Lecture Notes in Finance 2 (MiQE/F, MSc course at UNISG)

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Contents

Warning: a few of the tables and figures are reused in later chapters. This can mess up the references, so that the text refers to a table/figure in another chapter. No worries: it is really the same table/figure. Still, I promise to fix this some day.

Chapter 15

Forwards and Futures

Main References: Elton, Gruber, Brown, and Goetzmann (2014) 24 and Hull (2009) 5 and 8–9
Additional references: McDonald (2014) 6–8

15.1 Derivatives

Remark 15.1 *(On the notation) The notation is kept short. The current period is assumed to be $t = 0$ and the derivative expires in $t = m$. Time subscripts and indicators of time to maturity are typically suppressed, unless strictly needed in the context. For instance, instead of $F_0(m)$ we often use F denote the forward price (contracted in $t = 0$, expiring in m) and similarly for interest rates (y instead of $y_0(m)$). Also, instead of S_0 we use S, but we keep the subscript on S_m.*

Derivatives are assets whose payoff depend on some underlying asset (for instance, the stock of a company). The most common derivatives are futures contracts (or similarly, forward contracts) and options. Sometimes, options depend not directly on the underlying, but on the price of a futures contract on the underlying. See Figure 15.1.

Derivatives are in zero net supply, so a contract must be issued (a short position) by someone for an investor to be able to buy it (long position). For that reason, gains and losses on derivatives markets sum to zero.

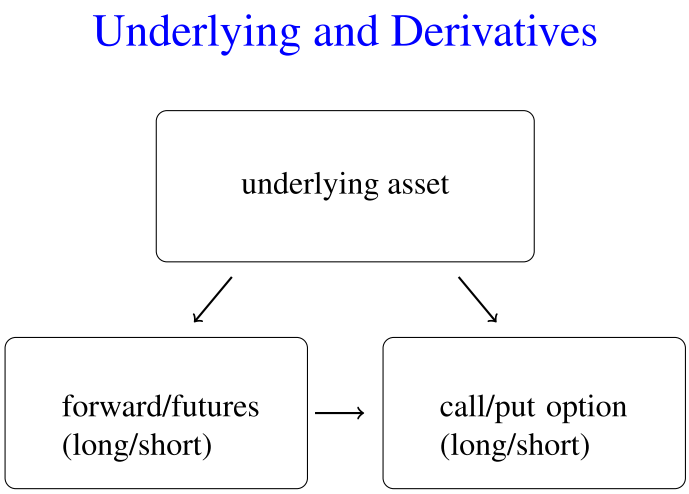


Figure 15.1: Derivatives on an underlying asset

15.2 Present Value

The present value of Z units paid m periods (years) into the future is

$$\text{PV}(Z) = (1 + Y)^{-m} Z, \text{ or} \tag{15.1}$$

$$= e^{-my} Z, \tag{15.2}$$

where Y is effective spot interest rate for a loan until m periods ahead, and y is the continuously compounded m-period interest rate ($y = \ln(1 + Y)$). As usual, the interest rates are expressed as the rate per year, so m should be also expressed in terms of years. On notation: trade time subscripts (on spot prices, forward prices, interest rates, etc) are mostly suppressed in these notes, except when strictly needed for clarity.

Example 15.2 *(Present value) With $y = 0.05$ and $m = 3/4$ we have the present value $e^{-0.05 \times 3/4} Z \approx 0.963Z$.*

15.3 Forward Contracts

15.3.1 Definition of a Forward Contract

A forward contract specifies (among other things) the expiration date, which asset should be delivered, and how much that should be paid for it, the forward price F. See Figure 15.2 for an illustration.

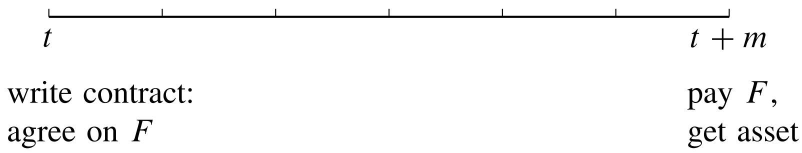


Figure 15.2: Timing convention of forward contract

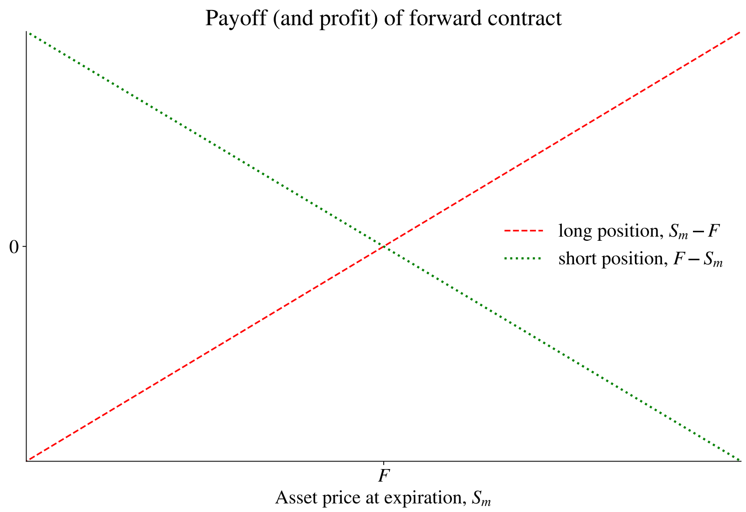


Figure 15.3: Profit (payoff) of forward contract at expiration

The profit (payoff) of a forward contract at expiration is very straightforward. Let S_m be the price (on the spot market) of the underlying asset at expiration (in m). Then, for the *buyer* of a forward contract

$$\text{payoff of a forward contract} = S_m - F. \tag{15.3}$$

At expiration, the owner of the forward contract pays F to get the asset, sells it immediately on spot market for S_m. See Figure 15.3 for an illustration of the payoff (at expiration) as a function of the underlying price, S_m. Similarly, the payoff for the *seller* of a forward contract is $F - S_m$ (she buys the asset on spot market for S_m, gets F for asset according to the contract). This sums to *zero*, irrespective of the value of the underlying asset.

15.3.2 Forward-Spot Parity

A forward contract entails both a right (to get the underlying asset at expiration) and an obligation (to pay the forward price at expiration), so it is perhaps not obvious what the value of it is. However, a no-arbitrage argument shows that the following proposition must hold (in the absence of trading costs). With trading costs, the proposition is still a good approximation.

Proposition 15.3 *(Forward-spot parity, no dividends) The present value of the forward price, F, contracted in $t = 0$ (but to be paid in m) on an asset without dividends equals the spot price:*

$$e^{-my}F = S, \text{ so} \tag{15.4}$$

$$F = e^{my}S, \tag{15.5}$$

where S is the spot price in $t = 0$ and y is m-period spot interest rate in $t = 0$.

With a positive interest rate, the forward price is higher that today's underlying price. The intuition is that the forward contract is like buying the underlying asset on credit—$e^{-my}F$ can be thought of as a prepaid forward contract. If you prefer effective interest rates, then the expression reads $F = (1 + Y)^m S$.

Example 15.4 *(Forward-spot parity) With $y = 0.05$, $m = 3/4$ and $S = 100$ we have the forward price $e^{0.05\times3/4}100 \approx 103.82$.*

Proof. (of Proposition 15.3) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy one unit of the asset at the price S. Both portfolios give one asset at expiration, so they must have the same costs today. ■

Proposition 15.5 *(Forward-spot parity, continuous dividends) When the dividend is paid continuously as the rate δ (of the price of the underlying asset), then*

$$e^{-my}F = Se^{-\delta m}, \text{ so} \tag{15.6}$$

$$F = Se^{m(y-\delta)} \tag{15.7}$$

Notice that the dividends decrease the futures price. The intuition is that the forward contract does not give the right to these dividends so its value is the underlying asset value stripped of the present value of the dividends.

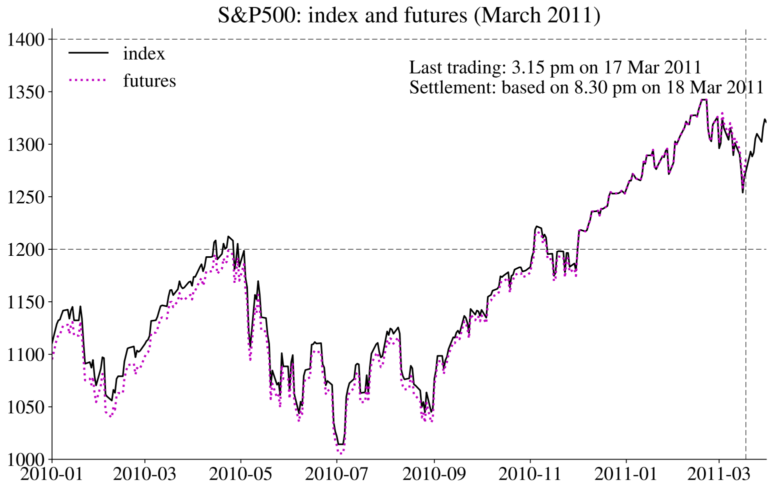


Figure 15.4: S&P 500 index level and futures

Proof. (*of Proposition 15.5) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy $e^{-\delta m}$ units of the asset at the price $e^{-\delta m}S$, and then collect dividends and reinvest them in the asset. Both portfolios give one asset at expiration, so they must have the same costs today. ■

Example 15.6 *(Forward-spot parity) With* $y = 0.05$, $m = 0.75$ *and* $S = 100$ *we have the forward price* $F = e^{0.75 \times 0.05}100 \approx 103.82$. *Instead with a continuous dividend rate of* $\delta = 0.01$, *we get* $F = e^{0.75 \times (0.05 - 0.01)}100 \approx 103.04$.

Figure 15.4 provides an example of how the underlying price and the futures price (on S&P 500) developed over a year. Notice how the futures prices converges to the underlying price at expiration of the futures. Before it can deviate because of delayed payment (+) and no part in dividend payments (−).

Proposition 15.7 *(*Forward-spot parity, discrete dividends) Suppose the underlying asset pays the dividend* d_i *at* m_i $(i = 1, ..., n)$ *periods into the future (but before the expiration date of the forward contract). To do the proper discounting, let* $y(m_i)$ *be today's* m_i*-period interest rate. If the dividends are known already today, then the forward price satisfies*

$$e^{-my(m)}F = S - \sum\nolimits_{i=1}^{n} e^{-m_i y(m_i)} d_i \text{, so} \tag{15.8}$$

$$F = e^{my(m)}S - e^{my(m)} \sum\nolimits_{i=1}^{n} e^{-m_i y(m_i)} d_i. \tag{15.9}$$

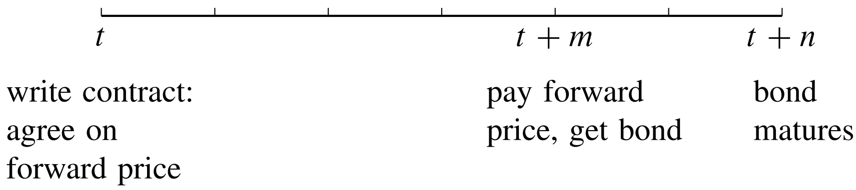


Figure 15.5: Timing convention of forward contract on a bond

Proof. (*of Proposition 15.7) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy one unit of the asset at the price S and sell the rights to the known dividends at the present value of the dividends. Both portfolios give one asset at expiration, so they must have the same costs today. ■

15.3.3 Application of the Forward-Spot Parity: Forward Price of a Bond

Consider a forward contract (expiring in m) on a discount (zero coupon) bond that matures in n (assuming $n > m$). See Figure 15.5 for an illustration.

By the forward spot parity (15.5) and the definition of a present value (15.3), today's forward price is

$$\begin{aligned} F &= e^{my(m)}B(n) \\ &= B(n)/B(m), \end{aligned} \tag{15.10}$$

where $B(n)$ is the price of an n-period bond today and $B(m) = e^{-my(m)}$ is the price of an m-period bond (with a face value of 1).

Example 15.8 *(Forward price of a bond) Let* $(m, n, B(m), B(n)) = (5, 7, 0.779, 0.657)$. *Then,* $F = 0.657/0.779 \approx 0.843$.

15.3.4 Application of the Forward-Spot Parity: Forward Price of Foreign Currency

Let S be the price (measured in domestic currency) of foreign currency. Investing in foreign currency effectively means investing in a foreign interest bearing instrument which earns the continuous interest rate ("dividend") y^*. Use $\delta = y^*$ in (15.7)

$$F = Se^{m(y-y^*)}. \tag{15.11}$$

This is called the *covered interest rate parity* (CIP). The price is quoted at the forward price F, or as the forward premium $F - S$. The premium is sometimes multiplied by 10,000 to give the premium in "pips." For instance, with $F = 1.22$ and $S = 1.20$, we have 200 pips.

Notice that $F > S$ (a positive premium) means that $y > y^*$. That is, if the domestic interest rate is higher than the foreign interest rate, then the forward price (of foreign currency) is higher than the spot price. In this way, the extra yield from the domestic interest rate is exactly matched by the "forward appreciation" of the foreign currency—to make the return the same.

Example 15.9 *(CIP) With* $S = 1.20, m = 1, y = 0.0665$ *and* $y^* = 0.05$ *we have*

$$F = 1.20e^{0.0165} = 1.22.$$

Buying one unit of foreign currency costs 1.20 and after one year we have $e^{0.05} = 1.0513$ *units of foreign currency, which are (when converted with* $F = 1.22$*) worth* $1.0513 \times 1.22 = 1.2826$ *in domestic currency. Since we invested* 1.20*, the gross return is* $1.2826/1.20 = 1.0688$*, which equals* $\exp(0.0665)$.

15.3.5 The Return on Holding a Forward Contract until Expiration

Suppose you enter a forward contract in $t = 0$ and and hold it until it expires in $t = m$. You do not pay anything up front in $t = 0$, but you have pledged to pay F in $t = m$, which has a present value of $e^{-my}F$. You could put this amount on a bank (money market) account and consider it as your investment. The payoff is clearly the value of the underlying asset at expiration: S_m. The gross return is therefore $1 + R = \frac{S_m}{F}e^{my}$.For an asset without dividends, the forward-spot parity (15.5) then shows that the gross return is just S_m/S.

15.3.6 The Value of an Old Forward Contract*

Consider a forward contract that expires in $t + m$, although the contract was written at some earlier point in time ($\tau < t$) and specified a forward price of F_τ (time subscripts are needed for the analysis here). The value of this contract in t is

$$W_t = e^{-ym}(F_t - F_\tau), \tag{15.12}$$

where F_t is today's forward price on the same underlying asset (and same expiration date). W_t is what someone would pay in order to buy that old forward contract. The intuition is that an owner of an old (τ) forward contract can short sell a new forward contract (t) and thereby cancel all risk—and stand to win $F_t - F_\tau$ at expiration. The present value of this is (15.12). Clearly, for a new contract ($t = \tau$), the value is zero.

Proof. (15.12) An investor sells (issues) a forward contract in t. At expiration, this will give $F_t - S_{t+m}$, where S_{t+m} is the price of the underlying asset at expiration. If she buys an old forward contract (paying W_t today), the payoff of that is $S_{t+m} - F_\tau$ at expiration. Hence, the total portfolio has the payoff $F_t - F_\tau$, which is riskfree so it must earn the riskfree rate: $(F_t - F_\tau)/W_t = e^{ym}$. Rearrange to get (15.12). ■

Remark 15.10 *("Return" on a forward contract over a short holding period*) In a traditional forward contract there is no up-front payment, so it is tricky to define a return. However, we can define a kind of return by considering pre-paid forward contracts.*

15.4 Forwards versus Futures

A forward contract is typically a private contract between two investors—and can therefore be tailor made. A futures contract is similar to a forward contract (write contract, get something later at a pre-determined price), but is typically traded on an exchange—and is therefore standardized (amount, maturity, settlement process). The settlement is either cash settlement or physical settlement. The latter is not used for synthetic/complex assets like equity indices (since it would involve considerable trading costs).

Another important difference is that a forward contract is settled at expiration, whereas a futures contract is settled daily (*marking-to-market*). This essentially means that gains and losses (because of price changes) are transferred between issuer and owner daily—but kept at an interest bearing account at the exchange. The counter parties have to post an initial margin—and the marking-to-market then adds to/subtracts from the margin account. If the amount decreases below a certain level (maintenance margin), then a margin call is issued to the investor—informing him/her to add cash to the margin account. See Example 15.12.

The margin requirements for an investor is governed by his/her overall portfolio (for instance, it is smaller if the portfolio includes negatively correlated positions) and is set by statistical measurements of the portfolio risk (see the *SPAN* system applied at CME and other exchanges).

If interest rates change randomly over time (and they do), the rate at which the money on the margin account is invested at (refinanced) will be different from the rate when the futures was issued. This risk of this happening is reflected in the futures price.

The proposition below shows that, if the interest rate path was non-stochastic (and there is no counter party risk), then the forward and futures prices would be the same. In practice, the difference between forward and futures prices is typically small.

Proposition 15.11 *(Forward vs. futures prices, non-stochastic interest rates) The forward and futures prices would be the same (a) if there were no counter party risk; (b) and if the interest rate only changed in a non-stochastic way.*

Proof. (of Proposition 15.11) To simplify the notation, let $t = 0$ and $m = 2$. Also, let r_s continuously compounded rate at which you accumulate interest on the margin account between days s and $s+1$ ($r = y/365$) and f_s be the futures price on day s. *Strategy A*: have e^{-r_1} long futures contracts on (the end of) day 0, pre-commit to increase it to 1 on day 1 and keep all settlements on the margin account. This gives

Day (s)	Settlement	Futures Position (end of day)	On Margin Account (end of day)
0		e^{-r_1}	0
1	$e^{-r_1}(f_1 - f_0) = A$	1	A
2	$f_2 - f_1 = B$	0	$e^{r_1}A + B$

The end-value of strategy A is therefore $f_2 - f_0$, which equals $S_2 - f_0$ since the value at expiration is the value of the underlying asset. *Strategy B*: be long one forward contracts, which gives a payoff on day 2 of $S_2 - F_0$. Both strategies take on exactly the same risk, so the prices must be the same: $f_0 = F_0$. (The proof relies on knowing r_1 already on day 0.) ■

Example 15.12 *(Margin account) Margin account of a buyer (holder) of a futures contract (maintenance margin = 0.75×initial margin) could be as follows (assuming a zero interest rate):*

Day	*Futures price*	*Daily gain*	*Posting of margin*	*Margin account*
0	100		4	4
1	99	−1		3
2	97	−2	2	3
3	99	2		5

On day 2, the investor received a margin call to add cash to the account—to make sure that the maintenance margin (here 3) is kept. Notice that the overall profit is the difference of what has been put into the margin account $(4+2)$ *and the final balance* (5)*, that is,* -1*. This is also the* cumulative *daily gain* $(-1-2+2=-1)$*. With marking to market this is all that happens: no payment of the futures price and no delivery of the underlying asset. However, it is equivalent to what happen without marking to market, since at expiration, the gain is* $99-100=-1$ *(futures = underlying at expiration).*

15.5 Appendix: Data Sources*

The data used in these lecture notes are from the following sources:

1. website of Kenneth French,
   `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`
2. Datastream
3. Federal Reserve Bank of St. Louis (FRED), `http://research.stlouisfed.org/fred2/`
4. website of Robert Shiller, `http://www.econ.yale.edu/~shiller/data.htm`
5. yahoo! finance, `http://finance.yahoo.com/`
6. OlsenData, `http://www.olsendata.com`

Chapter 16

Interest Rate Calculations

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9; Fabozzi (2004); Blake (1990) 3–5; and Campbell, Lo, and MacKinlay (1997) 10

16.1 Zero Coupon (discount or bullet) Bonds

16.1.1 Zero Coupon Bond Basics

Remark 16.1 *(On the notation) These notes often write B and Y instead of $B_t(m)$ and $Y_t(m)$, unless the indicator for the trading date (t) and/or maturity (m) are important in the specific context.*

Consider a zero coupon bond which costs $B(m)$ in t and gives the face value in $t+m$. See Figure 16.1 for an illustration.

The gross return (payoff divided by price) from investing in this bond is $1/B(m)$, since the face value is here normalized to unity. The relation between the *bond price* $B(m)$ and

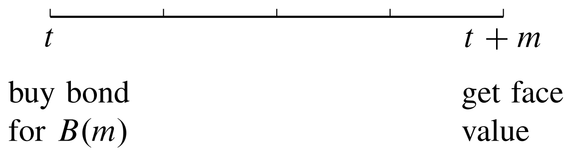


Figure 16.1: Timing convention of zero coupon bond

the *effective (spot) interest rate* $Y(m)$ is

$$\frac{1}{B(m)} = [1 + Y(m)]^m, \tag{16.1}$$

$$B(m) = [1 + Y(m)]^{-m}, \tag{16.2}$$

$$Y(m) = B(m)^{-1/m} - 1. \tag{16.3}$$

The interest rate is therefore an annualized rate of return from investing $B(m)$ and receiving the face value (here normalized to 1) m periods later. Another way to think about this is that if we invest the amount $B(m)$ by buying one bond, then after m periods we get $B(m)$ times the interest rate factor, that is, $B(m)\,[1 + Y(m)]^m = 1$. Notice that you can calculate the *present value* (of getting Z in $t+m$) as $B\,(m)\,Z$.

Example 16.2 *(Effective rates) Let the period length be a year (which is the most common convention for interest rates). Consider a six-month bill so* $m = 0.5$*. Suppose* $B = 0.95$*. From (16.1) we then have that*

$$\frac{1}{0.95} = (1+Y)^{0.5}, \text{ so } Y \approx 0.108.$$

Remark 16.3 *(A face value of 100) In case the face value is X (say, 100) instead of* 1*, then the bond price will be X times higher than with a face value of 1, so the left hand side of (16.1) will be $X/B(m)$ and give the same interest rate. In practice, bond quotes are typically expressed in percentages (like 97, often leaving out the % sign) of the face value, whereas the discussion here effectively uses the fraction of the face value (like 0.97).*

The relation between the rate and the price is clearly non-linear—and depends on the time to maturity (m): prices on long maturity bonds are more sensitive to interest rate changes than prices on short maturity bonds. See Figure 16.2 for an illustration.

We also have the following relation between the bond price and the *continuously compounded interest rate*

$$\frac{1}{B} = \exp(my), \tag{16.4}$$

$$B = \exp(-my), \tag{16.5}$$

$$y = -\ln B/m. \tag{16.6}$$

Example 16.4 *(Continuously compounded rate) Using the numbers as in Example 16.2, (16.4) gives*

$$\frac{1}{0.95} = \exp(0.5y), \text{ so } y \approx 0.103.$$

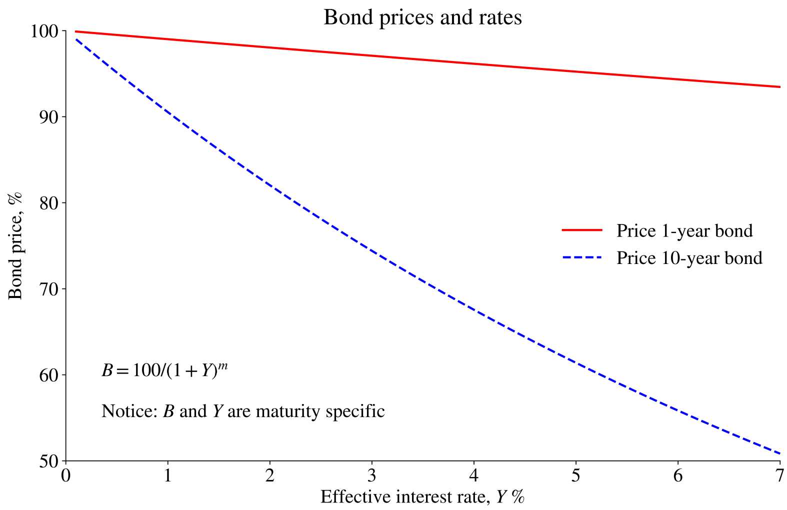


Figure 16.2: Interest rate vs. bond price

Some fixed income instruments (in particular inter bank loans, LIBOR/EURIBOR) are quoted in terms of a *simple interest rate*, $\tilde{Y}$. The "price" of a deposit that gives unity at maturity is then related to the simple interest rate according to

$$B = \frac{1}{1 + m\tilde{Y}}, \text{ or} \tag{16.7}$$

$$\tilde{Y} = \frac{1/B - 1}{m}. \tag{16.8}$$

Example 16.5 *(Simple rates) Consider a six-month bill so $m = 0.5$. Suppose $B = 0.95$. From (16.7) we then have that*

$$0.95 = \frac{1}{1 + 0.5 \times \tilde{Y}}, \text{ so } \tilde{Y} \approx 0.105.$$

Remark 16.6 *(The transformation from one type of rate to the other*) We have*

$$\begin{aligned}
y &= \ln(1+Y) \text{ and } y = \ln\left(1+m\tilde{Y}\right)/m, \\
Y &= \exp(y) - 1 \text{ and } Y = (1+m\tilde{Y})^{1/m} - 1 \\
\tilde{Y} &= [(1+Y)^m - 1]/m \text{ and } \tilde{Y} = [\exp(y) - 1]/m.
\end{aligned}$$

The different interest rates (effective, continuously compounded and simple) are typically very similar, except for very high rates. See Figure 16.3 for an illustration.

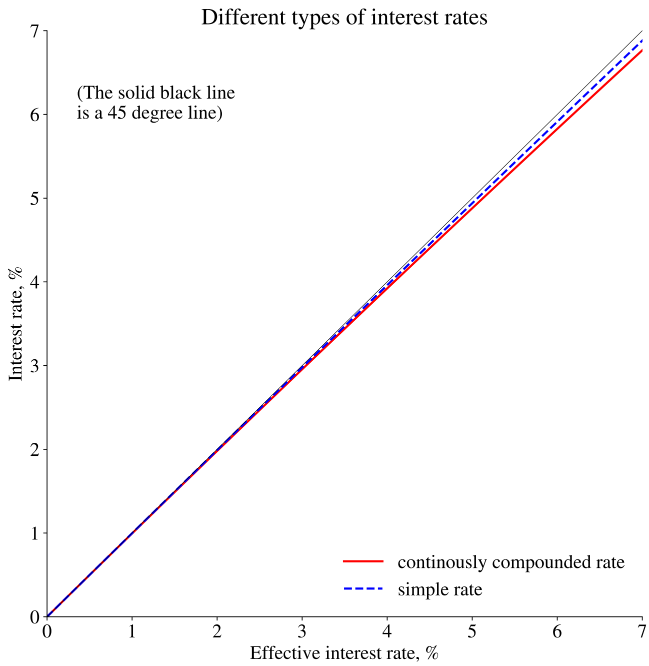


Figure 16.3: Different types of interest rates

Example 16.7 *(Different interest rates) With $m = 1/2$, $Y = 0.108$, $y = 0.103$ and* $\tilde{Y} = 0.106$

$$1.053 \approx (1 + 0.108)^{0.5} \approx \exp(0.5 \times 0.103) \approx 1 + 0.5 \times 0.105.$$

16.1.2 The Return from Holding a Zero Coupon Bond

The log *return from holding a zero coupon bond until maturity* is my. This follows directly from the definition of the log interest rate (see (16.4)).

The log *return from holding a zero coupon bond* from t to $t + s$ is clearly the relative change of the bond price

$$r_{t+s} = \ln \frac{B_{t+s}(m-s)}{B_t(m)}, \tag{16.9}$$

where the subscripts indicate the trading date (previously suppressed). Notice that the bond's maturity decreases with time: in this case from m to $m - s$. (This is a return over s periods and it is not rewritten on a "per period" basis as interest rates are.) In simplified notation (dropping the indicator of the maturity), the right hand side is just $\ln(B_{t+s}/B_t)$. Clearly, at maturity (so $s = m$) the bond price is 1, so (16.9) becomes $-\ln B_t(m) = my$.

Example 16.8 *(Bond return) If the bond price decreases from* 0.95 *to* 0.86*, then (16.9) gives the log return*

$$\ln \frac{0.86}{0.95} = -0.1.$$

Substituting for the bond prices in (16.9), and using a simplified notation (by dropping the indicator of the maturity), gives

$$r_{t+s} = -m(y_{t+s} - y_t) + s y_{t+s}. \tag{16.10}$$

We use this expression to study some special cases—to highlight the key properties of zero coupon bond returns.

Remark 16.9 *(*(16.10) in more precise notation)...is*

$$r_{t+s} = -m[y_{t+s}(m-s) - y_t(m)] + s y_{t+s}(m-s),$$

where $y_{t+s}(m-s)$ is the interest rate determined (traded) on date $t+s$ for an $m-s$ period loan.

The first special case is *a very short holding period* (s is very small). The second term in (16.10) is then virtually zero, so we can write

$$r_{t+s} = -m(y_{t+s} - y_t). \tag{16.11}$$

This is clearly negative if the interest rate change is positive—and more so if the maturity (m) is long. See Figure 16.4 for an empirical illustration (although it refers to bonds with coupons).

Example 16.10 *(Bond returns vs interest rate changes) Suppose that, over a split second (so the time to maturity is virtually unchanged), the interest rates for all maturities increase from 0.5% to 1.5%. Using (16.4) gives the following bond prices*

	1-year bond	*10-year bond*
at 0.5%	$e^{-1\times 0.005} = 0.995$	$e^{-10\times 0.005} = 0.951$
at 1.5%	$e^{-1\times 0.015} = 0.985$	$e^{-10\times 0.015} = 0.861$
Change in logs (%)	-1%	-10%

Using (16.11) directly gives the same: $-1 \times 0.01 = -0.01$ *and* $-10 \times 0.01 = -0.1$.

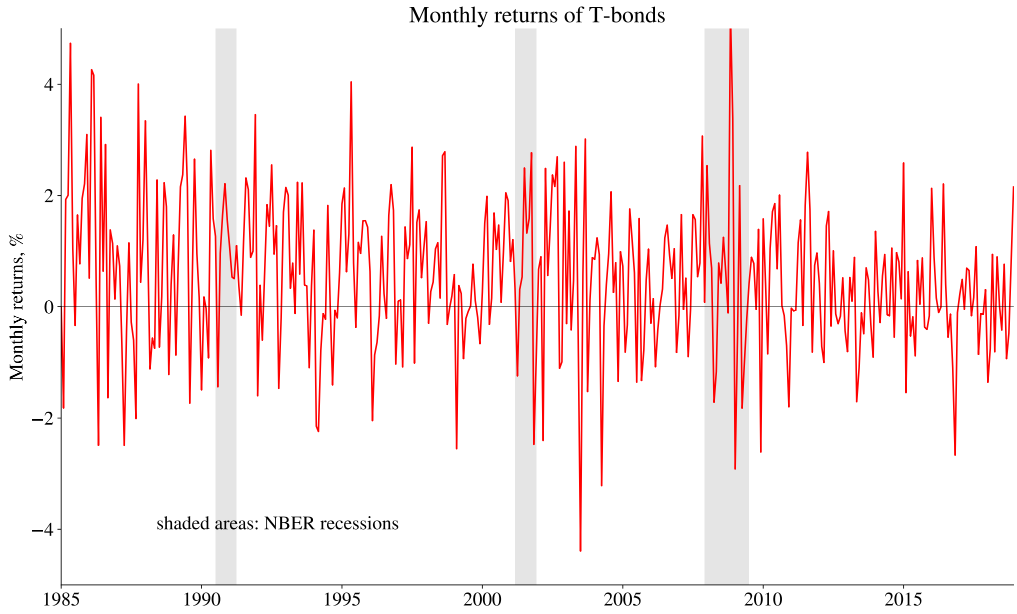


Figure 16.4: Returns on an index of U.S. Treasury bonds

The second special case is *an unchanged flat yield curve*. In this case, all interest rates in (16.10) are the same (denoted y), so we get

$$r_{t+s} = sy \tag{16.12}$$

which is just the holding period times the interest rate. The reason is simply that the bond starts out as a m-maturity bond, but becomes an $(m - s)$-maturity bond—and the latter has a higher price. See Figure 16.5.

16.2 Forward Rates

16.2.1 Definition of Forward Rates

A forward contract on a bond can be used to lock in an interest rate for an investment over a future period. Consider "buying" a forward contract in t: it stipulates what you have to pay in $t + m$ (the forward price) and that you then get a discount bond that pays the face value (here normalized to 1) at time $t + n$. See Figure 16.6 for an illustration.

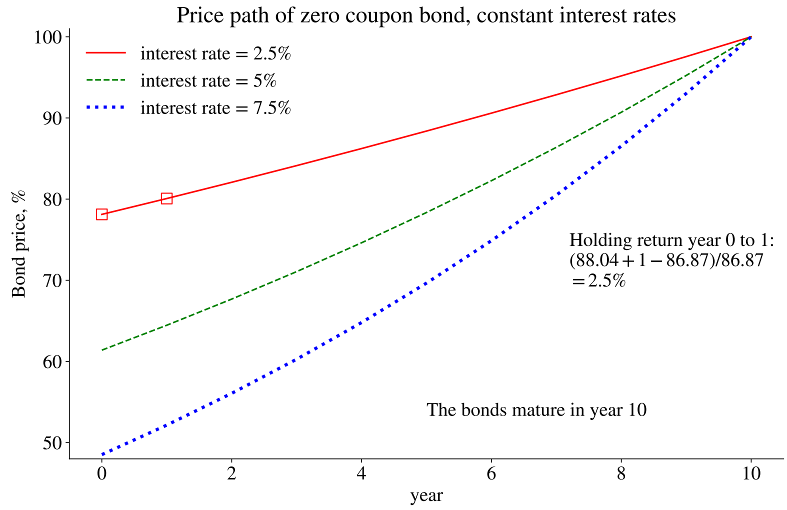


Figure 16.5: The price of a zero coupon bond maturing in year 10

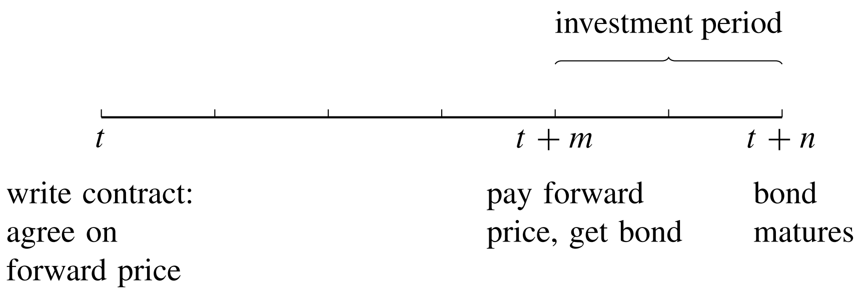


Figure 16.6: Timing convention of forward contract

16.2.2 Implied Forward Rates

The forward-spot parity implies that the forward price is

$$F = [1 + Y(m)]^m \, B(n). \tag{16.13}$$

Buying a forward contract is effectively an investment from $t + m$ to $t + n$, that is, over $n - m$ periods. The gross return (which is known already in t) is $1/F$. We define a per period effective rate of return, a *forward rate*, Γ, analogous with an interest rate as

$$\frac{1}{F} = (1 + \Gamma)^{n-m}. \tag{16.14}$$

Notice that Γ is the per-period rate of return over $t + m$ to $t + n$ that can be guaranteed in t.

By using the relation between bond prices and yields (16.1), the forward rate can be written

$$\Gamma = \frac{[1 + Y(n)]^{n/(n-m)}}{[1 + Y(m)]^{m/(n-m)}} - 1. \tag{16.15}$$

This shows that the forward rate depends on both interest rates, and thus, the general shape of the yield curve. Actually, the forward rate can be seen as the "marginal cost" of making a loan longer. See Figure 16.7 for an illustration.

Example 16.11 *(Forward rate) Let $m = 0.5$ (six months) and $n = 0.75$ (nine months), and suppose that $Y(0.5) = 0.04$ and $Y(0.75) = 0.05$. Then (16.15) gives*

$$\Gamma = \frac{(1 + 0.05)^{0.75/0.25}}{(1 + 0.04)^{0.5/0.25}} - 1 \approx 0.07.$$

See Figure 16.7 for an illustration.

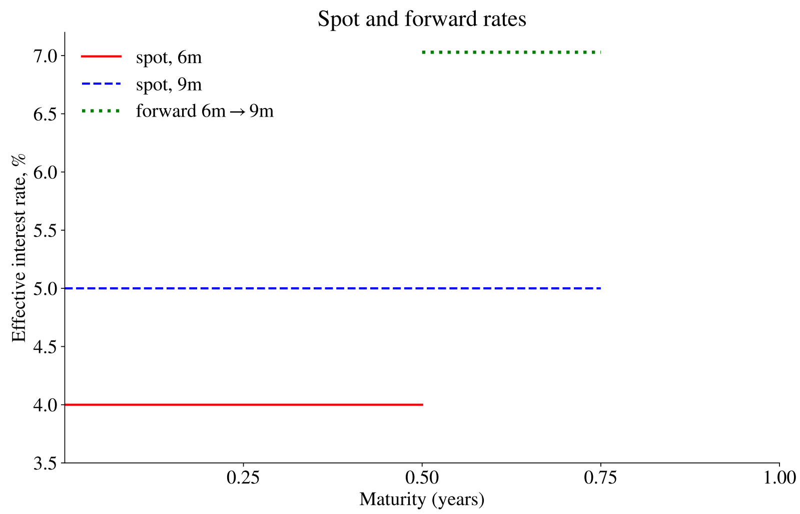


Figure 16.7: Spot and forward rates

Remark 16.12 *(Forward Rate Agreement) An* FRA *is an over-the-counter contract that guarantees an interest rate during a future period. The FRA does not involve any lending/borrowing—only compensation for the deviation of the future interest rate (typically LIBOR) from the agreed forward rate. An FRA can be emulated by a portfolio of zero-coupon bonds, similarly to a forward contract.*

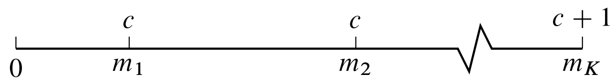


Figure 16.8: Timing convention of coupon bond

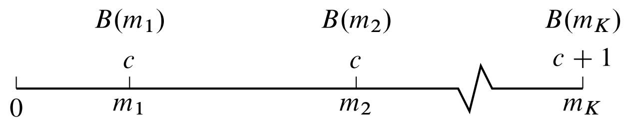


Figure 16.9: Using zero-coupon bonds to value a coupon bond

Remark 16.13 *(Alternative way of deriving the forward rate*) Rearrange (16.15) as*

$$[1+Y(m)]^m\,(1+\Gamma)^{n-m} = [1+Y(n)]^n\,.$$

This says that compounding $1+Y(m)$ *over m periods and then* $1+\Gamma$ *for* $n-m$ *periods should give the same amount as compounding the long rate,* $1+Y(n)$*, over n periods.*

16.3 Coupon Bonds

Remark 16.14 *(On the notation) These notes often use* P *instead of* $P_t(c, m_1, ..., m_K)$ *to denote the price of a coupon bond unless the indicator for the trading date* (t)*, coupon rate* (c) *and periods until coupon payments* $m_1, ..., m_K$ *are important in the specific context.*

16.3.1 Coupon Bond Basics

Consider a bond which pays coupons, c, for K periods (at $t+m_1, t+m_2, .., t+m_K$), and also the "face" (or "par" value, here normalized to 1) at maturity, $t+m_K$. See Figure 16.8 for an illustration.

The coupon bond can be thought of as a portfolio of zero coupon bonds: c maturing in $t+m_1$, another c in $t+m_2$,..., and $c+1$ in $t+m_K$. The price of the coupon bond, P, must therefore equal the price of the portfolio

$$P = \sum_{k=1}^{K-1} B(m_k)c + (c+1)B(m_K) \tag{16.16}$$

where $B(m_k)$ is the price of a zero coupon bond maturing m_k periods later. This is illustrated in Figure 16.9.

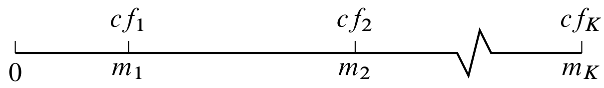


Figure 16.10: Timing convention of bond portfolio

We can apply the same valuation principle for more complicated cash flow processes, for instance, for a portfolio of bonds. Suppose the bond portfolio pays the cash flow cf_k in m_k periods from now, as illustrated in Figure 16.10. This cash flow includes both coupon payments and face values. The pricing formula (16.16) can then be generalised to

$$P = \sum_{k=1}^{K} B(m_k) cf_k. \tag{16.17}$$

Clearly, setting $cf_k = c$ for $k \leq K - 1$ and $cf_K = c + 1$ gives (16.16).

Example 16.15 *(Coupon bond prices) For the bonds with 1 and 2 periods until maturity, (16.16) can clearly be written*

$$\begin{bmatrix} P(1) \\ P(2) \end{bmatrix} = \begin{bmatrix} c(1)+1 & 0 \\ c(2) & c(2)+1 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \end{bmatrix},$$

where we use $P(m)$ and $c(m)$ to indicate the price and coupon rate for the m-period coupon bond. For instance, $(B(1), c(1)) = (0.95, 0)$ and $(B(2), c(2)) = (0.90, 0.06)$ we have that

$$\begin{bmatrix} P(1) \\ P(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.06 & 1.06 \end{bmatrix} \begin{bmatrix} 0.95 \\ 0.90 \end{bmatrix} \text{ gives } \begin{bmatrix} P(1) \\ P(2) \end{bmatrix} \approx \begin{bmatrix} 0.95 \\ 1.01 \end{bmatrix}.$$

Example 16.16 *(Coupon bond price at par) Suppose $B(1) = 1/1.06$ and $B(2) = 1/1.091^2$. The price of a bond with a 9% annual coupon with two years to maturity is then*

$$\frac{0.09}{1.06} + \frac{0.09}{1.091^2} + \frac{1}{1.091^2} \approx 1.$$

This bond is (approximately) sold "at par", that is, the bond price equals the face (or par) value (which is 1 in this case).

Remark 16.17 *("Bootstrapping") Reconsider Example 16.15, but suppose we instead have information about prices (and coupons) of the coupon bonds—and that we want* *to know the implied prices of the zero coupon bonds. This can be done by solving the equations for $B(1)$ and $B(2)$. That means we solve*

$$\begin{bmatrix} 0.95 \\ 1.01 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.06 & 1.06 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \end{bmatrix} \text{ to get } \begin{bmatrix} B(1) \\ B(2) \end{bmatrix} \approx \begin{bmatrix} 0.95 \\ 0.90 \end{bmatrix}.$$

(More details on bootstrapping are given in a special section of the lecture notes.)

Using the relation between (zero coupon) bond prices and (zero coupon) spot interest rates in (16.1), the bond price can also be written

$$P = \sum_{k=1}^{K} \frac{c}{[1 + Y(m_k)]^{m_k}} + \frac{1}{[1 + Y(m_K)]^{m_K}}. \tag{16.18}$$

This shows that coupon bond price is just the present value of the cash flow (from coupons and payment of the face value), but where the discounting is made by the different spot interest rates. In these calculations, P is the full (invoice) price of the bond—which can differ from quoted prices (also called "clean prices") by an accrued interest rate term.

