PART 1 THEORY

I Wave function

1.1 The Schrödinger Equation

Classical mechanics

$$m\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0$$

give $x(0)$, $\left.\frac{\partial x}{\partial t}\right|_{t=0}$

have $x(t)$

quantum mechanics

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \quad (\Psi = \Psi(x,t))$$

$\hbar \equiv \frac{h}{2\pi}$ is very small $\sim 10^{-34}\,Js$

give $\Psi(x,0)$

have $\Psi(x,t)$

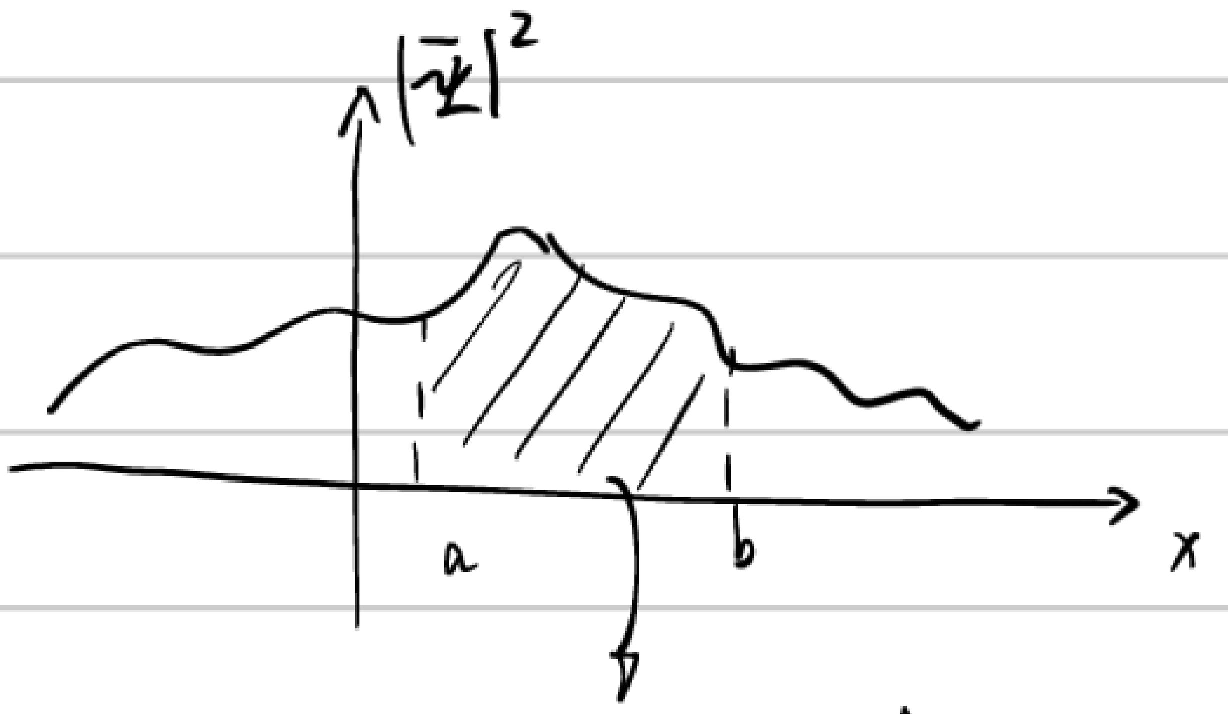
1.2 Statistical Interpretation (统计解释)

What does Ψ mean?

it's wave function of a particle

it means

$$\int_a^b |\Psi(x,t)|\,dx = \left\{ \text{probability of finding the particle between } a \,\&\, b \text{ at } t \right\}$$



probability → introduces a kind of indeterminacy into Q.M

Suppose at time T, you find a particle at point C.

Question: where it was before T?

