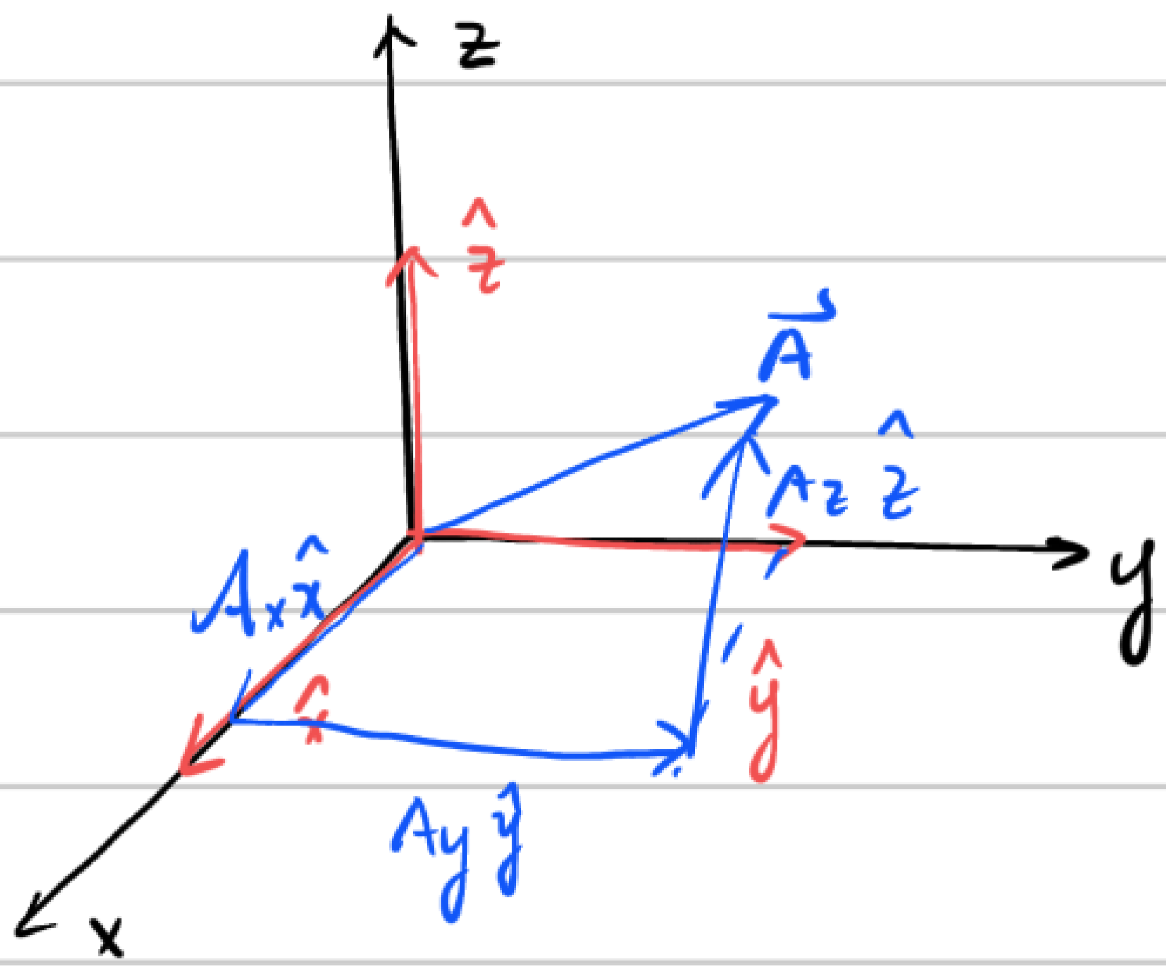
$A(B\cos\theta_1 + C\cos\theta_2) = A\,|\vec{B}+\vec{C}|\cos\theta$ $\qquad$ $A(B\sin\theta_1\hat{n} + C\sin\theta_2\hat{n}) = A\,|\vec{B}+\vec{C}|\sin\theta\,\hat{n}$

1.2

Not



1.1.2 Vector Algebra : Component Form



$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

A_x, A_y, A_z : components

$$A_x = \vec{A}\cdot\hat{x} \quad A_y = \vec{A}\cdot\hat{y} \quad A_z = \vec{A}\cdot\hat{z}$$

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{x} + (A_y+B_y)\hat{y} + (A_z+B_z)\hat{z}$$

$$a\vec{A} = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}$$

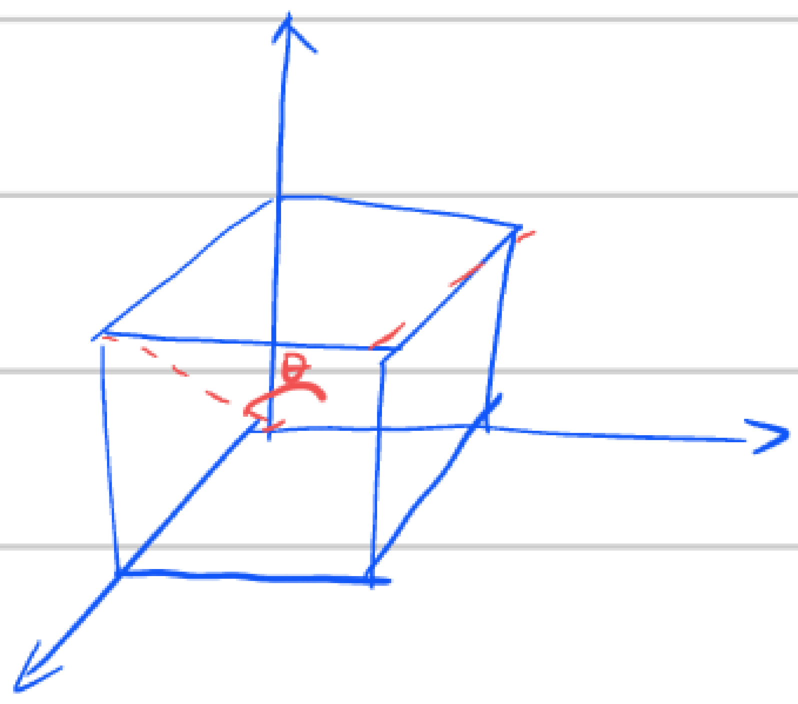
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} A_x & B_x & \hat{i} \\ A_y & B_y & \hat{j} \\ A_z & B_z & \hat{k} \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