For a more general cash flow process we instead get

$$P = \sum_{k=1}^{K} \frac{cf_k}{[1 + Y(m_k)]^{m_k}}. \tag{16.19}$$

Example 16.18 *(Coupon bond prices II) Example 16.15 can be expressed in terms of interest rates (instead of zero coupon bond prices)*

$$\begin{bmatrix} P(1) \\ P(2) \end{bmatrix} = \begin{bmatrix} c(1) + 1 & 0 \\ c(2) & c(2) + 1 \end{bmatrix} \begin{bmatrix} 1/[1 + Y(1)] \\ 1/[1 + Y(2)]^2 \end{bmatrix}.$$

The zero-coupon prices imply that $Y(1) \approx 5.3\%$ and $Y(2) \approx 5.4\%$ so

$$\begin{bmatrix} P(1) \\ P(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.06 & 1.06 \end{bmatrix} \begin{bmatrix} 1/1.053 \\ 1/1.054^2 \end{bmatrix}$$

gives the same coupon bond prices as before.

Remark 16.19 *(STRIPS, Separate Trading of Registered Interest and Principal of Securities*) A coupon bond can be split up into its embedded zero coupon bonds—and traded separately (as zero coupon bonds).*

16.3.2 Coupon Bond Pricing with a Flat Yield Curve*

The special (admittedly unrealistic) case when all spot rates are the same (flat yield curve) is interesting since it provides good intuition for how coupon bond prices are determined. In particular, if the next coupon payment is one period ahead ($m_k = k$), then (16.18) with $Y(m_k) = Y > 0$ becomes

$$P = 1 + \frac{c - Y}{Y}[1 - (1 + Y)^{-K}], \tag{16.20}$$

where Y is the (common) spot rate and K is the maturity. The term in square brackets is positive (assuming $Y > 0$ and $K > 0$), so when the interest rate (which then equals the yield to maturity, see below) is below the coupon rate, then the bond price is above the face value (since $c - Y > 0$), and vice versa. When $c = Y$ the bond trades at par, that is, the bond price equals the face value (here normalized to unity).

Example 16.20 *(of (16.20)) With $c = 5\%$, $Y = 2.5\%$ and $K = 10$ we get $P = 1.22$. Instead, with $c = 1\%$ we get 0.87.*

16.3.3 Yield to Maturity

The effective *yield to maturity* (also called redemption yield), θ, on a coupon bond is the internal rate of return which solves

$$P = \sum_{k=1}^{K} \frac{c}{(1 + \theta)^{m_k}} + \frac{1}{(1 + \theta)^{m_K}}, \tag{16.21}$$

where the bond pays coupons, c, at $m_1, m_2, ..., m_K$ periods ahead. This equation can be solved (numerically) for θ. Bonds are often quoted in terms of the yield to maturity (instead of the price). For a *par bond* (the bond price equals the face value, here 1), the yield to maturity equals the coupon rate. For a zero coupon bond, the yield to maturity equals the spot interest rate.

For a more general cash flow process we instead get

$$P = \sum_{k=1}^{K} \frac{cf_k}{(1 + \theta)^{m_k}}. \tag{16.22}$$

Example 16.21 *(Yield to maturity) A 4% (annual coupon) bond with 2 years to maturity.*

Suppose the price is 1.019. The yield to maturity is 3% since it solves

$$1.019 \approx \frac{0.04}{1+0.03} + \frac{1.04}{(1+0.03)^2}.$$

Example 16.22 *(Yield to maturity of a par bond) A 9% (annual coupon) par bond (price of 1) with 2 years to maturity. The yield to maturity is 9% since*

$$\frac{0.09}{1+0.09} + \frac{1.09}{(1+0.09)^2} = 1.$$

Example 16.23 *(From ytm to spot interest rates) If the ytm on a 1 year bond is* 6%*, then* $Y(1) = 6\%$*. (To see this, compare (16.19) to (16.22) when* $K = 1$*). Combined with a 2 year* 9% *coupon trading at par, Example 16.16 shows that* $Y(2) = 9.1\%$*.*

Example 16.24 *(Yield to maturity of a portfolio) A 1-year discount bond with a ytm (effective interest rate) of 7% has the price* $1/1.07$ *and a 3-year discount bond with a ytm of 10% has the price* $1/1.1^3$*. A portfolio with one of each bond has a ytm*

$$\frac{1}{1.07} + \frac{1}{1.1^3} = \frac{1}{1+\theta} + \frac{1}{(1+\theta)^3}, \text{ with } \theta \approx 0.091.$$

This is clearly not the average ytm of the two bonds. It would be, however, if the yield curve was flat.

16.3.4 The Return from Holding a Coupon Bond until Maturity

To calculate the *return from holding a coupon bond until maturity* we need to specify how the coupons are reinvested. The next proposition presents a special case.

Proposition 16.25 *(Return from holding a coupon bond until maturity, a special case) If all coupons are reinvested in assets that generate returns equal to the bond's yield to maturity* θ*, then the (annualized) rate of return is* θ*.*

However, this is mostly a theoretical case, since it requires that we (in the future) can reinvest at today's yield to maturity.

Proof. (of Proposition 16.25) Consider a 2-period coupon bond with ytm θ. From (16.21), the price of the bond is

$$P = \frac{c}{1+\theta} + \frac{c+1}{(1+\theta)^2}.$$

If we can reinvest the first coupon payment to give the return θ, it is worth $c(1+\theta)$ are maturity—and we also receive $c+1$ at maturity. Divide the end value with the initial investment (the bond price P)

$$\frac{c(1+\theta)+c+1}{c/(1+\theta)+(c+1)/(1+\theta)^2}=(1+\theta)^2.$$

■

Another special case is when the coupons are reinvested via forward contracts (agreed on at the time the bond is bought). This means that the investor buys the bond now and receives nothing until maturity—and he/she knows already now much will be received at maturity. This is just like he/she had bought a zero-coupon bond. Indeed, no-arbitrage arguments show that the return (from now to maturity) is indeed the spot interest on a zero-coupon bond. This is summarised in the following proposition.

Proposition 16.26 *(Return from holding a coupon bond until maturity, another special case) If the coupons are reinvested by forward contracts, then the (annualized) return on holding the bond until maturity is the current spot rate (on a zero coupon bond with the same maturity).*

Notice that this holds irrespective of the coupon rate. For this reason, it can well be said that coupons do not really matter for returns. With other assumptions about how the coupons are reinvested, the result is different (but typically not very much so).

Proof. (of Proposition 16.26) Consider a 2-period coupon bond. From (16.18), the price of the bond is

$$P_t=B_t(1)c+B_t(2)(c+1).$$

From (16.14), we know that the forward contract for the first coupon has the gross return (until maturity) $B_t(1)/B_t(2)$. The value of the reinvested coupon and the face value at maturity is then

$$\frac{B_t(1)}{B_t(2)}c+c+1.$$

Dividing by the first equation (the investment) gives $1/B_t(2)$ so the return on buying and holding (and reinvesting the coupons) this coupon bond is the same as the 2-period spot interest rate. (The extension to more periods is straightforward.) ■

Example 16.27 *(Holding a coupon bond until maturity) Suppose that the spot (zero coupon) interest rates are 4% for one year to maturity and 5% for 2 years to maturity (the zero* *coupon bond prices are* $B(1) = 0.962$ *and* $B(2) = 0.907$*). A 3% coupon bond with 2 years to maturity must have the current price*

$$\frac{0.03}{1.04} + \frac{0.03 + 1}{1.05^2} \approx 0.963.$$

However, the value of the bond portfolio at maturity, if the coupon is reinvested by a forward contract, is

$$0.03 \times \frac{0.962}{0.907} + 0.03 + 1 \approx 1.062,$$

so the gross return over two years is approximately $1.062/0.963 \approx 1.102$*. Compare that to* $(1 + 0.05)$^2*, which is approximately the same (some small rounding differences).*

16.3.5 The Return from Selling a Coupon Bond before Maturity 1

The gross *return from holding a coupon bond from t to $t + s$* depends on both the price development on the bond and the value in $t + s$ of the (reinvested) coupon payments received between t and $t + s$

$$1 + R_{t+s} = \frac{P_{t+s} + \text{value}_{t+s}(\text{coupon payments})}{P_t}, \tag{16.23}$$

where the subscripts indicate the trading date.

When there are changes in the interest rate level and we sell the bond before maturity, then the capital gains/losses (P_{t+s}/P_t) often dominate: lower interest rates mean capital gains and vice versa (just like for zero coupon bonds). For long-maturity bonds, the effects can be considerable. See Figure 16.11 for an illustration and Figure 16.4 for an empirical example.

Example 16.28 *(Playing the yield curve) (a) On 1 Oct 2015:1y LIBOR is 0.86% and 5y T-bond rate is 1.37%; (b) you believe the 1y rate will not change much over the next 5 years; (c) buy 5y T-bond ("receiving 1.37%") and finance it by 1y borrowing (paying 0.86% at most); (d) next year you roll over your debt; (e) works well unless the 1y rates increase. Orange County went bankrupt by doing something similar in 1994 (using a very leveraged position in inverse floaters, whose coupons are inversely related to the short interest rate). Unfortunately (for Orange County), short rates went up.*

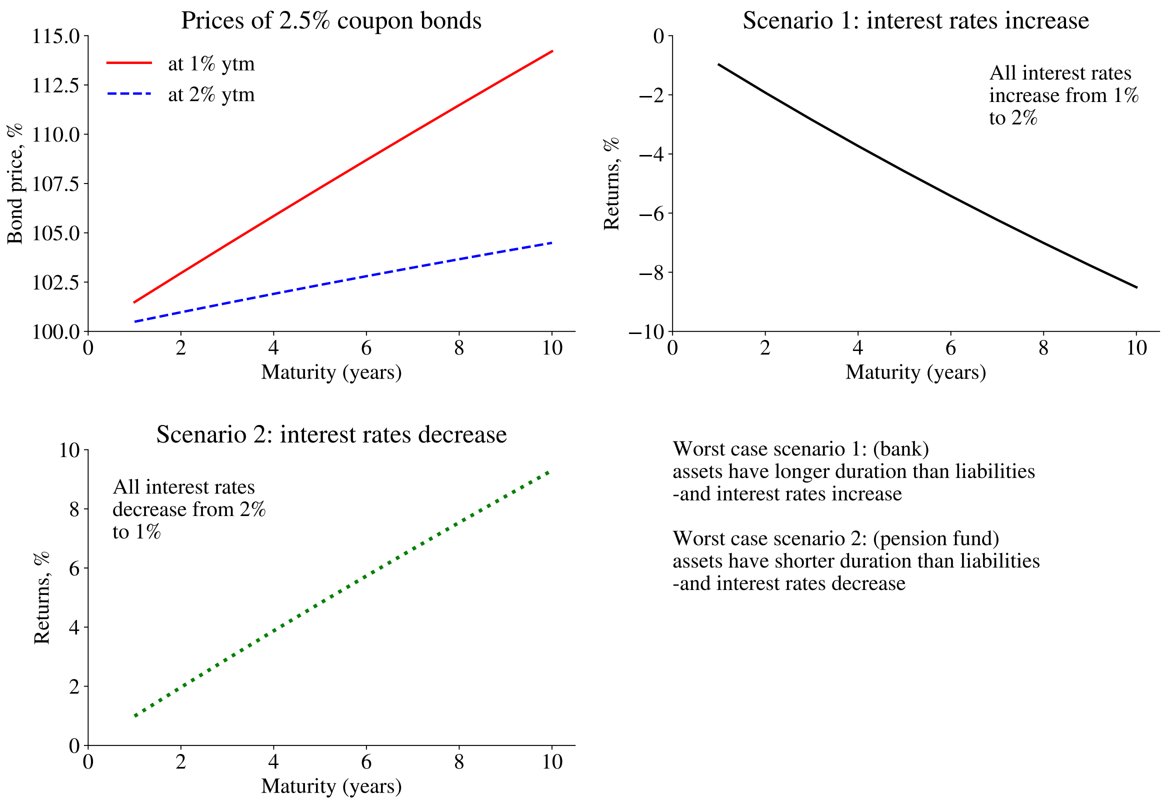


Figure 16.11: Gains and losses from interest rate changes

16.3.6 The Return from Selling a Coupon Bond before Maturity 2: Using Forwards

If the coupons are "locked in" by forwards, then the bond is effectively transformed into a zero-coupon bond. The next proposition shows the implication for the holding return.

Proposition 16.29 *(Bond holding return, a special case) Suppose we reinvest the coupons with forward contracts—as if we were going to hold the bond until maturity m_K. Holding the bond until $t+s$ ($s \leq m_K$) gives the total gross return $1+R_{t+s} = B_{t+s}(m_K - s)/B_t(m_K)$, where $B_t(m)$ denotes the price of an m-period zero coupon bond in t. This implies that the portfolio has the same return as an m_K-period zero coupon bond bought in t, which becomes an $m_K - s$ zero coupon bond in $t+s$.*

For instance, with a 3-period bond, the gross return on holding the bond for one period is $B_{t+1}(2)/B_t(3)$, while the gross return from holding it for two periods is $B_{t+2}(1)/B_t(3)$.

Clearly, the strategy to reinvest the coupons with forward contracts essentially turns this into an m_K-period zero coupon bond (where you invest in t but do not receive any payoffs until $t + m_K$). The return of the strategy is thus the same as on holding this zero coupon bond for s periods. Once again, with other assumptions about how the coupons are reinvested, the result is different.

Proof. (of Proposition 16.29*) Consider a 3-period coupon bond which we hold for 1 period. Enter forward contracts like in the proof of Proposition 16.26. The value of this portfolio in $t + 1$ must be the present value of the value at maturity, that is,

$$B_{t+1}(2)\left[\frac{B_t(1)}{B_t(3)}c + \frac{B_t(2)}{B_t(3)}c + c + 1\right],$$

where $B_{t+1}(2)$ denotes the price in $t + 1$ of a two-period zero coupon bond. Dividing by the bond price in t

$$P_t = B_t(1)c + B_t(2)c + B_t(3)(c + 1)$$

gives the gross return

$$1 + R_{t+1} = B_{t+1}(2)/B_t(3).$$

■

Example 16.30 *(Holding a coupon bond for one period) Use the same numbers as in Example 16.27 and assume that the interest rates are unchanged. The present value in* $t + 1$ *of the value at maturity is*

$$0.962 \times 1.062 = 1.022.$$

Dividing by the bond price P_t*, the gross return is*

$$\frac{1.022}{0.963} \approx 1.06.$$

Using Proposition 16.29 directly gives $B_{t+1}(1)/B_t(2)$*, which is approximately the same. Instead, if the interest rates change so* $B_{t+1}(1) = 0.957$*, then the return is* $0.957 \times 1.062/0.963 \approx 1.055$*, which is the same as* $B_{t+1}(1)/B_t(2)$*.*

Notice that in the *special case* of holding the bond until maturity ($s = m_K$), then Proposition 16.29) shows that $1 + R_{t+s} = 1/B(m_K)$ (since $B_{t+s}(0) = 1$), which is the same result as in Proposition 16.26). In this case, the bond earns the spot interest rate $Y(m_K)$ per period.

Also, notice that in the very *special case* of a flat and unchanged yield curve (with the

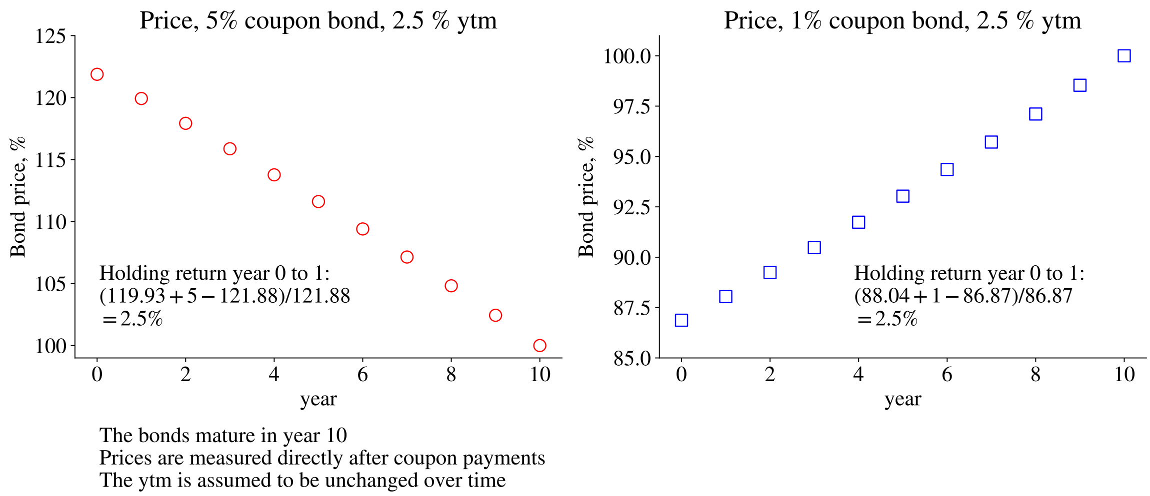


Figure 16.12: Bond price and yield to maturity

interest rate Y for all maturities), then then Proposition 16.29) shows that the return is

$$1 + R_{t+s} = (1 + Y)^s, \tag{16.24}$$

so the return is just cumulated interest rates. See Figure 16.12 for an illustration.

Remark 16.31 *(Realized forwards*) Sometimes another set of assumptions (labelled "realized forwards") is used to analyse the return on holding a coupon bond. In this case, the coupons are reinvested at the spot rates prevailing at the time of the coupon payment. However, it is assumed that those future spot rates will actually be equal to today's forward rates (hence "realized"). This is clearly unrealistic, but can be used to gauge the expected return on holding the bond, at least if today's forwards are close approximations of the expected future spot rates. The result is similar to Proposition 16.29.*

16.3.7 Par Yield*

A par yield is the coupon rate at which a bond would trade at par (that is, have a price equal to the face value). Setting $P = 1$ in (16.16) and solving for the implied coupon rate gives

$$c = \frac{1}{\sum_{k=1}^{K} B(m_k)} \left[1 - B(m_K)\right], \text{ or} \tag{16.25}$$

$$= \frac{1}{\sum_{k=1}^{K} \frac{1}{[1+Y(m_k)]^{m_k}}} \left[1 - \frac{1}{[1+Y(m_K)]^{m_K}}\right]. \tag{16.26}$$

Typically, this is very similar to the zero coupon rates: see Figure 16.13.

Example 16.32 *Suppose $B(1) = 0.95$ and $B(2) = 0.90$. We then have*

$$1 = (0.95 + 0.9)c + 0.9, \text{ so } c = \frac{1}{0.95 + 0.9}(1 - 0.9) \approx 0.054.$$

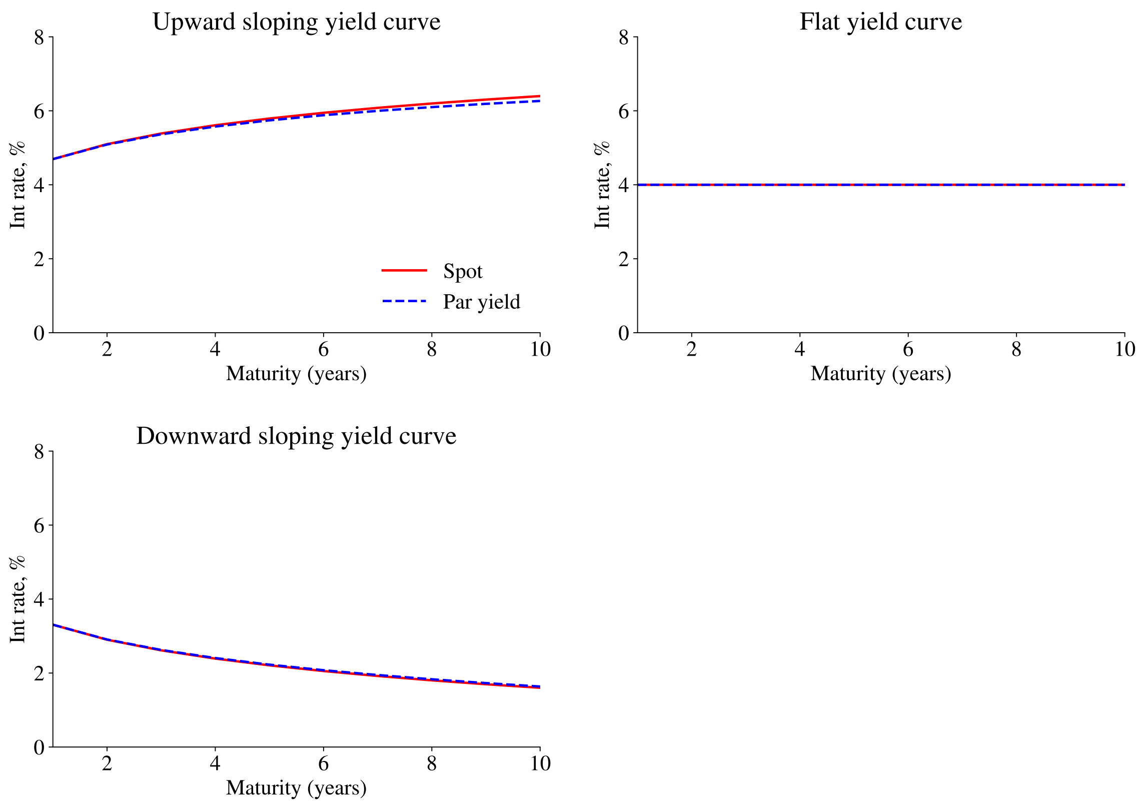


Figure 16.13: Spot and par yield curve

16.4 Swap and Repo

16.4.1 Swap

A (plain vanilla) interest rate swap involves a sequence of payments over the life time (maturity) of the contract: for each tenor (that is, sub period, for instance a quarter) it pays the floating market rate (say, the 3-month Libor) in return for a fixed *swap rate*. Notice that the floating rate is market determined at the beginning of the tenor, but paid at the end of the same tenor.

Split up the time until maturity m into m/h intervals of length h, for instance, a 1-year maturity into 4 quarters. Denote the floating simple h-period interest rate by $\tilde{Y}_t$. Let today be $t = 0$, so the next payment is $t = h$, then second payment is in $t = 2h$ according to

$$\begin{array}{cc} \underline{\text{period}} & \underline{\text{payment}} \\ h & h(\tilde{Y}_0 - R) \\ 2h & h(\tilde{Y}_h - R) \\ 3h & h(\tilde{Y}_{2h} - R) \\ 4h & h(\tilde{Y}_{3h} - R), \end{array} \tag{16.27}$$

where R is the fixed swap rate determined in t (as part of the swap contract). See Figures 16.14–16.15 for illustrations.

More generally, in period sh, the swap contract pays

$$h(\tilde{Y}_{(s-1)h} - R) \text{ for } s = 1, 2, ..., m/h \tag{16.28}$$

The swap contract can be replicated by a sequence of forward contracts, which implies that the swap rate must therefore be (assuming no default or liquidity premia)

$$R = \frac{1}{h}\frac{1 - B(m)}{\sum_{s=1}^{m/h} B(sh)}. \tag{16.29}$$

Example 16.33 *(Swap rate) Consider a one-year swap contract with quarterly periods ($m = 1, h = 1/4$). (16.29) is then*

$$R = 4\frac{1 - B(1)}{B(1/4) + B(1/2) + B(3/4) + B(1)}.$$

With the bond prices (0.99,0.98,0.97,0.96) we have

$$R = 4\frac{1 - 0.96}{0.99 + 0.98 + 0.97 + 0.96} \approx 4.1\%.$$

1-year interest rate swap

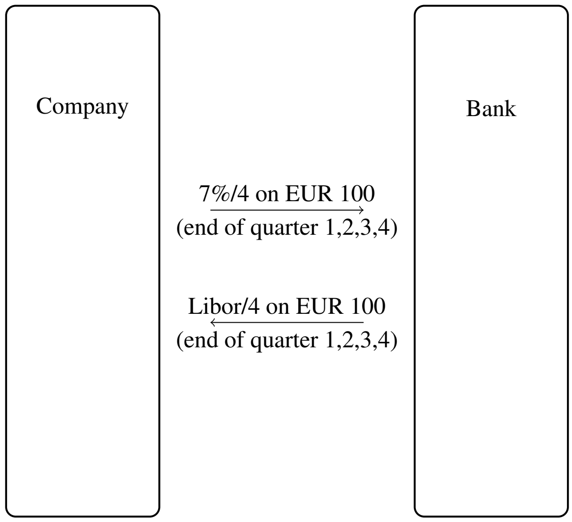


Figure 16.14: Interest rate swap

An *Overnight Indexed Swap* (OIS) is a swap contract where the floating rate is tied to an index of floating rates (for instance, federal funds rates in the U.S., EONIA in Europe—which is a weighted average of all overnight unsecured interbank lending transactions). Since the OIS has very little risk (as the face value or notional never changes hands—only the interest payment is risked in case of default), it is little affected by interbank risk premia. The quote is in terms of the fixed (swap) rate—which typically stays fairly close to secured lending rates like repo rates.

Proof. (of (16.29)*) Notice that a simple forward rate $\tilde{\Gamma}$ for an investment from $(s-1)h$ to sh satisfies

$$h\tilde{\Gamma}[(s-1)h, sh] = \frac{B[(s-1)h]}{B(sh)} - 1.$$

Rewrite (16.28) by replacing $h(\tilde{Y}_{(s-1)h}$ with the cost according to the forward contract, $h\tilde{\Gamma}$,

$$\frac{B[(s-1)h]}{B(sh)} - 1 - hR \text{ for } s = 1, 2, ..., m/h.$$

The present value of this cash flow stream is found by multiplying each term by $B(sh)$ and summing across $s = 1, 2, ..., m/h$

$$\sum\nolimits_{s=1}^{m/h} B(sh) \left\{ \left[\frac{B[(s-1)h]}{B(sh)} - 1 \right] - hR \right\}.$$

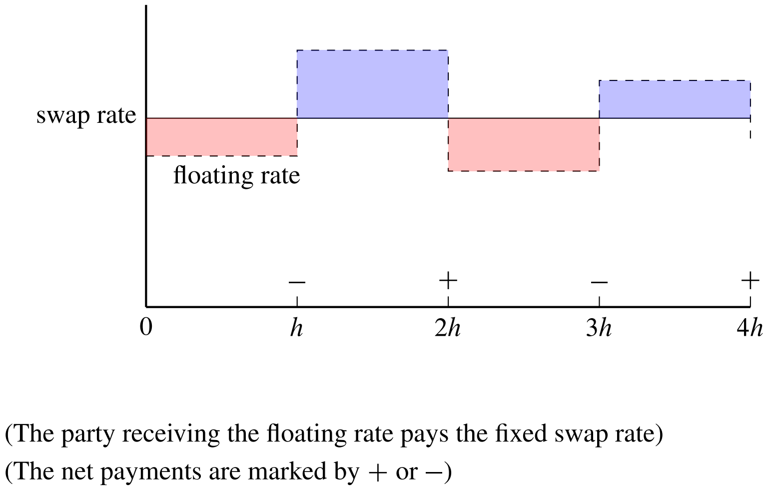


Figure 16.15: Timing convention of interest rate swap

Notice that this can be rewritten as

$$1 - B(m) = hR\sum\nolimits_{s=1}^{m/h} B(sh),$$

where we have used the fact that $B(0) = 1$. Since it is riskfree (assuming no default and liquidity premia) this present value should be zero, which gives (16.29). ■

16.4.2 Repo

A *Repo* (Repurchase agreement) is a way of borrowing against a collateral. Suppose bank A sells a security to bank B, but there is an agreement that bank A will buy back the security at some fixed point in time (the next day, after a week, etc.)—at a price that is predetermined. This means that bank A gets a loan against a collateral (the asset)—and pays an interest rate (final buy price/initial sell price minus one). See Figure 16.16. Bank B is said to have made a reverse repo.

The repo clearly means that bank B has "borrowed" the security—which can then be sold to someone else. This is a way of shortening the security, so the repo rate is low if there is a demand for shortening the security. A *haircut* (of 3%, say) means that the collateral (security) has market value that is 3% higher than the price agreed in the repo. This provides a safety margin to the lender—since the market price of the security could decrease over the life span of the repo.

Example 16.34 *(Long-short bond portfolio). First, buy bond X and use it as collateral*

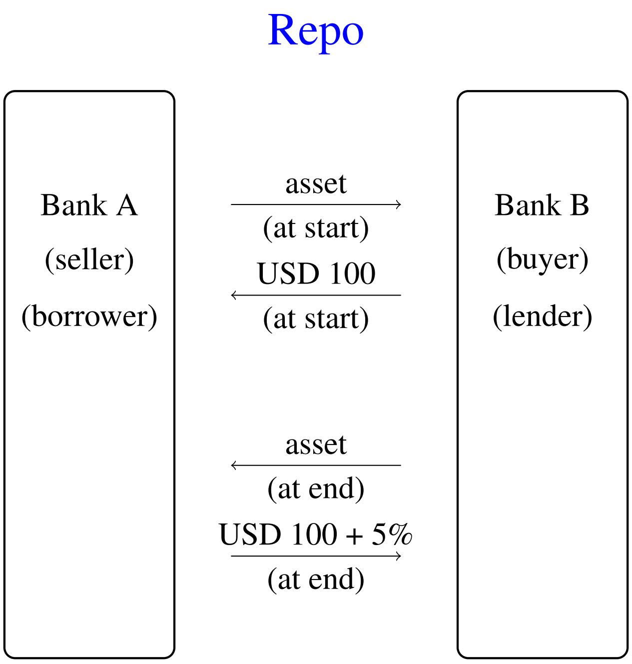


Figure 16.16: Repo

in a repo (the repo borrowing finances the purchase of the bond). Second, enter a reverse repo where bond Y is used as collateral and sell the bond (selling provides cash for the repo lending).

16.5 Estimating the Yield Curve*

The (zero coupon) spot rate curve is of particular interest: it helps us price any bond or portfolio of bonds—and it has a clear economic meaning ("the price of time").

In some cases, the spot rate curve is actually observable—for instance from swaps and STRIPS. In other cases, the instruments traded on the market include some zero coupon instruments (bills) for short maturities (up to a year or so), but perhaps only coupon bonds for longer maturities. This means that the spot rate curve needs to be calculated (or estimated). This section describes different methods for doing that.

16.5.1 Direct Calculation of the Yield Curve ("Bootstrapping")

We can sometimes calculate large portions of the yield curve directly from bond prices by a method called "bootstrapping."

The basic idea is to recursively use the fact that a coupon bond is a portfolio of discount (zero coupon) bonds. For instance, suppose we have a one-period coupon bond, here denoted $P(1)$, which by (16.16) must have the price

$$P(1) = B(1)[c(1) + 1], \tag{16.30}$$

where we use $c(1)$ to indicate the coupon value of this particular bond. The equation immediately gives the price of a one-period discount bond, $B(1)$. In this setting the discount bond prices, $B(m)$, are also called a *discount function* (considered as a function of m).

Suppose we also have a two-period coupon bond, which pays the coupon $c(2)$ in $t+1$ and $t+2$ as well as the principal in $t+2$, with the price (see (16.16))

$$P(2) = B(1)c(2) + B(2)[c(2) + 1]. \tag{16.31}$$

The two period discount function, $B(2)$, can be calculated from this equation since it is the only unknown. We can then move on to the three-period bond,

$$P(3) = B(1)c(3) + B(2)c(3) + B(3)[c(3) + 1] \tag{16.32}$$

to calculate $B(3)$, and so forth. Finally, we can use (16.1) to transform these zero coupon bond prices to spot interest rates.

Remark 16.35 *(Numerical calculation of the bootstrap) Equations (16.30)–(16.32) can clearly be written*

$$\begin{bmatrix} P(1) \\ P(2) \\ P(3) \end{bmatrix} = \begin{bmatrix} c(1)+1 & 0 & 0 \\ c(2) & c(2)+1 & 0 \\ c(3) & c(3) & c(3)+1 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \\ B(3) \end{bmatrix},$$

which is a recursive (triangular) system of equations.

Example 16.36 *(Bootstrapping) Suppose we know that $B(1) = 0.95$ and that the price of a bond with a 6% annual coupon with two years to maturity is 1.01. Since the coupon bond must be priced as*

$$0.95 \times 0.06 + B(2) \times 0.06 + B(2) = 1.01,$$

we can solve for the price of a two-period zero coupon bond as $B(2) \approx 0.90$. The spot interest rates are then $Y(1) \approx 0.053$ and $Y(2) \approx 0.054$. In this case the system of *equations is*

$$\begin{bmatrix} 0.95 \\ 1.01 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.06 & 1.06 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \end{bmatrix}.$$

Unfortunately, the bootstrap approach is tricky to use. First, there are typically gaps between the available maturities (at least outside the US treasury market). One way around that is to interpolate. Second (and quite the opposite), there may be several bonds with the same maturity but with different coupons/prices, so it is hard to calculate a unique yield curve. This could be solved by forming an average across the different bonds or by simply excluding some data. Alternatively, we use another method than the bootstrap (see below).

16.5.2 Estimating the Yield Curve with Regression Analysis

Recall equation (16.16) which expresses the coupon bond price in terms of a series of discount bond prices. It is reproduced here

$$P = \sum_{k=1}^{K} B(m_k)c + B(m_K). \tag{16.33}$$

If we attach some random error to the bond prices, then this looks very similar to regression equation: the coupon bond price is the dependent variable; the coupons are the regressors, and the discount function (discount bond prices) are the coefficients to estimate—perhaps with OLS. This is a way of overcoming the second problem discussed above since multiple bonds with the same maturity, but different coupons, are just additional data points in the estimation.

The first problem mentioned above, gaps in the term structure of available bonds, is harder to deal with. If there are more coupon dates than bonds, then we cannot estimate all the necessary zero coupon bond prices from data (fewer data points than coefficients). The way around this is to decrease the number of coefficients by assuming that the discount function, $B(m)$, is a linear combination of some J predefined functions of maturity, $g_1(m)$,..., $g_J(m)$,

$$B(m) = 1 + \sum_{j=1}^{J} a_j g_j(m), \tag{16.34}$$

where $g_j(0) = 0$ since $B(0) = 1$ (the price of a bond maturing today is one).

Once the $g_j(m)$ functions are specified, (16.34) is substituted into (16.16) and the j coefficients a_1,..., a_j are estimated by minimizing the squared pricing error (see, for instance, Campbell, Lo, and MacKinlay (1997) 10).

One possible choice of $g_j(m)$ functions is a polynomial, $g_j(m) = m^j$. Another common choice is to make the discount bond price a spline (see McCulloch (1975)).

Example 16.37 *(Quadratic discount function) With a quadratic discount function*

$$B(m) = a_0 + a_1 m + a_2 m^2,$$

we get from (16.16)

$$\begin{aligned} P(m_K) &= \textstyle\sum_{k=1}^K B(m_k)c + B(m_K) \\ &= \textstyle\sum_{k=1}^K \left(a_0 + a_1 m_k + a_2 m_k^2\right) c + \left(a_0 + a_1 m_K + a_2 m_K^2\right). \end{aligned}$$

Collect all constants (that does not depend on m) into a first regressor, then all terms that are linear in m into a second regressor and finally all terms that are quadratic in m into a third regressor

$$P(m_K) = a_0 \underbrace{(Kc+1)}_{\text{term 0}} + a_1 \underbrace{(c\textstyle\sum_{k=1}^K m_k + m_K)}_{\text{term 1}} + a_2 \underbrace{(c\textstyle\sum_{k=1}^K m_k^2 + m_K^2)}_{\text{term 2}}.$$

For a 1-year bonds that pays no coupons and a 2-year bond that pays a 6% coupons at $m_1 = 1$ and $m_2 = 2$, we have the following matrix of regressors (the bonds are on different rows)

Bond ↓	term 0	term 1	term 2
1-year, 0%	1	1	1
2-year, 6%	$2 \times 0.06 + 1 = 1.12$	$0.06 \times (1+2) + 2 = 2.18$	$0.06 \times (1^2 + 2^2) + 2^2 = 4.30.$

The a_0, a_1, and a_2 can be estimated by OLS if we have data on at least two bonds. This method can, however, lead to large errors in the fitted yields (if not the prices). See Figure 16.17 for an example.

Example 16.38 *(Cubic discount function*) With a cubic discount function*

$$B(m) = a_0 + a_1 m + a_2 m^2 + a_3 m^3,$$

we get

$$P(m_K) = a_0 (Kc+1) + a_1 \left(c\textstyle\sum_{k=1}^K m_k + m_K\right) + a_2 \left(c\textstyle\sum_{k=1}^K m_k^2 + m_K^2\right) + a_3 \left(c\textstyle\sum_{k=1}^K m_k^3 + m_K^3\right).$$

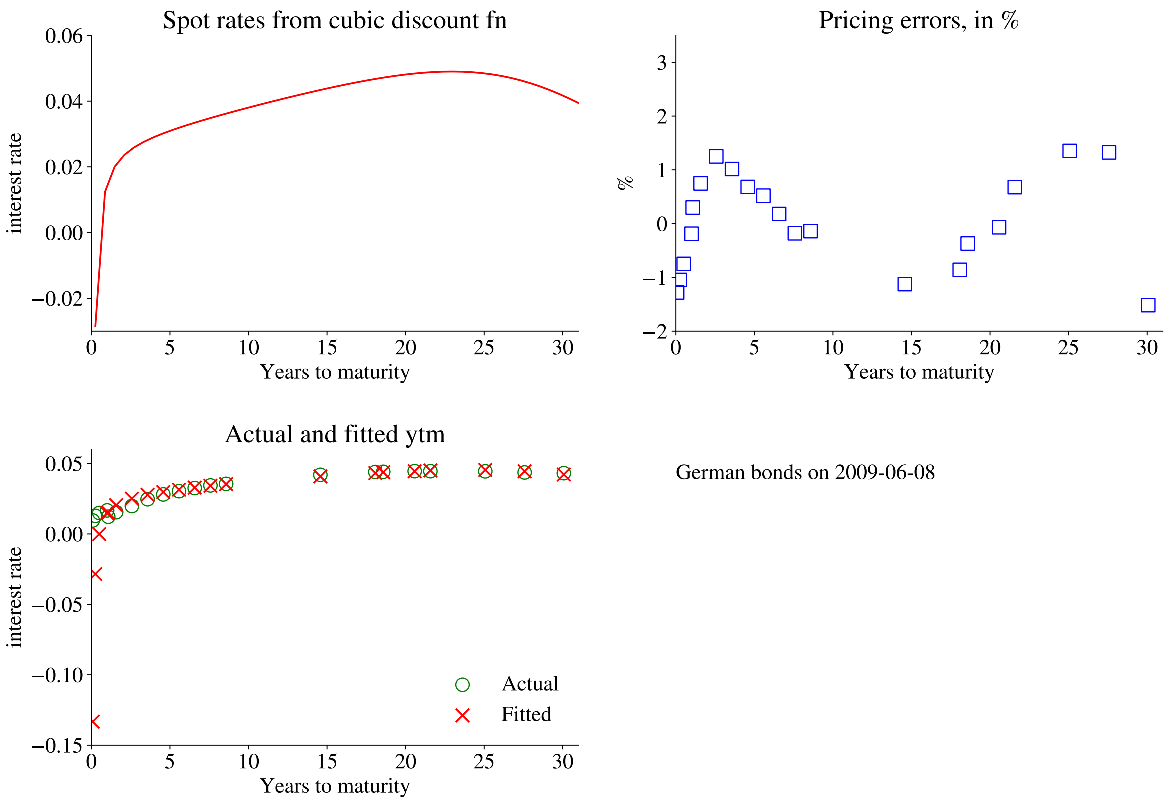


Figure 16.17: Estimated yield curves

16.5.3 Estimating a Parametric Forward Rate Curve*

Yet another approach to estimating the yield curve is to start by specifying a function for the instantaneous forward rate curve, and then calculate what this implies for the discount bond prices (discount function). (These will typically be complicated and not satisfy the simple linear structure in (16.34).)

Let $f(m)$ denote the instantaneous forward rate with time to settlement m. The *extended Nelson and Siegel forward rate function* (Svensson (1995)) is

$$f(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \tag{16.35}$$

where $\beta_0, \beta_1, \beta_2, \tau_1, \beta_3, \tau_2$ are parameters (β_0, τ_1 and τ_2 must be positive, and $\beta_0 + \beta_1$ must also be positive—see below). The original Nelson and Siegel function sets $\beta_3 = 0$.

Note that in either case

$$\lim_{m\to 0} f(m) = \beta_0 + \beta_1, \text{ and}$$
$$\lim_{m\to\infty} f(m) = \beta_0,$$

so $\beta_0 + \beta_1$ corresponds to the current very short spot interest rate (an overnight rate, say) and β_0 to the forward rate with settlement very far in the future (the asymptote).

The spot rate implied by (16.35) is (integrate as in (16.54) to see that)

$$y(m) = \beta_0 + \beta_1 \frac{1-\exp(-m/\tau_1)}{m/\tau_1} + \beta_2 \left[\frac{1-\exp(-m/\tau_1)}{m/\tau_1} - \exp\left(-\frac{m}{\tau_1}\right)\right] + \beta_3 \left[\frac{1-\exp(-m/\tau_2)}{m/\tau_2} - \exp\left(-\frac{m}{\tau_2}\right)\right]. \tag{16.36}$$

One way of estimating the parameters in (16.35) is to substitute (16.36) for the spot rate in (16.4), and then minimize the sum of the squared price errors (differences between actual and fitted prices), perhaps with 1/maturity (or 1/modified duration) as the weight for the squared error (a practice used by many central banks). Alternatively, one could minimize the sum of the squared yield errors (differences between actual and fitted yield to maturity). See Figure 16.18 for an illustration.

16.5.4 Par Yield Curve

When many bonds are traded at (approximately) par, the par yield curve (16.25) can be obtained by just plotting the coupon rates. In practice, the yield to maturity is used instead (to partly compensate for the fact that the bonds are only approximately at par)—and the gaps (across maturities) are filled by interpolation. (Recall that for a par bond, the yield to maturity equals the coupon rate.) This is basically the way the Constant Maturity Treasury yield curve, published by the US Treasury, is constructed.

16.5.5 Swap Rate Curve

The swap rates for different maturities can also be used to construct a yield curve.