1. realist position: at C. hidden variable can complete Q.M

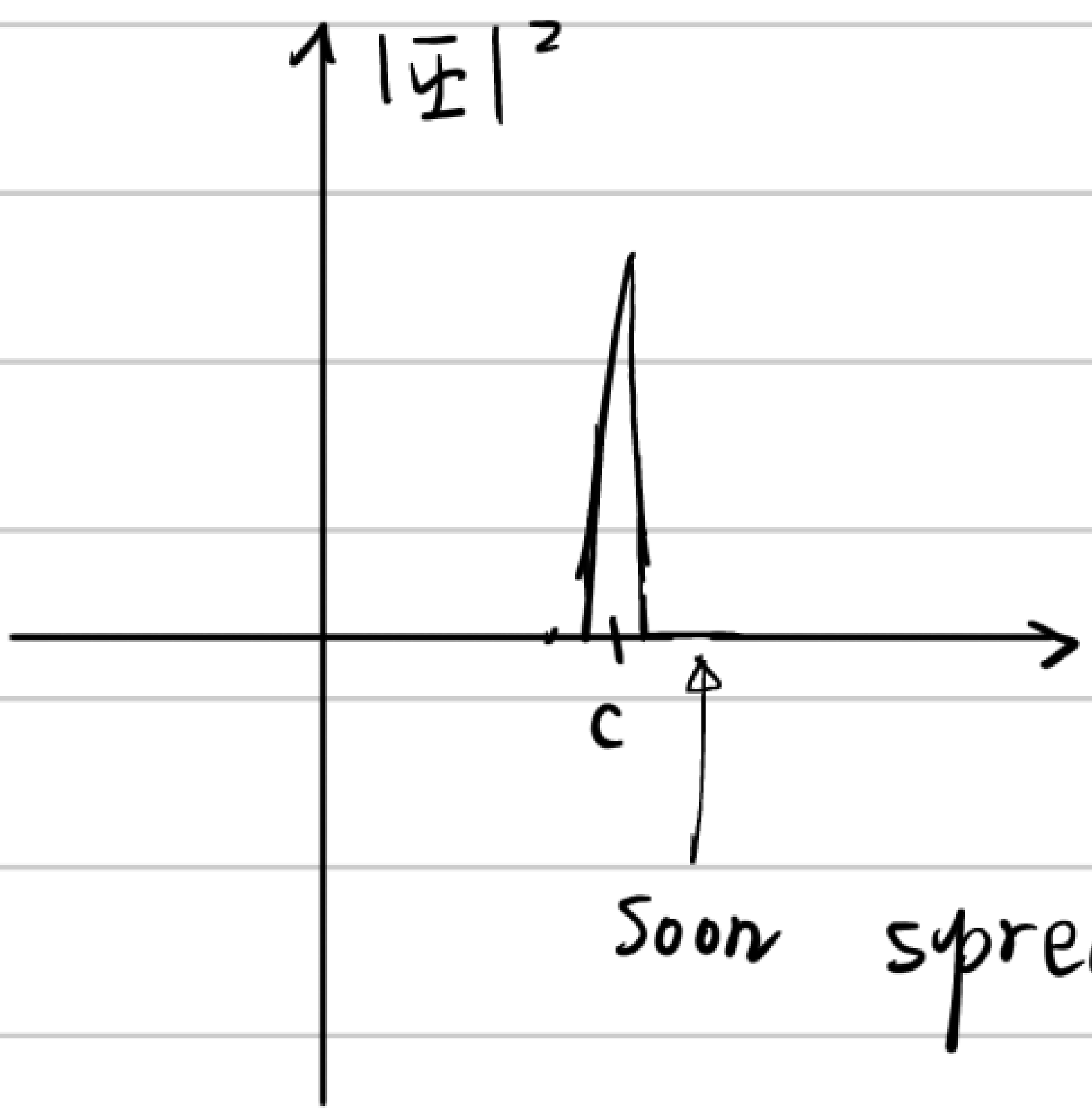
2. orthodox position: the particle wasn't really anywhere
It's the measurement compel the particle to assume a definition position. (Copenhagen interpretation)

3. agnostic position: Refuse to answer

- 1964 John Bell show that agnostic position is wrong.