eg. 1.2



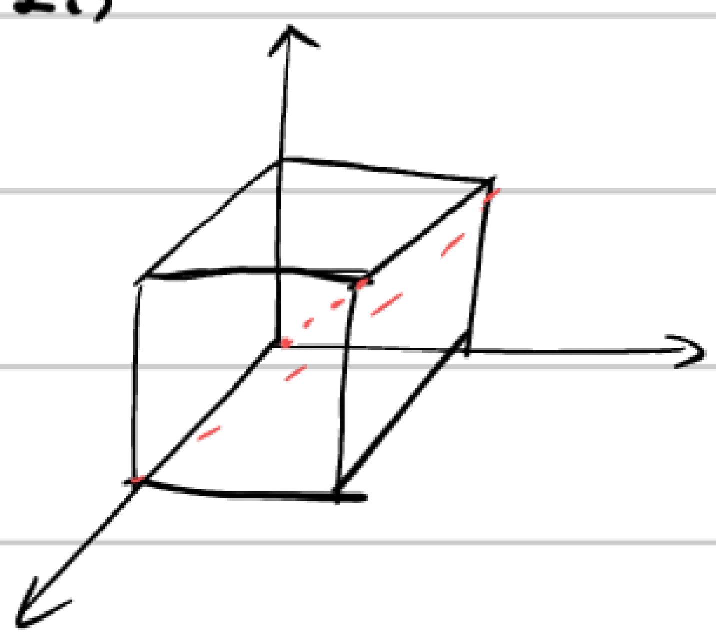
Find θ

$\vec{A} = (1,0,1) \quad \vec{B} = (0,1,1)$

$\vec{A}\cdot\vec{B} = 1 = 2\cos\theta \Rightarrow \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$

Problems

1.3



$\vec{A} = (1,1,1) \quad \vec{B} = (-1,1,1)$

$\vec{A}\cdot\vec{B} = 1 = 3\cos\theta \Rightarrow \cos\theta = \frac{1}{3}$

1.4

$\vec{A} = (-1,2,0) \quad \vec{B} = (-1,0,3)$

$$\vec{A}\times\vec{B} = \begin{vmatrix} -1 & -1 & \hat{i} \\ 2 & 0 & \hat{j} \\ 0 & 3 & \hat{k} \end{vmatrix} = (6,3,2)$$

$$\vec{n} = \frac{(6,3,2)}{\sqrt{6^2+3^2+2^2}} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

1.1.3 Triple Product

Scalar triple product

(i) $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} B_x & C_x & A_x \\ B_y & C_y & A_y \\ B_z & C_z & A_z \end{vmatrix}$$

(ii) Triple vector product

BAC-CAB rule:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

And $(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\times\vec{B}) = -\left(\vec{A}(\vec{C}\cdot\vec{B}) - \vec{B}(\vec{C}\cdot\vec{A})\right)$

$$= \vec{B}(\vec{C}\cdot\vec{A}) - \vec{A}(\vec{C}\cdot\vec{B})$$

eg

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = -\vec{C}\cdot((\vec{A}\times\vec{B})\times\vec{D})$$
$$= +\vec{C}\cdot(\vec{D}\times(\vec{A}\times\vec{B}))$$
$$= +\vec{C}\,(\vec{A}(\vec{D}\cdot\vec{B}) - \vec{B}\cdot(\vec{D}\cdot\vec{A}))$$
$$= -(\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D}) + (\vec{A}\cdot\vec{C})(\vec{D}\cdot\vec{B})$$

Problems:

1.5 $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

left: $\vec{A}\times(\vec{B}\times\vec{C})$

$$= \vec{A}\times\begin{vmatrix} B_x & C_x & \hat{i} \\ B_y & C_y & \hat{j} \\ B_z & C_z & \hat{k} \end{vmatrix}$$

$$= \vec{A}\times(B_yC_z - B_zC_y,\ B_zC_x - B_xC_z,\ B_xC_y - B_yC_x)$$

$$= \begin{vmatrix} A_x & B_yC_z - B_zC_y & \hat{i} \\ A_y & B_zC_x - B_xC_z & \hat{j} \\ A_z & B_xC_y - B_yC_x & \hat{k} \end{vmatrix} = (\underbrace{A_yB_xC_y}_{1} - \underbrace{A_yB_yC_x}_{3} - \underbrace{A_zB_zC_x}_{4} + \underbrace{A_zB_xC_z}_{2},$$
$$A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x,$$
$$A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y)$$

right: $\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$= (\cancel{B_x A_x C_x} + \underbrace{B_x A_y C_y}_{①} + \underbrace{B_x A_z C_z}_{②} - \cancel{A_x B_x C_x} - \underbrace{C_x B_y A_y}_{③} - \underbrace{C_x B_z A_z}_{④},$

$\cdots$

$\cdots)$

QED

1.6 $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B})$

$= \cancel{\vec{B}(\vec{A}\cdot\vec{C})} - \cancel{\vec{C}(\vec{A}\cdot\vec{B})}$