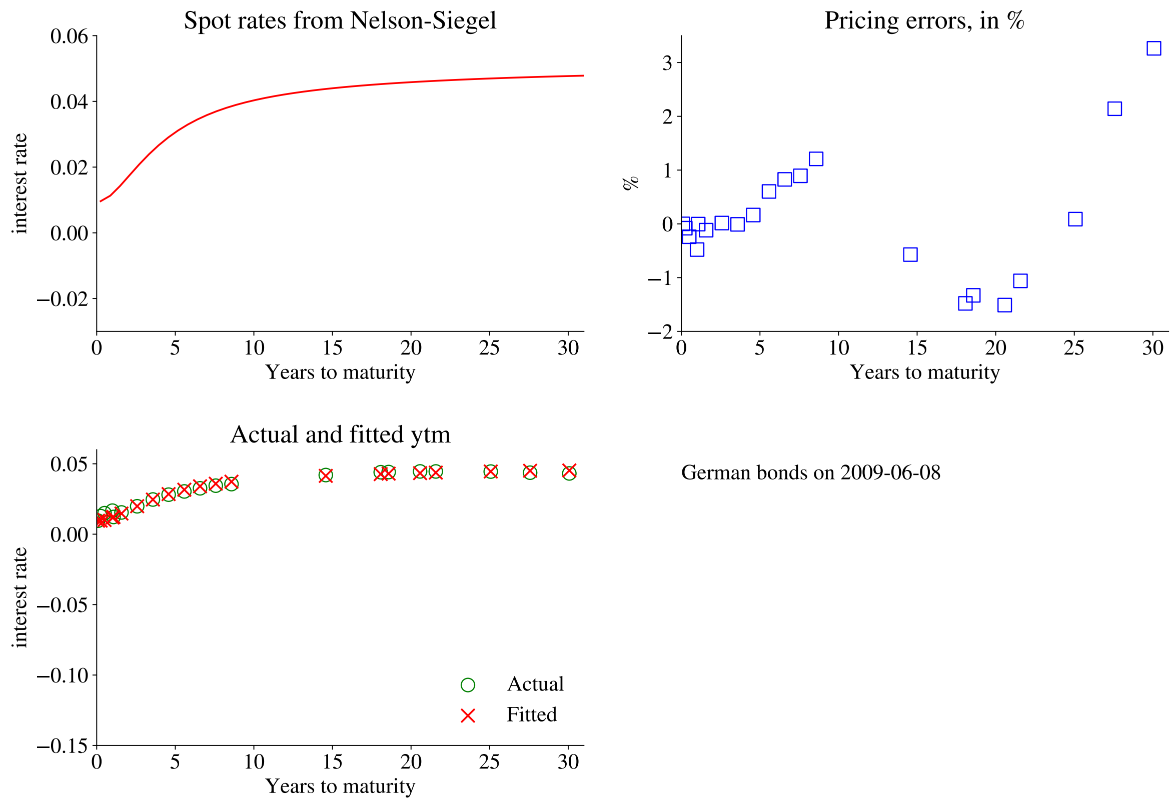


Figure 16.18: Estimated yield curves

16.6 Conventions on Important Markets*

16.6.1 Compounding Frequency

Suppose the interest rate r is compounded 2 times per year. This means that the amount B invested at the beginning of the year gives $B \times (1 + r/2)$ after six months—which is reinvested and therefore gives $B \times (1 + r/2)(1 + r/2)$ after another six months (at the end of the year). To make this payoff equal to unity (as we have used as our convention) it must be the case that the bond price $B = 1/(1 + r/2)^2$. By comparing with the definition of the effective interest rate (with annual compounding) in (16.1) we have

$$\frac{1}{B} = \left(1 + \frac{r}{2}\right)^2 = 1 + Y, \tag{16.37}$$

where Y is the annual effective interest rate.

This shows how we can transform from semi-annual compounding to annual compounding (and vice versa).

More generally, with compounding n times per year, we have

$$\frac{1}{B} = \left(1 + \frac{r}{n}\right)^n = 1 + Y. \tag{16.38}$$

Clearly, as $n \to \infty$, the expression in (16.38) goes to e^r, where r is the continuously compounded rate.

16.6.2 US Treasury Notes and Bonds

The convention for *US Treasury notes and bonds* (issued with maturities longer than one year) is that coupons are paid semi-annually (as half the quoted coupon rate), and that yields are semi-annual effective yields. (This applies also to most US corporate bonds and UK Treasury bonds.)

Remark 16.39 *(Bond price quotes*) On the U.S. Treasury bond market, the bond price quotes are often not in a decimal form. Instead, the quoted prices (for a face value of 100) use fractions of 4, 8, 26, 32 and 62 as in*

$$\begin{aligned}
&\textit{91-21 means } 91 + 21/32 \approx 91.6562\\
&\textit{91-21+ means } 91 + 21/32 + 1/64 \approx 91.6719\\
&\textit{91-21}\tfrac{3}{4} \textit{ means } 91 + (21 + 3/4)/32 \approx 91.6797\\
&\textit{91-213 means } 91 + (21 + 3/8)/32 \approx 91.6680.
\end{aligned}$$

However, both are quoted on an annual basis by multiplying by two. The quoted *yield to maturity*, ϕ, solves

$$P = \sum_{k=1}^{K} \frac{c/2}{(1 + \phi/2)^{n_k}} + \frac{1}{(1 + \phi/2)^{n_K}}, \tag{16.39}$$

where the bond pays coupons $c/2$, at $n_1, n_2, ..., n_K$ half-years ahead. By using (16.37), the yield quoted, ϕ, can be expressed in terms of an annual effective interest rate.

Example 16.40 *A 9% US Treasury bond (the coupon rate is 9%, paid out as 4.5% semi-annually) with a yield to maturity of 7%, and one year to maturity has the price*

$$\frac{0.09/2}{1 + 0.07/2} + \frac{0.09/2}{(1 + 0.07/2)^2} + \frac{1}{(1 + 0.07/2)^2} = 1.019.$$

From (16.37), we get that the yield to maturity rate expressed as an annual effective interest is $(1 + 0.035)^2 - 1 \approx 0.071$.

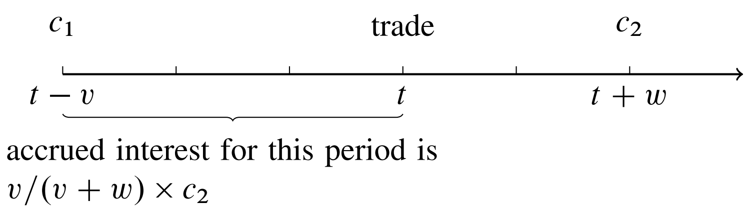


Figure 16.19: Accrued interest

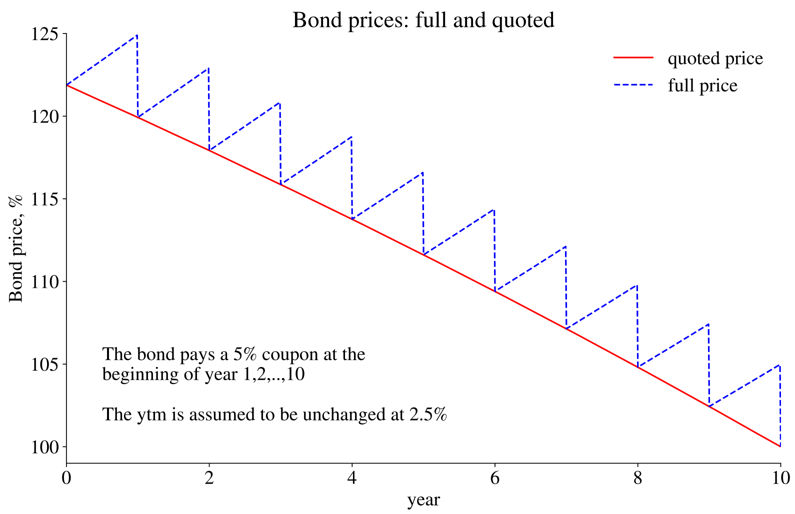


Figure 16.20: Full and quoted bond prices

16.6.3 Accrued Interest on Bonds

The quotes of bond prices (as opposed to yields) are not the full price (also called the dirty price, invoice price, or cash price) the investor pays. Instead, the full price is

$$\text{full price} = \text{quoted price} + \text{accrued interest.}$$

The buyer of the bond (buying in t) will typically get the next coupon (trading is "cum-dividend"). The accrued interest is the faction of that next coupon that has been accrued during the period the seller owned the bond. It is calculated as

$$\text{accrued interest} = \text{next coupon} \times \text{days since last coupon/days between coupons.}$$

For instance, for most US bonds, the next coupon is half the coupon rate and the days between coupons are 182.5. See Figures 16.19–16.20.

16.6.4 US Treasury Bills

Discount Yield

US Treasury bills have no coupons and are issued in 3, 6, 9, and 12 months maturities—but the time to maturity does of course change over time. They are quoted in terms of the (banker's) *discount yield*, Y_{db}, which satisfies

$$B = 1 - mY_{db}, \text{ where } m = \text{days}/360, \text{ so} \tag{16.40}$$

$$Y_{db} = (1 - B)/m. \tag{16.41}$$

Notice the convention of m =days/360. (If the face value is different from one, then we have $Y_{db} = [\text{face}-B]/(\text{face}\times m)$.)

From (16.1) and (16.40) it is the clear that the effective interest rate and the continuously compounded interest rates can be solved as

$$Y = [1 - mY_{db}]^{-1/m} - 1 \tag{16.42}$$

$$y = -\ln(1 - mY_{db})/m. \tag{16.43}$$

Example 16.41 *A T-bill with 44 days to maturity and a quoted discount yield of 6.21% has the price* $1-(44/360)\times 0.0621 \approx 0.992$. *The effective interest rate is* $[1-(44/360)\times 0.0621]^{-360/44} - 1 \approx 6.43\%$.

16.6.5 LIBOR and EURIBOR

The LIBOR (London Interbank Offer Rate) and the EURIBOR (Euro Interbank Offered Rate) are the simple interest rate on a short term loan without coupons. It is quoted as a simple annual interest rate, using a "actual/360" day count—with the exception of pounds which are quoted "actual/365." This means that borrowing one dollar for 150 days at a 6% LIBOR requires the payment of $0.06 \times 150/360$ dollars in interest at maturity. Rescaling to make the payment at maturity equal to unity (which is the convention used in these lecture notes), the loan must be $1/(1 + 0.06 \times 150/360)$—which is the "price" of a deposit that gives unity in 150 days.

16.6.6 European Bond Markets

The major continental European bond markets (in particular, France and Germany) typically have annual coupons and the accrued interest is calculated according to the "actual/actual" convention, that is, as

$$\text{accrued interest} = \text{next coupon} \times \text{days since last coupon}/365 \text{ (or 366)}.$$

(The computation is slightly more complicated for the UK and the Scandinavian countries, since they have ex-dividend periods.)

16.7 Other Instruments

16.7.1 Collateralized Debt Obligations

CDO is a repackaging of a set of assets ("collaterals," typically bonds) where the claims (payouts) are tranched (have different priorities). See Figure 16.21 for an example.

CDOs are created for two main reasons. First, it is a way for the issuer (typically a bank), to "package and sell off,":that, it is a way to shrink the balance sheet for the bank (securitisation) but still earn a fee. Second, a CDO transforms risky bonds to (a) some safe bonds and (b) some very risky ones. This opens up new possibilities for investors. For instance, it may allow risk averse investors (including pension funds) to invest into the safe tranches, while they would otherwise not dare (or be allowed to) invest into the original bonds.

It is clear that the correlation of the defaults of the bonds in the CDO is important. The idea of tranching (in particular, to regard the senior tranche as safe) depends on the assumption that not all underlying debts default at the same time. Underestimating the correlation can lead to serious overpricing of the senior tranches—as was often the case just before the financial crisis 2008–9.

Another important aspect of the CDO is whether the originator (bank) holds the junior trance or not. If it does, then it has the incentives to screen the borrowers/monitor the loans, otherwise not.

16.7.2 Credit Default Swaps

A credit default swap is an insurance against default on a bond (for instance, Greek government bonds). See Figure 16.22 for an example.

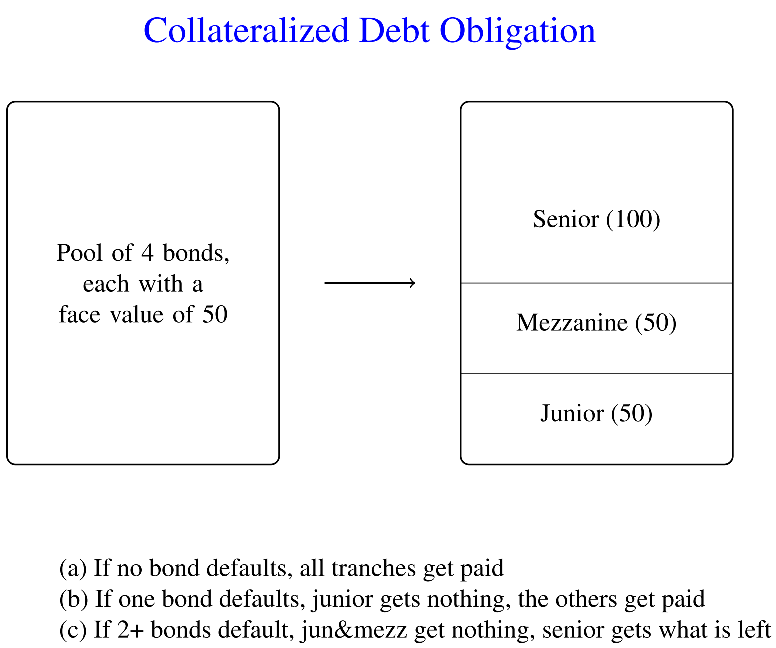


Figure 16.21: Collateralized Debt Obligation

If you hold a portfolio of one risky bond and a CDS on it, then you effectively own a riskfree bond. The other way around is to buy one riskfree bond and issue a CDS, which gives effectively the same as owning the risky bond. This simple observation is the key to understanding how the CSD ("insurance") premium is determined.

year	Probability of survival through year t	Probability of default in year t	Expected spread payment	Expected payment from insurance	Expected PV of net payment
1	0.98	0.02	$0.98s$	0.02×0.6	$0.98s - 0.012$
2	0.95	0.03	$0.95s$	0.03×0.6	$0.95s - 0.018$
Sum					$1.93s - 0.03$

Table 16.1: Example of the payment flows of a 2-year CDS with an assumed recovery rate of 0.4 and a riskfree interest rate of zero. The CDS spread is denoted s.

16.7.3 Inflation-Indexed Bonds*

Reference: Deacon and Derry (1998)

Consider an inflation-indexed coupon bond issued in t, which has both coupons and

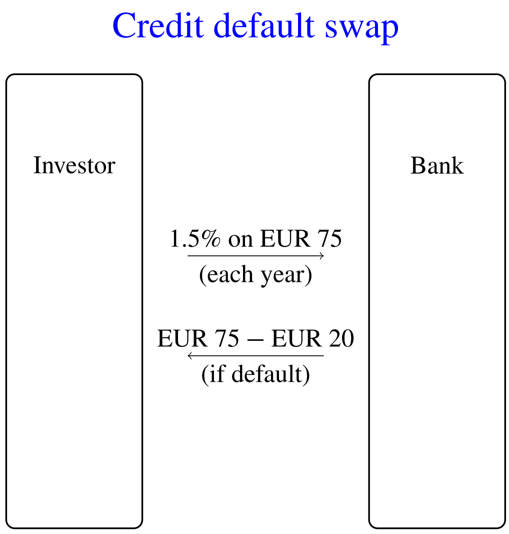


Figure 16.22: Credit default swap

principal adjusted for inflation up to the period of payment (this is called "capital indexed," which is the most common type). Let Q_t be the value of the relevant price index (typically a CPI) in period t. The coupon payments are cQ_{t+m_1-l}/Q_{t-l} at $t+m_1$, cQ_{t+m_2-l}/Q_{t-l} at $t+m_2$, and so forth—and also the principal is paid as Q_{t+m_K-l}/Q_{t-l} in $t+m_K$.

The lag factor l is the *indexation lag*. There are two reasons for this lag. First, the convention on many markets is that the bond price is quoted disregarding accrued interest (clean price). The typical case is as follows. The next coupon payment is m_1 periods ahead. The buyer of the bond in t will get this coupon (trading is "cum-dividend"). The full price the buyer pays to the seller in t is therefore

$$\text{full price} = \text{quoted price} + \text{accrued interest},$$

where the accrued interest is typically the coupon payment times the fraction of this coupon period that has already passed. To pay this accrued interest, we have to know the next coupon payment, that is, cQ_{t+m_1-l}/Q_{t-l}; in t we must know the price level in $t+m_1-\dot{l}$. This mean that $l \geq m_1$ must always hold: the indexation lag must be at least as long as the time between coupon payments (six months in the UK).

Second, it takes time to calculate and publish price indices. Suppose we learn to know Q_s in $s+k$. This means that the indexation lag must be an additional k periods, $l \geq m_1 + k$, so it uses a known price level. For instance, in the UK, the indexation lag is

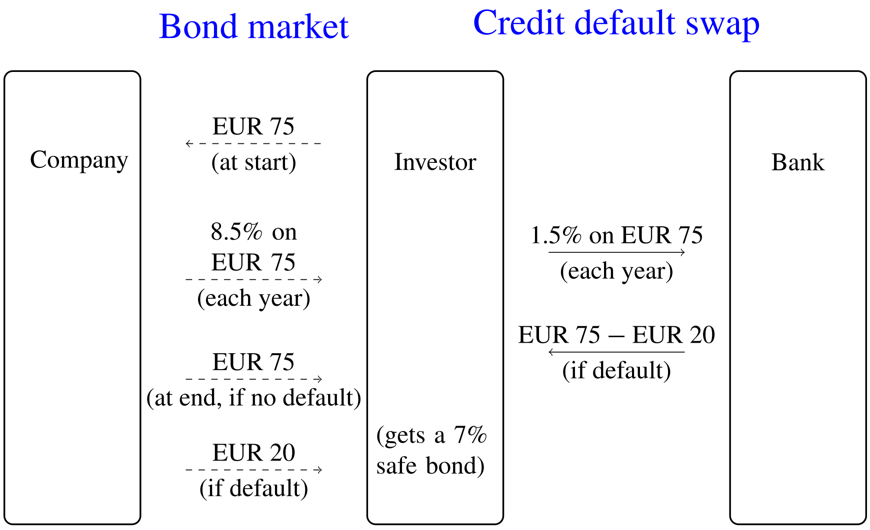


Figure 16.23: Credit default swap

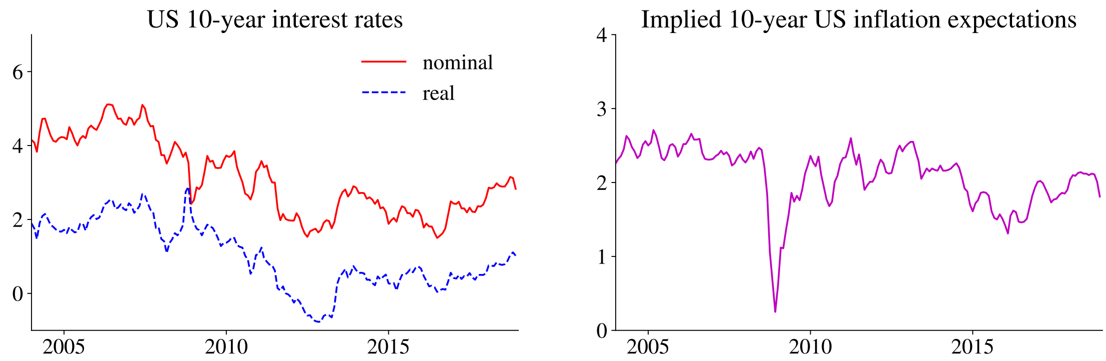


Figure 16.24: US nominal and real interest rates

8 months.

To simplify matters in the rest of this section, suppose the indexation lag is zero. Use (16.16), modified to allow for different coupons, to price the inflation-indexed bond. To further simplify, suppose that bonds do not have any risk premia (clearly a strong assumption), so that the bond price equals the discounted expected payoffs

$$P = \sum_{k=1}^{K} \frac{c \, \mathrm{E}_t \, Q_{t+m_k}/Q_t}{[1+Y(m_k)]^{m_k}} + \frac{\mathrm{E}_t \, Q_{t+m_K}/Q_t}{[1+Y(m_K)]^{m_K}}. \tag{16.44}$$

The Fisher equation is

$$[1+Y(m)]^m = [1+R(m)]^m \frac{\mathrm{E}_t\, Q_{t+m}}{Q_t}, \tag{16.45}$$

where R is the real interest rate. It splits up the gross nominal return in the bond into a gross real return and gross inflation rate. Notice that the Fisher equation assumes that there is no risk premia, which is a strong assumption.

Use (16.45) to rewrite (16.44) as

$$\begin{aligned} P &= \sum_{k=1}^{K} \frac{c\, \mathrm{E}_t\, Q_{t+m_k}/Q_t}{[1+R(m_k)]^{m_k}\, \mathrm{E}_t\, Q_{t+m_k}/Q_t} + \frac{\mathrm{E}_t\, Q_{t+m_K}/Q_t}{[1+R(m_K)]^{m_K}\, \mathrm{E}_t\, Q_{t+m_K}/Q_t} \\ &= \sum_{k=1}^{K} \frac{c}{[1+R(m_k)]^{m_k}} + \frac{1}{[1+R(m_K)]^{m_K}} \end{aligned} \tag{16.46}$$

With a set of inflation-indexed bonds, we could therefore estimate a *real yield curve*, that is, how $R(m)$ depends on m. If the Fisher equation indeed holds, then the difference between a nominal interest rate and a real interest rate can be interpreted as a measure of the market's inflation expectations (often called the "break-even inflation rate").

16.8 Appendix: More on Forward Rates*

16.8.1 Forward Rate as a Marginal Cost*

Split up the time until n into n/h intervals of length h (see Figure 16.25). Then, the n-period spot rate equals the geometric average of the h-period forward rates over t to $t+ns$

$$\begin{aligned} 1+Y(n) &= [1+\Gamma(0,h)]^{h/n} \times [1+\Gamma(h,2h)]^{h/n} \times \ldots \times [1+\Gamma(n-h,n)]^{h/n} \\ &= \prod\nolimits_{s=0}^{n/m-1} \{1+\Gamma[sh,(s+1)h]\}^{h/n}. \end{aligned} \tag{16.47}$$

This means that the forward rate can be seen as the "marginal cost" of making a loan longer. See Figure 16.26 for an illustration.

Proof. (of (16.47)) Let $n=2m$ and use (16.14) for forward contracts between 0 to m and m to $2m$

$$\frac{1}{B(m)/B(0)} = [1+\Gamma(0,m)]^m \text{ and } \frac{1}{B(2m)/B(m)} = [1+\Gamma(m,2m)]^m.$$

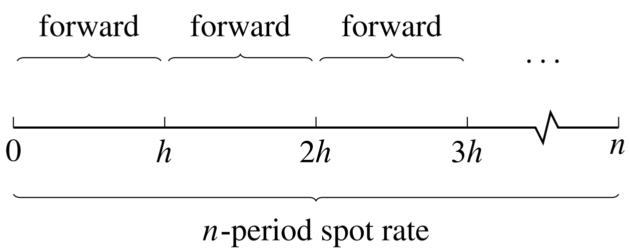


Figure 16.25: Forward contracts for several future periods

Multiply and simplify to get

$$\frac{1}{B(n)} = [1 + \Gamma(0, m)]^m \times [1 + \Gamma(m, 2m)]^m .$$

Raise to the power of $1/n$ to get the interest rate

$$1 + Y(n) = [1 + \Gamma(0, m)]^{m/n} \times [1 + \Gamma(m, 2m)]^{m/n} .$$

■

Example 16.42 *(Spot as average forward rate) In the previous example, (16.47) gives, using $\Gamma(0, 1) = Y(1)$,*

$$1.04^{1/2}1.06^{1/2} \approx 1.05,$$

which indeed equals $1 + Y(2)$.

16.8.2 Continuously Compounded and Simple Forward Rates*

Taking logs of $1 + \Gamma(m, n)$ in (16.15) we get the continuously compounded forward rate

$$f(m, n) = \frac{1}{n - m} \ln \frac{B(m)}{B(n)} = \frac{ny(n) - my(m)}{n - m}. \tag{16.48}$$

Conversely, the n-period (continuously compounded) spot rate equals the average (continuously compounded) forward rate (take logs of 16.47)

$$y(m) = \frac{h}{n} \sum\nolimits_{s=0}^{n/h-1} f[sh, (s + 1)h]. \tag{16.49}$$

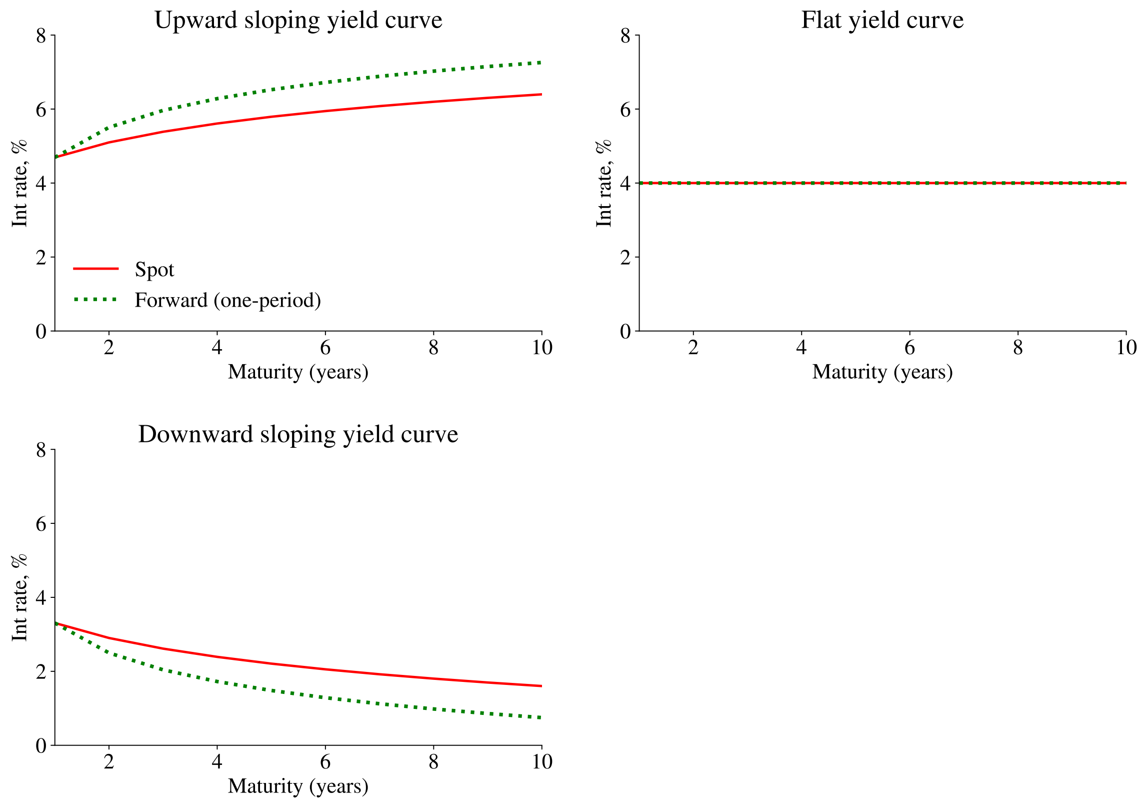


Figure 16.26: Spot and forward rates

A simple forward rate (used on interbank markets) is defined as

$$\frac{1}{B(n)/B(m)} = 1 + (n-m)\tilde{\Gamma}(m,n), \text{ so} \tag{16.50}$$

$$\tilde{\Gamma}(m,n) = \frac{1}{n-m}\left[\frac{B(m)}{B(n)} - 1\right] = \frac{n\tilde{Y}(n) - m\tilde{Y}(m)}{(n-m)[1+m\tilde{Y}(m)]}. \tag{16.51}$$

16.8.3 Instantaneous Forward Rates*

The instantaneous forward rate, $f(m)$, is defined as the limit when the maturity date of the bond approaches the settlement date of the forward contract, $n \to m$. This can be thought of as a forward "overnight" rate m periods ahead in time. From (16.48) it is

$$\begin{aligned} f(m) &= \lim_{n \to m} f(m,n) && (16.52) \\ &= \lim_{n \to m} \frac{n-m}{n-m} y(n) - \lim_{n \to m} \frac{m\,[y(m) - y(n)]}{n-m} \\ &= y(m) + m \frac{dy(m)}{dm}. && (16.53) \end{aligned}$$

Conversely, the average of the forward rates over t to $t+n$ is the spot rate, which we see by integrating (16.53) to get

$$y\,(n) = \frac{1}{n} \int_0^n f(s)ds. \tag{16.54}$$

Equations (16.53) and (16.54) show that the difference between the forward and spot rates, $f(n) - y(n)$, is proportional to the slope of the yield curve.

Proof. (of (16.54)) Integrating the first term on the right hand side of (16.53) over $[0, n]$ gives $\int_0^n y(s)ds$. Integrating (by parts) the second term on the right hand side of (16.53) over $[0, n]$, $\int_0^n s\frac{dy(s)}{ds}ds$, gives $ny(n) - \int_0^n y(s)ds$. Adding the two terms gives $ny(n)$. ■

16.9 Appendix: More Details on Bond Conventions*

16.9.1 Bond Equivalent Yields on US Bonds

The financial press typically quotes a bond equivalent yield for T-bills—in an attempt to make the yields comparable. The bond equivalent yield is the coupon (and yield to maturity) of a par bond that would give the same yield as the T-bill. For a T-bill with at most half a year to maturity, this gives a simple interest rate, but for longer T-bills the expression is more complicated.

We first analyse *a T-bill with more than half a year to maturity*. Consider a coupon bond with face value B (which equals the current price of the T-bill), semi-annual coupon $c/2$ and the same yield to maturity. Since the coupon and the yield to maturity are the same, the "clean price" of the bond (the price to pay if the seller gets to keep the accrued interest on the first coupon payment) equals the face value (here B): it is traded at par. Notice that the latter means that the buyer gets the following fraction of the next coupon payment (which is $B \times c/2$): the fraction of a half year until the next coupon payment (or (days to next coupon)$\times 2/365$).

When the T-bill has more than half a year to maturity, then the bond has two coupon payments left (including the maturity). At maturity, the owner will have the following: *(i)* the principal plus final coupon, $B \times (1 + c/2)$; *(ii)* the part of the first coupon that belongs to the current owner, $d = B \times 2n \times c/2$, where n =(days to next coupon)/365; and *(iii)* the interest on d when reinvested at the semi-annual rate $c/2$ for half a year, $d \times c/2$.

To get the same return as on the T-bill, the owner of the coupon bond must get a value of one at maturity (the return is then $1/B$), or

$$1 = B \times [1 + c/2 + 2n \times c/2 \times (1 + c/2)]. \tag{16.55}$$

Solving for c gives the bond equivalent yield

$$c = \frac{\sqrt{2n/B + 1/4 - n + n^2} - n - 1/2}{n}. \tag{16.56}$$

Example 16.43 *A T-bill with 212 days to maturity and a quoted discount yield of 5.9% has the price $1 - (212/360) \times 0.059 \approx 0.965$. There must be $212 - 182 = 30$ days to the next coupon payment, so $n = 30/365$. The bond equivalent yield is the c such that*

$$c = \frac{\sqrt{2\,(30/365)\,/0.965 + 1/4 - (30/365) + (30/365)^2} - (30/365) - 1/2}{(30/365)} \approx 6.2\%$$

Remark 16.44 *If we define $h =$ (days to maturity)$/365$, then $n = h - 1/2$ and we can rearrange (16.56) as*

$$c = \frac{2\sqrt{h^2 + (2h-1)(1/B-1)} - 2h}{2h-1}.$$

This is the expression in McDonald (2014) Appendix 9.A and Blake (1990) 4.2.

We now apply the same logic to a *T-bill with at most half a year to maturity*. The bond then only has the final coupon left (which is split with the previous owner), and the face value (which is not split). In particular, there is no reinvestment. In this case, (16.55) simplifies to

$$1 = B \times (1 + 2n \times c/2). \tag{16.57}$$

Solving for c (and using the fact that $n = h =$ (days to maturity)$/365$) gives

$$c = \frac{1/B - 1}{h} \text{ or} \tag{16.58}$$

$$B = \frac{1}{1 + h \times c}. \tag{16.59}$$

Example 16.45 *A T-bill with 44 days to maturity and a quoted discount yield of 6.21% has the price* $1 - (44/360) \times 0.0621 \approx 0.992$*. The bond equivalent yield is the* c *such that*

$$0.992 = \frac{1}{1 + \frac{44}{360}c} \text{ or } c = 6.6\%.$$

Remark 16.46 *There are two other, but equivalent, expressions for the bond equivalent yield for maturities of at most half a year (see, for instance, McDonald (2014) Appendix 9.A). The first is*

$$c_1 = \frac{1 - B}{B}\frac{1}{m}.$$

Substituting for B *using (16.59) shows that* $c_1 = c$*. The second is*

$$c_2 = \frac{365 \times Y_{db}}{360 - Y_{db} \times days}.$$

Substituting for Y_{db} *using (16.41) shows that* $c_2 = c_1 = c$*.*

Chapter 17

Hedging Bonds

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9

17.1 Bond Hedging

Suppose we want to hedge against price movements of a bond portfolio or a liability stream. (This is also called immunization.) See Figure 17.1 for a motivation.

The basic idea is to form a total portfolio of both that bond portfolio/liability stream and some other bonds that makes the overall portfolio "immune" to changes in the interest rate level. To simplify, the analysis is focused on changes over a short time period and we often make strong assumptions about how the yield curve changes (for instance, only parallel movements).

Example 17.1 *(Why a liability is not hedged by putting its present value on a bank account) Suppose our liability is an annuity that pays 0.2 every year (starting a year from now) for 10 years. At a 5% interest rate, the present value is*

$$\sum\nolimits_{k=1}^{10} \frac{0.2}{1.05^k} = 1.54.$$

Instead, with an interest rate of 3%, the present value is

$$\sum\nolimits_{k=1}^{10} \frac{0.2}{1.03^k} = 1.71.$$

Putting 1.54 on a bank account will not cover the liability payments if we only get a 3% interest rate.

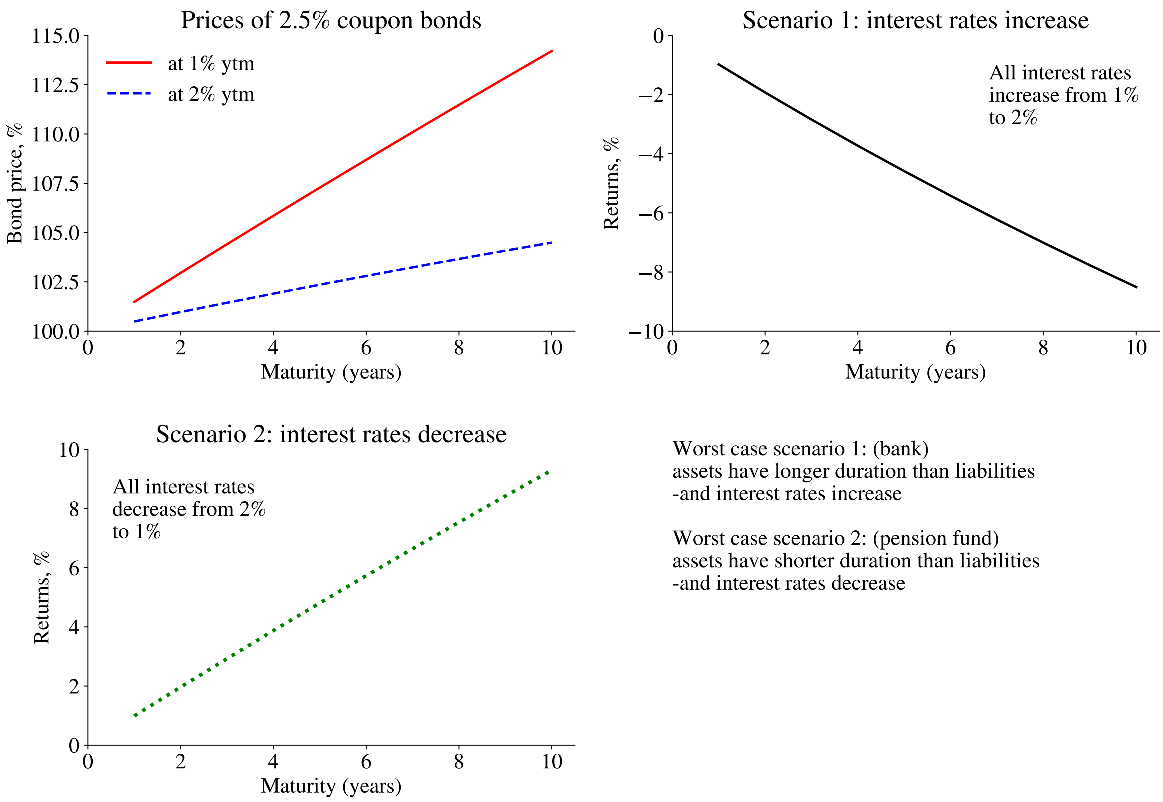


Figure 17.1: Gains and losses at interest rate changes

17.2 Duration: Definitions

The "duration" of a coupon bond is used to analyse how the bond price will change in response to changes in the yield curve. This section gives the definitions of the most commonly used duration measures.

Consider a bond portfolio with the cash flows cf_k as illustrated in Figure 17.2. Recall

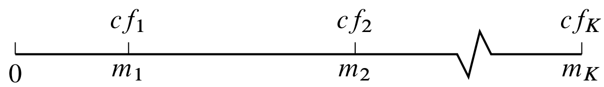


Figure 17.2: Timing convention of bond portfolio

that the price P and the yield to maturity θ are related according to

$$P = \sum_{k=1}^{K} \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.1}$$

The derivative of the price with respect to its yield to maturity is

$$\frac{dP(\theta)}{d\theta} = -\frac{1}{1+\theta} \sum_{k=1}^{K} m_k \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.2}$$

The *dollar duration*, $D^{\$}$, is typically defined as the negative of this derivative

$$D^{\$} = -\frac{dP(\theta)}{d\theta} \tag{17.3}$$

$$= \frac{1}{1+\theta} \sum_{k=1}^{K} m_k \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.4}$$

To calculate the dollar duration $D^{\$}$ you need all the cash flows and the times to them (cf_k and m_k for $k = 1$ to K) and also the yield to maturity (θ). The latter is typically calculated by (numerically) solving (17.1) for θ.

The change of the price, ΔP, due to a small change in the yield, $\Delta\theta$, is approximately

$$\Delta P \approx \frac{dP(\theta)}{d\theta} \times \Delta\theta \tag{17.5}$$

$$= -D^{\$} \times \Delta\theta \tag{17.6}$$

This says that an increase in the interest rate (more precisely, the yield to maturity, θ) translates into a decrease in the bond price—and more so if the duration ($D^{\$}$) is long.

It is common to divide the dollar duration by the price, P, to get the *adjusted (or modified) duration*, D^a,

$$D^a = D^{\$}/P. \tag{17.7}$$

By dividing both sides of (17.6) by the bond price and using the definition of the adjusted duration we see that the relative (percentage) change of the bond price due to a small change in the yield is approximately

$$\frac{\Delta P}{P} \approx -D^a \times \Delta\theta \tag{17.8}$$

It is also common to multiply the dollar duration by $(1+\theta)/P$ to get *Macaulay's* *duration*, D^M,

$$D^M = D^\$(1+\theta)/P \tag{17.9}$$

$$= \sum_{k=1}^{K} w_k m_k, \text{ where } w_k = \frac{cf_k}{(1+\theta)^{m_k} P}. \tag{17.10}$$

Notice that Macaulay's duration is a weighted average of the time to the coupon (and face value) payments $(m_1, m_2, ..., m_K)$. The weight w_k is the fraction of the bond price accounted for by the payment in m_k. The weights sum to unity.

Macaulay's duration is therefore an average "time to payment" of the bond. For instance, for a zero coupon bond, Macaulay's duration is the time to maturity. For bonds with coupons, Macaulay's duration is less than the time to maturity—and this effect is more pronounced at high coupon rates and at high yields to maturity. This is illustrated in Figure 17.3.

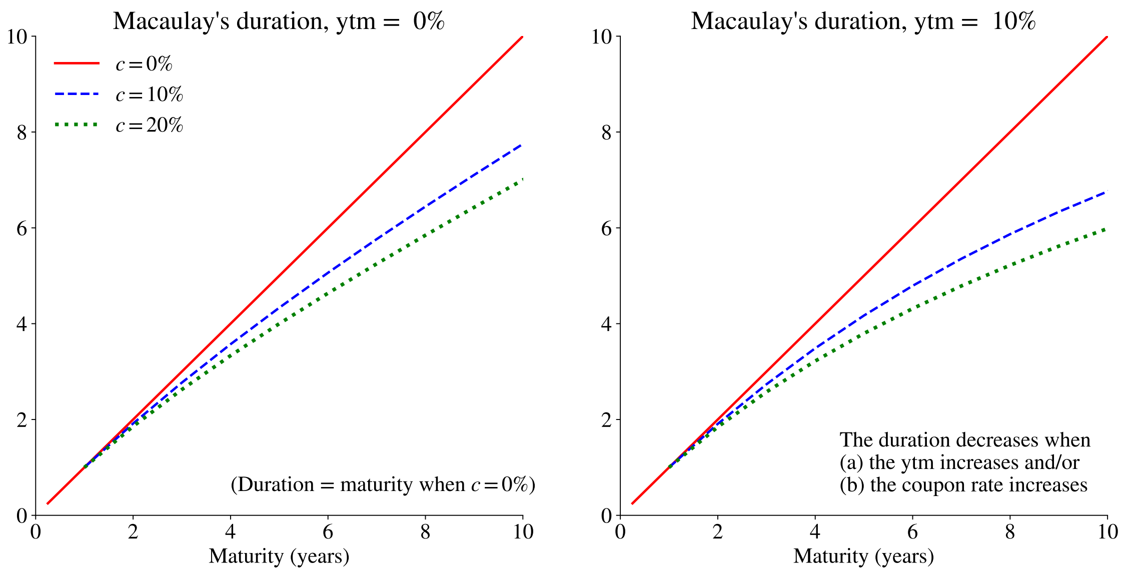


Figure 17.3: Macaulay's duration

Example 17.2 *(Duration) The liability in Example 17.1 has a ytm of 5% if all interest rates are 5%. The dollar duration is*

$$D^\$ = \frac{1}{1.05}\sum_{k=1}^{10} k \frac{0.2}{1.05^k} = 7.5$$

and Macaulay's duration is

$$D^M = \sum_{k=1}^{10} k \frac{0.2}{1.05^k \times 1.54} = 5.1.$$

By multiplying both sides of (17.6) by $(1+\theta)/P$ and using the definition of Macaulay's duration we see that the relative (percentage) change of the bond price due to a small relative (percentage) change in the yield is approximately

$$\frac{\Delta P}{P} \approx -D^M \times \frac{\Delta\theta}{1+\theta}. \tag{17.11}$$

The term last term, $\Delta\theta/(1+\theta)$, is the relative change in the gross yield—since $\Delta\theta = \Delta(1+\theta)$.

Example 17.3 *(Approximate price change) When the ytm changes from 5% to 3%, then (17.11) says that the liability in Examples 17.1 and 17.2 has a relative value change*

$$\frac{\Delta P}{P} = -5.1 \times \frac{-0.02}{1.05} \approx 0.097.$$

From Example 17.1, we know that the exact change is $(1.71 - 1.54)/1.54 = 0.105$.