* A repeated measurement (on the same particle) must return the same value

orthdox: the wave function collapses upon measurement

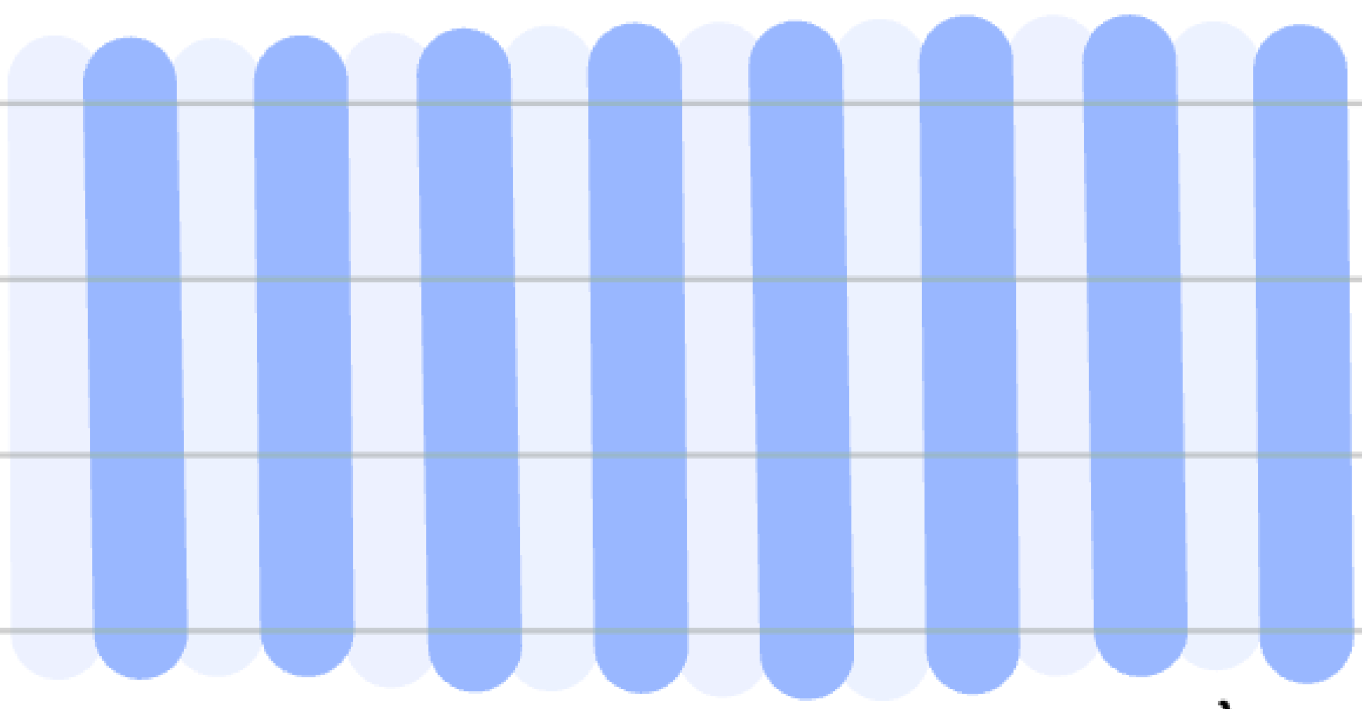


Soon spreads again (so the second measurement must be made quickly) e.g 1.1

Electron Interference

We have particles, how can we check the wave nature (if it does exist)?

We use the interference (a signature of classic waves)



the constructively interference make the $\int |\Psi|^2 dx$ bigger so that if we are patient enough we can see more electrons

$\int |\Psi|^2 dx$ is smaller.

if

a. close off one slit $\Rightarrow$ Change the B.C of Schrödinger eqn

b. try to detect which slit each electron passes through

$\Rightarrow$ collapse of wave function

$\Downarrow$ Both leads to

the inference pattern disappears

But the electrons interferes with itself, passing both slits

1.3 Probability

1.3.1 Discrete Variables

$$P(j) = \frac{N(j)}{N}$$

$$\sum_j P(j) = 1$$

Most probable : $P(j)$ is max

median data (中位数)

average (mean):

$$\langle j \rangle = \frac{\sum j N(j)}{N} = \sum j P(j)$$

expectation value

$$\langle j^2 \rangle = \sum j^2 P(j)$$

Generally

$$\boxed{\langle f(j) \rangle = \sum f(j) P(j)}$$

$$\Delta j \equiv j - \langle j \rangle$$

$$\langle \Delta j \rangle = \sum \Delta j P(j) = \sum (j - \langle j \rangle) P(j) = \sum j P_j - \langle j \rangle \sum P(j)$$

$$= \langle j \rangle - \langle j \rangle \cdot 1 = 0$$

meaningless

so we square the Δj to avoid this problem

$\sigma^2 \equiv \langle (\Delta j)^2 \rangle$ (variance of the distribution)

σ is called is called the standard deviation

↳ the customary measure of the spread about $\langle j \rangle$

σ is big  small : 

$$\sigma^2 = \langle (j - \langle j \rangle)^2 \rangle$$
$$= \langle j^2 + \langle j \rangle^2 - 2j\langle j \rangle \rangle$$
$$= \langle j^2 \rangle + \langle j \rangle^2 - 2\langle j \rangle^2$$
$$= \langle j^2 \rangle - \langle j \rangle^2$$

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

1 3 $\langle j^2 \rangle = 5$

$\langle j \rangle^2 = 4$

1.3.2 Continous Variables

only sensible thing is the probability in some interval.