$+ \cancel{\vec{C}(\vec{B}\cdot\vec{A})} - \cancel{\vec{A}(\vec{B}\cdot\vec{C})}$

$+ \cancel{\vec{A}(\vec{C}\cdot\vec{B})} - \cancel{\vec{B}(\vec{C}\cdot\vec{A})}$

$= \vec{0}$

when $\vec{B}\times(\vec{C}\times\vec{A}) = 0$

$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$

$\Rightarrow \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\times\vec{C}$

1.1.4 Position, Displacement, and Separation Vectors

(i) A point can be described by (x, y, z)
the vector from origin O to that point is called
the position vector

$$\vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

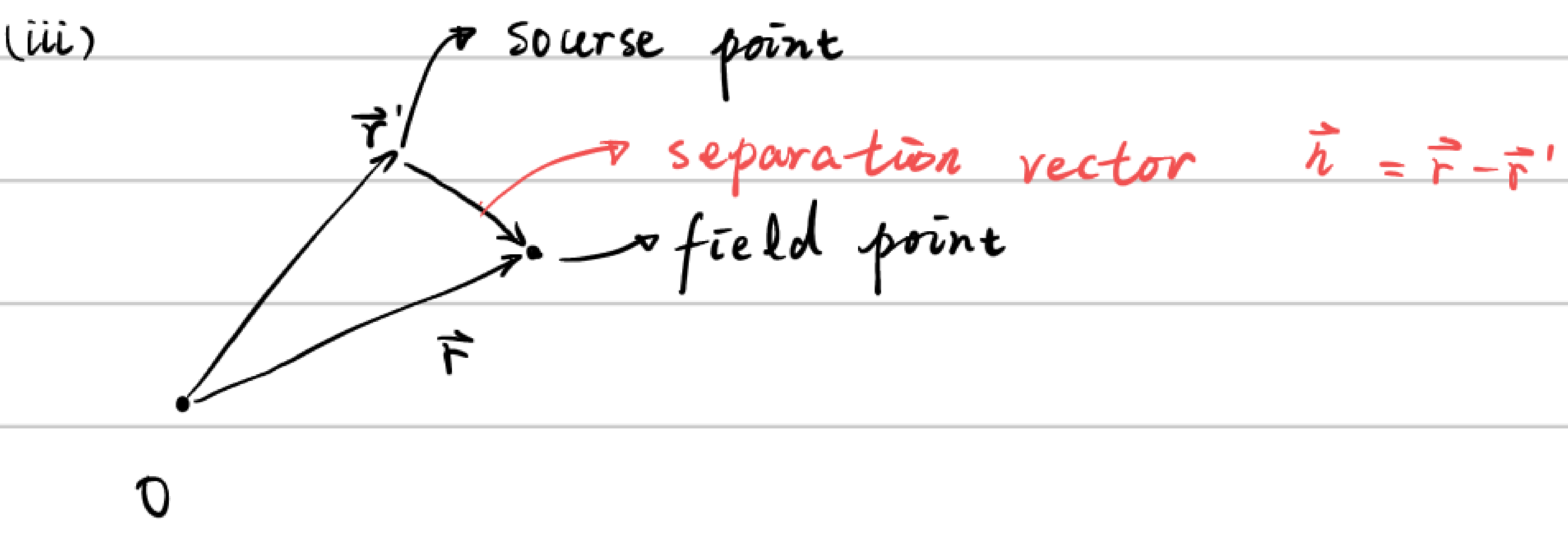
$$\hat{r} = \frac{\vec{r}}{r}$$

(ii) the infinitesimal displacement vector from (x, y, z) to $(x+dx, y+dy, z+dz)$

is $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

(we could call it $d\vec{r}$, but it's useful to have a special notation for it)

(iii)



$\imath = |\vec{\imath}| = |\vec{r} - \vec{r}'|$ $\qquad \hat{\imath} = \dfrac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$

Problems:

1.7

$\vec{\imath} = (4-2, 6-8, 8-7) = (2, -2, 1)$

$\imath = \sqrt{4+4+1} = 3$

$\hat{\imath} = \dfrac{\vec{\imath}}{3} = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$

1.1.5 How vectors transform

What does the "direction" of a vector mean?