17.2.1 Duration in Special Cases*

Remark 17.4 *(Duration of a zero coupon bond) For a zero-coupon bond with a face value of unity and maturity of m, the price is* $B = 1/(1+\theta)^m$, *where* θ *is the yield to maturity. The duration measures are*

$$D^\$ = \frac{m}{1+\theta} B,\ D^a = \frac{m}{1+\theta}, \text{ and } D^M = m.$$

In particular, Macaulay's duration is the same as the maturity.

The duration of a portfolio of bonds can be calculated by the general formulas above. However, there are easier ways when the bonds have the same ytm—which is an unlikely case. Still, the result is summarised in the next proposition.

Proposition 17.5 *(Duration of a portfolio) If the yield to maturities of bond* i *and* j *(with prices denoted by* P_i *and* P_j*) are the same, then a portfolio of both bonds has the dollar duration* $D_i^\$ + D_j^\$$ *and the Macaulay's duration* $P_i/(P_i+P_j)D_i^M + P_j/(P_i+P_j)D_j^M$ *(the value weighted average of the different Macaulay's durations). If the ytms are different, this does not hold.*

Proof. (Duration of a portfolio) The first part of the proposition is intuitive since the dollar duration is linear in the cash flows, see (17.4). For the second part of the proposition, multiply the dollar duration $D_i^\$ + D_i^\$$ by (1+ytm) and divide by the portfolio value $(P_i + P_j)$. This is Macaulay's duration of the portfolio. Now, rewrite by using $D^\$ = PD^M/(1+\theta)$ to get the result in the proposition. ∎

17.3 Using Duration to Improve the Hedging of a Bond Portfolio

17.3.1 Basic Setup

Suppose we want to hedge a liability.

A liability is the same as being short one unit of a bond (portfolio) with price P_L. We will hedge this portfolio by buying v units of a bond portfolio (denoted H) with price P_H. The value of the overall position is then

$$V = vP_H + M - P_L, \tag{17.12}$$

where M is a short-term money market account. The choice of M is typically such that the initial (on the first day of the hedging) value of V is zero. The subsequent value of the money market account will change as payments are made/received and the valuation of the bonds change (the positions are marked-to-market every day). The portfolio will typically have to be rebalanced over time in order to stay hedged.

In a first step, we choose which bond portfolio to use as bond H. Choosing a bond portfolio with a duration similar to the liability is typically a good idea. In a second step, we find v so that vP_H and P_L are equally sensitive to the main risk: changes in the general interest rate level.

One way of hedging is to hold a bond portfolio so as to *match every cash flow* of the liability, so portfolio L and H are identical (and $v = 1$ and $M = 0$). However, that may well be both difficult and costly (transaction costs). The following analysis will therefore focus on a case where we buy some simpler bond portfolio H to use as a hedge.

Example 17.6 *(Cash flow matching) To match each cash flow of the liability in Example 17.1, we need to buy 0.2 1-year zero coupon bonds, 0.2 2-year zero coupon bond etc.*

Remark 17.7 *(Overall portfolio value over several subperiods*) Start by creating a portfolio with a zero initial value*

$$0 = v_t P_{H,t} + M_t - P_{L,t}, \text{ so } M_t = 0 - v_t P_{H,t} + P_{L,t},$$

where M_t is the amount held in a money market account (almost zero duration) with an interest rate Y_t. In $t+1$ (say, one day later), this portfolio is worth

$$V_{t+1} = v_t(P_{H,t+1} + cf_{H,t+1}) + M_t(1+Y_t)^m - (P_{L,t+1} + cf_{L,t+1}),$$

where $cf_{H,t+s}$ and $cf_{L,t+s}$ are the coupon payments (or any other cash flows) and the bond prices are measured after coupons. Notice that the interest factor is $(1+Y_t)^m$ where Y_t is the effective interest rate and m is the length of time between t and $t+1$ ($m = 1/365$ for days, $m = 1/12$ for months). After rebalancing in $t+1$, we need v_{t+1} units of bond H and we are still short one bond L, so the balance on the money market account is

$$M_{t+1} = V_{t+1} - v_{t+1}P_{H,t+1} + P_{L,t+1}.$$

This is similar to the expression for M_t in the first equation, except that V_{t+1} may be non-zero. The value of the portfolio in $t+s$ is computed as in the second equation, but with subscripts advanced one period.

Using the approximate relation of the bond price change (17.6), we have that the change of value (due to a sudden change in the interest rates) of the overall position is

$$\Delta V = v\Delta P_H - \Delta P_L \tag{17.13}$$

$$\approx -vD_H^M P_H \times \frac{\Delta\theta_H}{1+\theta_H} + D_L^M P_L \times \frac{\Delta\theta_L}{1+\theta_L}, \tag{17.14}$$

where the durations are Macaulay's duration.

Several of the hedging approaches (discussed below) assume that $\Delta\theta_H = \Delta\theta_L$, that is, a parallel shift of the yield curve. The weakness of that assumption is also discussed below.

17.3.2 Duration Matching

Suppose we choose a hedge bond with the same duration at the liability ($D_H^M = D_L^M$) and that we invest the same amount in the hedge bond as the value of the liability ($vP_H = P_L$). This means that the initial position on the money market account is zero.

Suppose the yield curve shifts up in a parallel fashion (so $\Delta\theta_L/(1+\theta_L) = \Delta\theta_H/(1+\theta_H)$) as in Figure 17.4. Using (17.14) then gives

$$\frac{\Delta V}{P_L} \approx 0, \tag{17.15}$$

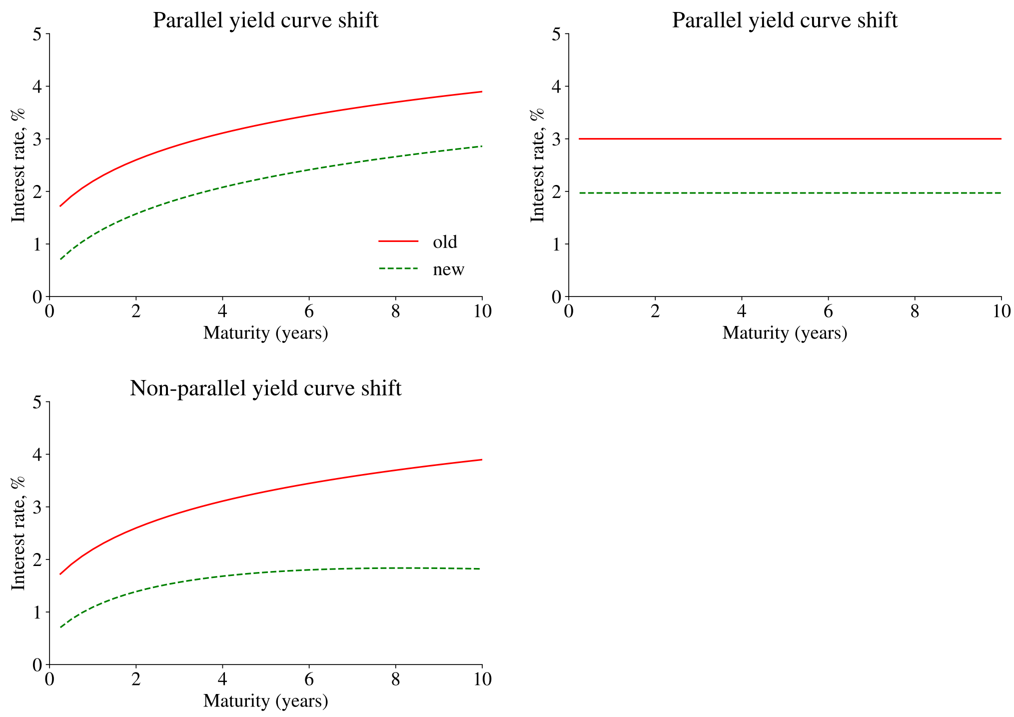


Figure 17.4: Yield curve shifts

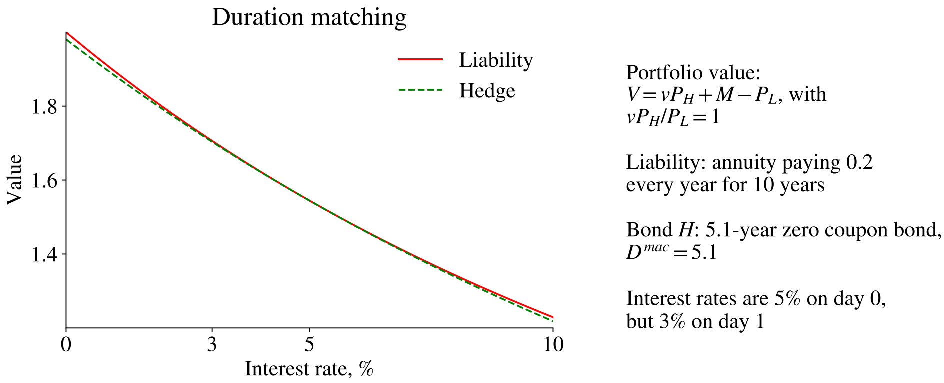
so the duration hedge makes the overall portfolio (approximately) immune to interest rate changes.

As interest rates change, the duration does too. This means that a hedge bond that was suitable (had the same duration as the liability) in t may not be suitable in a later period. The portfolio needs to be rebalanced over time.

Example 17.8 *(Duration matching) Figure 17.5 illustrates a case where a liability stream is hedged by a (here, zero-coupon) bond with the same duration. This appears to give a very precise hedge (the value of V stays very close to zero). Notice, however, that the duration of the liability changes as the interest rates do, so we must rebalance to be immune to further interest rate changes.*

17.3.3 Naive Hedging

Suppose we just set $v = P_L/P_H$, so the amount invested in bond H equals the value of our liability, but we do not pay any attention to the durations. This means that the initial position on the money market account is zero.



	θ	D_L^M	P_H	v	vP_H	M	$-P_L$	V	vP_H/P_L
Day 0	0.05	5.10	0.78	1.98	1.54	0.00	-1.54	0.00	1.00
Day 1 (m-to-m)	0.03	5.26	0.86	1.98	1.70	0.00	-1.71	-0.00	1.00

Rebalancing here requires switching hedge bond (or applying a duration hedge)

Figure 17.5: Example of duration matching. "m-to-m" stands for the marking-to-market stage

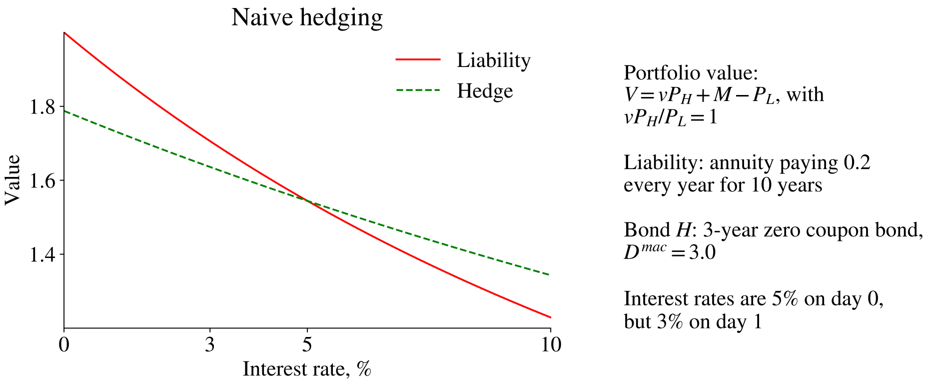
Using (17.14) and assuming that the yield curve shifts up in a parallel fashion (so $\Delta\theta_L/(1+\theta_L) = \Delta\theta_H/(1+\theta_H)$) gives

$$\frac{\Delta V}{P_L} \approx (D_L^M - D_H^M) \times \frac{\Delta\theta}{1+\theta}. \tag{17.16}$$

For instance, suppose interest rates decrease ($\Delta\theta < 0$) and the duration of the liability is longer than of the hedge bond ($D_L^M > D_H^M$). Then, the portfolio will lose money. See Figure 17.6 for an example. The reason is that the value of the liability goes up more than the value of the hedge bond—as longer bonds are more sensitive to interest rate changes than short bonds.

Example 17.9 *(Naive hedging) Figure 17.6 shows a case of naive hedging when we have a duration mismatch. This makes the overall portfolio (V) sensitive to interest rate changes. In this case, interest rates decrease (from 5% to 3%) so the liability increases more in value than the hedge bond (which has too low duration—and is thus not sufficiently sensitive to interest rate changes). We face losses. In terms of (17.16),we have* $D_L^M - D_H^M > 0$ *and* $\Delta\theta < 0$.

Remark 17.10 *(Effect of yield curve shift on a bank) A bank typically has liabilities with short duration (deposits, inter-bank lending) and assets with long duration (loans to companies and households), so* $D_L^M - D_H^M < 0$. *Equation (17.16) shows that an increase in*



	θ	D_L^M	P_H	v	vP_H	M	$-P_L$	V	vP_H/P_L
Day 0	0.05	5.10	0.86	1.79	1.54	0.00	-1.54	0.00	1.00
Day 1 (m-to-m)	0.03	5.26	0.92	1.79	1.64	0.00	-1.71	-0.07	0.96
Day 1 (rebal)	0.03	5.26	0.92	1.86	1.71	-0.07	-1.71	-0.07	1.00

Figure 17.6: Example of naive hedging

the interest rate level will hurt the bank ($D_L^M - D_H^M < 0$ *and* $\Delta\theta > 0$) *since the assets decrease more than the liabilities. This can also be phrased as follows: the bank has fixed incomes from the loans it has made, but it now needs to refinance itself (deposits and inter-bank loans) at a higher cost.*

17.3.4 Duration Hedging

Instead of the naive hedge, suppose we instead choose

$$v = \frac{D_L^M}{D_H^M} \times \frac{P_L}{P_H}, \text{ so} \tag{17.17}$$

$$\frac{vP_H}{P_L} = \frac{D_L^M}{D_H^M}. \tag{17.18}$$

Again, consider the case of $D_L^M > D_H^M$. The second equation shows that the *amount* invested into the hedge bond (vP_H) now exceeds the value of the liability. In this way, we compensate for the hedge bond's lower interest rate sensitivity by having a larger exposure to it. The initial position on the money market account is typically nonzero. As in the other cases, the portfolio typically needs to be rebalanced over time.

Combine (17.14) and the hedge ratio (17.17) to get

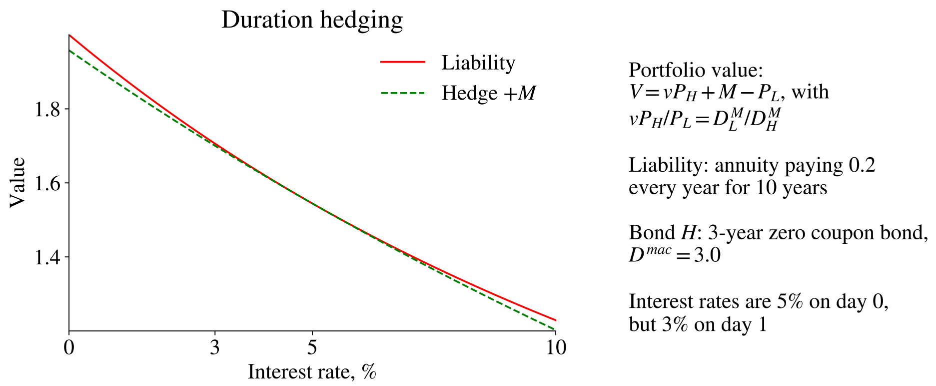
$$\frac{\Delta V}{P_L} \approx D_L^M \times \left(\frac{\Delta\theta_L}{1+\theta_L} - \frac{\Delta\theta_H}{1+\theta_H} \right). \tag{17.19}$$

Suppose the yield curve shifts up in a parallel fashion (so $\Delta\theta_L/(1+\theta_L) = \Delta\theta_H/(1+\theta_H)$). Then, (17.19) shows that the overall portfolio value will not change ($\Delta V/P_L \approx 0$). See Figure 17.7 for an example how the duration hedging works.

The intuition is that the price of a bond with long duration is more sensitive to a yield curve shift than the price of a short bond. Therefore, to hedge a bond with a long duration (as bond L) we need to buy more of the bond with a short duration (bond H).

Example 17.11 *(Duration hedging) Figure 17.7 illustrates a case where we have a duration mismatch (similar to the case of naive hedging), but where this compensated for by a hedge ratio that takes the mismatch into account. The hedge bond has a too low duration, we therefore take a larger position in it (the amount invested into the hedge bond, vP_H, is much larger than the value of the liability)—so as to increase the interest rate sensitivity of the position.*

See Figures 17.8–17.9 for an empirical illustration.



	θ	D_L^H	P_H	v	vP_H	M	$-P_L$	V	vP_H/P_L
Day 0	0.05	5.10	0.86	3.04	2.62	-1.08	-1.54	0.00	1.70
Day 1 (m-to-m)	0.03	5.26	0.92	3.04	2.78	-1.08	-1.71	-0.01	1.63
Day 1 (rebal)	0.03	5.26	0.92	3.27	2.99	-1.29	-1.71	-0.01	1.75

Figure 17.7: Example of duration hedging

Remark 17.12 *(Using the dollar duration instead*) Recall that $D^M = D^\$(1+\theta)/P$), so (17.14) can be rewritten as*

$$\Delta V \approx -vD_H^\$ \times \Delta\theta_H + D_L^\$ \times \Delta\theta_L.$$

Set $\Delta V = 0$ *to get the hedge ratio* $v = \frac{D_L^\$}{D_H^\$} \times \frac{\Delta\theta_L}{\Delta\theta_H}$*.If we assume that both yields change equally much, then* $v = \frac{D_L^\$}{D_H^\$}$.

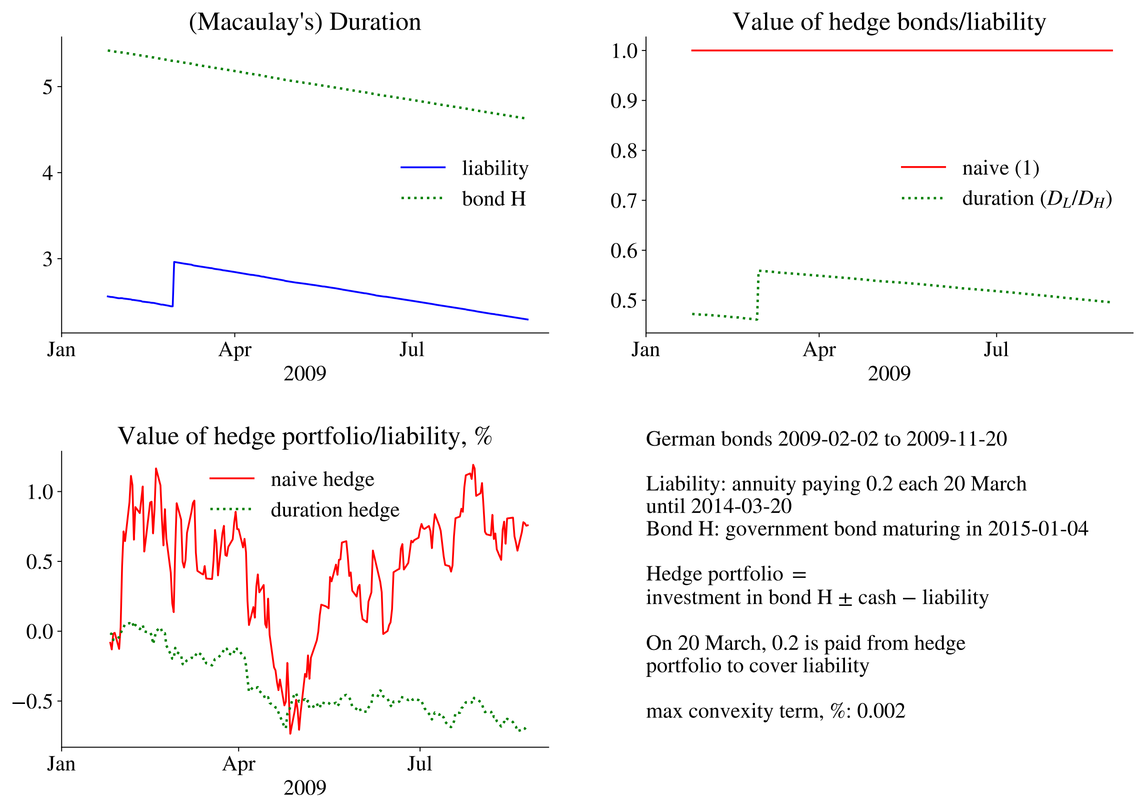


Figure 17.8: Duration hedging

17.4 Problems with Duration Hedging

17.4.1 Problem 1: Approximation Error

The formula for the price change (17.6) is only exact for infinitesimal yield changes—and the approximation error could potentially be large when the yield changes are substantial.

The formula is really a first-order Taylor approximation of the form

$$\Delta P \approx \frac{dP}{d\theta} \times \Delta\theta. \tag{17.20}$$

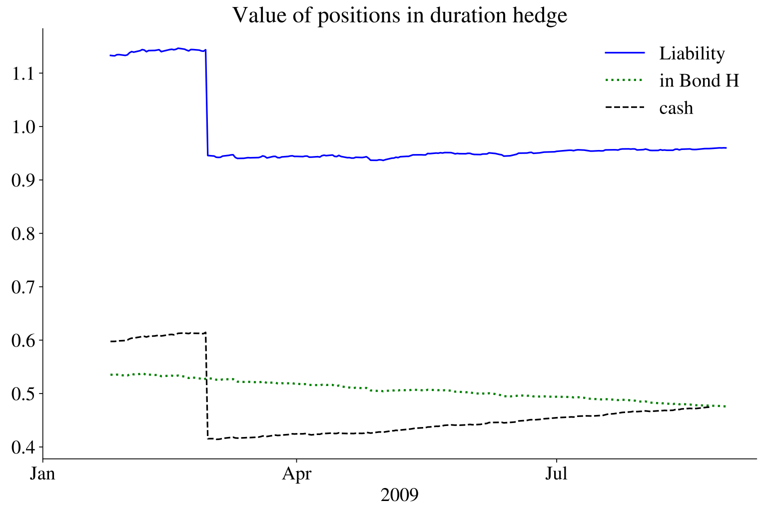


Figure 17.9: Duration hedging

Obviously, a second-order Taylor approximation is more precise. It would be

$$\Delta P \approx \frac{dP}{d\theta} \times \Delta\theta + \frac{1}{2}\frac{d^2 P}{d\theta^2} \times (\Delta\theta)^2. \tag{17.21}$$

where the last term includes the second derivative of the bond price with respect to the yield to maturity. The second derivative is easily calculated to be

$$\frac{d^2 P}{d\theta^2} = \textstyle\sum_{k=1}^{K} m_k(m_k+1)\frac{cf_k}{(1+\theta)^{m_k+2}}. \tag{17.22}$$

Dividing (17.21) by the bond price and using (17.8) gives

$$\frac{\Delta P}{P} \approx -D^M \times \frac{\Delta\theta}{1+\theta} + \frac{1}{2}C \times (\Delta\theta)^2, \tag{17.23}$$

where C (often called "convexity") is the second derivative in (17.21) divided by the bond price. It can be shown that the convexity is positive, but decreasing in the coupon rate — for a given ytm and maturity. (The convexity is actually increasing in the coupon rate for a given ytm and modified duration.) See Figure 17.7 for an illustration of the non-linear effect.

Choosing the hedging bond (portfolio) so that it has a similar convexity to the bond to be hedged may make the hedge more precise.

Example 17.13 *(Convexity) The convexity of the 10-year n bond in Example 17.1 is*

(when interest rates are 2%)

$$C = \frac{1}{1.54}\sum_{k=1}^{10} k(k+1)\frac{0.2}{1.05^{k+2}} \approx 35.6.$$

If interest rates increase from 5% to 3%, then the second-order term in (17.23) is

$$\frac{35.6}{2} \times 0.02^2 = 0.007,$$

which is fairly small compared to the duration effect (see Example 17.3).

17.4.2 Problem 2: Changing Cash Flows

The duration measures assume that the times when the coupons and the face value are paid are unaffected by the yield change. That is true for many instruments (like most government bonds), but not for callable bonds—and effectively not for bonds whose risk premium depends on the interest rate level as most corporate bonds do (as the interest rate level affects the default risk).

17.4.3 Problem 3: Yield Curve Changes vs. Changes in Yields to Maturity

The probably most important problem with using duration for hedging is that the hedge ratio in (17.24) depends on how the yields change—and that is not known when we construct the hedging portfolio.

The ideal case for duration hedging is when the yields (to maturity) move in parallel. This will be the case, for instance, if the yield curve is flat (across maturities)—and the only movements are parallel shifts up and down. In reality, *most* (but not all) movements in the yield curve are parallel. Often the short interest rates move more (in response to news) than long rates.

Equation (17.19) shows how the value of the overall portfolio depends on the yields of the liability and the hedge bond. For instance, suppose the yield curve changes from being flat to being downward sloping and the hedging bond has shorter duration than the liability. In this case, the overall portfolio loses value. The reason is that the value of the hedging portfolio increases less (the yield decreases less) in price than the liability. See Figure 17.4 for an illustration.

To overcome this problem, the hedge ratio should be (set $\Delta V = 0$ in (17.14))

$$v = \frac{D_L^M}{D_H^M} \times \frac{P_L}{P_H} \times \frac{\Delta\theta_L/(1+\theta_L)}{\Delta\theta_H/(1+\theta_H)}. \qquad (17.24)$$

This is indeed the same as the duration hedging (17.17) if all changes of the yield curve are parallel shifts (last term in (17.24) is unity). However, the relative frequencies of the yield curve movements (level, slope, curvature) seem to change over time (according to business cycle conditions and monetary policy regime). This suggests that the ability of a simple duration matching to provide a hedge is different in different time periods and different markets.

Explicit models of how the entire yield curve moves in response to a small number of factors have implications for how the two yields will change—which may vary across instruments and time. This would allow us to also model how the last term in (17.24) would react to the drivers of the yield curve—and this provide a more precise hedge ratio.

Chapter 18

Interest Rate Models

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9

18.1 Empirical Properties of Yield Curves

Yield curves (in the US and most other developed countries) tend to have the following features (see Figures 18.1–18.2 for some examples).

First, most of the time, the yield curve is upward sloping. This is only consistent expectations hypothesis if short rates are expected to be higher in the future. This means that short rates should (most of the time) be increasing over time—which contradicts empirical evidence. It is more likely that long rates tend to be high because of risk premia.

Second, the yield curve changes over time. It is common to describe the movements in terms of three "factors": level, slope, and curvature. One way of measuring these factors is by defining

$$\begin{aligned}
\text{Level}_t &= y_{10y} \\
\text{Slope}_t &= y_{10y} - y_{3m} \\
\text{Curvature}_t &= (y_{2y} - y_{3m}) - (y_{10y} - y_{2y}).
\end{aligned} \tag{18.1}$$

This means that we measure the level by a long rate, the slope by the difference between a long and a short rate—and the curvature (or rather, concavity) by how much the medium/short spread exceeds the long/medium spread. For instance, if the yield curve is hump shaped (so y_{2y} is higher than both y_{3m} and y_{10y}), then the curvature measure is positive. In contrast, when the yield curve is U-shaped (so y_{2y} is lower than both y_{3m} and y_{10y}), then the curvature measure is negative. See Figure 18.3 for an example.

Most evidence on US data suggests that changes in the level dominate—perhaps accounting for 80–90% of the total variation in yields. The slope comes second (perhaps accounting for 10%), and hump third (accounting for a few percent).

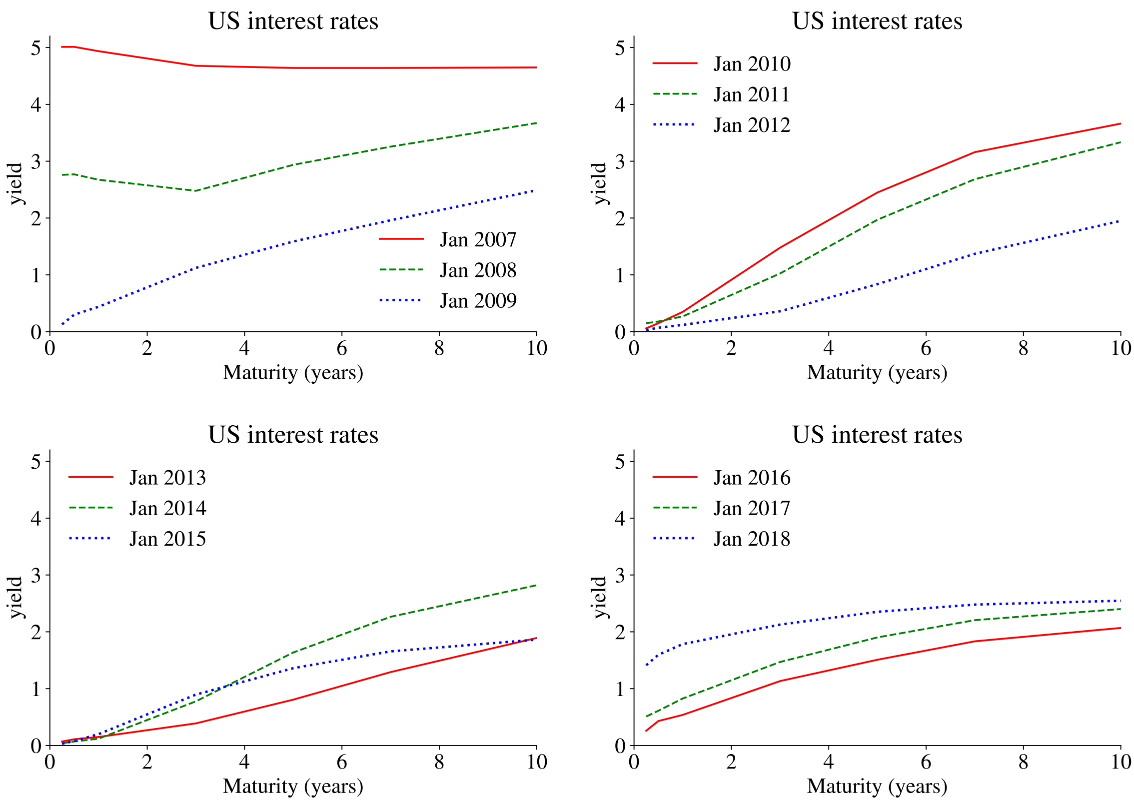


Figure 18.1: Estimated yield curves

Interest rates are strongly related to business cycle conditions, so it often makes sense to include macro economic data in the modelling. See Figure 18.4 for how the term spreads are related to recessions: the term spreads are often very small (or even negative) at the beginning of recessions and then increase towards the end of recessions.

18.2 Yield Curve Models

Remark 18.1 *(Different time scales*) With monthly data it is natural to think of time periods as months, that is, there is a month between t and $t+1$, so n periods mean n months. In contrast, most bond (and option) pricing has the convention that periods are years. A conversion is therefore required. (Alternatively, we could use a much heavier*

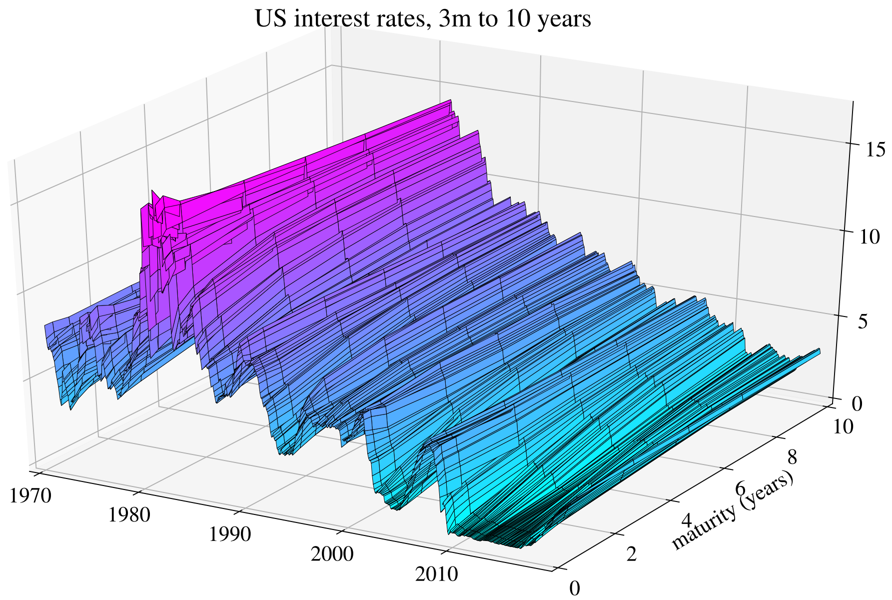


Figure 18.2: US yield curves

notation for the data to account for that it is sampled at $1/12$ *years apart. We avoid that here.)*

18.2.1 The Expectations Hypothesis of Interest Rates

The expectations hypothesis of interest rates says that long bonds have no, or possibly constant, risk premia. The empirical evidence is mixed, so the expectations hypothesis is best thought of as a rough, although convenient, approximation.

The expectations hypothesis says that the n-period spot rate equals the average of the 1-period (the shortest maturity) rates over t to $t+n$

$$y_t(n) = \lambda(n) + \frac{1}{n}\textstyle\sum_{s=0}^{n-1} \mathrm{E}_t\, r_{t+s}, \tag{18.2}$$

where r_t is short hand notation for the 1-period rate. The period length (from t to $t+1$) should correspond to the maturity of the short interest rate. For instance, if r_t is a 1-month rate, then we are working with monthly periods so $y_t(120)$ means today's 120-month (10 year) interest rate. This requires some care when using $y_t(n)$ in bond pricing formulas (see below for a discussion).

Example 18.2 *(The expectations hypothesis) Suppose the* $(r_t, \mathrm{E}_t\, r_{t+1}, \mathrm{E}_t\, r_{t+2}) = (3\%, 2\%, 2\%)$

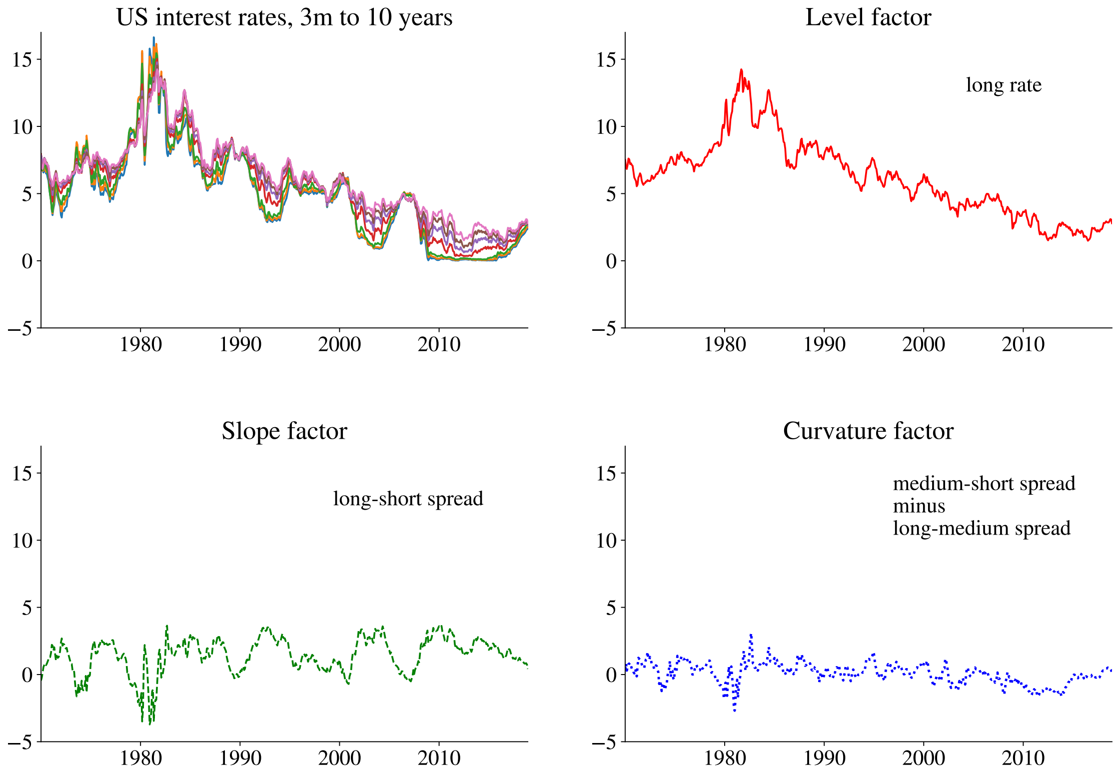


Figure 18.3: US yield curves: level, slope and curvature

are the expected 1-month rates, then the 3-month rate is $\lambda(3) + 2.33\%$.

The expectations hypothesis allows for constant risk premia ($\lambda(n) \neq 0$), which may differ across maturities (n). If $\lambda(n) = 0$, then the *pure* expectations hypothesis is said to hold. See Figure 18.5 for an illustration.

18.2.2 Risk Premia

There are several reasons for why bonds should have risk premia. First, long bonds are risky for investors who do not intend to keep them until maturity—and will therefore have term premia. Second, some bonds are not traded much (for instance, off-the-run bonds and many index-linked bonds)—so they are likely to have liquidity premia. Third, the real return of a long bond is very sensitive to inflation changes—probably more than equity. Bonds are therefore likely to have inflation risk premia.

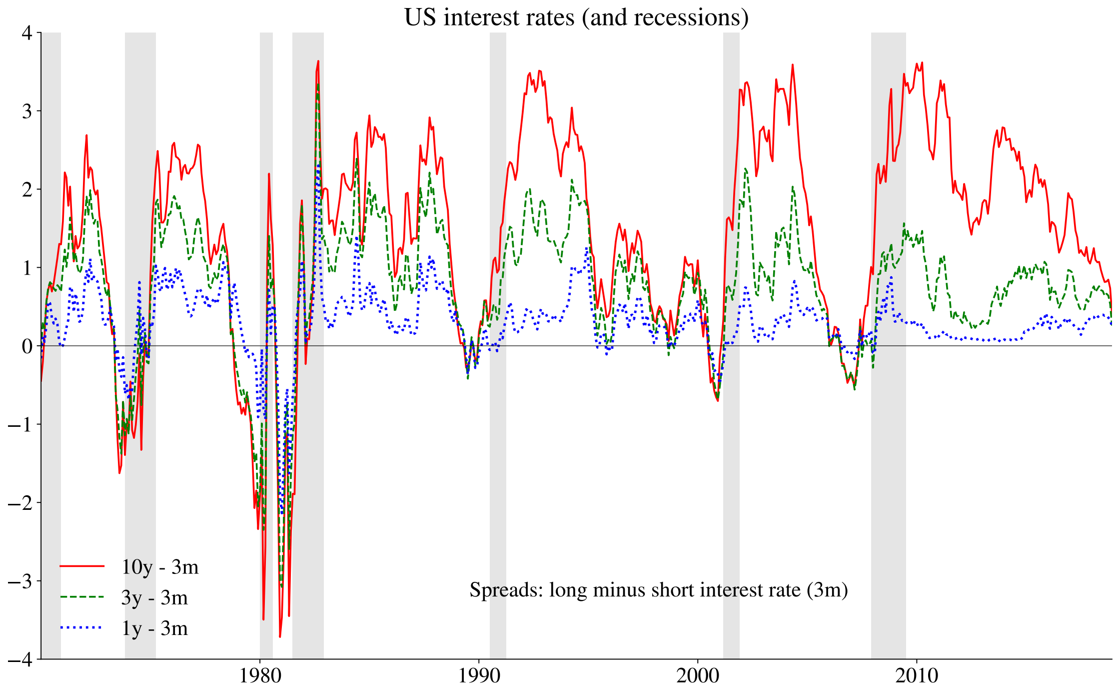


Figure 18.4: US term spreads (over a 3m T-bill)

18.2.3 A Simple One-Factor Model: The Vasicek Model

The Vasicek model assumes that the short interest rate is an AR(1). To simplify, I will crudely assume that there are some unspecified constant risk premia. The more general formulation (not used here) derives the risk premia in terms of the mean reversion and volatility of the short rate.

To simplify the notation, let the short rate, r_t, follow an AR(1)

$$r_{t+1} - \mu = \rho (r_t - \mu) + \varepsilon_{t+1}, \tag{18.3}$$

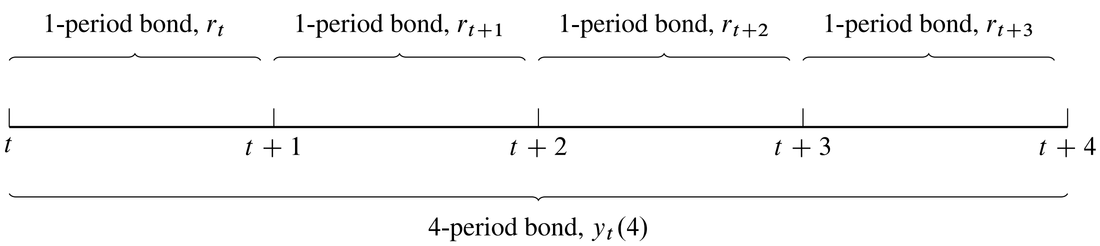


Figure 18.5: Timing for expectations hypothesis

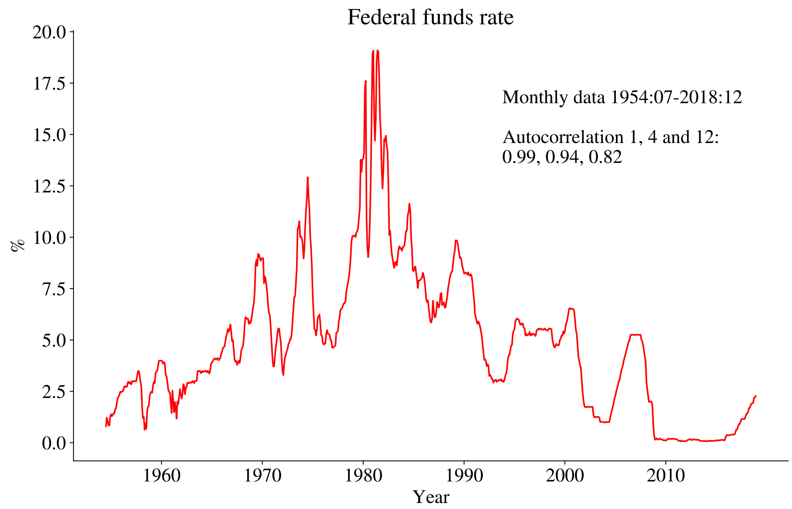


Figure 18.6: Federal funds rate, monthly data

where μ is the mean. Notice that the short interest rate r_t is typically annualized (even if it is a 1-month rate). Thus, μ is the average (annualized) short interest rate, and ρ is the correlation of r_t and r_{t-1}. See 18.6 for an illustration.