$$\left\{ \begin{array}{l} \text{probability that the variable} \\ \text{lies between } x \text{ and } (x+dx) \end{array} \right\} = \underbrace{\rho(x)\,dx}_{}$$

probability density

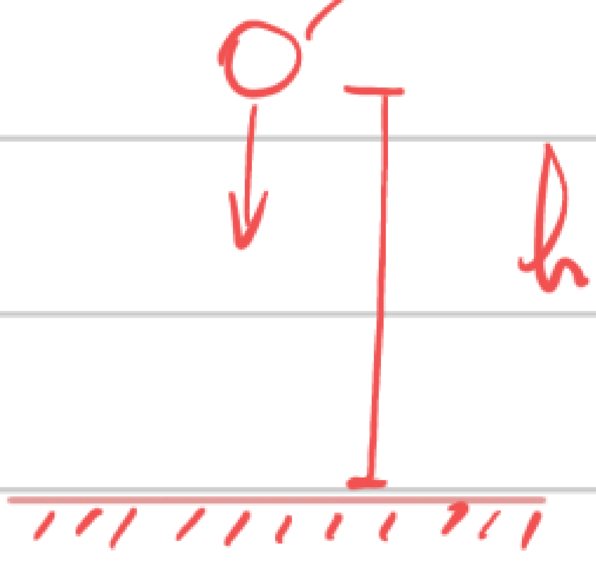
$$P_{ab} = \int_a^b \rho(x)\,dx \ ; \ \int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

$\langle x\rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx \qquad \langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)\,dx$

$\sigma^2 \equiv \langle x^2\rangle - \langle x\rangle^2$

eg 1.2

start at rest



take 1 million photos, then what is the average of all distances in these pictures?

Solution

$x(t) = \frac{g}{2}t^2 \Rightarrow t = \sqrt{\frac{g}{2x}}$

$\frac{dx}{dt} = gt \Rightarrow dt = \frac{dx}{gt}$

$T = \sqrt{2h/g} \Rightarrow$ The probability that a photo taken between t and $(t+dt)$ is $\frac{dt}{T}$

so the distance between x & $x+dx$

$\frac{dt}{T} = \frac{dx}{gt}\sqrt{\frac{g}{2h}} = dx\sqrt{\frac{1}{2\cancel{g}h}\cdot\frac{\cancel{g}}{2x}} = dx\sqrt{\frac{1}{4xh}}$

$\rho(x) = \frac{1}{2\sqrt{xh}} \quad (0 \le x \le h)$ (check $\int_0^h \rho(x)dx = \frac{1}{\sqrt{h}}x^{1/2}\Big|_0^h = 1$)

$\Rightarrow \langle x\rangle = \int_0^h x\frac{1}{2\sqrt{h}}x^{-\frac{1}{2}}dx = \frac{2}{3}\frac{1}{2\sqrt{h}}x^{3/2}\Big|_0^h = \frac{1}{3}h$

△ $x \to 0$, $\rho(x)$ is infinite, but $\int_0^{+\varepsilon}\rho\,dx$ is finite.

Problems:

1.1

1.2 $\rho(x) = \frac{1}{2\sqrt{h}} \frac{1}{\sqrt{x}}$

(a) $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$= \text{sqrt}\left(\int_0^h \frac{1}{2\sqrt{h}} x^{3/2} dx - \left(\int_0^h \frac{1}{2\sqrt{h}} x^{1/2} dx\right)^2 \right)$

$= \text{sqrt}\left(\frac{2}{5} \frac{1}{2\sqrt{h}} h^{5/2} - \left(\frac{h}{3}\right)^2 \right)$

$= \text{sqrt}\left(\frac{h^2}{5} - \frac{h^2}{9} \right)$

$= \text{sqrt}\left(\frac{4}{45} h^2 \right)$

$= \frac{2}{15}\sqrt{5}\, h \approx 0.2981\, h$

(b) $\langle x\rangle = \frac{h}{3}$

$P_1 = \int_{\frac{h}{3}-\sigma}^{\frac{h}{3}+\sigma} \rho(x) dx = \int_{\frac{h}{3}-\sigma}^{\frac{h}{3}+\sigma} \frac{1}{2\sqrt{h}} \frac{1}{\sqrt{x}} dx = 2 \cdot \frac{1}{2\sqrt{h}} x^{1/2} = \left(\frac{1}{3}+0.2981\right)^{1/2} - \left(\frac{1}{3}-0.2981\right)^{1/2}$