if we use (N_x, N_y, N_z) to reprent a barrel of fruit

N_x means N_x pears

N_y - - - - - - apples

N_z - - - - - - bananas

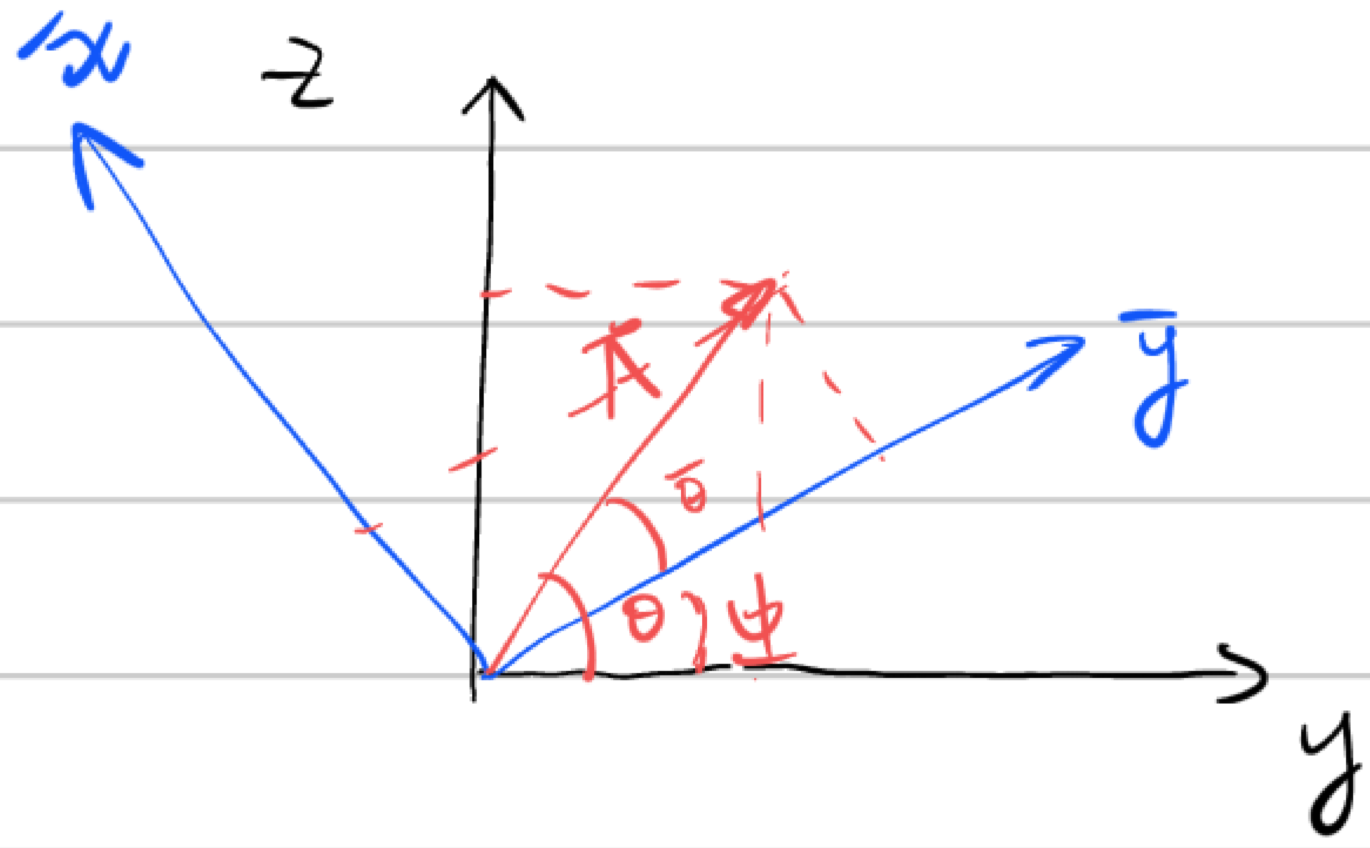
you can also add (N_x, N_y, N_z) & (M_x, M_y, M_z)

However, it's NOT a vector!

For N does not transform properly.

The coordinate frame we use to describe positions in space is of course entirely arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another.

Suppose, $\bar{x}, \bar{y}, \bar{z}$ system is rotated by angle ϕ, about the $x = \bar{x}$ axes.



$$\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta - \phi) = (A\cos\theta)\cos\phi + (A\sin\theta)\sin\phi$$
$$= y\cos\phi + z\sin\phi$$

$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta - \phi) = (A\sin\theta)\cos\phi - (A\cos\theta)\sin\phi$$
$$= z\cos\phi - y\sin\phi$$

$$\Rightarrow \begin{pmatrix} \bar{A}_y \\ \bar{A}_x \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

more generally

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{xy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

or

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j \quad (1 = x \quad 2 = y \quad 3 = z)$$

By the way, a (second-rank) tensor is a quantity with nine components

$$\bar{T}_{ij} = R_{ik} R_{jl} T_{ij}$$

Problems:

1.8 (a) $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$
$= \bar{A}_x \bar{B}_x + \bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z$

2-dim rotation

$\Rightarrow \quad A_x = \bar{A}_x \quad B_x = \bar{B}_x$

$\Rightarrow \quad A_x B_x = \bar{A}_x \bar{B}_x$

$$\Rightarrow \quad A_y B_y + A_z B_z = \bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z$$