Remark 18.3 *(Alternative formulation of (18.3)*) The process is sometimes specified in terms of changes as*

$$r_{t+1} - r_t = a\,(\mu - r_t) + \varepsilon_{t+1}.$$

Clearly, this can be written

$$r_{t+1} - \mu = (1 - a)\,(r_t - \mu) + \varepsilon_{t+1},$$

where $1 - a$ *corresponds to* ρ*. With* $0 < a < 1$ *(that is, with* $0 < \rho < 1$*) the process is mean reverting.*

The forecast for $t + s$ is

$$\mathrm{E}_t\, r_{t+s} = (1 - \rho^s)\,\mu + \rho^s r_t. \tag{18.4}$$

Notice that when r_t is a 1-month rate, then (18.4) is today's expectation of the 1-month rate s months ahead. See Figure 18.7.

Example 18.4 *(Predictions from an AR(1)) With* $\mu = 0.05$*,* $\rho = 0.975$ *and* $r_t = 0.07$*, then* $\mathrm{E}_t\, r_{t+50} = 0.72 \times 0.05 + 0.28 \times 0.07 = 0.056$.

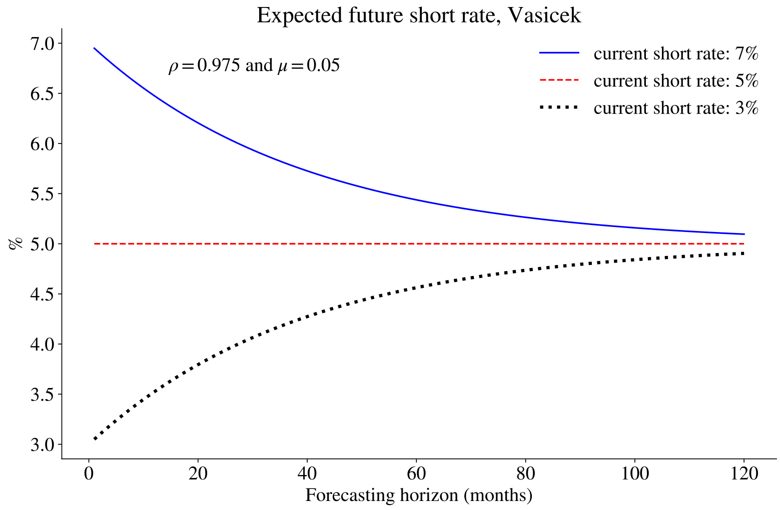


Figure 18.7: Expected future short rate in Vasicek model, for different initial short rates

Remark 18.5 *(Calibrating the AR(1) to data*) Notice that (18.3) implies that* $\text{Corr}(r_t, r_{t-s}) = \rho^s$*, so we could thus estimate* ρ *by* $\text{Corr}(r_t, r_{t-s})^{1/s}$*. If the AR(1) is a very good fit to data, then it should not matter (much) if you use* $s = 1$ *or* $s = 12$ *(say). In practice, the results may well differ. For instance, suppose monthly data gives* $\text{Corr}(r_t, r_{t-1}) = 0.99$ *but* $\text{Corr}(r_t, r_{t-12}) = 0.80$*, which imply* $\rho = 0.99$ *and* $\rho = 0.982$ *respectively. This matters for the pricing of long-maturity bonds: with 120 months (10 years) we get* $0.99^{120} = 0.3$ *while* $0.98^{120} = 0.09$*. Which value we choose to use depends on whether we are most interested in the short maturities (use* $\rho = 0.99$*) or the long maturities (use* $\rho = 0.982$*).*

We now assume that the expectations hypothesis holds for continuously compounded rates. Using this in (18.2) gives the long interest rate. For instance, the two period (annualized, continuously compounded) rate is

$$\begin{aligned} y_t(2) &= \lambda(2) + \frac{1}{2}\left[r_t + (1-\rho)\,\mu + \rho r_t\right] \\ &= \lambda(2) + \mu\,(1-\rho)\,/2 + r_t\,(1+\rho)\,/2. \end{aligned} \tag{18.5}$$

The general expression for a maturity of n periods is

$$\begin{aligned} y_t(n) &= a(n) + b(n) r_t, \text{ where} \\ a(n) &= \lambda(n) + \mu\left[1 - b(n)\right] \text{ and} \\ b(n) &= (1 + \rho + \ldots + \rho^{n-1})/n = (1-\rho^n)/[(1-\rho)n]. \end{aligned} \tag{18.6}$$

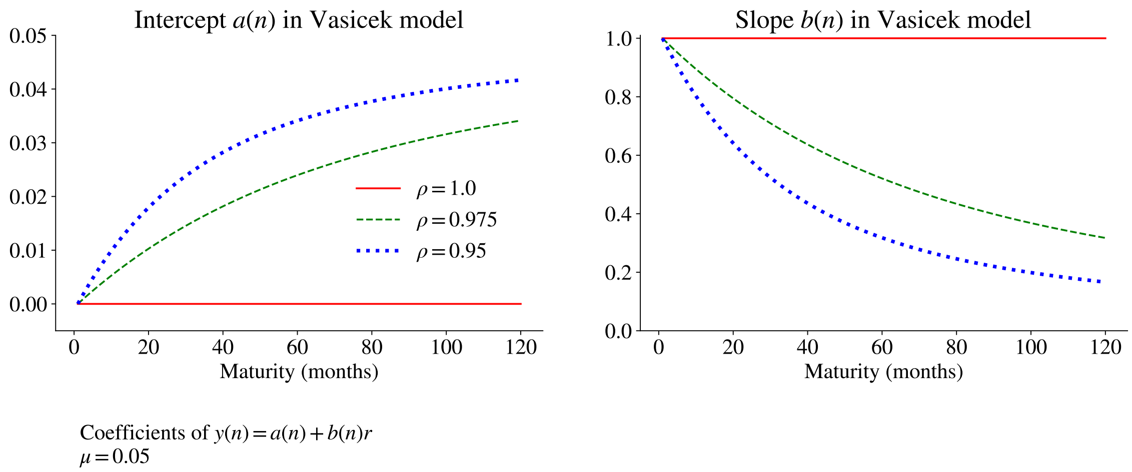


Figure 18.8: Intercept and slope in the Vasicek model

Again, notice that the period length is defined by the maturituty of the short rate. For instance, when r_t is a 1-month rate, $y_t(120)$ is a 120-month (10 year) rate.

In this model, all movements of the yield curve are driven by the short rate, so it is a *one-factor model*. The shifts of the yield curve are parallel if $\rho = 1$ (the random walk model) since then $b(n) = 1$ in (18.6), so we get

$$y_t(n) = \lambda(n) + r_t, \text{ if } \rho = 1. \tag{18.7}$$

For lower values of ρ, the short rate process is mean-reverting, so the expected future short rates (and therefore the current long rates) are always closer to the mean than the current short rate. See Figures 18.8–18.9 for an illustration.

Example 18.6 *(Vasicek model) For $\rho = 0.975$ and $\mu = 0.05$, (18.6) gives (assuming no risk premia)*

$$\begin{bmatrix} y_t(1) \\ y_t(2) \\ y_t(3) \\ y_t(4) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.062 \\ 0.124 \\ 0.184 \end{bmatrix} \frac{1}{100} + \begin{bmatrix} 1 \\ 0.988 \\ 0.975 \\ 0.963 \end{bmatrix} r_t.$$

18.3 The Vasicek Model: Hedging a Bond

The Vasicek model allows us to calculate (or rather estimate) the proper way of *hedging a bond portfolio*.

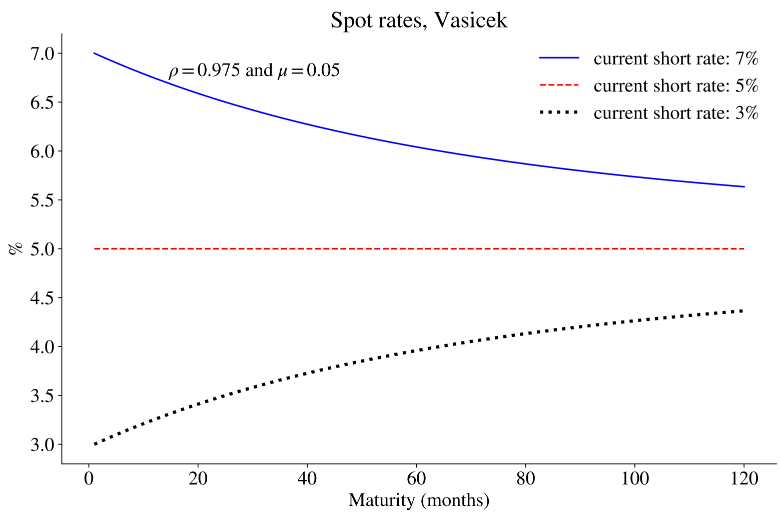


Figure 18.9: Vasicek model, spot rates for different initial short rates

The change of the hedge portfolio is

$$\Delta V = v\Delta P_H - \Delta P_L, \tag{18.8}$$

and a bond price can be calculated as

$$P = \sum_{k=1}^{K} \frac{cf_k}{\exp[m_k y(m_k)]}, \tag{18.9}$$

where cf_k is the cash flow at $t + m_k$ and $y(m_k)$ is the continuously compounded interest

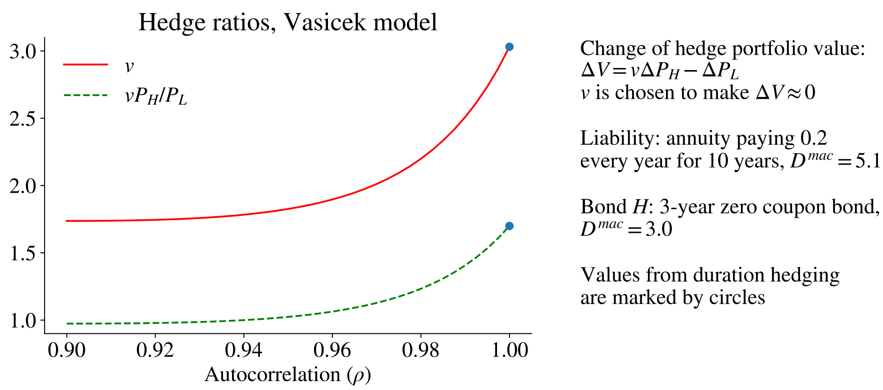


Figure 18.10: Hedge ratios in the Vasicek model

between t and $t + m_k$. Notice that time is here measured in *years* since the interest rates $y(m_k)$ are annualized rates.

Once we know the parameters of the Vasicek model, it is straightforward to calculate (at least numerically) what ΔP_H and ΔP_L are—as a function of the change in the short rate interest rate (Δr_t). In practice, this can be done by the following steps.

1. For an initial value of the short interest rate r, use (18.6) to calculate all spot rates $y(m_k)$ needed in (18.9). Notice that the periods in the Vasicek model (n) might be shorter than years. For instance, if $m_k = (0.5, 1, 10)$ years but the Vasicek model is for monthly data, then you calculate $y(n)$ for $n = (6, 12, 120)$ months according to (18.6) and then use them for $m_k = (0.5, 1, 10)$ in (18.9).

2. Use the spot interest rates $y(m)$ to calculate the prices of the hedge bond the liability according to (18.9).

3. Redo points 1 and 2, but starting from another short rate.

4. Calculate the difference of the prices at the two different short rates ($\Delta P_H, \Delta P_L$). We then set v so that $\Delta V = 0$, that is, $v = \Delta P_L / \Delta P_H$.

This approach is a way of finding the sensitivity of P_L and P_H to the driver of the yield curve (r), and can thus be interpreted as a kind of "duration."

Remark 18.7 *(Duration hedging with the Vasicek model*) The Vasicek model can also be used to calculate the yield changes in a duration hedge. Recall that the following value (dollars) invested into a hedge bond is (H) relative to the value of the liability (L) should provide a good hedge:*

$$vP_H/P_L = \frac{D_L^M}{D_H^M} \times \frac{\Delta\theta_L/(1+\theta_L)}{\Delta\theta_H/(1+\theta_H)}$$

where D_i^M is Macaulay's duration, θ_i the yield to maturity and P_i the price of bond i. In the typical duration hedge we assume that all yield curve moments are parallel, so the last term in this expression equals one. Follow the same steps as above, but also calculate the durations (only at the initial short interest rate) and the yield to maturities. Then calculate vP_H/P_L according to the equation above. The results are very similar to the easier approach discussed above.

Figure 18.10 gives an illustration. The hedge ratio v converges to the duration hedge ratio as the autocorrelation (ρ) in the short rate process (18.3) increases towards unity: in that limiting case all yield curve movements are indeed parallel. For lower values of the autocorrelation, the hedge ratio is lower. The main reason is that mean-reversion, that is, low autocorrelation makes interest rates on long maturity bonds (here, the liability) move less than interest rates on short bonds. This means that the value of the liability is not that sensitive to the short rate (when ρ is low)—and hence we need not invest so much into the (shorter maturity) hedge bond.

Notice, however, that all one-factor models (not least the Vasicek model) imply that all yields are perfectly correlated (there is a common single driving force) and only fairly limited yield curve movements are possible. *Multi-factor models* overcome most of those limitations. For instance, the model in Nelson and Siegel (1987) is a two-factor model.

18.4 Interest Rates and Macroeconomics*

This section outlines several (not mutually exclusive) macroeconomic approaches to modelling the yield curve.

18.4.1 The Fisher Equation and Index-Linked Bonds

Let π_{t+n} be the one period inflation rate over t to $t+n$ and $y_t^r(n)$ the n-period continuously compounded real interest rate (an interest rate measured in purchasing power terms).

The Fisher equation (here in the form of continuously compounded rates) says that the nominal interest rate includes compensation both for inflation expectations, $\mathrm{E}_t\,\pi_{t+n}$, the real interest rate, $y_t^r(n)$, and possibly a constant (across time) risk premium, $\psi(n)$,

$$y_t(n) = \mathrm{E}_t\,\pi_{t+n} + y_t^r(n) + \psi(n). \tag{18.10}$$

Example 18.8 *(Fisher equation) Suppose the nominal interest rate is* $y(n) = 0.07$*, the real interest rate is* $y^r(n) = 0.03$*, and the nominal bond has no risk premium (*$\psi = 0$*), then the expected inflation is* $\mathrm{E}_t\,\pi_{t+n} = 0.04$.

The same type of relation holds for forward rates. The Fisher equation suggests a framework for analysing nominal interest rates in terms of real interest rates and inflation expectations. This is commonly used for long rates. Information about real interest rates

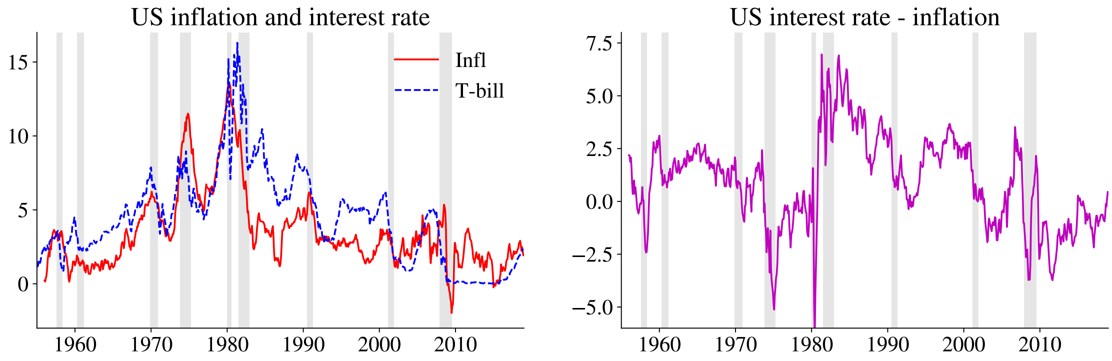


Figure 18.11: US inflation and 3-month interest rate

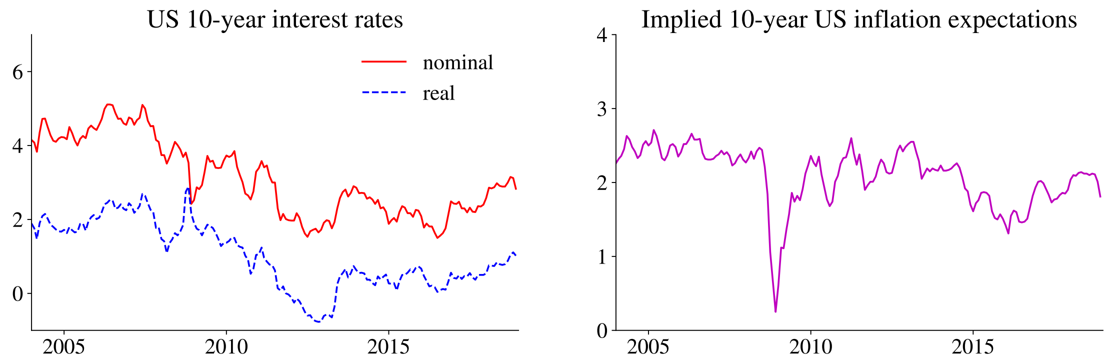


Figure 18.12: US nominal and real interest rates

can be elicited from *index-linked bonds*, that is, bonds which give automatic compensation for actual inflation.

Empirical results typically indicate that there are non-trivial movements in the real interest rate and/or risk premia—especially for short forecasting horizons. This holds also when inflation expectations, as measured by surveys, are used as the dependent variable. Inflation expectations seem to vary by less than the interest rate. It is therefore not straightforward to extract inflation expectations from nominal interest rates.

The Fisher equation could also be embedded in a macro model to construct a sophisticated (and complicated) model of the yield curve. This involves using macro theory/empirics to model how real interest rates and inflation expectations (for different maturities) depend on the state of the economy.

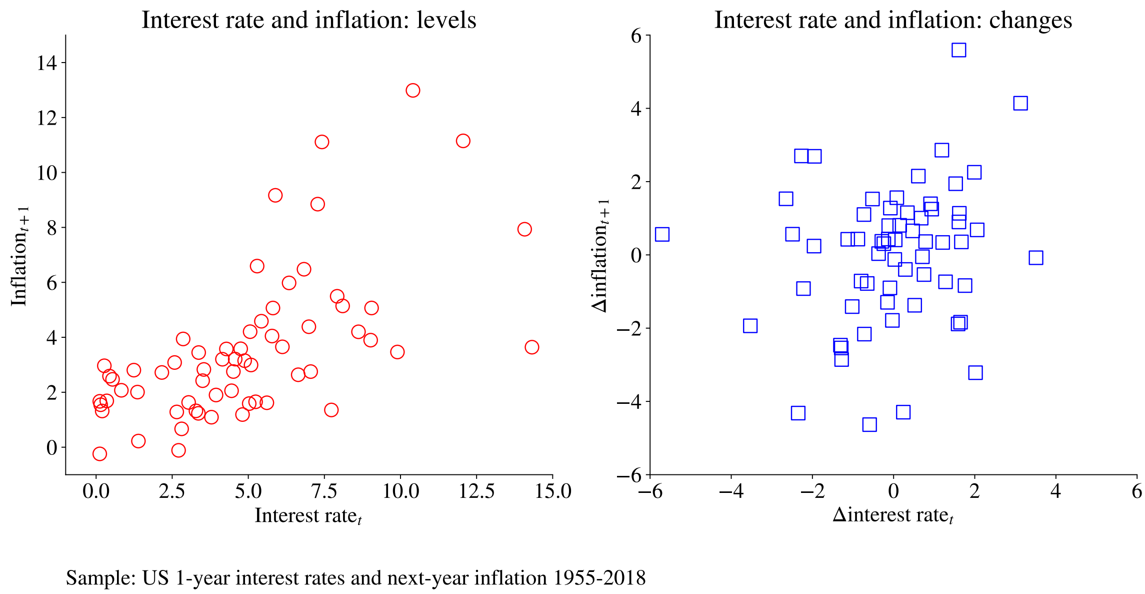


Figure 18.13: US nominal interest rates and subsequent inflation

18.4.2 The Expectations Hypothesis of Interest Rates

The expectations hypothesis of interest rates says that long interest rates equal an average of expected future short rates, possibly with a constant (across time, not maturities) risk premium as in (18.2). Alternatively, that forward rates equal expected future spot rates.

The expectations hypothesis is often used to calculate implied "forecasts" of future short interest rates. For instance, suppose the central bank increases its policy rate (typically a very short rate, at most a week or two). This is likely to affect also longer interest rates, but how is another matter. Let us consider a few different cases. For simplicity we assume that risk premia are unaffected by this move in the policy rate.

Example 18.9 *(Macroeconomics and the Pricing of Long-maturity Bonds I) An example of a macro based approach:*

- *Suppose the macroeconomic outlook becomes worse, inflation expectations down*
- *the central bank is expected to keep short interest rates low for an extended period*
- *the expectation hypothesis*$\Rightarrow$*interest rates on long bonds decrease today (*$\Leftrightarrow$*bond prices up)*

Example 18.10 *(Macroeconomics and the Pricing of Long-maturity Bonds II) an alternative, more asset market focused approach, but with the same result:*

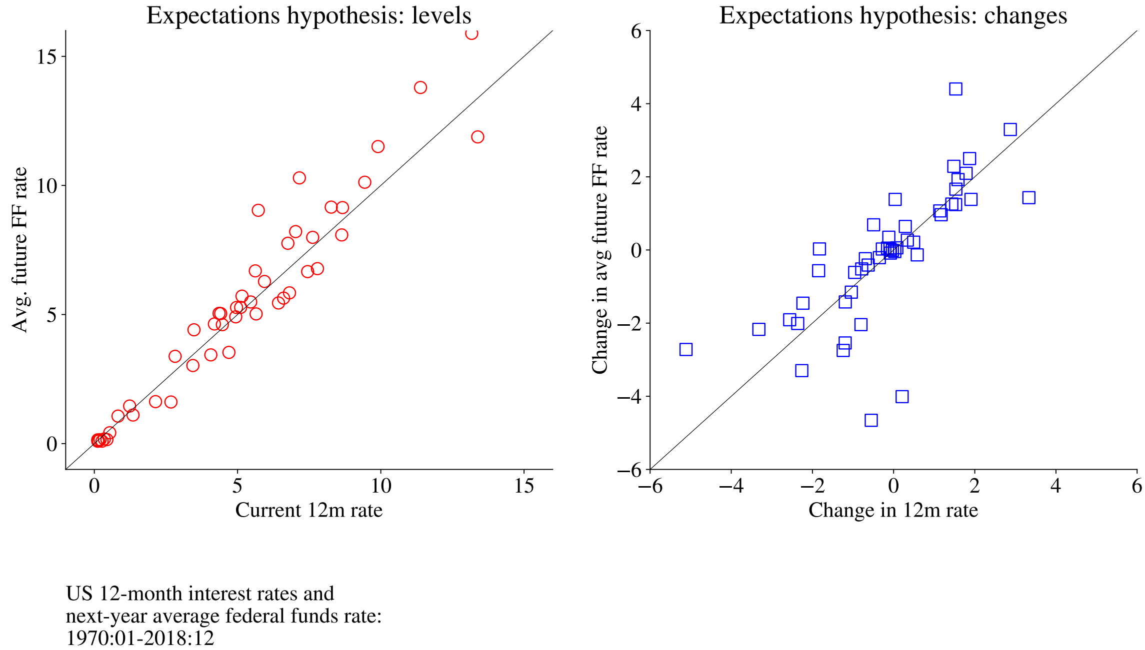


Figure 18.14: US 12-month interest and average federal funds rate (next 12 months)

- *Suppose the macroeconomic outlook becomes worse, inflation expectations down*
- *(a) inflation expectations down: nominal bonds more attractive; (b) growth down: bonds more attractive than stocks*
- *demand for long bonds up⇒ prices of long bonds up (⇔long interest rates down)*

First, one possibility is that only the very short interest rates change, and that all longer interest rates stay unchanged. This would happen if the policy move was well anticipated.

Second, another possibility is that most long interest rates increase. Under the expectations hypothesis of interest rates the interpretation is that the market now expects high short interest rates also in the future. That is, that the central bank will not reverse its policy action in the foreseeable future. If we are willing to assume that the real interest rate was not affected by the policy move, then one possible interpretation is that the central bank has received information about a long-lasting inflation pressure.

Third, and finally, short rates may increase, but really long interest rates decrease. A common interpretation of this scenario is that the central bank has become more inflation averse. It therefore raises the policy rate to bring down inflation. If the market believes that it will succeed, then it follows that it will eventually be possible to lower interest rates (when inflation and inflation expectations are lower).

The expectations hypothesis has been tested many times, typically by an ex post linear regression (realized interest rates regressed on lagged forward rates). The results often reject the expectations hypothesis, but they depend on how the test is done. It is not clear, however, if the rejection is due to systematic risk premia or to fairly small samples (compared to the long swings in interest rates). The expectations hypothesis gets more support when survey data on interest rate expectations is used instead on realized interest rates.

18.4.3 A New-Keynesian Model of Monetary Policy

Monetary policy is a crucial part of the macroeconomic picture these days, so it is important to understand how monetary policy is formed. It has not always been this way: there are long periods when many countries adopted a very simple (or so it seemed) monetary policy by pegging the currency to another currency. Macroeconomic policy was then synonymous with fiscal policy. Recently, the roles have changed.

Modern macro models are often smaller than the older macroeconometric models and they pay more attention to both the supply side of the economy and the role of expectations. These models try to capture the key elements in the way central banks (and most other observers) reason about the interaction between inflation, output, and monetary policy.

In these models, inflation depends on expected future inflation (some prices are set today for a long period and will therefore be affected by expectations about future costs and competitors' prices), lagged inflation, and a "Phillips effect" where an *output gap* (output less trend output) affects price setting via demand pressure. For instance, inflation (π_t) is often modelled as

$$\pi_t = \alpha \,\mathrm{E}_t \pi_{t+1} + \beta \pi_{t-1} + \phi x_t + \varepsilon_{\pi t}, \tag{18.11}$$

where x_t is the output gap and $\varepsilon_{\pi t}$ can be interpreted as "cost push" shocks (wage demands, oil price shocks). This equation can be said to represent the supply side of the economy and it is typically derived from a model where firms with some market power want to equate marginal revenues and marginal costs, but choose to change prices only gradually.

The demand side of the economy is modelled from consumers' savings decision, where the trade off between consumption today and tomorrow depends on the real interest rates. Simplifying by setting consumption equal to output we get something like the following equation for the output gap

$$x_t = x_{t-1} - \gamma(i_t - \mathrm{E}_t \pi_{t+1}) + u_t, \tag{18.12}$$

where i_t is the nominal interest rate (set by the central bank) and u_t is a shock to demand. Note that the expected *real* interest rate affects demand (negatively).

In some cases, the real exchange rate is added to both (18.11) and (18.12), capturing price increases on imported goods and foreign demand for exports, respectively. The exchange rate is then linked to the rest of the model via an assumption of uncovered interest rate parity (that is, expected exchange rate depreciation equals the interest rate differential).

Some of the important features of this simple model are: *(i)* inflation expectations matter for today's inflation (think about wage inflation), *(ii)* the instrument for monetary policy, the short interest rate i_t, can ultimately affect inflation only via the output gap; *(iii)* it is the real, not the nominal, interest rate that matters for demand.

To make the model operational, two more things must be added: the monetary policy (the way the interest rate is set) and the expectations in (18.11)–(18.12) must be specified.

It is common to assume that the central bank has some instrument rule like the famous "Taylor rule"

$$i_t = \theta_0 + 0.5x_t + 1.5\pi_t + v_t. \tag{18.13}$$

The residual v_t is a "monetary policy shock," which picks up factors left out of the model (for instance, the central bank's concern for the banking sector or simply changes in the central bank's objectives). This simple reaction function has been able to track US monetary policy fairly well over the last decade or so. Another approach to find a policy rule is to assume that the central bank has some loss function that it minimizes by choosing a policy rule. This loss function is often a weighted average of the variance of inflation and the variance of the output gap. The policy rule is the solution of the minimization problem, and can often look more complicated than the Taylor rule. However, there is one interesting special case. Suppose the central bank wants to minimize the (unconditional) variance of inflation. The formal optimization problem is then

$$\min_{i_t} \mathrm{Var}(\pi_t), \text{ subject to (18.11) and (18.12).} \tag{18.14}$$

The solution is then that the interest rate should be set so that actual inflation is zero (here the mean) in every period. If the model is changed so there is a time lag between the interest rate decision and its effect on inflation (for instance, by letting inflation in (18.11) react to x_{t-1} instead of x_t), then the interest rate should be set so that the conditional expectation of next period's inflation is zero (the mean), $\mathrm{E}_t\, \pi_{t+1} = 0$. This type of "rule" is used in much of the monetary policy debate.

The expectations in (18.11)–(18.12) can be handled in many ways. The perhaps most straightforward way is to assume that the expectations about the future equal the current value of the same variable (a "random walk"). A more satisfactory way is to use survey data on inflation expectations. Finally, many model builders assume that expectations are "rational" (or "model consistent") in the sense that the expectation equals the best guess we could do under the assumption that the model is correct. This latter approach typically requires a sophisticated way of solving the model (as the model both generates the best guesses and depends on them).

18.5 Forecasting Interest Rates*

The expectations hypothesis of interest suggests that current long rates can help predict future short rates. Empirically, this has some (moderately strong) support. However, there are also a number of other forecasting approaches.

18.5.1 Forecasting Monetary Policy or Inflation?

There is a two-way causality: inflation and the real economy (which depend on the real interest rate) affect monetary policy, and monetary policy can surely affect inflation and the real economy. This makes it difficult to analyse and forecast interest rates. However, for short term forecasting, the emphasis is typically on forecasting the next monetary policy move. Long run forecasting relies more on understanding the determinants of real interest rates and inflation, which depends on the general business cycle prospects, but also on the long run stance of monetary policy ("tough on inflation or not?").

18.5.2 Interest Rate Forecasts by Analysts

Kolb and Stekler (1996) use a semi-annual survey of (12 to 40) professional analysts' interest rate forecasts published in Wall Street Journal. The (6 months ahead) forecasts are for the 6-month T-bill rate and the yield on 30-year government bonds. The paper studies four questions, and I summarize the findings below.

1. Q. Is the distribution of the forecasts (across forecasters) at any point in time symmetric? (Analysed by first testing if the sample distribution could be drawn from

a normal distribution; if not, then checking asymmetry (skewness).) A. Yes, in most periods. (The authors argue why this makes the median forecast a meaningful representation of a "consensus forecast.")

2. Q. Are all forecasters equally good (in terms of ranking of (absolute?) forecast error)? A. Yes for the 90-day T-bill rate; No for the long bond yield.

3. Q. Are some forecasters systematically better (in terms of absolute forecast error)? (Analysed by checking if the absolute forecast error is below the median more than 50% of the time) A. Yes.

4. Q. Do the forecasts predict the direction of change of the interest rate? (Analysed by checking if the forecast gets the sign of the change right more than 50% of the time.) A. No.

18.5.3 Market Positions as Interest Rate Forecasts

Hartzmark (1991) has data on daily futures positions of large traders on eight different markets, including futures on 90-day T-bills and on government bonds. He uses this data to see if the traders changed their position in the right direction compared to realized prices (in the future) and if they did so consistently over time.

The results indicate that these large investors in T-bills and bond futures did no better than an uninformed guess of the direction of change of the bill and bond prices. He gets essentially the same results if the size of the change in the position and in the price are also taken into account.

There is of course a distribution of how well the different investors do, but it looks much like one generated from random guesses (uninformed forecasts). The investors change places in this distribution over time: there is very little evidence that successful investors continue to be successful over long periods.

18.6 Risk Premia on Fixed Income Markets

There are many different types of risk premia on fixed income markets.

Nominal bonds are risky in real terms, and are therefore likely to carry *inflation risk premia*. Long bonds are risky because their market values fluctuate over time, so they probably have *term premia*. Corporate bonds and some government bonds (in particular, from developing countries) have *default risk premia*, depending on the risk for default.

Interbank rates may be higher than T-bill of the same maturity for the same reason (see the TED spread, the spread between 3-month Libor and T-bill rates) and illiquid bonds may carry *liquidity premia* (see the spread between off-the run and on-the-run bonds).

Figures 18.15–18.17 provide some examples.

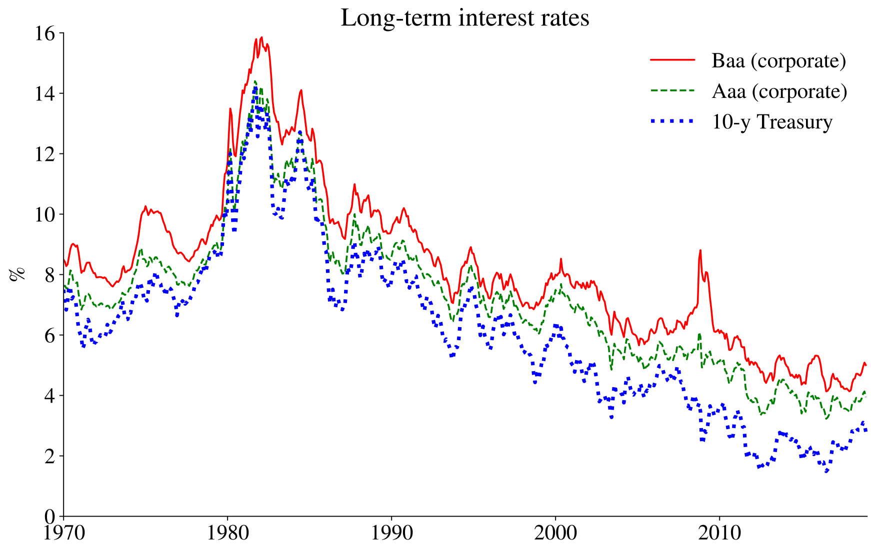


Figure 18.15: US interest rates

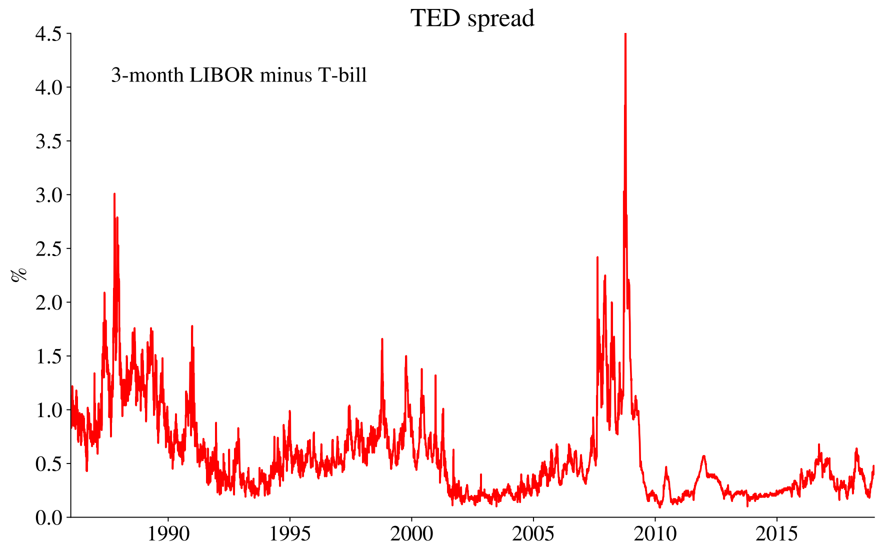


Figure 18.16: TED spread

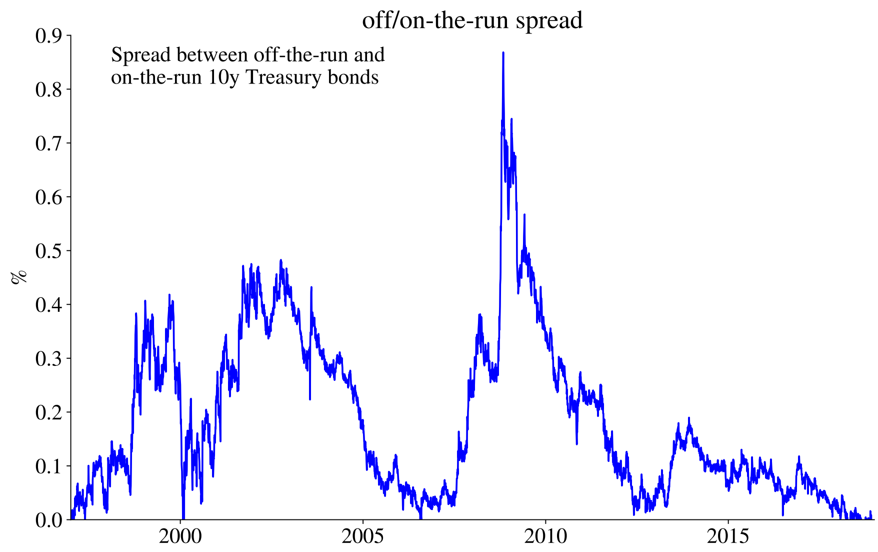


Figure 18.17: Off-the-run liquidity premium

Chapter 19

Basic Properties of Options

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23
Additional references: McDonald (2014) 11–12

19.1 Derivatives

Derivatives are assets whose payoff depend on some underlying asset (for instance, shares of a company). The most common derivatives are futures contracts (or similarly, forward contracts) and options. Sometimes, options depend not directly on the underlying, but on the price of a futures contract on the underlying. See Figure 19.1.

Derivatives are in zero net supply, so a contract must be issued (a short position) by someone for an investor to be able to buy it (long position). For that reason, gains and losses on derivatives markets sum to zero.

19.2 Introduction to Options

Remark 19.1 *(On the notation) The notation is kept short. The current period is assumed to be $t = 0$ and the derivative expires in $t = m$. The current price of the underlying is denoted S (rather than S_0), the forward price according to a contract agreed on now and expiring in $t = m$ is F (rather than $F_0(m)$) and the continuously compounded interest between $t = 0$ and $t = m$ is y (rather than $y_0(m)$). However, to avoid confusion, the price of the underlying asset at expiration is denoted S_m.*

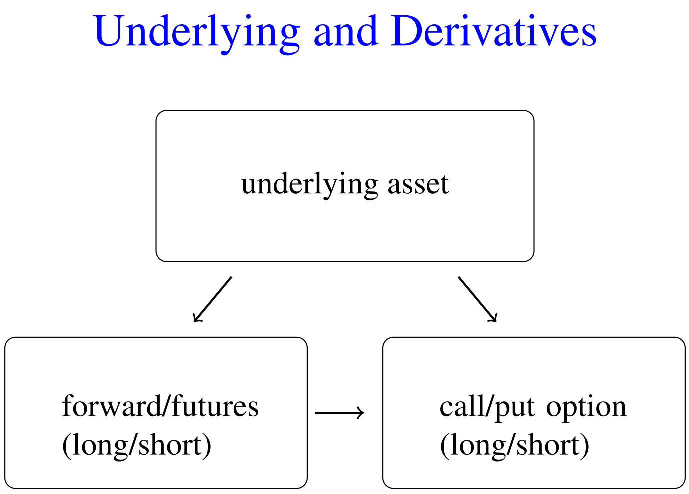


Figure 19.1: Derivatives on an underlying asset

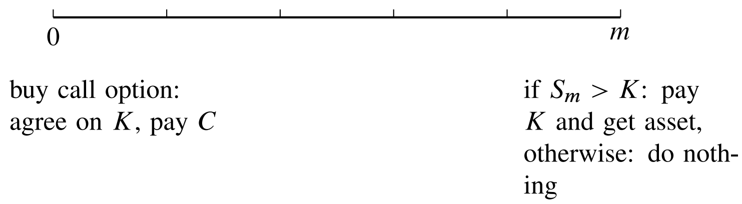


Figure 19.2: Timing convention of a European call option contract

19.2.1 Definition of European Calls and Puts

A European *call* option contract traded in $t = 0$ stipulates that the owner of the contract has the *right* (but not the obligation) to buy one unit of the underlying asset ("exercise the option") from the issuer of the option on the expiration date $t = m$ at the strike price K. Compare with a forward contract where the owner must exercise. See Figure 19.2 for the timing convention.

To the owner of a call option, the payoff at expiration is either zero (if the owner does not exercise) or the value the underlying asset S_m minus the strike price K (if the owner exercises). For a rational investor (who only exercises if $S_m \geq K$), the payoff is thus

$$\text{call payoff}_m = \max\left(0, S_m - K\right). \tag{19.1}$$

Clearly, an owner of a call option benefits from a high price of the underlying asset.

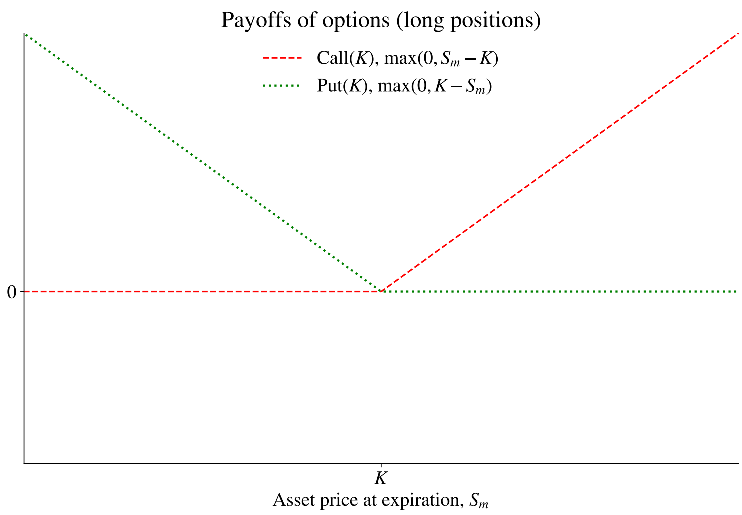


Figure 19.3: Payoffs of options, long positions

Example 19.2 *(Call option payoffs) With $K = 5$ we have*

S_m	*Exercise*	*Payoff*
4.5	*no*	0
5.5	*yes*	$5.5 - 5 = 0.5$

The profit at expiration is thus

$$\text{call profit}_m = \text{call payoff}_m - e^{my}C, \tag{19.2}$$

where C is the call price, typically paid in $t = 0$. (To simplify the notation, the time subscript on C is suppressed, but we could write C_0 when required.) The e^{my} factor captures the capital cost of paying the option price already on the trade date (think: borrow C in $t = 0$ and repay with interest, $e^{my}C$, on the expiration date).

See Figure 19.4 for an illustration. Notice that the price of the option (C) is always paid, irrespective of whether the option is exercised or not.

Remark 19.3 *(In-the-money*) An option that would be profitable to exercise is called* in-the-money*; an option that would be unprofitable to exercise is called* out-of-the-money*—and an option that would just break even is called at-the-money.*

The payoff of the issuer is the mirror image of the owner's payoff: the owner's gain is the issuer's loss: a *zero sum game*. See Figures 19.3 for an illustration. This zero sum game property is true both for the payoff at exercise as well as the for the profit.