≈ 0.6069

$P = 1 - P_1 = 0.3931$

1.3 (a) $\int \rho(x) dx = A \int e^{-\lambda(x-a)^2} dx = 1$

$\Rightarrow A = 1 \left(\int e^{-\lambda(x-a)^2} dx \right)^{-1}$

$= 1 \cdot \sqrt{\frac{\lambda}{\pi}} = \sqrt{\frac{\lambda}{\pi}}$

(b) $\rho(x) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}$

$$\langle x \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}}\, x e^{-\lambda (x-a)^2}\, dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2}\, dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} (x-a) e^{-\lambda(x-a)^2} dx + a \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx \right)$$

$$= \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} t e^{-\lambda t^2} dt + a \sqrt{\frac{\pi}{\lambda}} \right)$$

$$= \sqrt{\frac{\lambda}{\pi}} \left(\underbrace{-\frac{1}{2\lambda} e^{-\lambda t^2} \Big|_{-\infty}^{\infty}}_{0} + a \sqrt{\frac{\pi}{\lambda}} \right.$$

$$= a$$

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda (x-a)^2} dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} (x^2 - 2ax + a^2)\, e^{-\lambda (x-a)^2} dx \right) + 2a \langle x \rangle - a^2 \cdot 1$$

$$= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} t^2 e^{-\lambda t^2} dt + a^2$$

$$= \sqrt{\frac{\lambda}{\pi}} \frac{1}{-2\lambda} \int_{-\infty}^{\infty} t\, \underbrace{\left(-2\lambda t e^{-\lambda t^2} dt\right)}_{d e^{-\lambda t^2}} + a^2$$

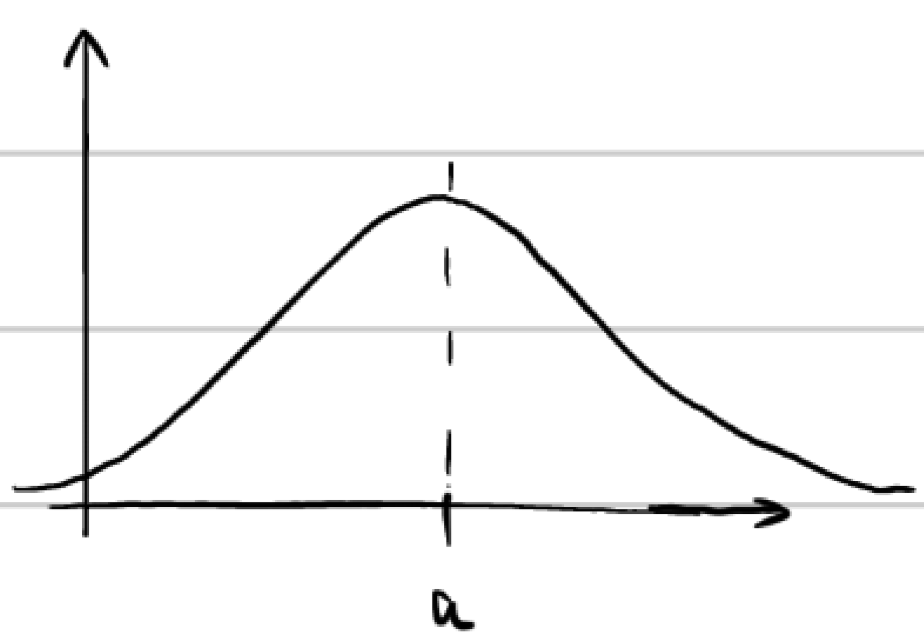
$$= \sqrt{\frac{\lambda}{\pi}} \frac{1}{-2\lambda} \left(\underbrace{t e^{-\lambda t^2} \Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} e^{-\lambda t^2} dt \right) + a^2$$

$$= \sqrt{\frac{\lambda}{\pi}} \frac{1}{-2\lambda} \cdot -\sqrt{\frac{\pi}{\lambda}} + a^2$$

$$= a^2 + \frac{1}{2\lambda}$$

$$\sigma = \sqrt{a^2 + \frac{1}{2\lambda} - a^2} = \sqrt{\frac{1}{2\lambda}}$$

(c)



1.4 Normalization

$|\Psi|^2$ is the probability density for finding a particle at point x, time t.

So naturally

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

If $\Psi_1(x)$ is the solution of

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

any complex constant

then $A\Psi_1$ should also be the solution, too.

So the function $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$ cannot give the physical solution Ψ, we must do $\int_{-\infty}^{\infty} A\Psi \, dx = 1$ to get A then we have real solution, this process is called normalizing the wave function

some funtion's integral is infinite, or some function is $\Psi = 0$, these non-normalizable solutions can not present particles.

Physically realizable states correspond to the square-integrable solutions.