(b) $A^2 = \vec{A}\cdot\vec{A} = \vec{A}'\vec{A}'$

$$= \vec{A}^T \overleftrightarrow{R}^T \overleftrightarrow{R} \vec{A}$$

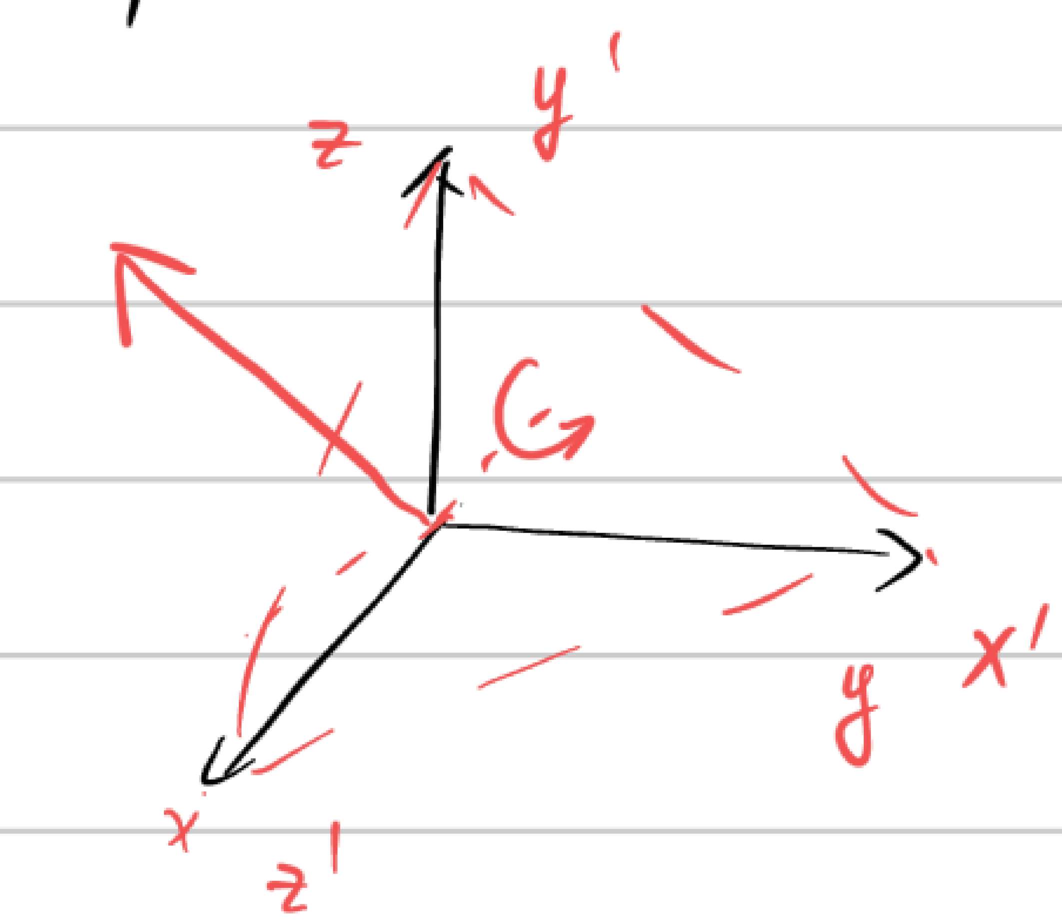
$$= \vec{A}^T \vec{A}$$

Because the $\vec{A}$ is arbitrary.

$$\overleftrightarrow{R}^T \overleftrightarrow{R} = \overleftrightarrow{I}$$

$$\Rightarrow \quad \delta_{ij} = R_{ki} R_{kj}$$

1.9



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

1.10

(a) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ The vector can move, too.

(b) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

(c) $\vec{A} \times \vec{B} = \vec{C}$

$$\begin{pmatrix} C_x' \\ C_y' \\ C_z' \end{pmatrix} = \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix}$$

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{E}$$

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

力矩，角动量 → Angular momentum

↳ torque

(d) $\vec{A} \cdot (\vec{B} \times \vec{C}) = q$

$q' = -q$

1.2 Differential calculus

1.2.1 "Ordinary" Derivatives

Suppose we have $f(x)$

$\frac{df}{dx}$ tells us how rapidly the function $f(x)$ varies when we change x by a tiny amount, dx.

$$df = \left(\frac{df}{dx}\right) dx$$

a proportionality factor

Geometrical Interpretation: $\frac{df}{dx}$ is the slope of the graph of f versus x.

1.2.2 Gradient

We have $T(x,y,z)$ stands for the temp of (x,y,z) in the room

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= (\vec{\nabla} T) \cdot (d\vec{l})$$

$$\vec{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$ is the gradient

Geometrical interpretation of gradient

$$dT = (\vec{\nabla} T) \cdot d\vec{l} = |\vec{\nabla} T|\,|dl|\cos\theta$$

if we fix $|d\vec{l}|$, vary θ.

the maximum change in T evidentally occurs when $\theta = 0$

$\Rightarrow$ $\vec{\nabla} T$ points out the direction of maximum increase of the function T

Moreover, $|\vec{\nabla} T|$ gives the slope along this maximum direction.

eg. 1.3

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{\nabla} r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

Problems:

1.11

(a) $\vec{\nabla} f = (2x, 3y^2, 4z^3)$

(b) $\vec{\nabla} f = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3)$

(c) $\vec{\nabla} f = (e^x \sin y \ln z, e^x \cos y \ln z, e^x \sin y \frac{1}{z})$

1.12

(a) $\vec{\nabla} h = 10(2y - 6x - 18, 2x - 8y + 28)$

Let $\vec{\nabla} h = 0$

$$\begin{cases} 2y-6x-18=0 \\ 2x-8y+28=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=-2 \\ y=3 \end{cases}$$

the top is located at $(-2,3)$

(b) $h_{max} = h(-2,3) = 10(-\cancel{1/2} - 12 - \cancel{36} + \cancel{36} + 84 + \cancel{1/2})$

$= 720$

(c) $\vec{\nabla} h(1,1) = 10(-22, 22)$

$= (-220, 220)$

$|\vec{\nabla} h| = 220\sqrt{2}$

1.13

(a) $\vec{\nabla}(\eta^2) = \vec{\nabla}(\vec{\eta}\cdot\vec{\eta})$

$= \vec{\nabla}((x-x')^2 + (y-y')^2 + (z-z')^2)$

$= (2(x-x'), 2(y-y'), 2(z-z'))$

$= 2\vec{\eta}$

(b) $\vec{\nabla}(\frac{1}{\eta}) = \vec{\nabla}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)$

$= \left(-\frac{1}{2}\frac{2(x-x')}{\eta^3}, -\frac{1}{2}\frac{2(y-y')}{\eta^3}, -\frac{1}{2}\frac{2(z-z')}{\eta^3}\right)$

$$= -\frac{\vec{r}}{r^3}$$

$$= -\frac{\hat{r}}{r^2}$$

(c) $\vec{\nabla}(r^n) = \vec{\nabla}\left(\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{n/2}\right)$

$$= \frac{n}{2}\, r^{2\cdot\left(\frac{n}{2}-1\right)} \quad (2(x-x'),\ 2(y-y'),\ 2(z-z'))$$

$$= n\, r^{n-2}\, \vec{r}$$

$$= n\, r^{n-1}\, \hat{r}$$

1.14

$$\begin{cases} \bar{y} = y\cos\phi + z\sin\phi \\ \bar{z} = -y\sin\phi + z\cos\phi \end{cases}$$

$$\Rightarrow \begin{cases} \bar{y}\cos\phi = y\cos^2\phi + z\sin\phi\cos\phi \\ \bar{z}\sin\phi = -y\sin^2\phi + z\cos\phi\sin\phi \end{cases}$$