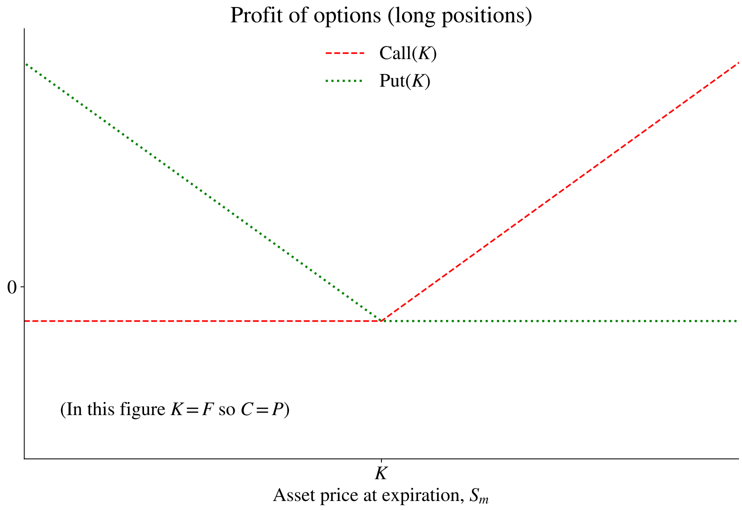


Figure 19.4: Profit of options, long positions

Remark 19.4 *(Margin requirements*) A buyer of an option does not have to post any margin, but a issuer typically does. The reason is that a default of the issuer could create a loss for the option owner (if the option is worth exercising). In contrast, the default of the owner cannot create a loss for the issuer.*

A *put* option instead gives the owner of the contract the right to sell one unit of the underlying asset. The put price is here denoted by P. An owner of a put option benefits from a low price of the underlying asset (buy the asset cheaply and exercise the right to sell for K). The payoff is

$$\text{put payoff}_m = \max\left(0, K - S_m\right). \tag{19.3}$$

Example 19.5 *(Put option payoffs) With $K = 5$ we have*

S_m	*Exercise*	*Payoff*
4.5	*yes*	$5 - 4.5 = 0.5$
5.5	*no*	0

Figures 19.6–19.7 illustrate the trade intensity of options with different strike prices (but same expiration and underlying asset). It seems clear that most of the trade is in out-of-the-money options (high strike prices for the calls and low strike prices for the puts). Figure 19.8 shows that most of the trade is close to expiration, and there is a seasonality pattern related to rolling over the investment from other (expired) options. Figure 19.9

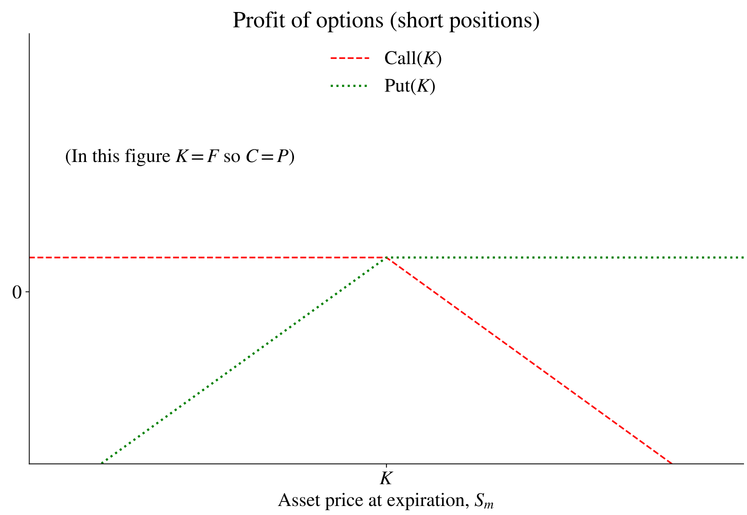


Figure 19.5: Profit of options, short positions

shows how the trading volume at CBOE has developed over time. The volume grew up to the financial crisis, decreased somewhat during the crisis and has stabilized since. The ratio of traded put contracts to traded call contracts in Figure 19.9 is sometimes used to gauge market nervousness. The idea is that investors will demand put contracts if they want to insure against a stock market decline.

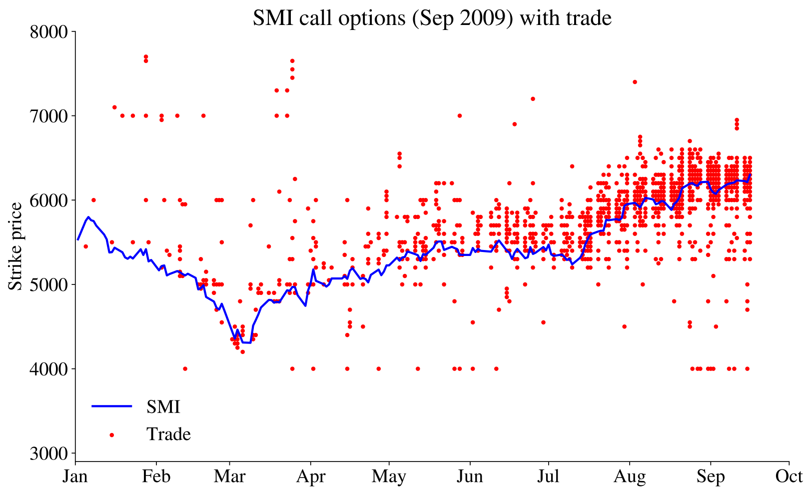


Figure 19.6: Traded options

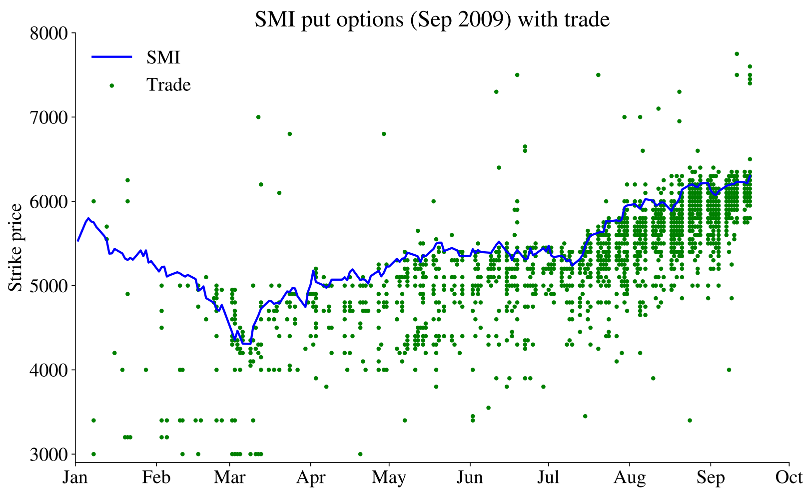


Figure 19.7: Traded options

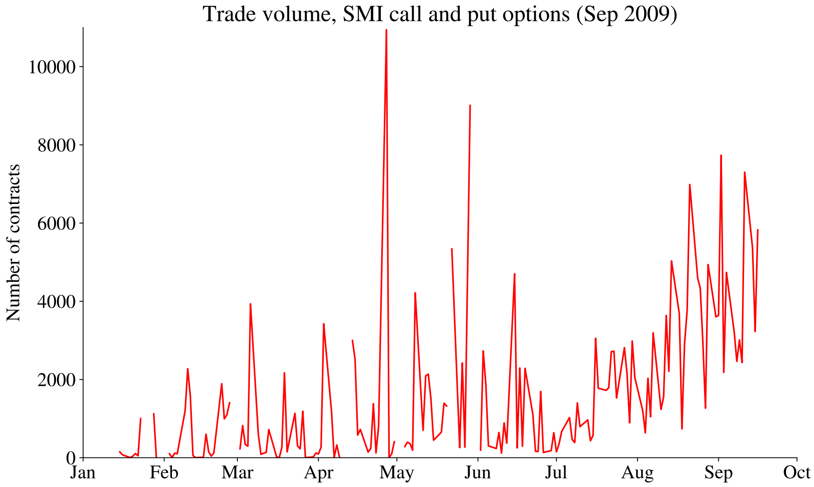


Figure 19.8: Option trade volume

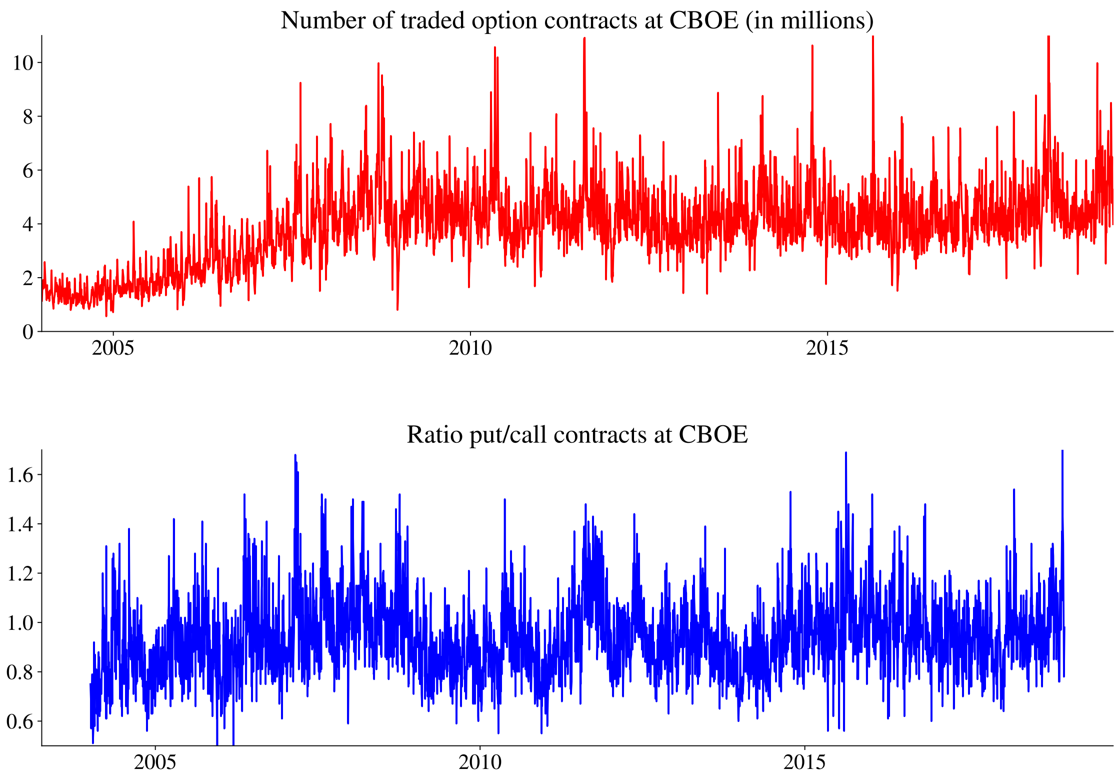


Figure 19.9: Option trade volume

19.2.2 Options Are Risky Assets

The buyer always stands the risk of getting a zero payoff, that is, a return of -100%. For instance, the net return on a European call option is

$$\text{return on call}_m = \frac{\max\left(0, S_m - K\right)}{C} - 1, \tag{19.4}$$

where C is the call option price. Whenever the option isn't exercised, the whole investment is lost (and the return is -100%). However, for the owner of an option the risk is limited to the initial option price. In contrast, the option issuer can lose much more than that. For these reasons, options issuers need to post margin, while option buyers do not.

It is clear that option returns cannot be normally (or even lognormally) distributed: the density function has a spike at -100% (whose probability is the same as the probability of $S_m \leq K$). This means, that we cannot motivate "mean-variance" pricing of options by referring to a normal distribution of the return. (This does not rule out mean-variance pricing, which could be motivated by, for instance, mean-variance preferences.)

19.2.3 Basic Properties of Option Prices

Options prices depend on many things, but there are some fairly general results

First, *call option prices are decreasing in the strike price*, while put options prices are increasing in the strike price, see Figure 19.10. The intuition is illustrated in Figure 19.11 which illustrates the perceived (by the market) distribution of the asset price at expiration. Notice that a higher strike price means that an owner of a call option will have to pay more in case of exercise—and there is also a lower chance of exercise. (Later sections provide a more formal proof on the same property.)

Second, both *call and put option prices are typically increasing in the (perceived) uncertainty* of the future price of the underlying asset, see Figure 19.12. The intuition is illustrated in Figure 19.13, which shows that a wider dispersion of the distribution increases the probability of a really high price of the underlying asset (although the figure is constructed to have the same probability of exercise in the two cases). Of course, it also increases the probability of a really low asset price, but that is of no concern since the call option payoff is bounded below at zero.

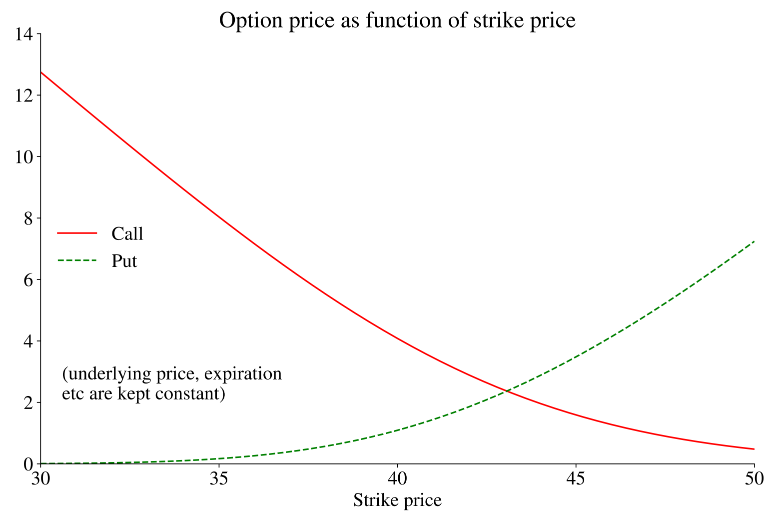


Figure 19.10: Option price as a function of the strike price

19.3 Financial Engineering

19.3.1 Hedging a Future Purchase of the Underlying

Suppose your firm needs to buy oil next year, since you use oil to run the factory. The oil price fluctuates, so it is a source of risk. One way of hedging this risk is to enter a forward contract: this provides a complete insurance. Alternatively, could buy a call option: in this case you know that you will be able to buy oil for the strike price or less: it hedges against the risk up the oil price increasing, but keeps the chance of gaining from a lower price.

19.3.2 Replicating a Forward

Options markets are often very liquid—and are therefore useful for constructing replicating portfolios. Let call(K) - put(K) for $K = F$ (the forward price) be short hand notation for portfolio that is long on call option with strike price $K = F$ and short one put option with the same strike price. This portfolio replicates a forward contract, so it is a synthetic forward. Clearly, we can then replicate a short position in a forward contract by selling such a portfolio. See Figure 19.14.

Example 19.6 *(Payoff of a syntthetic forward) With $K = 5$, we have the differences of*

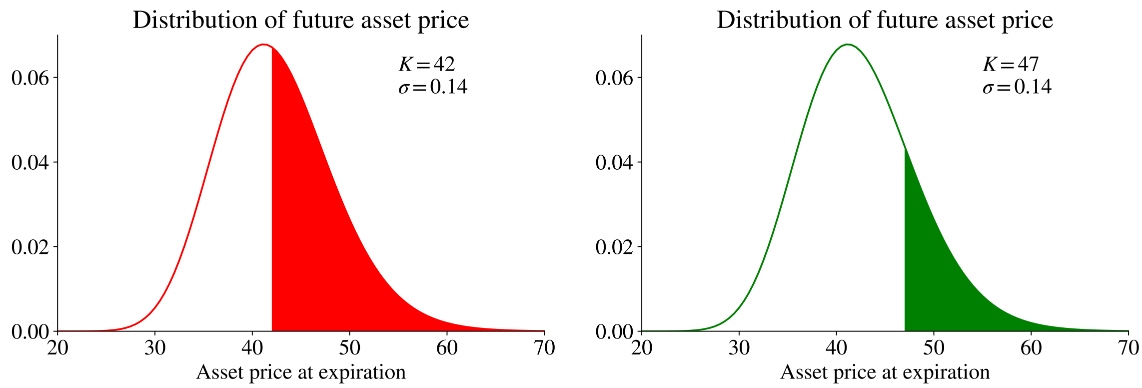


Figure 19.11: Distribution of future asset price

the payoffs in Examples 19.2 and 19.5, that is,

S_m	*Exercise call*	*Payoff from call*	*Exercise put*	*Payoff from put, short*	*Total Payoff*
4.5	*no*	0	*yes*	$-(5-4.5)$	-0.5
5.5	*yes*	$5.5-5$	*no*	0	0.5

To get the profit, subtract the difference of the call and put prices.

19.3.3 Portfolio Insurance

A *protective put* is a combination of a put and a position in the underlying asset. This allows the owner to capture the upside of the price movement (of the underlying), at the same time as insuring against the downside. This is indeed very similar to just buying a call option. See Figure 19.15.

19.3.4 Betting on Large Changes

An option is a bet on a change in a specific direction. Option portfolios can be constructed to instead make a bet on a large change in either direction (that is, high volatility): a *straddle* is call(K) + put(K), and a *strangle* is call(K_2) + put(K_1) where $K_1 < K_2$ where K_1 and K_2 are two different strike prices. See Figure 19.16.

Example 19.7 *(Payoff of a straddle) With $K = 5$, we have the sum of the payoffs in*

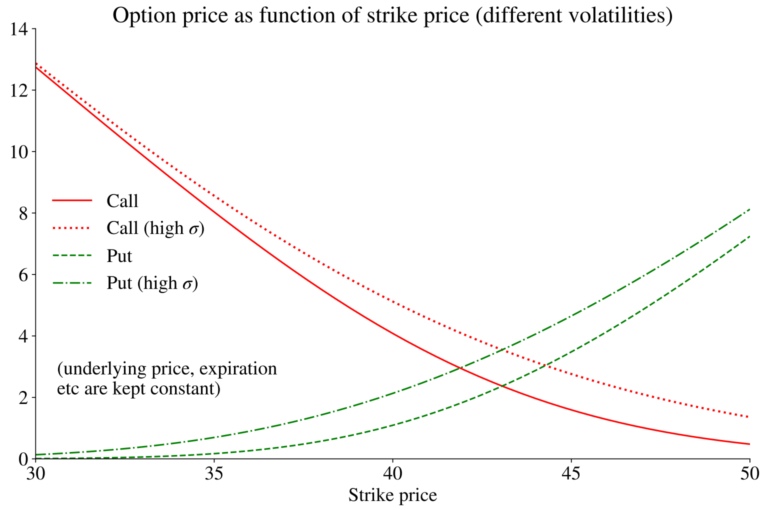


Figure 19.12: Option price as a function of the strike price

Examples 19.2 and 19.5, that is,

S_m	*Exercise call*	*Payoff from call*	*Exercise put*	*Payoff from put*	*Total Payoff*
4.5	*no*	0	*yes*	$5-4.5$	0.5
5.5	*yes*	$5.5-5$	*no*	0	0.5

To get the profit, subtract the sum of the call and put prices.

19.3.5 Putting a Collar on Losses and Gains

A *collared stock* is a combination of the underlying asset, a put with a low strike price (K_1) and a short call with a high strike price (K_2). This portfolio has a profit that increases one-for-one with the underlying asset as long as it is between K_1 and K_2. The losses for values of the underlying below K_1 are limited (by the put), and the gains for values above K_2 are also capped (by the short call). See Figure 19.17.

19.3.6 Betting on a Large Price Decrease

A variation on the synthetic short forward is the *collar*: $-\text{call}(K_2)+\text{put}(K_1)$ where $K_1 < K_2$. It also looks like a short position in a forward contract, except that the payoff is flat between the strike prices. Clearly, this is betting on a large price decrease. Selling a collar (or *reversal*) is instead a bet on a large price increase.

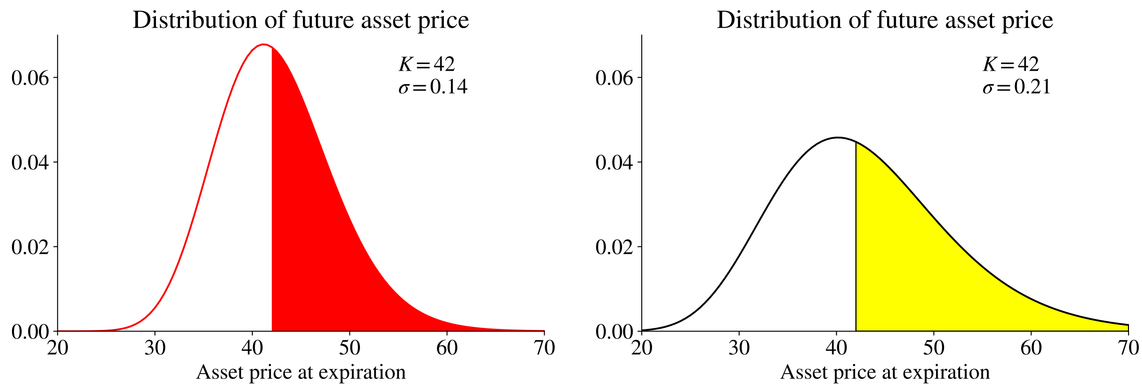


Figure 19.13: Distribution of future asset price

A collar (reversal) can be used to hedge a long (short) position in the underlying asset, except that there is no hedge between the strike prices. It provides insurance outside the strike prices. See Figure 19.18.

19.3.7 Betting On a Small Price Increase

To bet on a small increase in the price of the underlying asset we can use a *bull spread*: call(K_1) - call(K_2) where $K_1 < K_2$. This portfolio has flat payoffs outside the strike prices, but a payoff that increases with the underlying asset between them. Selling a bull spread creates a *bear spread*, which is a bet on a small decrease of the underlying price. (These spreads can also be constructed by combing puts.) See Figure 19.18.

19.4 Put-Call Parity for European Options

There is a tight link between European call and put prices. If you know one of them (and the forward price), then you can easily calculate what the other must be. The following proposition is more precise.

Proposition 19.8 *(Put-call parity for European options) The put-call parity for European options is*

$$C - P = e^{-my}(F - K), \tag{19.5}$$

where $e^{-my}(F - K)$ is the present value of the forward price minus the strike price.

Time subscripts and indicators of maturity have been suppressed to make the notation

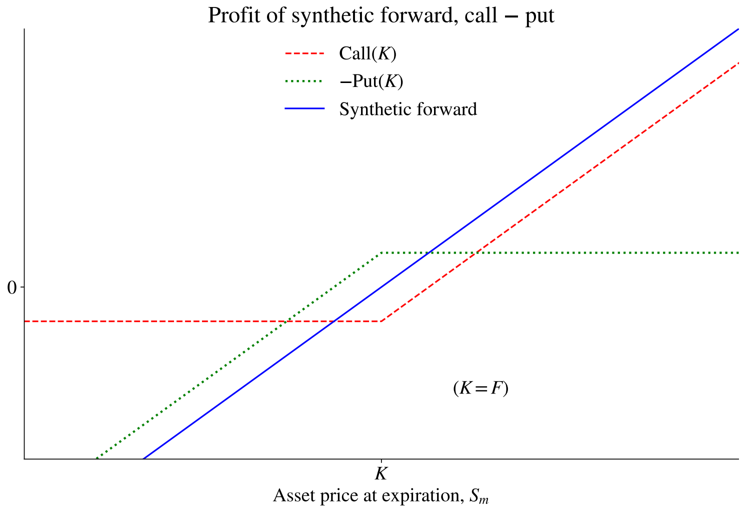


Figure 19.14: Profit of an option portfolio that replicates a forward contract

a bit easier. The parity holds irrespective of whether the underlying asset has dividends or not (since the expression uses the forward price).

The practical importance of the proposition is that it shows how to use two assets to replicate a third asset. For instance, we can combine a call option (with strike price $K = F$) and a forward contract to replicate a put option, or buy a call and sell a put (with strike price $K = F$) to replicate a forward contract. Transaction costs can cause (relatively small) deviations from the parity condition. See Figure 19.19 for an illustration.

Example 19.9 *(Put-call parity) Let* $S = 42, m = 1/2, y = 5\%, K = 38$. *If the underlying asset has no dividends, then* $F = e^{0.5\times0.05}42 = 43.06$. *With* $C = 5.5$, *(19.5) gives*

$$5.5 - P = e^{-0.5\times0.05}(43.06 - 38) \text{ or } P \approx 0.56.$$

Proof. (of Proposition 19.8) Portfolio A: buy one call option and sell one put option, both with the strike price K, at the cost $C - P$. This will with certainty give $S_m - K$ at maturity (since the call *or* the put will be exercised). Portfolio B: enter a forward contract and put $e^{-my}(F - K)$ in the bank (your cost). At expiration, get $S_m - F$ from the forward contract plus the $F - K$ that you have in the bank: $S_m - K$. Since the two portfolios give the same at expiration, they must have the same costs today. ■

This formula is very general, but a few special cases are of particular interest. First, when the underlying asset pays no dividends, then (19.5) together with the forward-spot

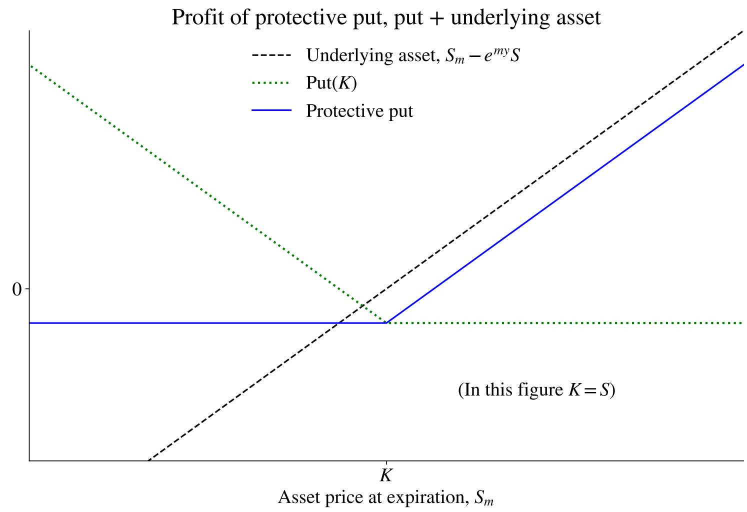


Figure 19.15: Profit of an option portfolio that insures the underlying asset

parity give

$$C - P = S - e^{-my}K \text{ if no dividends,} \tag{19.6}$$

$$C - P = S - \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i - e^{-my}K \text{ if dividends,} \tag{19.7}$$

$$C - P = Se^{-\delta m} - e^{-my}K \text{ if continuous dividend rate } \delta. \tag{19.8}$$

19.4.1 Put-Call Parity and Synthetic Replications*

The following remarks provides details on how two assets can be used to replicate a third—since they are all tied together by the put-call parity.

Remark 19.10 *(Synthetic forward) Buy one call and sell one put at a strike price that equals the forward price prevailing in t ($K = F$). By (19.5), the cost of this portfolio is zero. At expiration, it will give one unit of the underlying, at the cost K. Just like a forward contract. See Figure 19.20.*

Remark 19.11 *(Synthetic call option) Buy one forward and one put with strike price $K = F$. By (19.5), this has the price C. If, at expiration, $S_m < K$, then the forward pays off $S_m - F$ and the put option $K - S_m$. Since $K = F$, the sum is zero. Instead, if $S_m > K$, then the forward pays off $S_m - F$ and the put nothing. In either case, this is just like a call option with strike price K. See Figure 19.20.*

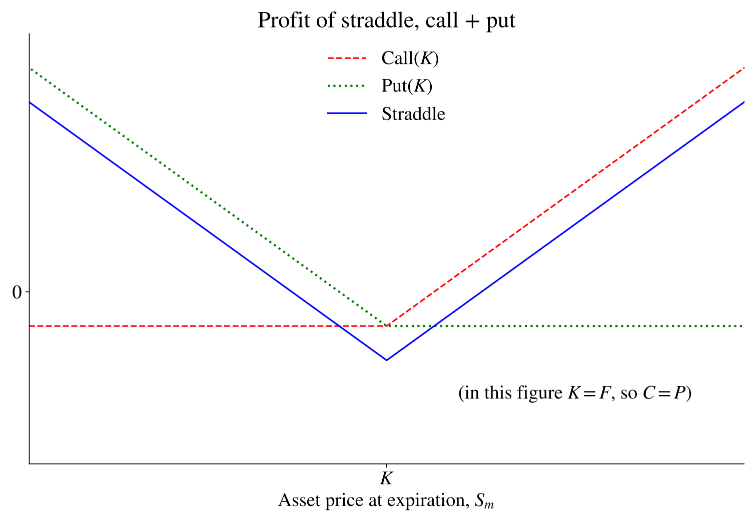


Figure 19.16: Profit of an option portfolio than bets on volatility

Remark 19.12 *(Synthetic put option) Buy one call with strike price $K = F$ and sell one forward. By (19.5), this has the price P. If, at expiration, $S_m < K$, then the call pays off nothing and the short forward $-(S_m - F)$. Since $K = F$, the sum is $K - S_m$. Instead, if $S_m > K$, then the call pays off $S_m - K$ and the short forward $-(S_m - F)$, which sums to zero. In either case, this is just like a put option with strike price K. See Figure 19.20*

19.5 Definition of American Calls and Puts

An American option is like a European option, except that it *can be exercised on any day* before or on the expiration date. This means that an American option has more rights than a European option and is therefore worth at least as much

$$C_A \geq C_E \text{ and } P_A \geq P_E, \tag{19.9}$$

where we use subscripts to distinguish between American and European options.

If the (American) option is exercised, then the immediate payoff is $\max(0, S - K)$ for a call and $\max(0, K - S)$ for a put, where S should be understood as the current price of the underlying. This means, that the options prices must (at any point in time) obey

$$\begin{aligned} C_A &\geq \max(0, S - K) \\ P_A &\geq \max(0, K - S). \end{aligned} \tag{19.10}$$

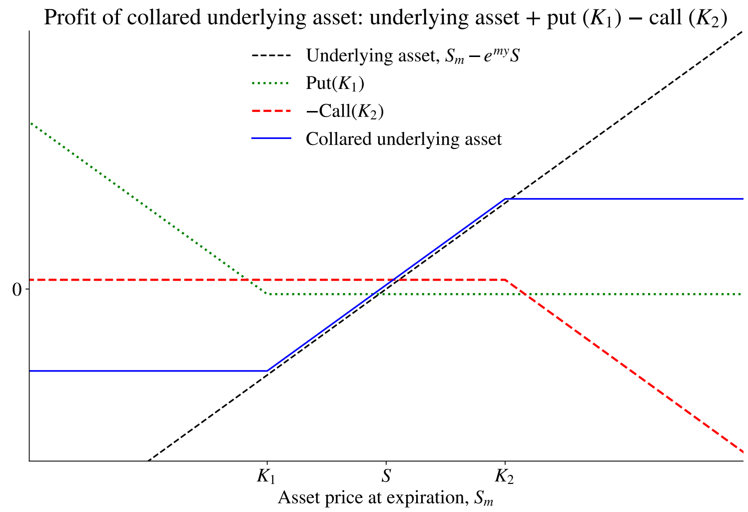


Figure 19.17: Profit of a collared underlying asset

The right hand sides are what you get by exercising the option now, and are called the "intrinsic values."

We will later demonstrate the following results. First, if there are no dividends, then it is never optimal to exercise an American call option early (such a call option will have the same price as a European call option), but it can still be optimal to exercise an American put option early. Second, if there are dividends, then the American call option should only be exercised just prior to the dividend payments, while an American put should perhaps also be exercised also at other times. See Figure 19.22.

Remark 19.13 *Figures 19.21 and 19.22 provide an example of how the futures price (on S&P 500), the intrinsic value of the option and the option price developed over a year. Notice how the futures prices converges to the index level at expiration of the futures. Before it can deviate because of delayed payment (+) and no part in dividend payments (−). Also notice that even options with zero intrinsic value (zero payoff if exercised now) can have fairly high option prices—at least if the time to expiration is long, but it converges to zero the expiration date gets closer..*

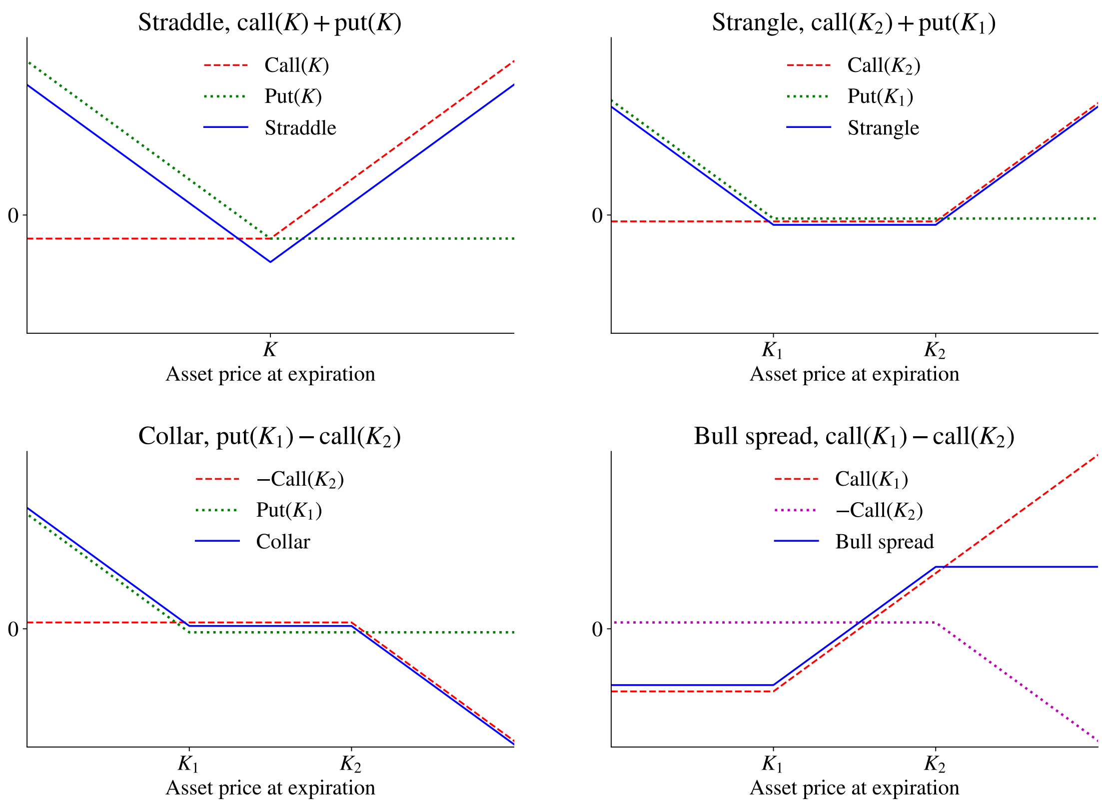


Figure 19.18: Profits of option portfolios

19.6 Pricing Bounds and Convexity of Pricing Functions

19.6.1 Pricing Bounds for (European and American) Call Options

The prices of call options must satisfy the following restrictions

$$C \leq e^{-my} F \leq S \tag{19.11}$$

$$0 \leq C \tag{19.12}$$

$$e^{-my}(F - K) \leq C. \tag{19.13}$$

These bounds hold for both American or European call options (we here use C to denote both of them.)

The motivations are basically as follows (the intuition based on European options, but the results extend to American options as well). First, a call option with a zero strike price ($K = 0$) would be the same as owning a prepaid forward contract (which is worth as much or less than the underlying asset). Whenever the strike price is higher, the call

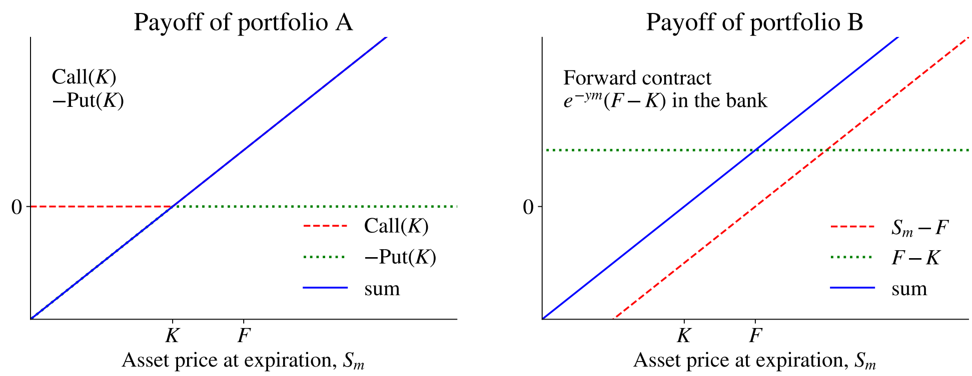


Figure 19.19: Put-call parity

price is lower. Second, the call option gives rights, not obligations: its price value cannot be negative. Third, the lowest possible value of a put option is zero, so the put-call parity (19.5) immediately gives that the call price must exceed the present value of $F - K$. (See below for an alternative proof.) Transaction costs can cause (relatively small) failures of the bounds.

Combining the bounds, we get

$$C \leq e^{-my}F \leq S \tag{19.14}$$

$$C \geq \max[0, e^{-my}(F - K)]. \tag{19.15}$$

In particular, for a financial asset without dividends (until expiration of the option), we have $\max(0, S - e^{-my}K) \leq C \leq S$. See Figures 19.23, 19.24 and 19.29 for illustrations.

Example 19.14 *(Pricing bounds for call option) Using the same parameters as in Example 19.9, we get* $C \leq 42$ *and*

$$C \geq \max[0, e^{-0.5\times 0.05}(43.06 - 38)] = 4.94.$$

Remark 19.15 *(The put price bounds in Figure 19.24) At very low strike prices, it is almost certain that the option will be exercised at expiration. Therefore, the present value of the cost, $C + e^{-ym}K$, must be almost equal to the present value of a forward contract, $e^{-ym}F$. Combining gives $C = e^{-my}(F - K)$. In contrast, at very high strike prices, the probability of exercise is almost zero—so the option price is too.*

Proof. (*of (19.13)) Portfolio A: one European call option and $e^{-my}K$ on a bank account. At expiration, this portfolio is worth S_m if the option is exercised, and K oth-

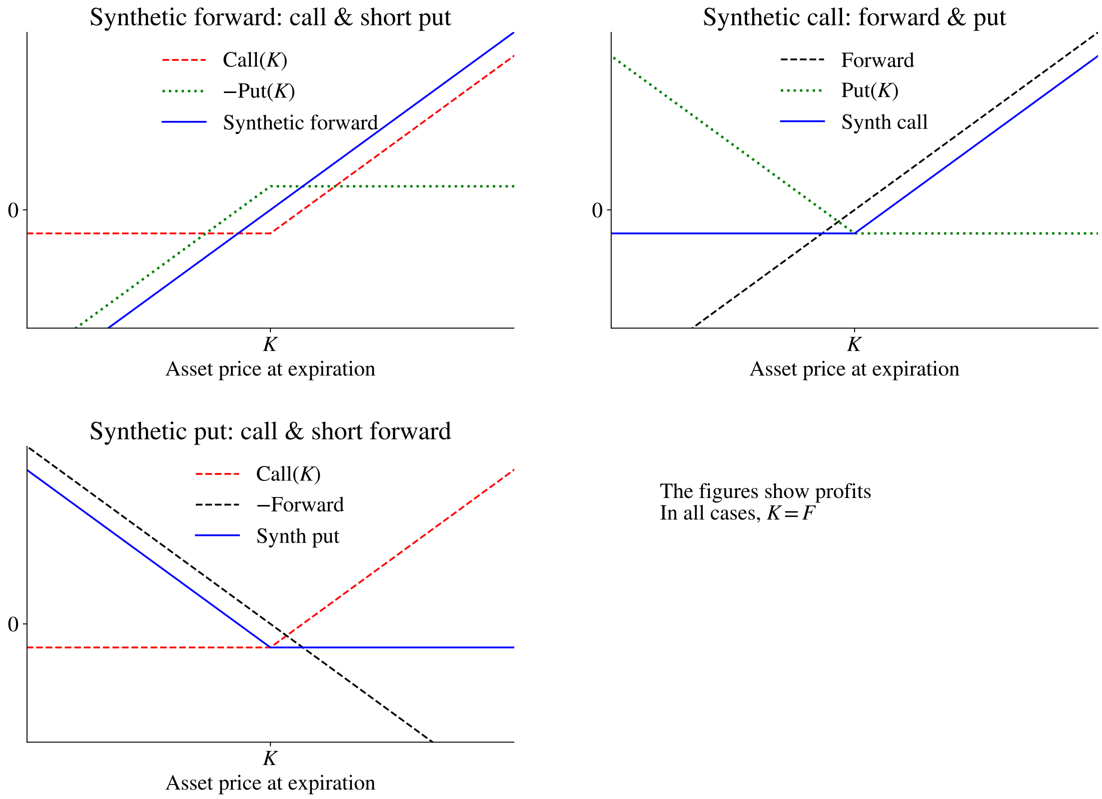


Figure 19.20: Synthetic replication

erwise: $\max(S_m, K)$. Portfolio B: one prepaid forward contract, which is worth S_m at expiration. (Since you pay $e^{-my}F$ now, there is no payment at expiration.) Clearly, portfolio A is always worth more at expiration, so it must also be worth more right now: $C_E + e^{-my}K \geq e^{-my}F$. Rearrange to get (19.13). Since $C_A \geq C_E$, the bound holds also for an American call option. ■

19.6.2 Pricing Bounds for (European and American) Put Options

The prices of American and European put options must satisfy the following restrictions

$$P_E \leq e^{-my}K \text{ and } P_A \leq K \tag{19.16}$$

$$0 \leq P_E \text{ and } 0 \leq P_A \tag{19.17}$$

$$e^{-my}(K - F) \leq P_E \text{ and } K - S \leq P_A. \tag{19.18}$$

See Figure 19.25.

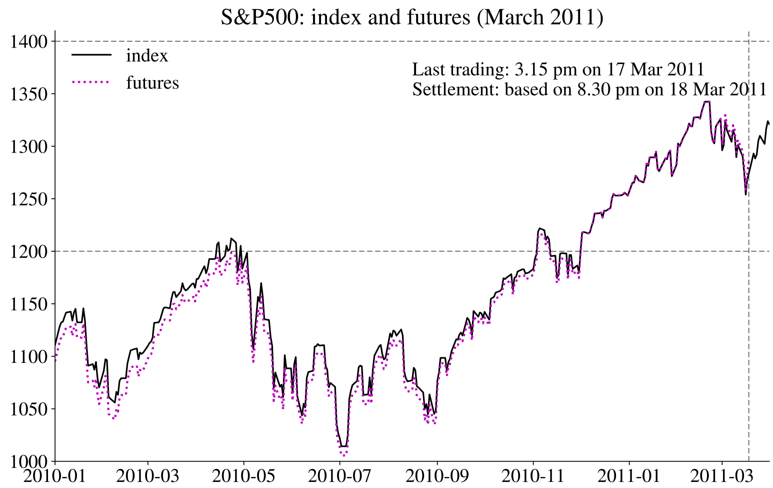


Figure 19.21: S&P 500 index level and futures

The motivations are as follows. First, the payoff from a put option is $\max(K - S, 0)$, so the maximum value is the strike price (when $S = 0$). For the European put, this payoff is received only at expiration, so the maximum value today is the present value of the strike price. Second, the put option gives rights, not obligations: its price value cannot be negative. Third, the lowest possible value of a call option is zero, so the put-call parity (19.5) immediately gives that the European put price must exceed the present value of $K - F$. (See below for an alternative proof.) In contrast, the American put can be exercised now so its value must be at least as high as the intrinsic value.