!!! $\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dx = 0$, this property protects the normaliztion is useful all the time

proof:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dt = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi|^2 dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

for $i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

$$\Rightarrow \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{V}{\hbar}\Psi$$

$$\Rightarrow \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{V}{\hbar}\Psi^*$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{V}{\hbar}\Psi^*\right)\Psi + \left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{V}{\hbar}\Psi\right)\Psi^*$$

$$= \frac{i\hbar}{2m}\left(\frac{\partial^2 \Psi}{\partial x^2}\Psi^* - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right)$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial x}\Psi^* - \frac{\partial \Psi^*}{\partial x}\Psi\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\left(\frac{\partial \Psi}{\partial x}\Psi^* - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right)$$

$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{\infty}|\Psi|^2 dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}|\Psi|^2 dx = \frac{i\hbar}{2m}\left(\frac{\partial \Psi}{\partial x}\Psi^* - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}$$

because of the requirement of normalization

$\Psi(\pm\infty) = 0$

$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{\infty}|\Psi|^2 dx = 0$$

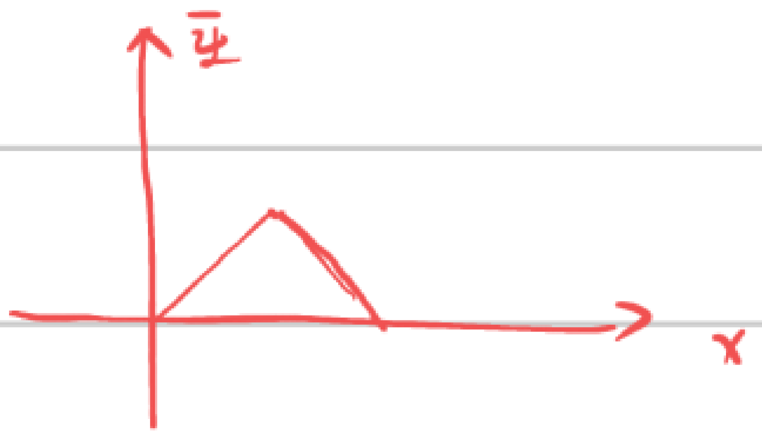
If Ψ is normalized at $t=0$, it stays normalized all the time

QED

Problems

1.4

sketch: $\Psi(x)$



(a) $$|\Psi|^2 = \begin{cases} \frac{A^2}{a^2}x^2 & 0 \le x \le a \\ A^2\frac{(b-x)^2}{(b-a)^2} & a < x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_0^a \frac{A^2}{a^2} x^2 dx + \int_a^b A^2 \frac{(b-x)^2}{(b-a)^2} dx$$

$$= A^2 \frac{1}{a^2} \cdot \frac{a^3}{3} + A^2 \frac{1}{(b-a)^2} \left(-\frac{1}{3} (b-x)^3\right)\Big|_a^b$$

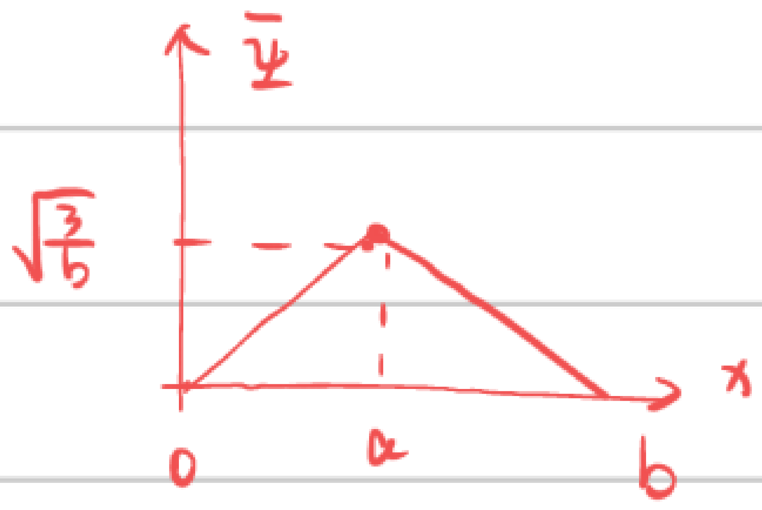
$$= A^2 \frac{a}{3} + A^2 \frac{1}{(b-a)^2} \frac{1}{3} (b-a)^3$$

$$= A^2 \frac{1}{3} (a + (b-a))$$

$$= A^2 \frac{b}{3} = 1$$

$$\Rightarrow A = \sqrt{\frac{3}{b}}$$

(b) sketch:



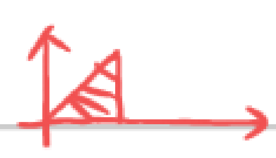
(c) at $[a-dx, a+dx]$

(d) $P = \int_0^a |\Psi(x)|^2 dx$

$$= \frac{1}{a^2} \frac{3}{b} \int_0^a x^2 dx$$

$$= \frac{3}{b} \frac{a}{3} = \frac{a}{b}$$

check: $b = 2a$ $\quad P = \frac{1}{2}$ 

$b = a$ $\quad P = 1$ 

(e) $\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi|^2 dx$

$$= \frac{1}{a^2}\frac{3}{b}\int_0^a x^3 dx + \frac{3}{b} \frac{1}{(b-a)^2} \int_a^b x (b-x)^2 dx$$

$$= \frac{3}{a^2 (b-a)^2 b} \left(\frac{a^4}{4} (b-a)^2 + \frac{a^2 b^2}{2} (b^2-a^2) - 2a^2 b \frac{b^3-a^3}{3} + \frac{a^2}{4} (b^4 - a^4) \right) dx$$

$$= \frac{3}{\cancel{a^2 (b-a)^2 b}} \left(\frac{1}{12} \cancel{a^2 (b-a)^2 b} (2a+b) \right)$$

$$= \frac{1}{4} (2a+b)$$

1.5 (a) Let $t=0$, $\Psi(x,0) = A e^{-\lambda|x|} \cdot 1$

just need to normalize $\Psi(x,0)$

$$\int_{-\infty}^{\infty} |A e^{-\lambda|x|}|^2 \, dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} \, dx$$

$$= A^2 \left(\int_{-\infty}^{0} e^{2\lambda x} \, dx + \int_{0}^{\infty} e^{-2\lambda x} \, dx \right)$$

$$= A^2 \left(\frac{1}{2\lambda} e^{2\lambda x} \Big|_{-\infty}^{0} + \frac{1}{-2\lambda} e^{-2\lambda x} \Big|_{0}^{\infty} \right)$$

$$= A^2 \left(\frac{1}{2\lambda}(1-0) + \frac{1}{-2\lambda}(0-1) \right)$$

$$= A^2 \frac{1}{\lambda} = 1$$

$$\Rightarrow A = \sqrt{\lambda}$$

$$\Rightarrow \Psi = \sqrt{\lambda}\, e^{-\lambda|x|} e^{-i\omega t}$$

$$|\Psi|^2 = \lambda e^{-2\lambda|x|}$$

(b) $$\langle x \rangle = \int_{-\infty}^{\infty} \lambda x e^{-2\lambda|x|} \, dx$$

$$= \lambda \left(\int_{-\infty}^{0} x e^{2\lambda x} \, dx + \int_{0}^{\infty} x e^{-2\lambda x} \, dx \right)$$

$$= \lambda \left(\frac{1}{2\lambda} \left(\cancelto{0}{x e^{2\lambda x} \Big|_{-\infty}^{0}} - \int_{-\infty}^{0} e^{2\lambda x} \, dx \right) + \frac{1}{-2\lambda} \left(\cancelto{0}{x e^{-2\lambda x} \Big|_{0}^{\infty}} - \int_{0}^{\infty} e^{-2\lambda x} \, dx \right) \right)$$

$$= \lambda \left(\frac{1}{2\lambda} \frac{1}{-2\lambda} 1 + \frac{1}{-2\lambda} \frac{1}{2\lambda} (-1) \right)$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \lambda e^{-2\lambda|x|} x^2 \, dx$$

$$= \lambda \left(\int_{-\infty}^{0} x^2 e^{2\lambda x} \, dx + \int_{0}^{\infty} x^2 e^{-2\lambda x} \, dx \right)$$