$\Rightarrow \quad y = \bar{y}\cos\phi - \bar{z}\sin\phi \Rightarrow \frac{\partial y}{\partial \bar{y}} = \cos\phi \quad \frac{\partial y}{\partial \bar{z}} = -\sin\phi$

Also we have $z = \bar{y}\sin\phi + \bar{z}\cos\phi \Rightarrow \frac{\partial z}{\partial \bar{y}} = \sin\phi \quad \frac{\partial z}{\partial \bar{z}} = \cos\phi$

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = -\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi$$

$$\Rightarrow \begin{pmatrix} \frac{\partial f}{\partial \bar{y}} \\ \frac{\partial f}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

1.2.3 The Del Operator

The gradient has a formal appearance of a vector $\vec{\nabla}$ multiply a scalar T

$$\vec{\nabla} T = \left(\hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z}\right) T$$

$$\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z}\right)$$

del operator

$\vec{\nabla}$ is a vector operator that acts upon T
not a vector multiply a scalar T

an ordinary vector $\vec{A}$	$\vec{\nabla}$
$\vec{A}a$	$\vec{\nabla} T$ (gradient)
$\vec{A}\cdot\vec{B}$	$\vec{\nabla}\cdot\vec{v}$ (divergence)
$\vec{A}\times\vec{B}$	$\vec{\nabla}\times\vec{v}$ (the curl

1.2.4 The Divergence

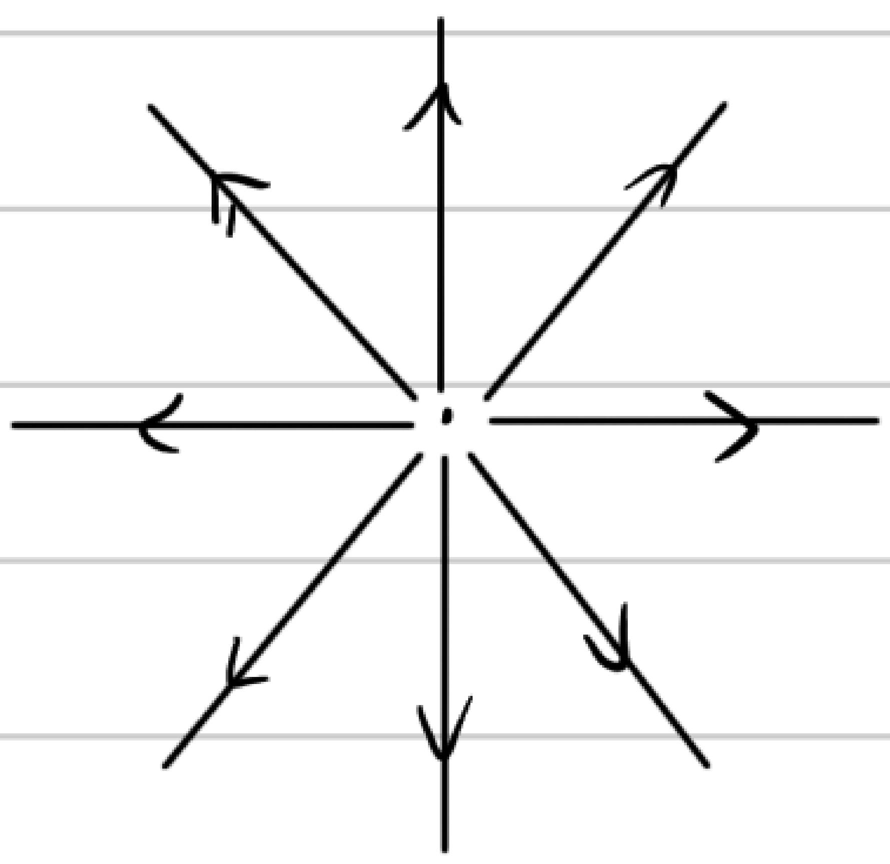
$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z}\right)\cdot(v_x, v_y, v_z)$$