Proof. (*of (19.18)) Portfolio A: one European put option and a prepaid forward contract. At expiration, this portfolio is worth K if the option is exercised, and S_m otherwise: $\max(K, S_m)$. Portfolio B: $e^{-my}K$ on a bank account, which is worth K at expiration. Clearly, portfolio A is always worth more at expiration, so it must also be worth more right now: $P_E + e^{-my}F \geq e^{-my}K$. Rearrange to get (19.18). Since $P_A \geq P_E$, the bound holds also for an American put option. ■

19.7 Early Exercise of American Options

This section discusses early exercise of American options. There are some cases where we can exclude early exercise, so the American option is priced as a European option. In other cases, we cannot exclude early exercise—but we may still be able to say something about when early exercise is likely. More precise answers will require building a model

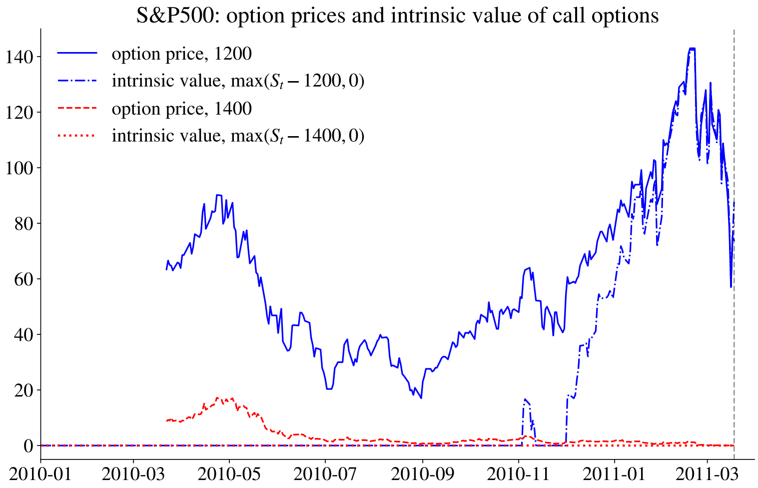


Figure 19.22: S&P 500 options

for the pricing. Clearly, the answer is then model dependent.

The key results are as follows (assuming interest rates are positive):

	without dividends	with dividends
Call	no early exercise	early exercise (at high S)
Put	early exercise (at low S)	early exercise

Proofs and details are found in the Appendix. (Negative interest rates means that you could plausibly have early exercise for all four types.)

Example 19.16 *(Bankruptcy, American put, no dividends) Suppose the underlying asset goes bankrupt, then $S = 0$ and it is known that it will stay at $S = 0$. Exercising the American put option now gives K, whereas waiting until expiration has a present value of $e^{-my}K$ (which is lower): early exercise is optimal.*

See Figures 19.26–19.28 for an illustration, based on a numerical solution (of a specific model, so the precise results are not general) for the price on an American put option. In particular, Figure 19.26 shows in which nodes early exercise is optimal: at low asset prices. Figure 19.28 illustrates that the American put price is close to the European put price when the asset price is low, that is, when early exercise is unlikely to happen in the near future. However, the American put price starts to increase above the European put price when the asset price is lower.

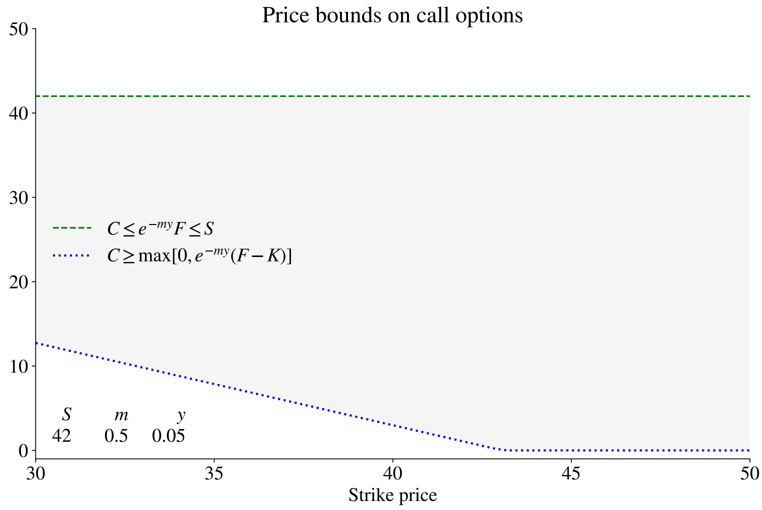


Figure 19.23: Call option price bounds as a function of the strike price

19.8 Appendix: Convexity of Option Prices*

Suppose we have American or European call options with different strike prices, $K_1 < K_2$. We the have the following price relations

$$C(K_2) - C(K_1) \leq 0 \tag{19.19}$$

$$\frac{C(K_2) - C(K_1)}{K_2 - K_1} \geq -1 \tag{19.20}$$

$$C\left[\lambda K_1 + (1-\lambda)K_2\right] \leq \lambda C(K_1) + (1-\lambda)C(K_2), \text{ for } 0 \leq \lambda \leq 1. \tag{19.21}$$

The first relation says that the call option price is decreasing in the strike price. The intuition is that a higher strike price means that an owner of a call option will have to pay more in case of exercise—and there is also a lower chance of exercise. The second relation says that change is smaller than the change in the strike price. The third relation says that the relation is convex. If these relations do not hold, then there are arbitrage opportunities (see the proofs below).

In other words, these three conditions say that we have the following partial derivatives (if they exist) of the call option price function

$$-1 \leq dC(K)/dK \leq 0 \text{ and } dC^2(K)/dK^2 \geq 0. \tag{19.22}$$

This means that the call option price is decreasing in the strike price, but slower than the

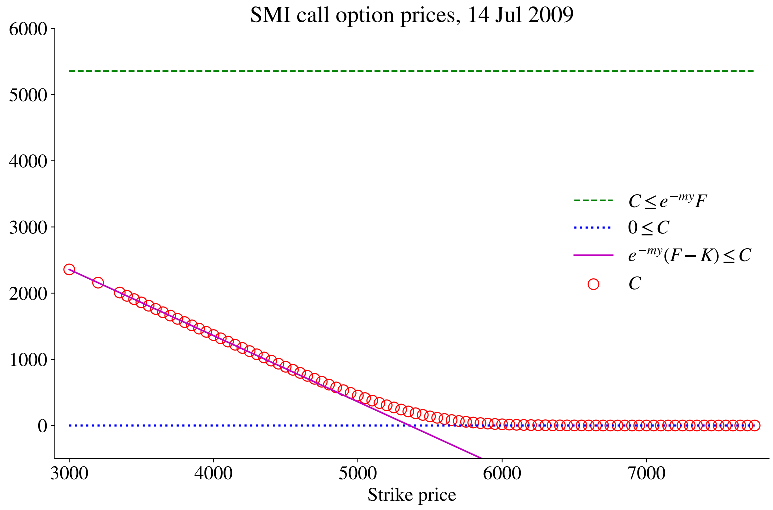


Figure 19.24: Prices and bounds for SMI options

strike price itself, but that the curve flattens out at high strike prices. See Figure 19.24 for an illustration.

Proof. (of (19.19)) If (19.19) was not true, so $C(K_2) > C(K_1)$, then a bull spread (buy $C(K_1)$ and sell $C(K_2)$), would have a negative price ($C(K_1) - C(K_2) < 0$). However, the payoff of a bull spread is

$$\max(0, S - K_1) - \max(0, S - K_2) = \begin{cases} 0 & \text{if } S \leq K_1 \\ S - K_1 & \text{if } K_1 < S \leq K_2 \\ K_2 - K_1 & \text{if } K_2 < S. \end{cases}$$

This would give a non-negative payoff for a negative asset price, which creates arbitrage opportunities. ■

Proof. (of (19.20)) If (19.20) was not true, so $C(K_1) - C(K_2) \geq K_2 - K_1$, then we can sell a bull spread (sell $C(K_1)$ and buy $C(K_2)$) and invest the proceeds in a T-bill (zero investment). The payoff at expiration (m period later) is then

$$\max(0, S-K_2)-\max(0, S-K_1) = \underbrace{[C(K_1) - C(K_2)]e^{rm}}_{>K_2-K_1} + \begin{cases} 0 & \text{if } S \leq K_1 \\ -(S - K_1) & \text{if } K_1 < S \leq K_2 \\ -(K_2 - K_1) & \text{if } K_2 < S. \end{cases}$$

In either case, there is a positive profit (recall that the initial investment is zero), which creates arbitrage opportunities. ■

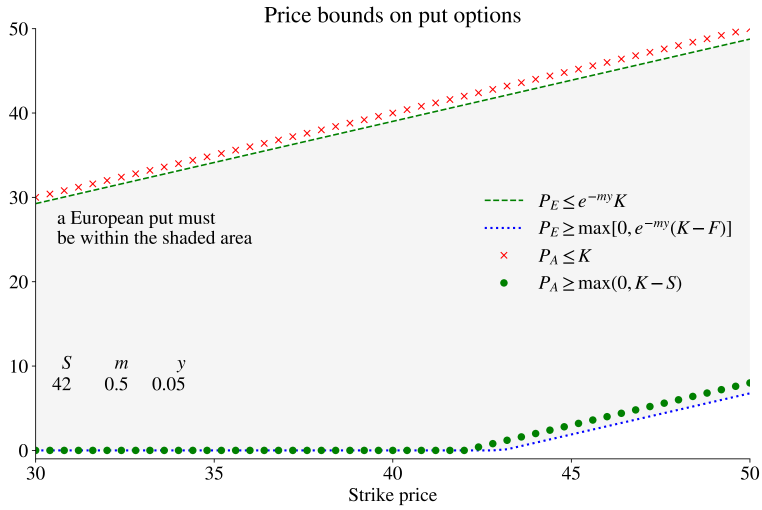


Figure 19.25: Put option price bounds as a function of the strike price

Proof. (of (19.21)) Let $\bar{K} = \lambda K_1 + (1-\lambda)K_2$. If (19.21) was not true, so $C(\bar{K}) > \lambda C(K_1) + (1-\lambda)C(K_2)$, then we can sell $C(\bar{K})$ and buy $\lambda C(K_1) + (1-\lambda)C(K_2)$ (zero investment). The payoff at expiration (m period later) is then

$$\lambda \max(0, S-K_1) - \max(0, S-\bar{K}) + (1-\lambda)\max(0, S-K_2)$$
$$= \begin{cases} 0 & = 0 & \text{if } S \leq K_1 \\ \lambda(S-K_1) & = \lambda(S-K_1) & \text{if } K_1 < S \leq \bar{K} \\ \lambda(S-K_1) - (S-\bar{K}) & = (1-\lambda)(S-K_2) & \text{if } \bar{K} < S \leq K_2 \\ \lambda(S-K_1) - (S-\bar{K}) + (1-\lambda)(S-K_1) & 0 & \text{if } K_2 < S, \end{cases}$$

where the second column uses the definition of $\bar{K}$. All payoffs are non-negative, and some are positive. Since the initial investment is zero, this creates arbitrage opportunities. ■

19.9 Appendix: Details on Early Exercise of American Options*

19.9.1 Early Exercise of American Call Options (No Dividends)

American call options on an asset without dividends (until expiration of the option) are not exercised early. The following proposition is more precise.

Proposition 19.17 *(No early exercise, American call, no dividends) An American call option on an asset without dividends should never be exercised early (if the interest rate*

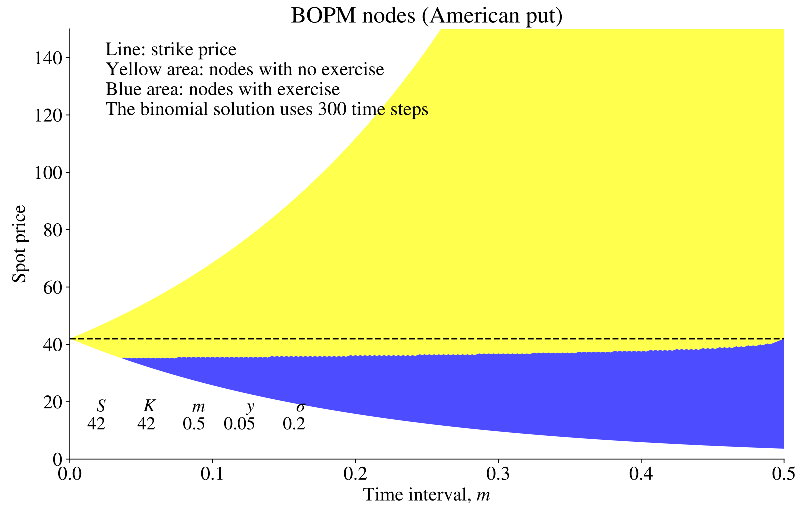


Figure 19.26: Numerical solution of an American put price (no dividends)

is positive). It therefore has the same price as a European call option.

See Figure 19.29 for an illustration of the fact that early exercise is not profitable since $C_A \geq C_E > \max(0, S - K)$.

Suppose that you are pretty sure that price of the underlying will drop tomorrow. The above proposition suggests that you should still not exercise the call option, but it might be sensible to sell the option today. If we exercised early, then we would effectively throw away the downside protection inherent in the call option and be left with the underlying asset and also pay the strike price now instead of later—neither of which is good (and which a potential buyer of the call option would be willing to pay for).

One way to think about this situation is as follows. If you think the underlying asset will drop in price soon, then you might be tempted to exercise now. If you do so and immediately sell the underlying, then you get $S_t - K$, which is worth $e^{y/365}(S_t - K)$ tomorrow. Rather, by keeping the option and shortening the underlying asset you will be better off. By shortening today you also get S_t, which is worth $e^{y/365}S_t$ tomorrow. Tomorrow, you need to close the short position: buy the asset for S_{t+1} or exercise the option (whichever is cheaper), so you pay $\min(K, S_{t+1})$, giving you a total value of at least $e^{y/365}S_t - \min(K, S_{t+1})$ tomorrow. (In case you do not exercise the option you also keep the value of it.) This is more than if you had exercised today. This shows that you should wait with exercising (and the same type of argument applies tomorrow).

Proof. (of Proposition 19.17) First, consider the case when $C_A > 0$. Then, (19.15)

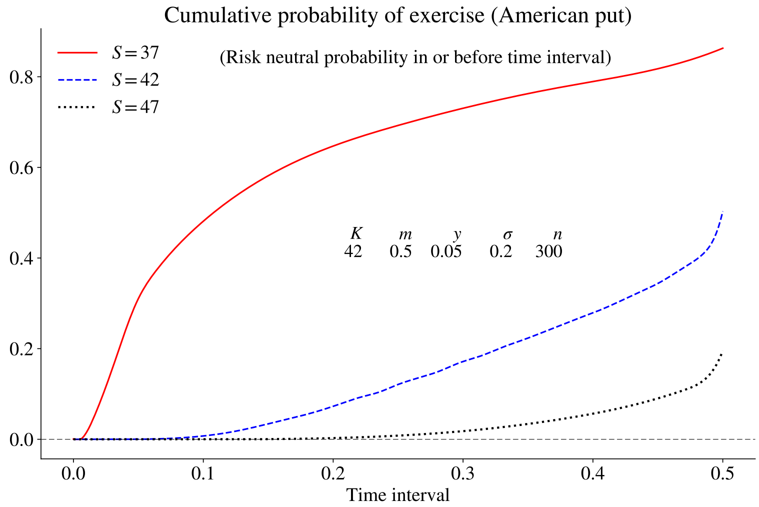


Figure 19.27: Probability of (early) exercise of American put option

shows that, as long as the interest rate is positive, $C_A > \max(0, S - K)$ for an underlying asset without dividends (since $S = e^{-my}F$ and $K > e^{-my}K$). This is higher than what exercising now gives, $S - K$, so exercising now is suboptimal. Second, if $C_A = 0$, then we know from (19.10) that $S \leq K$: (a) with $S < K$ exercise brings a loss; (b) while with $S = K$ and $C = 0$ we are indifferent (early exercise could thus happen, but it would be pointless). ■

Proof. (of Proposition 19.17, alternative*) First, consider the case when $C_A > 0$. Consider an investor who is willing to keep the underlying asset until tomorrow (at least). Clearly, such investors must exist, or else the underlying asset would not be worth its current price. Portfolio A: one American call option and $e^{-y/365}K$ on a bank account. Tomorrow, this portfolio is worth S_{t+1} if the option is exercised, and $K + C_{A,t+1}$ otherwise: $\max(S_{t+1}, K + C_{A,t+1})$. Portfolio B: one unit of the underlying asset, which is worth S_{t+1} tomorrow. Clearly, portfolio A is worth at least as much as B tomorrow, so it must also be worth at least as much right now: $C_A + e^{-y/365}K \geq S$. Rearranging gives $C_A \geq S - e^{-y/365}K$, and we also know that $C_A > 0$, so $C_A > \max(0, S - K)$ as long as the interest rate is positive. If you are not an investor who is willing to keep the underlying asset until tomorrow, then you should sell the option to such an investor. Second, if $C_A = 0$, then we know from (19.10) that $S \leq K$: (a) with $S < K$ exercise brings a loss; (b) while with $S = K$ and $C = 0$ we are indifferent (early exercise could thus happen, but it would be pointless). ■

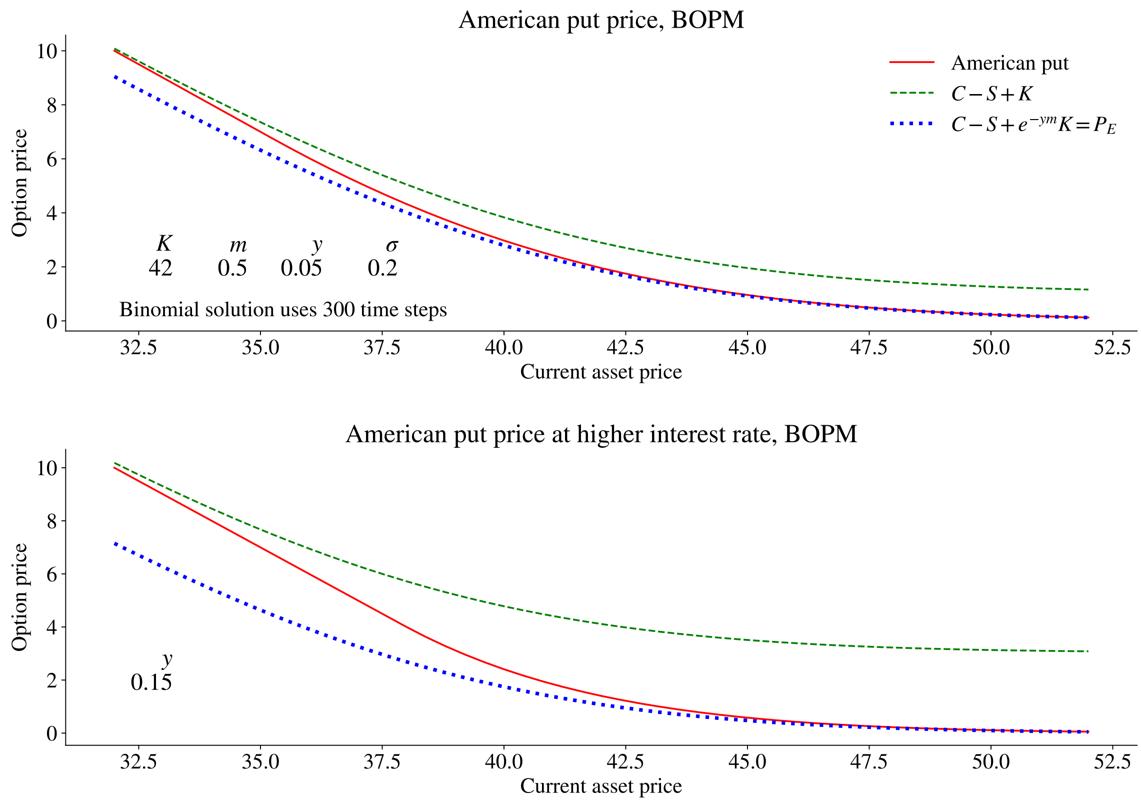


Figure 19.28: Numerical solution of an American put price (no dividends)

19.9.2 Early Exercise of American Put Options (No Dividends)*

American put options on an asset without dividends (until expiration of the option) may be exercised early. The following proposition is more precise.

Proposition 19.18 *(Early exercise, American put, no dividends) An American put option on an asset without dividends could be exercised early. However, we can rule out early exercise when* $P_E > \max(0, K - S)$*, since* $P_A \geq P_E$ *then implies that selling the option is better than exercising. From the put-call parity for European options, we notice that* $P_E > \max(0, K - S)$ *happens when* $C_E > (1 - e^{-my})K$*. For instance, this is always the case if the interest rate is zero—so there is no early exercise. This holds when* $K < S$ *(the put is out of the money), when* K *is slightly above* S *(the put is in the money, but not much), but not when the put is deep in the money. Hence, early exercise is only possible when the asset price is very low compared to the strike price.*

See Figure 19.30 for an illustration of the fact that early exercise is not profitable

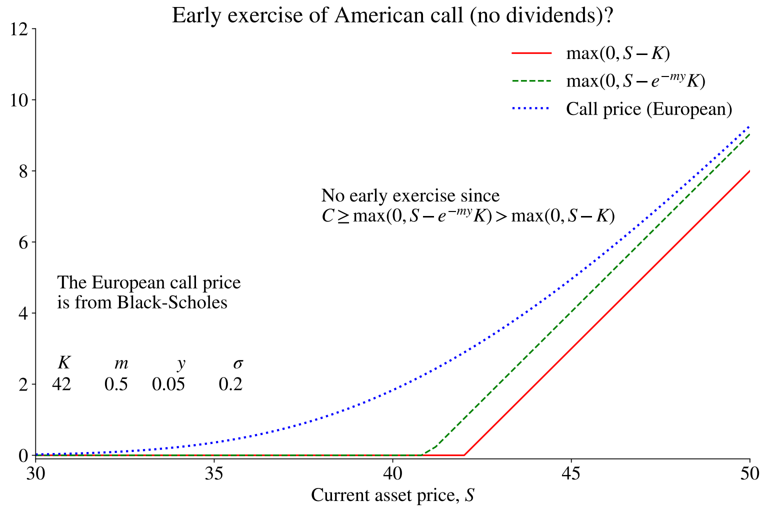


Figure 19.29: Early exercise of American call option (no dividends)

(since $P_E > \max(0, K - S)$) for high asset prices, but might be so for low asset prices (since $P_E < \max(0, K - S)$ means that $P_A < \max(0, K - S)$ is possible). Clearly, the proposition relies on having information about a European put price—or a good model of what the price should be. If we do not have that information, the proposition is not very useful—except in telling us that early exercise is more likely if the asset price is low and the interest rate high.

Example 19.19 *(Early exercise of American put option?) Using the same parameters as in Example 19.9, we have that $C_E > (1 - e^{-my})K$ is satisfied since*

$$5.5 > (1 - e^{-1/2 \times 0.05})38 = 0.94,$$

so there is no early exercise of the American put option. The reason is the put-call parity for European options (19.6) and the fact $P_A \geq P_E$ give

$$P_A \geq P_E = \underbrace{C_E}_{5.5} + K - S - \underbrace{(1 - e^{-my})K}_{0.94},$$

so selling the put option (getting P_A) gives the same as exercising $(K - S)$ plus at least $5.5 - 0.94 = 4.56$, so selling gives more than exercising. If, for some reason, we instead have $y = 35\%$ (so $(1 - e^{-my})K = (1 - e^{-1/2 \times 0.35})38 = 6.1$) but the same prices, then

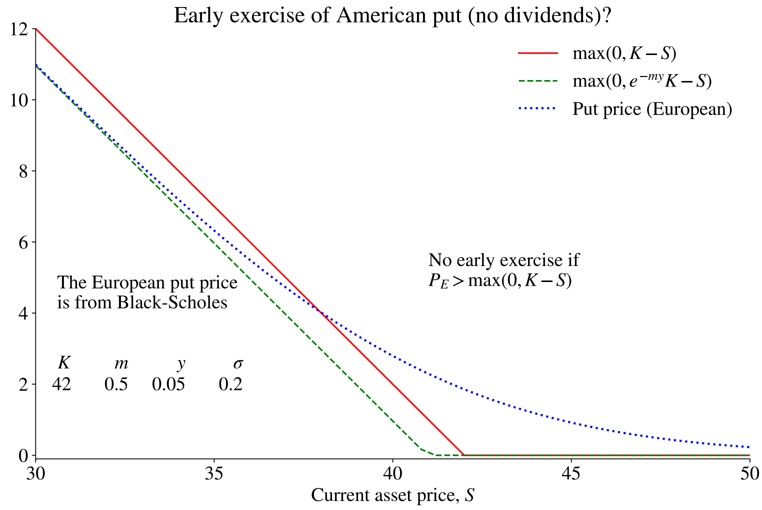


Figure 19.30: Early exercise of American put option (no dividends)

we would perhaps get early exercise. In particular, the expression above would say

$$P_A \geq P_E = \underbrace{C_E}_{5.5} + K - S - \underbrace{(1 - e^{-my})K}_{6.1},$$

so selling the put gives the same as exercising $(K - S)$ *plus at least* $5.5 - 6.1 = -0.6$*, so it's not sure that selling is better than exercising (could be or not).*

Proof. (of Proposition 19.18) To avoid early exercise, selling (getting P_A) should be more profitable than exercising (getting $K - S$), $P_A > K - S$. Put-call parity for European options (19.6) says

$$P_E = C_E + K - S - (1 - e^{-my})K.$$

If

$$C_E > (1 - e^{-my})K,$$

then $P_A \geq P_E > K - S$ so selling is better than exercising. This means that there is no early exercise if the European call price is high (high asset price compared to strike price), the strike price is low, or if the discounting until expiration is low (low interest rate or small time to expiration). For instance, with a zero interest rate, $P_A \geq C_E + K - S$, so there is never early exercise as long as $C_E > 0$. If these conditions are not satisfied, we cannot rule out early exercise. ■

19.9.3 Early Exercise of American Call and Put Options (Dividends)*

American call and put options on an asset with dividends (until expiration of the option) may be exercised early. The following propositions are more precise.

Proposition 19.20 *(Early exercise, American call, dividends) An American call option on an asset with dividends could be exercised early, especially just before a dividend payment and when the option is deep in-the-money (low strike price/high asset price). Conversely, there is no early exercise if $(1-e^{-my})K > \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i$, that is, with a high strike price and low present value of the dividends.*

Example 19.21 *(Early exercise, American call, dividends?) Suppose there is one dividend payment one month ahead: $D_1 = 0.95$ at $m_1 = 4/12$. If we use the same parameters as in Example 19.9, we then have*

$$(1-e^{-1/2\times0.05})38 = 0.94 > e^{-4/12\times0.05}0.95 = 0.93,$$

so we can rule out early exercise. However, if the dividend payment is at $m_1 = 1/12$, then we cannot.

Proof. (of Proposition 19.20) To avoid early exercise, selling (getting C_A) should be more profitable than exercising (getting $S-K$), $C_A > S-K$. Put-call parity for European options (19.7) says

$$C_E = S - K - \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i + (1-e^{-my})K + P_E.$$

If

$$(1-e^{-my})K > \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i,$$

and $P_E \geq 0$ (always true), then $C_A \geq C_E > S-K$: selling is better than early exercise. Hence, there is no early exercise if the present value of dividends is low, the strike price is high or if the discounting until expiration is large (high interest rate or long time to expiration). In the opposite case, we cannot rule out early exercise. ■

Proposition 19.22 *(Early exercise, American put, dividends) Early exercise is possible...*

19.10 Appendix: Put-Call Relation for American Options*

There is no put-call parity for American options. However, pricing bounds (based on the values of European options) can be derived.

Proposition 19.23 *(Put-call, American option, no dividend) For an American option on an asset without dividends, the put price must be inside the interval*

$$\underbrace{C_A - S + e^{-my} K}_{P_E} \leq P_A \leq \underbrace{C_A}_{C_E} - S + K. \tag{19.23}$$

The lower boundary is the European put price from (19.6). The reason is that the American and European call options have the same prices (the American call option on an asset without dividends is never exercised early—see Section 19.7). The upper bound is very similar, except that it involves the strike price, not its present value. Clearly, when the interest rate is low, then the interval is narrow—and with a zero interest rate it collapses to the put-call parity of European options. (The latter corresponds to the fact that an American put option on an asset without dividends is never exercised early if the interest rate is zero, see Section 19.7). The inequalities can be rewritten as

$$S - K \leq C_A - P_A \leq S - e^{-my} K, \tag{19.24}$$

which can be compared with the put-call parity for European options (19.5) which says that $C_E - P_E = S - e^{-my} K$.

See Figure 19.28 for an illustration.

Example 19.24 *(Bounds for an American put option) Using the same parameters as in Example 19.9, we get the following bounds for an American put option (no dividends)*

$$0.56 \leq P_A \leq 5.5 - 42 + 38 = 1.5.$$

Proof. (of Proposition 19.23) The lower boundary is the European put price (since $C_A = C_E$ when there are no dividends) and it is always true that $P_A \geq P_E$.

The upper boundary follows from the following argument where we compare two portfolios. Portfolio A: one call option with strike price K plus a deposit of K. Portfolio B: one put option plus one underlying asset. If the put option is held until expiration (the call is not exercised early), then portfolio A will be worth $\max(0, S_m - K) + e^{my} K$ in period m (where m is date of expiration), and portfolio B will be worth $\max(0, K - S_m) + S_m$, so portfolio A is worth (weakly) more. If, instead, the put is exercised earlier ($l < m$), then portfolio A will be worth $C_{A,l} + e^{ly} K$ in period l, and portfolio B will be worth $K - S_l + S_l = K$, so portfolio A is worth (weakly) more. In period 0 ($0 \leq l < m$) we don't know when/if the early exercise of the put will happen—but we know that in either case A portfolio will then be worth more than a portfolio B: portfolio A must therefore be worth (weakly) more than B already in 0: $C_{A,0} + K \geq P_{A,0} + S_0$, which is the upper bound in (19.23). ■

Proposition 19.25 *(Put-call, American option, dividends) With dividends, the upper boundary in (19.23) is changed by adding the present value of the dividend stream*

$$C_A - S + e^{-my} K \leq P_A \leq C_A - S + K + \sum\nolimits_{i=1}^{n} e^{-m_i y_t(m_i)} D_i. \tag{19.25}$$

Notice that the lower boundary is not equal to the European put price anymore (since $C_A \geq C_E$ and the present value of the dividends is not added). Together this means that the interval is wider with dividends than without dividends.

Proof. (of Proposition 19.25) The lower boundary follows from the following argument. Buy one call option, lend $e^{-my} K$, and sell one asset—the total value is $C_A + e^{-my} K - S$, which is the left hand side of (19.25). If the call is exercised prior to expiry, the payoff is $S - K + e^{-my} K - S = (e^{-my} - 1)K < 0$ which must be less than the value of the put whose value is nonnegative. If no early exercise, then the payoff at expiration is $\max(0, S - K) + K - S = \max(0, K - S)$ which is the same as the put payoff.

The upper boundary is a bit trickier, so we leave it for now. ■

Chapter 20

The Binomial Option Pricing Model

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23 and Hull (2009) 11
Additional references: McDonald (2014) 13–14

20.1 Overview of Option Pricing

There are basically two ways to model option prices: by some sort of factor model (like CAPM) or by a no-arbitrage argument. The latter is clearly much more precise, so it is typically preferred—when it works. These notes focus on a particularly simple case: when the underlying asset follows a binomial process.

20.2 The Basic Binomial Model

The binomial model, where the change of the price of the underlying asset can take only two values, is very stylized, but it is useful for establishing the key ideas of option pricing. It can also be transformed into a realistic model by cumulating many (short) subperiods. In the limit (as the subperiods become very many/very short) this binomial option pricing model (BOPM) converges to the well-known Black-Scholes model.

20.2.1 Binomial Process for the Stock Price

The binomial tree for the underlying asset starts at the current price S and has probability q of moving to Su in the next period and a probability of $1-q$ of moving to Sd. This is illustrated in Figure 20.1. These probabilities are the true ("natural") probabilities. Clearly, the expected value of the future asset price is $qSu + (1-q)Sd$.

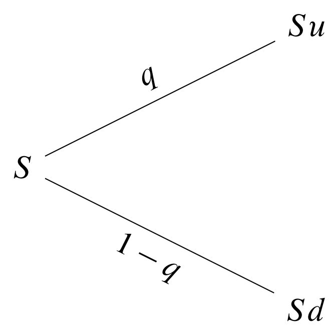


Figure 20.1: Binomial process for S

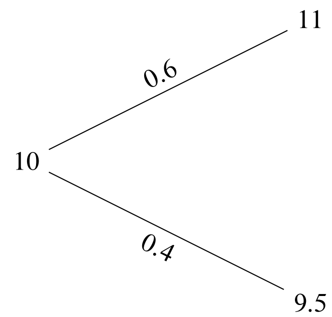


Figure 20.2: Numerical example of a binomial process for S

Example 20.1 *(Binomial process) Suppose $S = 10, u = 1.1, d = 0.95$, and $q = 0.6$. Then, the process has a 60% probability of increasing from 10 to 11 and a 40% probability of decreasing to 9.5. See Figure 20.2.*

We take it for granted that

$$u > e^{yh} > d. \tag{20.1}$$

If this condition is not satisfied, then there are a trivial arbitrage opportunities. For instance, if $e^{yh} > u$, then we could shorten the stock and buy bonds: this would guarantee a positive payoff for a zero investment (an arbitrage possibility).

20.2.2 No-Arbitrage Pricing of a Derivative

Basic Setup

Consider a derivative asset that will be worth f_u in case we end up at Su and f_d if we end up at Sd. Notice that f_u is just the notation for the value (price) of the derivative in

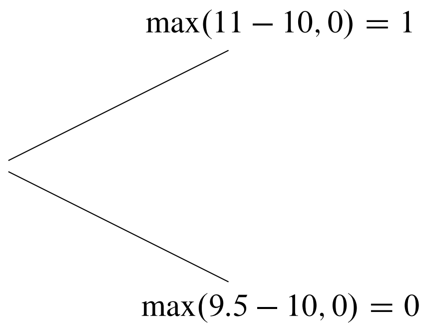


Figure 20.3: Numerical example of call option payoff

the up state (it should *not* be read as f times u).

As an example, suppose the derivative is a call option with strike price K and that the next period is the expiration date. Then,

$$f_u = \max(Su - K, 0) \text{ and } f_d = \max(Sd - K, 0). \tag{20.2}$$

Example 20.2 *(European call option) With the parameters in Example 20.1, equation (20.2) shows that a European call option with strike price of 10 has*

$$f_u = \max(11 - 10, 0) = 1 \textit{ and } f_d = \max(9.5 - 10, 0) = 0,$$

while a strike price of 9 gives

$$f_u = \max(11 - 9, 0) = 2 \textit{ and } f_d = \max(9.5 - 9, 0) = 0.5.$$

See Figure 20.3.

Step 1: Construct a Riskfree Portfolio

We now use a no-arbitrage argument to derive what today's price of the derivative (denoted f) must be.

Construct the following portfolio with Δ of the underlying asset and of -1 of the derivative replicates a safe asset:

$$\begin{aligned} &\Delta \text{ of the underlying asset, and} \\ &-1 \text{ of the derivative.} \end{aligned} \tag{20.3}$$

For a given value of Δ, the payoff of the portfolio in the next period is $\Delta Su - f_u$ in the "up" state and $\Delta Sd - f_d$ in the "down" state. To make the portfolio riskfree, Δ must be such that the payoff is the same in both states

$$\Delta Su - f_u = \Delta Sd - f_d, \text{ so}$$
$$\Delta = \frac{f_u - f_d}{S(u-d)}. \tag{20.4}$$

With this choice of Δ (also called the "delta hedge") the portfolio is riskfree. For future reference, we can also notice that Δ looks like $\partial f / \partial S$: the possible change of the derivative value $(f_u - f_d)$ as a fraction of the possible change of the underlying asset $(Su - Sd)$.

Example 20.3 *(European call option) Continuing Example 20.2 we get*

$$\Delta = \frac{1-0}{10(1.1-0.95)} = \frac{2}{3} \textit{ for } K = 10.$$

The payoff of this portfolio is indeed safe. For instance, for the $K = 10$ *option the value in the up state is* $\frac{2}{3} \cdot 11 - 1 = 19/3$ *and in the down state* $\frac{2}{3} \cdot 9.5 - 0 = 19/3$. *For a* $K = 9$ *call option,* $\Delta = \frac{2-0.5}{10(1.1-0.95)} = 1$.

Step 2: Make the Return of the Portfolio Equal to the Riskfree Rate

Since the choice of Δ in (20.4) makes the portfolio safe, it must have same return as the riskfree asset. This will help us determine what today's price of the derivative (f) is. The gross return on the riskfree from now until expiration (h periods later) is e^{yh} for the riskfree asset (y is the continuously compounded interest rate). For our portfolio, it is the portfolio value at expiration (same in the up and down states) divided by the price of the portfolio today $(\Delta S - f)$. Equating the two gross returns gives

$$\frac{\Delta Su - f_u}{\Delta S - f} = e^{yh}, \tag{20.5}$$

where we still keep the Δ notation (to save ink), but assume that Δ is determined as in (20.4).

Solve for the (current) price of the derivative, f, and use the value of Δ from (20.4)

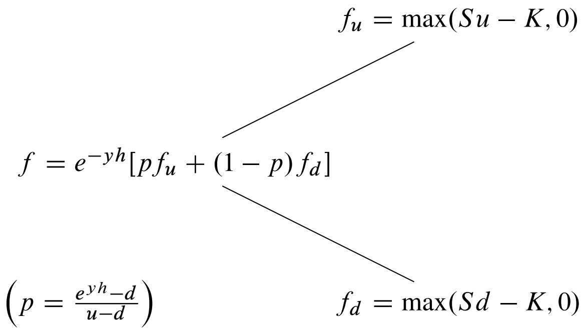


Figure 20.4: Solving for a call option price

that ensures that the portfolio is riskfree

$$
\begin{aligned}
f &= \Delta S(1 - e^{-yh}u) + e^{-yh} f_u & (20.6)\\
&= \frac{f_u - f_d}{u - d}(1 - e^{-yh}u) + e^{-yh} f_u & (20.7)\\
&= e^{-yh}\left[pf_u + (1-p)\, f_d\right] \text{ with } p = \frac{e^{yh} - d}{u - d} & (20.8)\\
&= e^{-yh}\, \mathrm{E}^*(\text{future payoff of derivative}) & (20.9)
\end{aligned}
$$

Equations (20.7)–(20.9) are alternative ways to write the price of the derivative.

Equation (20.7) shows what the price of the derivative must be—and is written in terms of the possible outcomes and the interest rate. Notice that neither probabilities (of the different outcomes), nor risk preferences enter this expression—since we have used a no-arbitrage argument to price this derivative. This works (that is, we can construct a riskfree portfolio) because we have as many relevant assets (riskfree and underlying risky asset) as there are possible outcomes (up or down).

Equation (20.8) shows that the current price of the derivative is the discounted value (e^{-yh}) times what looks like an expectation of the payoff of the derivative. This expression is quite useful since we can think of p as a "risk neutral probability"—although it is not a probability in the usual sense: it is just a convenient construction.

Notice that p does not depend on which derivative (with the same underlying asset) we consider: p depends on the underlying asset (and the interest rate), not the derivative. Under the restrictions in (20.1), $0 < p < 1$, as any "probability" should be. This interpretation is highlighted in (20.9), where E^* stands for the expectations according to the *risk neutral distribution* (more about that later). The computation in (20.8) is illustrated

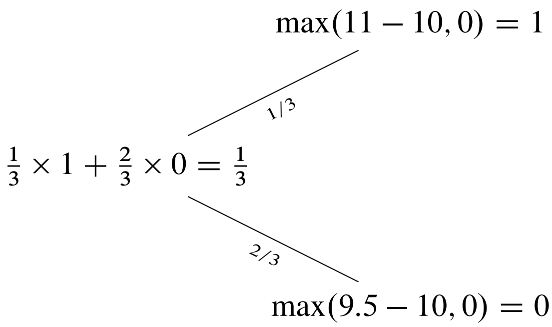


Figure 20.5: Numerical example of call option price, zero interest rate

in Figure 20.4.

Example 20.4 *(European call option) Continuing Example 20.2 and assuming that $y = 0$, equation (20.8) gives the price of a call option with strike price 10 as*

$$\begin{aligned} f &= e^{-0}\left[p1 + (1-p)\,0\right] \text{ with } p = \frac{1-0.95}{1.1-0.95} = 1/3 \\ &= 1/3. \end{aligned}$$

See Figure 20.5. For the call option with a strike price of 9, we get

$$f = e^{-0}\left[(1/3) \times 2 + (2/3) \times (1/2)\right] = 1.$$

20.2.3 Applying the No-Arbitrage Pricing on Different Derivatives

This section discusses how we apply the pricing formula (20.8) to some special derivatives.

Consider *the underlying asset itself*. It is clearly a trivial derivative with $f_u = Su$ and $f_d = Sd$. According to (20.8) the current price of the underlying asset should be

$$S = e^{-yh}\left[pSu + (1-p)\,Sd\right]. \tag{20.10}$$

This looks (again) like a discounted expected future payoff.

Example 20.5 *(The underlying asset itself) Continuing Example 20.4, equation (20.10) gives*

$$S = e^{-0}\left[(1/3) \times 11 + (2/3) \times 9.5\right] = 10.$$

A *forward contract* has a zero current price (nothing is paid until expiry), and the payoff at expiry is $f_u = Su - F$ in the up state (the value of the underlying asset minus the forward price) and $f_d = Sd - F$ in the down state. Using this in (20.8) gives

$$0 = e^{-yh} \left[p \left(Su - F \right) + (1 - p) \left(Sd - F \right) \right], \text{ so} \tag{20.11}$$

$$F = pSu + (1 - p) \, Sd. \tag{20.12}$$

This shows that the mean of the risk neutral distribution equals the forward price. Combining (20.10) and (20.12) clearly gives the spot-forward parity, $F = e^{yh} S$.

Example 20.6 *(A forward contract) Continuing Example 20.4, we get* $F = 10$ *(same as in Example 20.5) since the interest rate is zero).*

An "*Arrow-Debreu asset*" (a sort of theoretical derivative often used in asset pricing models) pays off one unit in the up state and zero otherwise ($f_u = 1$ and $f_d = 0$). This is also a so-called "cash-or-nothing" call option provided the up state means that the option is in the money ($Su > K$). From (20.8) we have

$$f = e^{-yh} p. \tag{20.13}$$

20.2.4 Replicating (and Hedging) a Derivative

The no-arbitrage argument in (20.4) was based on the fact that a portfolio of Δ of the underlying asset and of -1 of the derivative replicates a safe asset.