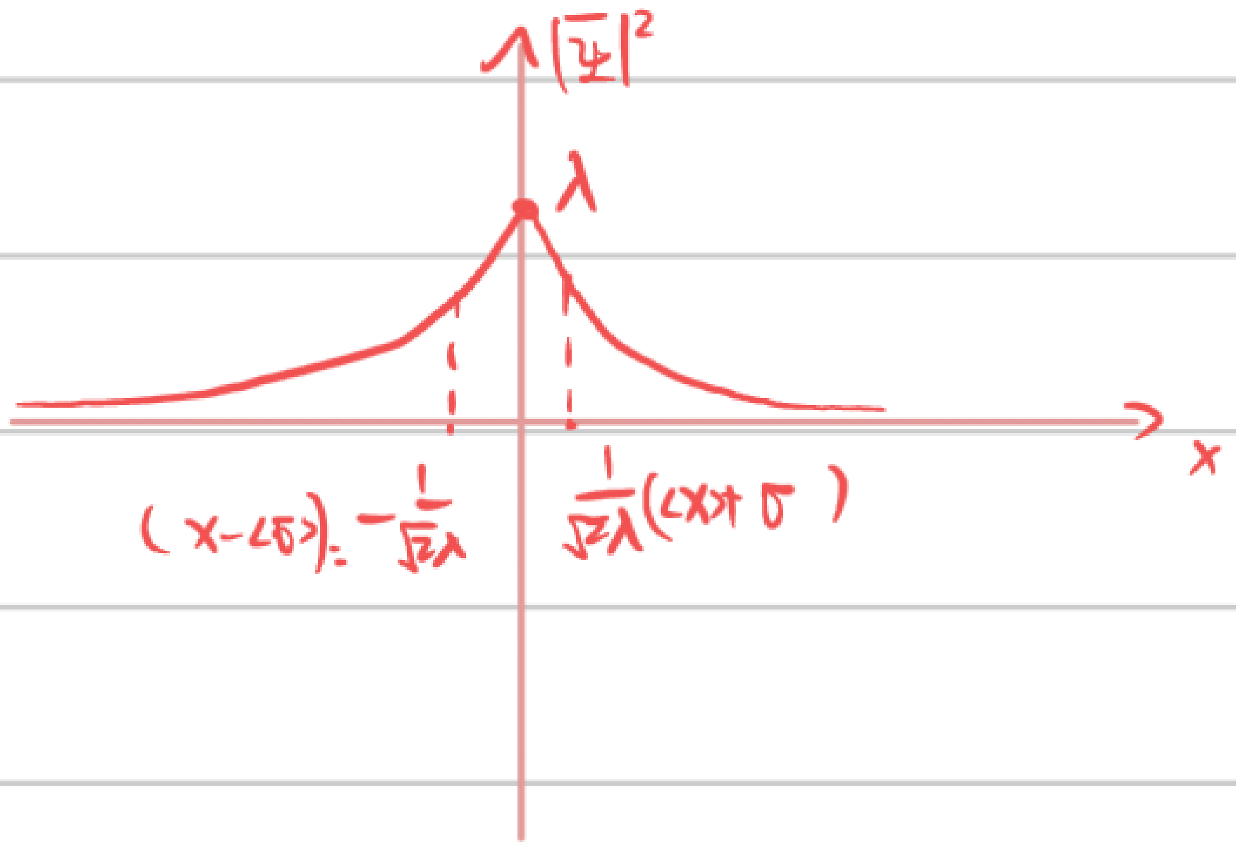
$$= 2\lambda \int_{0}^{\infty} x^2 e^{-2\lambda x} \, dx$$

$$= 2\lambda \frac{1}{-2\lambda} \left(\cancelto{0}{x^2 e^{-2\lambda x} \Big|_{0}^{\infty}} - \int_{0}^{\infty} 2x e^{-2\lambda x} \, dx \right)$$

$$= \cancel{2\lambda} \frac{1}{\cancel{-2\lambda}} 2\lambda \frac{1}{\cancel{-2\lambda}} \left(\cancelto{0}{x e^{-2\lambda x} \Big|_{0}^{\infty}} + \int_{0}^{\infty} e^{-2\lambda x} \, dx \right)$$

$$= \frac{1}{\lambda} \frac{1}{-2\lambda} (-1) = \frac{1}{2\lambda^2}$$

(c) $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle} = \frac{1}{\sqrt{2}\lambda}$



$$P_{in} = \int_{-\frac{1}{2\lambda}}^{\frac{1}{2\lambda}} \lambda e^{-2\lambda|x|}\,dx$$

$$= \lambda\left(\int_{-\frac{1}{\sqrt{2}\lambda}}^{0} e^{2\lambda x}\,dx + \int_0^{\frac{1}{\sqrt{2}\lambda}} e^{-2\lambda x}\,dx\right)$$

$$= \lambda\left(\frac{1}{2\lambda} e^{2\lambda x}\Big|_{-\frac{1}{\sqrt{2}\lambda}}^{0} + \frac{1}{-2\lambda} e^{-2\lambda x}\Big|_0^{\frac{1}{\sqrt{2}\lambda}}\right)$$

$$= \frac{1}{2} - \frac{e^{-\sqrt{2}}}{2} - \left(\frac{e^{-\sqrt{2}}}{2} - \frac{1}{2}\right)$$

$$= 1 - e^{-\sqrt{2}}$$

$$P_{out} = e^{-\sqrt{2}} \approx 0.2431$$