The result is a scalar

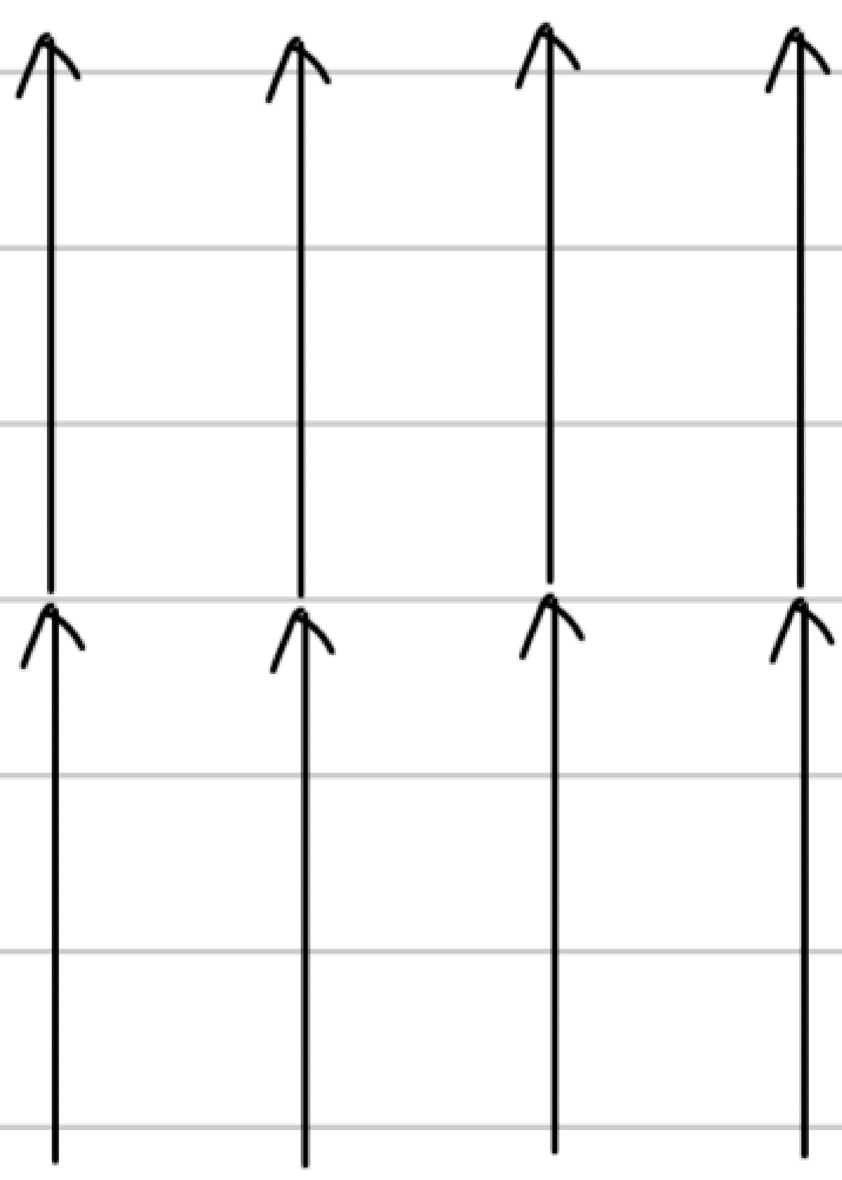
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Geometrical interpretation:

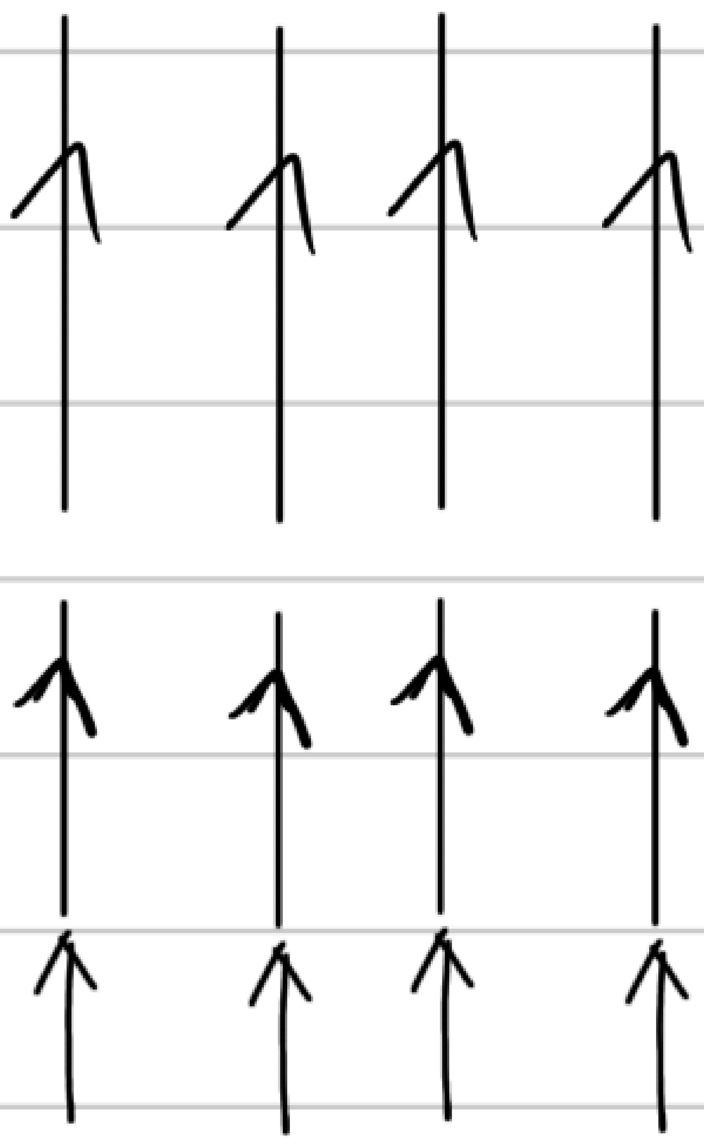
$\vec{\nabla}\cdot\vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point.



has a great positive divergence



zero divergence



has a positive divergence

eg $\vec{V}_a = (x, y, z)$ $\vec{V}_b = (0, 0, 1)$ $\vec{V}_c = (0, 0, z)$

$\vec{\nabla} \cdot \vec{V}_a = 1+1+1 = 3$ $\vec{\nabla} \cdot \vec{V}_b = 0$ $\vec{\nabla} \cdot \vec{V}_c = 1$

Problems

1.15

(a) $V_a = (x^2, 3xz^2, -2xz)$

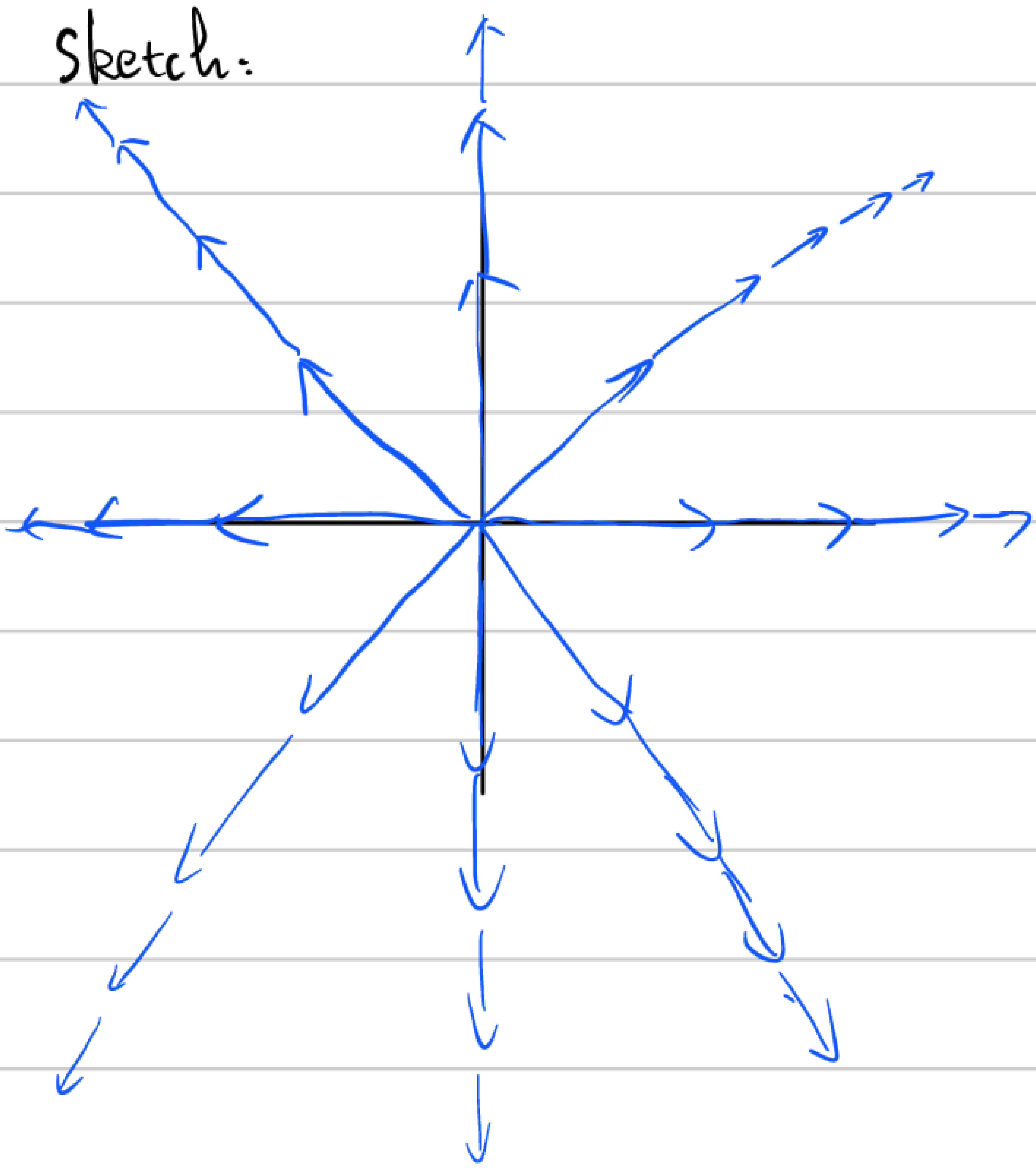
$\vec{\nabla} \cdot \vec{V}_a = 2x + 0 - 2x = 0$

(b) $\vec{\nabla} \cdot \vec{V}_b = y + 2z + 3x$

(c) $\vec{\nabla} \cdot \vec{V}_c = 0 + 2x + 2y = 2(x+y)$

1.16

Sketch:



$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2}$$

$$=\vec{\nabla}\cdot\left(\frac{x}{r^3},\ \frac{y}{r^3},\ \frac{z}{r^3}\right)$$

$$=\frac{3(x^2+y^2+z^2)^{3/2}-\frac{3}{2}r(2x^2+2y^2+2z^2)}{r^6}$$

$$=\frac{3r^3-3r^3}{r^6}$$

$$=0$$

explaination: I don't know.

1.17

$$\begin{pmatrix}\bar{V}_y\\ \bar{V}_z\end{pmatrix}=\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}V_y\\ V_z\end{pmatrix}$$

$$\vec{\nabla}\cdot\vec{v}=\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$$

$$\frac{\partial\bar{V}_y}{\partial\bar{y}}=\cos\phi\frac{\partial V_y}{\partial\bar{y}}+\sin\phi\frac{\partial V_z}{\partial\bar{y}}$$

$$=\cos\phi\left(\frac{\partial V_y}{\partial y}\frac{\partial y}{\partial\bar{y}}+\frac{\partial V_y}{\partial z}\frac{\partial z}{\partial\bar{y}}\right)+\sin\phi\left(\frac{\partial V_z}{\partial y}\frac{\partial y}{\partial\bar{y}}+\frac{\partial V_z}{\partial z}\frac{\partial z}{\partial\bar{y}}\right)$$

$$= \cos^2\phi \frac{\partial V_y}{\partial y} + \cos\phi \sin\phi \frac{\partial V_y}{\partial z} + \sin\phi \cos\phi \frac{\partial V_z}{\partial y}$$

$$+ \sin^2\phi \frac{\partial V_z}{\partial z}$$

$$\frac{\partial \bar{V}_z}{\partial \bar{z}} = + \sin^2\phi \frac{\partial V_y}{\partial y} - \sin\phi \cos\phi \frac{\partial V_y}{\partial z} - \sin\phi \cos\phi \frac{\partial V_z}{\partial y}$$

$$+ \cos^2\phi \frac{\partial V_z}{\partial z}$$

$$\Rightarrow \frac{\partial \bar{V}_y}{\partial \bar{y}} + \frac{\partial \bar{V}_z}{\partial \bar{z}} = (\sin^2\phi + \cos^2\phi) \frac{\partial V_y}{\partial y} + (\sin^2\phi + \cos^2\phi) \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$