This argument can be turned around to replicate the derivative by holding the following portfolio (these are values of the positions)

$$\begin{aligned} &\Delta S \text{ in the underlying asset, and} \\ &-e^{-yh} \left(\Delta Su - f_u \right) \text{ in a safe asset.} \end{aligned} \tag{20.14}$$

This means that we hold Δ stocks (each of which costs S) and borrow $e^{-yh} \left(\Delta Su - f_u \right)$ on the money market. This replicates the derivative's payoff. We can therefore hedge a short position in the derivative by portfolio (20.14).

Proof. (of that (20.14) replicates the derivative) The payoff of this portfolio in the up state is $\Delta Su - (\Delta Su - f_u) = f_u$ and in the down state it is $\Delta Sd - (\Delta Sd - f_d) = f_d$ (since $\Delta Su - f_u = \Delta Sd - f_d$). ■

Example 20.7 *(Replicating a call option) For the call option with a strike price of 10 and with a zero interest rate, we have (see Example 20.3)* $\Delta = 2/3$ *and*

$$-e^{-yh}(\Delta Su - f_u) = -1(\frac{2}{3} \times 10 \times 1.1 - 1) = -6\frac{1}{3},$$

so we borrow. The value of this portfolio at the starting node is $\frac{2}{3} \times 10 - 6\frac{1}{3} = \frac{1}{3}$, *just like the call option. In the up node, the value is* $\frac{2}{3} \times 11 - 6\frac{1}{3} = 1$ *and in the down node* $\frac{2}{3} \times 9.5 - 6\frac{1}{3} = 0$ *which are also the same as the call option.*

Remark 20.8 *(Alternative expression for the replicating portfolio) The portfolio (20.14) can also be written*

$$\begin{aligned} &\Delta S \text{ in the underlying asset, and} \\ &f - \Delta S \text{ in a safe asset,} \end{aligned}$$

where f *is the value of the derivative today. (To show this, substitute for* f *using (20.6).)*

20.2.5 Where is the Risk Premium?

We have used a no-arbitrage method to price the derivative. It works since the derivative is a redundant asset: it can be replicated by a portfolio of the underlying asset and a riskfree asset—and therefore must have the same price as this portfolio. This does not mean, however, that the option is in itself riskfree. In fact, options are typically very risky and therefore carry large risk premia. It may seem as if the pricing formula (20.8) is free from the preference parameters that would determine the risk premium. Not correct. The pricing formula contains the current asset price (through f_u and f_d) which is indeed affected by preference parameters.

Also, recall that we can replicate the derivative by holding a portfolio of the underlying asset and bonds, see (20.14). Clearly, this portfolio will incorporate a risk premium—and so must the derivative.

20.3 Interpretation of the Risk Neutral Probabilities

The relation between the true probabilities (q) and the risk neutral probabilities (p) depends whether the underlying asset has a risk premium or not.

When the underlying asset has an expected return (over t to $t + h$) that is higher than the risk free rate, that is, a positive risk premium, then

$$\frac{\mathrm{E}_t\, S_{t+h}}{S} > e^{yh} \text{ (with positive risk premium),} \tag{20.15}$$

where S denotes the current price. (For simplicity, we disregard dividends here.)

From the spot-forward parity for an asset without dividends, we know that $F = e^{yh} S$. Combining gives

$$\mathrm{E}_t\, S_{t+h} > F \text{ (with positive risk premium).} \tag{20.16}$$

In particular, the binomial process implies that the expected value of the future asset price is

$$\mathrm{E}_t\, S_{t+h} = qSu + (1-q)\, Sd, \tag{20.17}$$

where q is the natural probability of the up state.

At the same time, the risk neutral expected value equals the forward price (see (20.12))

$$F = pSu + (1-p)\, Sd. \tag{20.18}$$

For $\mathrm{E}_t\, S_{t+h} > F$ to hold, we must have

$$q > p \text{ (with positive risk premium).} \tag{20.19}$$

To understand the intuition for this, consider the alternative case when the risk premium is zero (so $\mathrm{E}_t\, S_{t+h} = F$), then

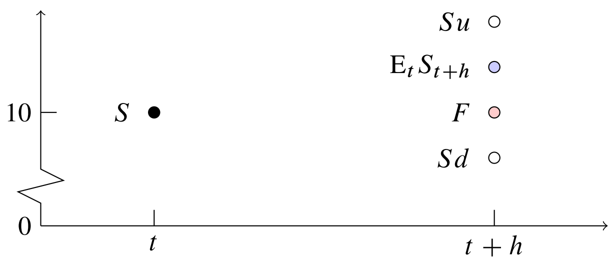
$$q = p \text{ (with no risk premium).} \tag{20.20}$$

The absence of a risk premium could either depend on (a) the asset has no systematic risk; or (b) that we have risk neutral investors. In either case, p equals the true probabilities—suggesting the name “risk neutral probability”for p.

Now, (20.19) is easier to interpret: both $\mathrm{E}_t\, S_{t+h}$ and F are averages of the same values (Sd and Su) and they only differ with respect to the probabilities. Clearly, for $\mathrm{E}_t\, S_{t+h}$ to exceed F, the former must have a higher probability for the high value (Su). See Figure 20.6 for an illustration. One interpretation is that a risk averse investor requires a higher probability of the up state (and thus a higher expected return) than a risk neutral investor.

Example 20.9 *(Natural versus risk neutral probability) With the parameters in Example 20.1*

$$\mathrm{E}_t\, S_{t+h} = 0.6 \times 11 + (1-0.6) \times 9.5 = 10.4.$$



Calculations:
$S = 10, Sd = 9.5, Su = 11$
$E_t S_{t+h} = 0.6 \times 11 + 0.4 \times 9.5 = 10.4$
$F = 1/3 \times 11 + 2/3 \times 9.5 = 10$

Figure 20.6: Risk premium and risk neutral probabilities

With $y = 0$, $F = S = 10$. In this case, the underlying asset indeed has a positive risk premium (see (20.16)), and $q = 0.6$ while $p = 1/3$. See Figure 20.6 for an illustration.

Proof. (of (20.19)) For (20.16) to hold, we need $qSu + (1-q)Sd > pSu + (1-p)Sd$. Subtract Sd from both sides to get $qS(u-d) > pS(u-d)$ and notice that $S(u-d) > 0$ to conclude that $q > p$ is required. ∎

20.4 Multi-Period Trees I: Basic Setup

20.4.1 The Binomial Tree for the Underlying Asset

In numerical applications, we chain a large number of up/down movement to create more realistic model properties of the underlying asset. This means that time to expiration is divided into many small time steps and that we can rebalance the portfolio at each of them.

Figure 20.7 is an illustration of a binomial tree with two subintervals and Figure 20.8 gives a numerical example. This tree has only three final nodes since $Sud = Sdu$: it is "recombining," which is very useful to keep the number of nodes manageable. This would not be the case if the up and down moves were different for different periods (non-iid price process).

Let m be the time to expiration of the derivative. With n short time intervals, the length of each interval is $h = m/n$, see Figure 20.9. Clearly, if we use more time steps, then each of them is shorter. The size of the up and down movements as well as the discounting

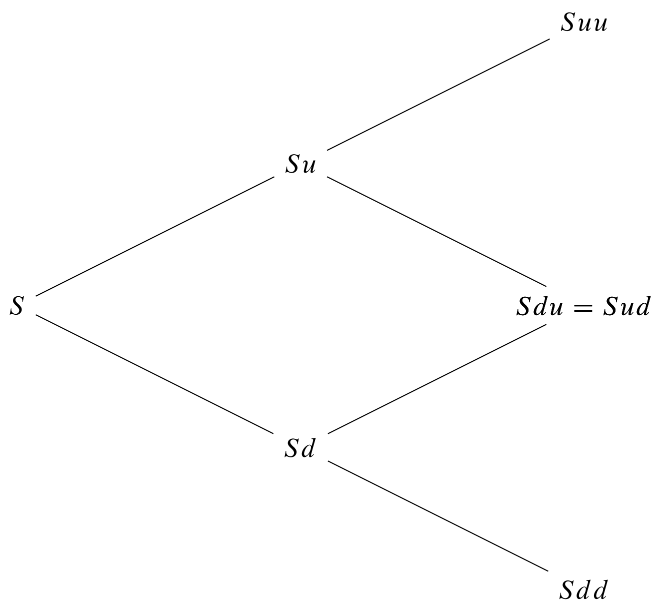


Figure 20.7: Binomial tree for underlying asset ($n = 2$)

must also be scaled by the number of time steps (details about this are discussed later).

Remark 20.10 *(Size of the binomial tree*) With n time steps, there are $n + 1$ different prices at the end nodes. Also, there are a total of $(n + 1)(n + 2)/2$ nodes. There are $n!/[(n - s)!s!]$ different ways to reach the sth node below the top node (where $x! = x \times (x-1) \times \ldots \times 1$). Summing across the nodes shows that the tree contains 2^n different paths. For instance, our (recombining) tree has*

n	*no. end nodes*	*no. total nodes*	*no. paths*
2	3	6	4
25	26	351	33,554,432
200	201	20,301	1.6×10^{60}

In contrast, a non-recombining tree has 2^n end nodes, that is, as many as there are paths in the recombining tree.

20.4.2 Using a Binomial Tree for Pricing European Options

We can now apply the pricing formula (20.8) to each "subtree," beginning at the end of the tree (time step 2) and working backwards towards the start of the tree (time step 0).

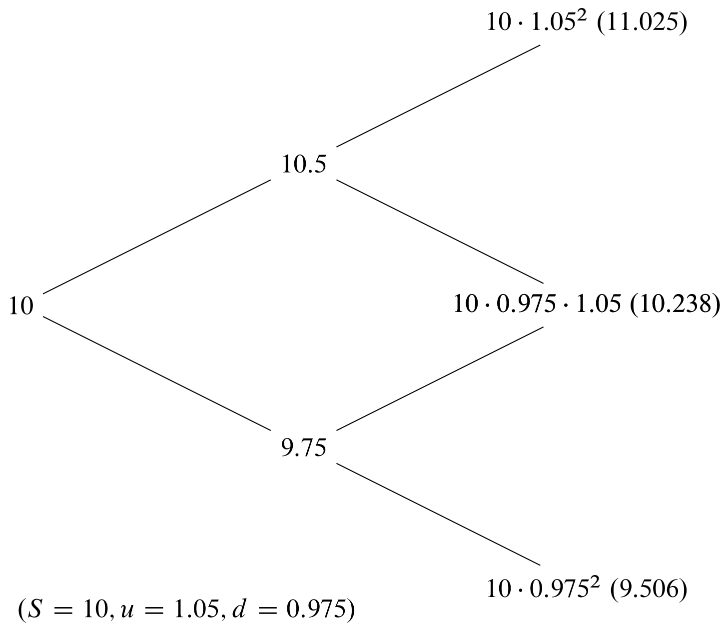


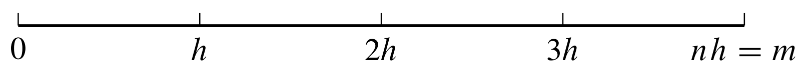
Figure 20.8: Numerical example of a binomial tree for underlying asset ($n = 2$)

Figure 20.10 illustrates the computations for a European call option with strike price K and two steps ($n = 2$) and Figure 20.11 gives a numerical example.

The structure of the tree for a European put option is the same as for a European call option, except at the end nodes, see Figure 20.12.

Example 20.11 *(Tree for a European put) For a put option with strike price* $K = 10$*, the values in Figure 20.11 would change to* $f = 0.219$*,* $(f_u, f_d) = (0, 0.329)$ *and* $(f_{uu}, f_{ud}, f_{dd}) = (0, 0, 0.494)$.

This recursive calculation (using a tree with two time steps as in Figure 20.10) gives



Expiry: $m = 1/2$ years
Steps: $n = 4$
Step length: $h = m/n = 1/8$ years

Figure 20.9: Steps to reach time to expiration m

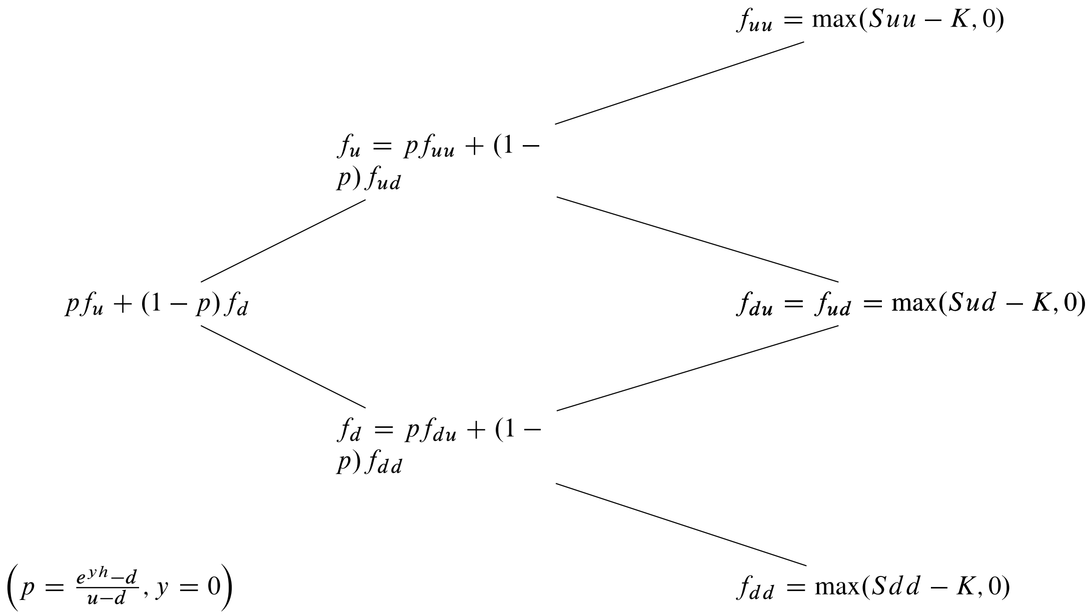


Figure 20.10: Binomial tree for European call option ($n = 2$), zero interest rate

the European option price

$$
\begin{aligned}
f &= e^{-yh}[pf_u + (1-p)f_d] \\
&= e^{-ym}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}],
\end{aligned}
\tag{20.21}
$$

since $2h = m$ (which means that $e^{-y2h} = e^{-ym}$).

Notice that p^2 is the risk-neutral probability of the payoff f_{uu}, $2p(1-p)$ of the payoff f_{ud} and $(1-p)^2$ of the payoff f_{dd} — see Figure 20.13. Therefore, (20.21) is a generalisation of (20.9):

$$
f = e^{-ym}\,\mathrm{E}^*(\text{payoff of derivative at expiration}), \tag{20.22}
$$

which says that the (european style) derivative is the present value of the risk-neutral expected payoff at expiration. For instance, the call payoff at expiration is $\max(0, S_m - K)$ and the put payoff is $\max(0, K - S_m)$.

Remark 20.12 *(The binomial distribution*) After n independent draws, the number of up moves (k) has the binomial pdf, $n!/[k!(n-k)!]p^k(1-p)^{n-k}$ for $k = 0, 1, ..., n$. For instance, with $n = 2$, we have p^2 for $k = 2$, $2p(1-p)$ for $k = 1$, and $(1-p)^2$ for $k = 0$.*

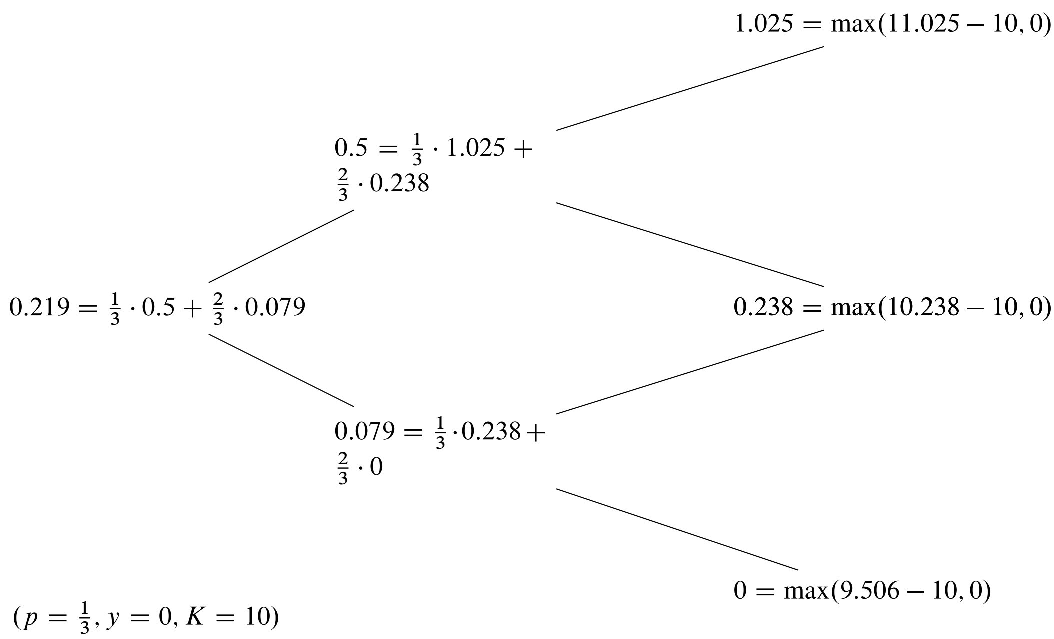


Figure 20.11: Numerical example of binomial tree for European call option ($n = 2$), zero interest rate. The underlying is described in Figure 20.8.

20.4.3 Using a Binomial Tree for Pricing American Options

The binomial tree we have used so far assumes that the derivative is "alive" until expiration. This is not necessarily the case for American options, so the approach needs to be modified to handle the possibility of early exercise.

Whenever you can exercise, the option value is the maximum of the exercise value and the value of keeping the option alive

$$\max(\text{value if exercised now,value of keeping an unexercised option}). \tag{20.23}$$

The value of an unexercised option is calculated as in (20.8): the present value of the risk neutral expected value in the next time step. This means that we solve this problem starting from the expiration date (just like for the European options), and calculate the value at each node—assuming (perhaps counterfactually) that the option has not already been exercised at an earlier time step. See Figure 20.14 for an illustration. Also, see Figure 20.15 for a numerical example. The nodes were exercise is optimal are indicated by bold.

Figure 20.16 illustrates the solution for an American put option on an asset without

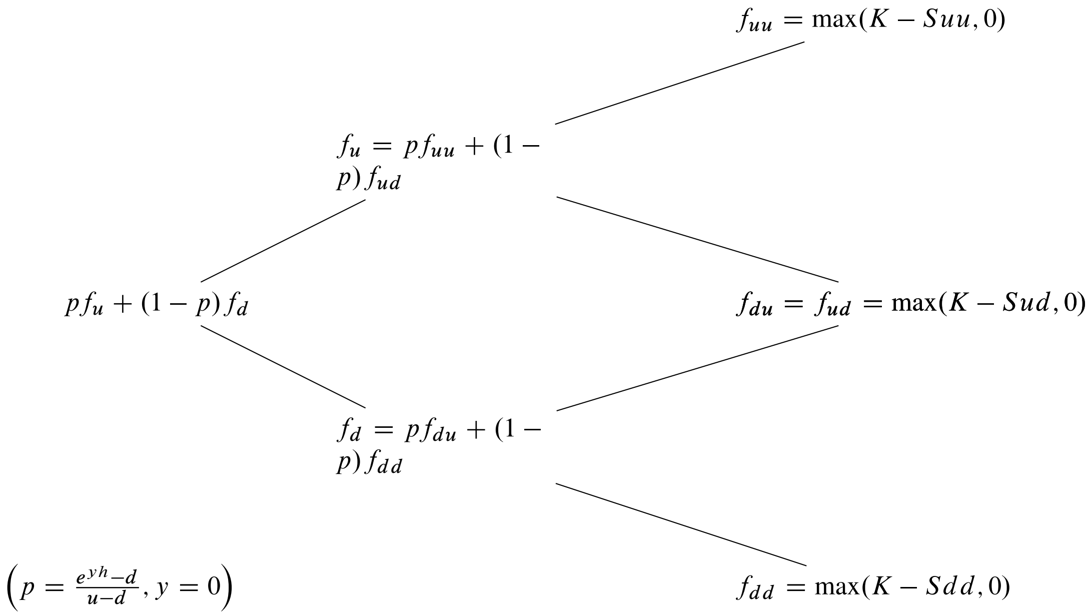


Figure 20.12: Binomial tree for a European put option ($n = 2$), zero interest rate

dividends (the details of the calculations will be discussed later). Notice that the American put price exceeds the European put price—and more so at low asset prices ($S < K$ is necessary, but not sufficient for early exercise) and high interest rates. The lower and upper limits on the put price are from the put-call "parity" (two inequalities) for American options. The call price C used in the figure is the same for European and American options (since there is no early exercise in this case).

20.5 Multi-Period Trees II: Calibrating the Tree

We now discuss how to construct a binomial tree (how to choose u and d) with many small time steps—so that it mimics the statistical properties of the underlying asset.

20.5.1 Mean and Variance of Data

Suppose you have a sample of log returns (r_τ for $\tau = 1$ to T) of the underlying asset—and that you are willing to assume that the *log returns are iid*. Calculate the sample mean

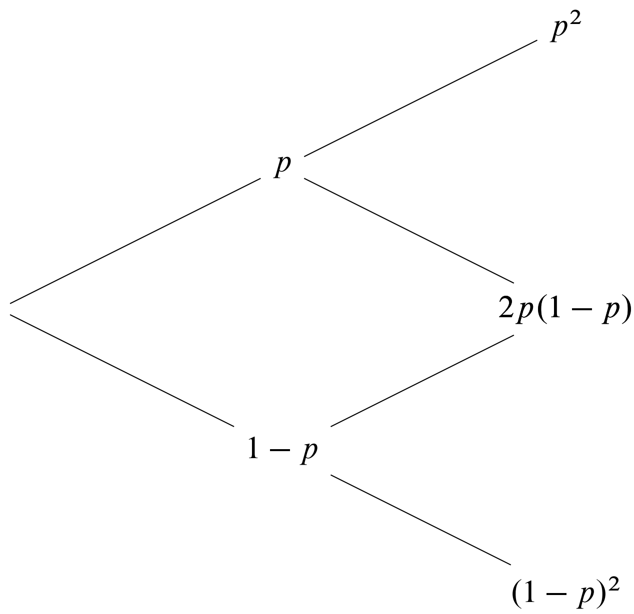


Figure 20.13: Probabilities of different nodes in a binomial tree

and variance and *annualize* them by dividing by the time length of your return periods (k)

$$\hat{\mu} = \frac{1}{k}\bar{r}_\tau \tag{20.24}$$

$$\hat{\sigma}^2 = \frac{1}{k}\widehat{\text{Var}}(r_\tau) \tag{20.25}$$

For instance, with daily return data $k = 1/252$ (only counting the trading days). Expressing the moments in terms of annualised numbers ($\hat{\mu}$ and $\hat{\sigma}^2$) helps relating to the binomial model—and to compare results across different sampling intervals.

Example 20.13 *(Variance for daily return) If the data is daily ($k = 1/252$, assuming 252 trading days per year) and the standard deviation is estimated to be* 0.0126*, then the annualised variance is* $\hat{\sigma}^2 = 0.0126^2 \times 252 \approx 0.2^2$ *and the annualized standard deviation is* $\hat{\sigma} \approx 0.0126 \times \sqrt{252} = 0.2$.

20.5.2 Mean and Variance according to the Binomial Model

Recall the binomial process (for instance, in Figure 20.1)

$$S_{t+h} = \begin{cases} Su & \text{with probability } q \\ Sd & \text{with probability } 1-q, \end{cases} \tag{20.26}$$

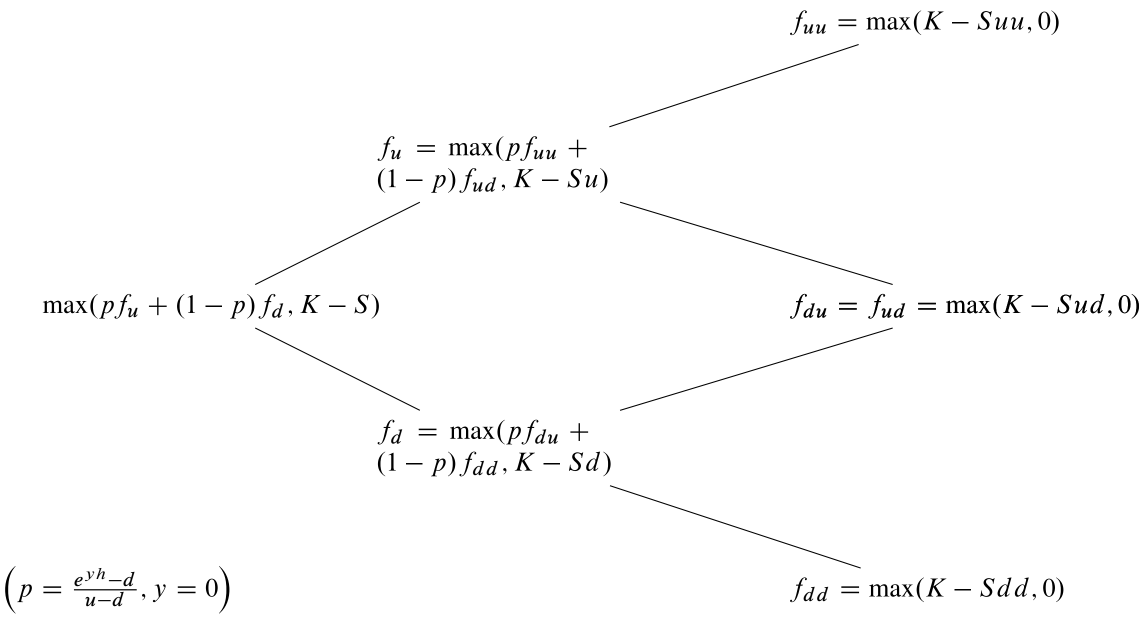


Figure 20.14: Binomial tree for an American put option ($n = 2$), zero interest rate

where S_{t+h} denotes the price in the next period (each period in the model is h years long). Clearly, this means that the log returns, $r_{t+h} = \ln S_{t+h} - \ln S_t$, follow

$$r_{t+h} = \begin{cases} \ln u & \text{with probability } q \\ \ln d & \text{with probability } 1-q. \end{cases} \tag{20.27}$$

Remark 20.14 *(Mean and variance of a binomial process) The mean of a (shifted) binomial process like (20.27) is* $q \ln u + (1-q) \ln d$ *and the variance is* $q(1-q)(\ln u - \ln d)^2$.

This binomial process implies that the *annualized* mean and variance of the asset returns should be (see Remark 20.14)

$$\text{annualized mean} = \frac{1}{h}[q \ln u + (1-q) \ln d], \tag{20.28}$$

$$\text{annualized variance} = \frac{1}{h} q(1-q)(\ln u - \ln d)^2. \tag{20.29}$$

Notice that the k in the sampling frequency and the h in the binomial tree need not be the same.

Example 20.15 *(Binomial process) Suppose* $S = 10, u = 1.1, d = 0.95$, *and* $q = 0.6$. *This gives an expected value of* $0.6 \times \ln 1.1 + 0.4 \times \ln 0.95 = 0.037$ *and a variance of*

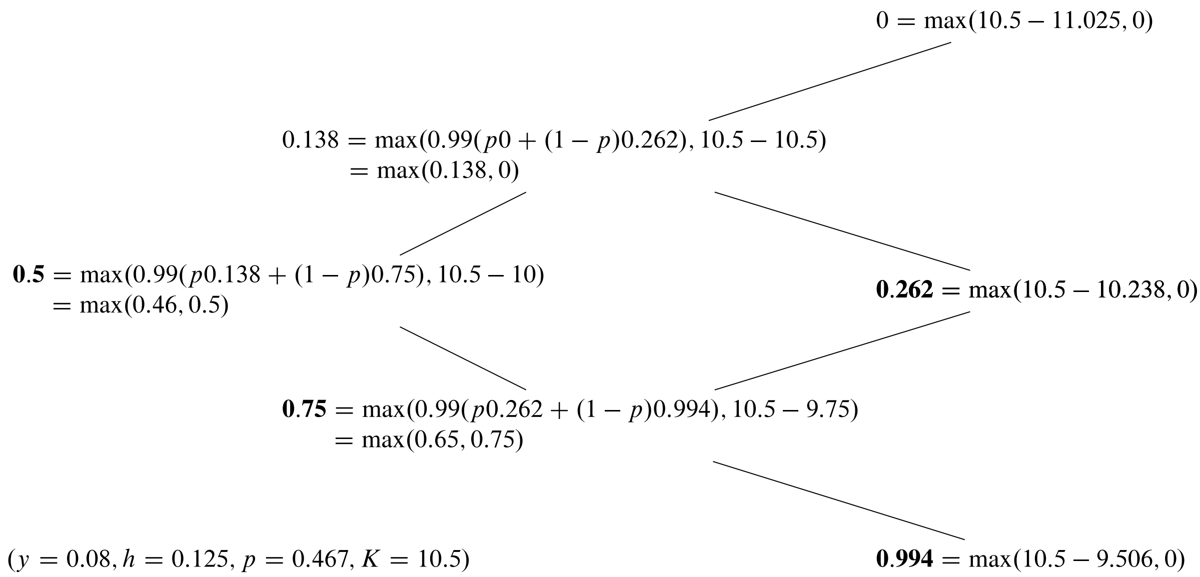


Figure 20.15: Numerical example of a binomial tree for an American put option ($n = 2$). Exercise is indicated by bold.

$0.6 \times 0.4 \times (\ln 1.1 - \ln 0.95)^2 = 0.0052$. *If the periods in the model are weeks* ($h = 1/52$) *, then the annualized mean is* $0.037 \times 52 \approx 1.9$ *and the annualized variance is* $0.0052 \times 52 \approx 0.27$.

20.5.3 Comparing Data and Model

There are three parameters (u, d, and q) which can be chosen to match the two moments, that is, to make the annualized mean and the variance from the model (20.28)–(20.29) equal to $(\hat{\mu}, \hat{\sigma}^2)$ from data in (20.24)–(20.25). We therefore have some free choices. The following is a common approach.

First, for any u and d (not yet decided), pick q to match the annualized sample mean, that is,

$$\hat{\mu} = \frac{1}{h} q \ln u + \frac{1}{h}(1-q) \ln d. \tag{20.30}$$

(The solution is $q = (\hat{\mu} h - \ln d)/(\ln u - \ln d)$.)

Strictly speaking, we do not need the physical probability q in order to price derivatives (the risk neutral probability is enough), but it is still useful to understand the logic of calibrating d and u (below).

Second, using the q from above, we now try to pick u and d to match the annualized

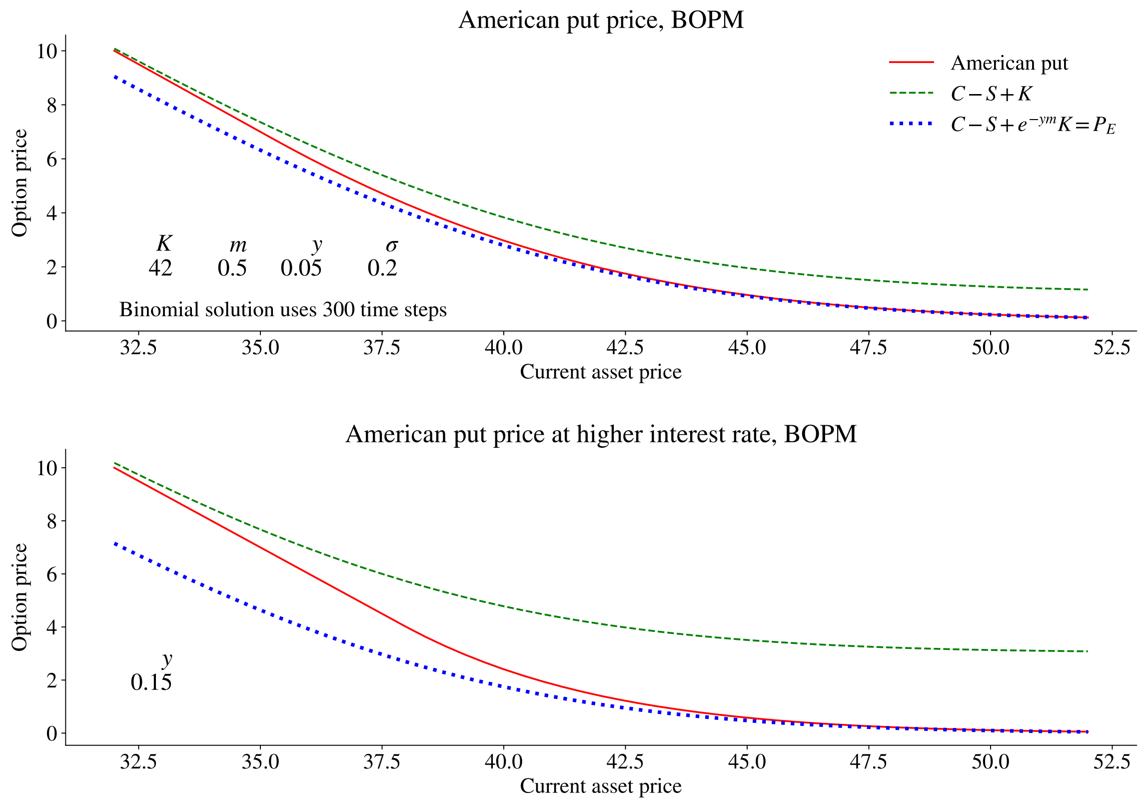


Figure 20.16: Numerical solution of an American put price

sample variance, that is, we try to match

$$\hat{\sigma}^2 = \frac{1}{h} q(1-q)(\ln u - \ln d)^2. \tag{20.31}$$

However, this is one equation with two unknowns (d and u), so we must impose further restrictions.

20.5.4 The CRR Approach

There are several ways to proceed to pick u and d, but the most common is the approach of Cox, Ross, and Rubinstein (1979) where

$$u = e^{\hat{\sigma}\sqrt{h}} \text{ and } d = e^{-\hat{\sigma}\sqrt{h}}. \tag{20.32}$$

There are other ways to construct the binomial tree, but they have similar properties.

Example 20.16 *(Parameters to binomial tree) With $h = 1/52$ and $\hat{\sigma} = 0.2$, (20.32) gives $u \approx 1.028$ and $d \approx 0.973$.*

Using (20.32) together with (20.30) and (20.31) gives (after some straightforward algebra) that the annualized variance of the binomial process is

$$\text{Annualized Var (binomial process)} = \hat{\sigma}^2 - \hat{\mu}^2 h. \tag{20.33}$$

This does not fit the volatility exactly because of the $\hat{\mu}^2 h$ term, but the approximation improves as h decreases (the number of time steps increases).

Notice that once we have the values of u and d, the pricing of derivatives does not use the natural probability of the up state (q).

However, we must ensure that (20.1) holds ($u > e^{yh} > d$, to rule arbitrage opportunities), that is,

$$e^{\hat{\sigma}\sqrt{h}} > e^{yh} > e^{-\hat{\sigma}\sqrt{h}}, \tag{20.34}$$

which requires $\hat{\sigma} > y\sqrt{h} > -\hat{\sigma}$. In practice, this means that h must be small (the number of step, n, large). Always check that this condition is satisfied.

Example 20.17 *(Checking parameters of binomial tree) With the parameters in Example 20.16 and assuming $y = 0.05$, we notice that $e^{yh} = e^{0.05/52} \approx 1.001$, so the requirement is fulfilled*

$$1.028 > 1.001 > 0.973.$$

20.6 Multi-period Trees III: Numerical Applications

20.6.1 Implementing the CRR Approach

We now use the CRR approach (20.32) together with (20.8) in each "subtree" of the large tree.

See Figures 20.17–20.18 for an illustration of how the parameters (p, u, d) and the resulting option price converge as the number of time steps increases (keeping the time to expiration fixed).

Figure 20.19 illustrates the calculations of the American put price for one current value of the underlying asset. The shaded areas show the location of the nodes (future prices of the underlying asset) that are used in the calculation—and at which nodes that early exercise will happen.

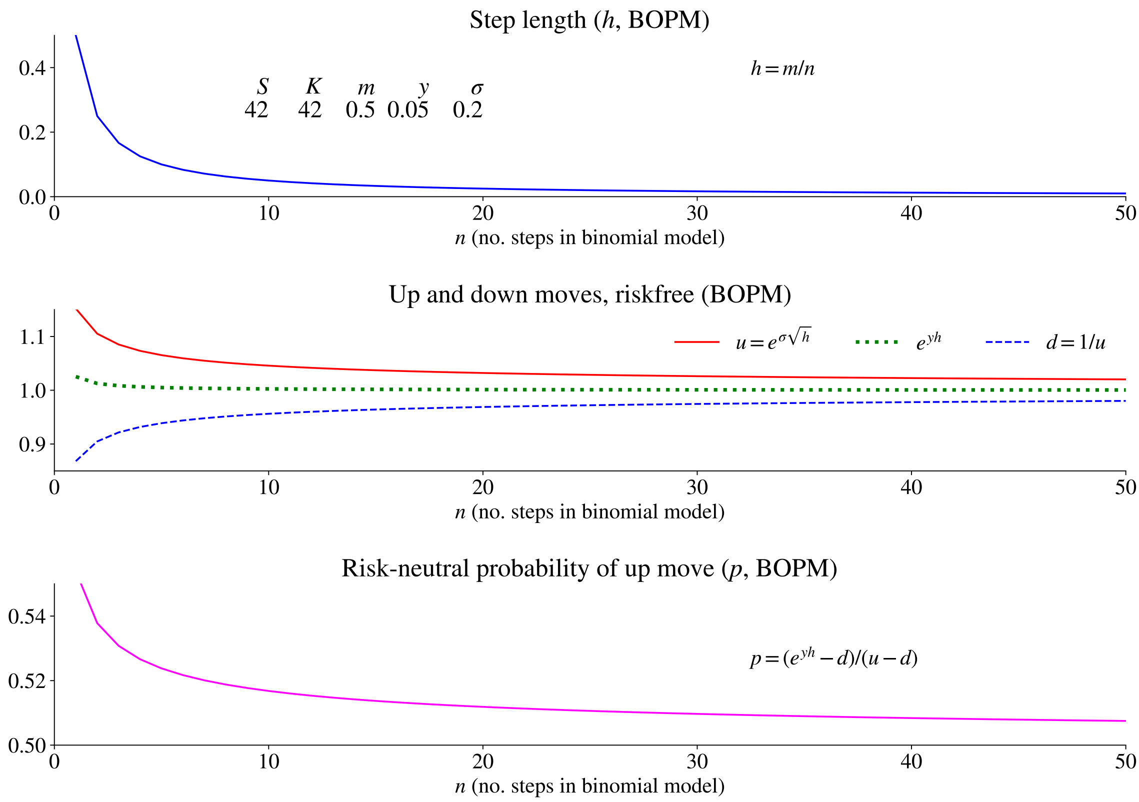


Figure 20.17: Convergence of the parameters in a binomial model

20.6.2 A Binomial Tree with Continuous Dividends*

It is straightforward to construct another tree that allows for continuous dividends, provided they are proportional to the asset price.

Suppose dividends are paid at the (known) continuous *rate* δ and let the up and down movements in the asset price reflect the ex-dividend price (S in the initial period t). Buying one unit of the underlying asset in the initial period costs S. If we move to the "up state" in the next period ($t + h$), then the owner first gets the dividend $Su(e^{\delta h} - 1)$ and can then sell the asset for the (ex-dividend) price Su: the total value is $Sue^{\delta h}$. Notice that the dividend is proportional to price in the same period. The "down state" is similar: just replace u by d.

We now construct a risk-free portfolio to find out how a derivative is priced in the initial period. First, to construct a riskfree portfolio, hold Δ of the underlying asset and -1 of the derivative. The payoff of the portfolio at expiry is $\Delta Sue^{\delta h} - f_u$ in the "up" state and $\Delta Sde^{\delta h} - f_d$ in the "down" state. To make the portfolio riskfree the delta must

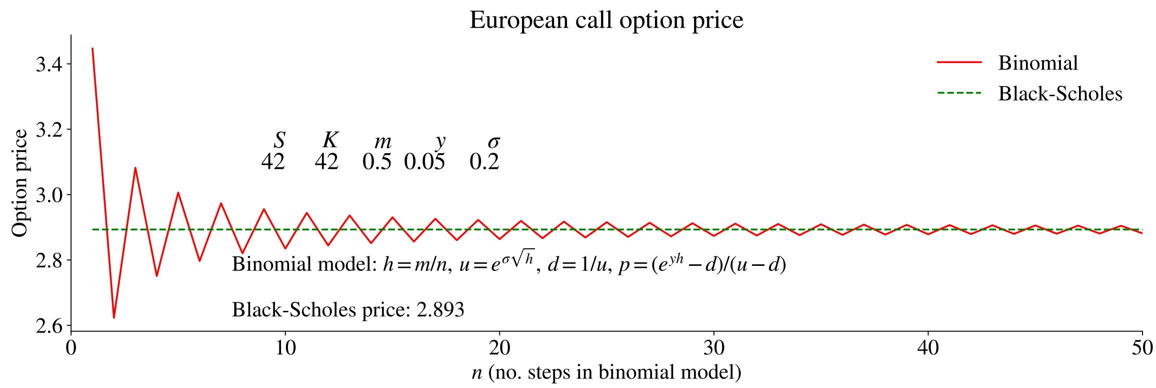


Figure 20.18: Convergence of the binomial price

be

$$\Delta = \frac{f_u - f_d}{Se^{\delta h}(u-d)}. \tag{20.35}$$

Second, to make the return of the portfolio equal to the riskfree rate, we set the present value of our riskfree portfolio equal to the cost of the portfolio

$$e^{-yh}[\Delta S e^{\delta h} u - f_u] = \Delta S - f. \tag{20.36}$$

Use (20.35) and rearrange as

$$f = \Delta S[1 - e^{(\delta - y)h} u] + e^{-yh} f_u \tag{20.37}$$

$$= \frac{f_u - f_d}{e^{\delta h}(u-d)}[1 - e^{(\delta - y)h} u] + e^{-yh} f_u \tag{20.38}$$

$$= e^{-yh}[p f_u + (1-p) f_d] \text{ with } p = \frac{e^{(y-\delta)h} - d}{u - d}. \tag{20.39}$$

With this new definition of p, the rest of the computations are as in the case without dividends. In particular, the drift of the asset price does not matter, so u and d can be chosen as before, for instance, as in (20.32).

Remark 20.18 *(Risk neutral drift with continuous dividends) With continuous dividends, the risk neutral expected value is* $\mathrm{E}_t^p\, S_{t+h}/S_t = e^{(y-\delta)h}$*, so the drift is* $(y-\delta)h$ *over the short time interval* h*.*

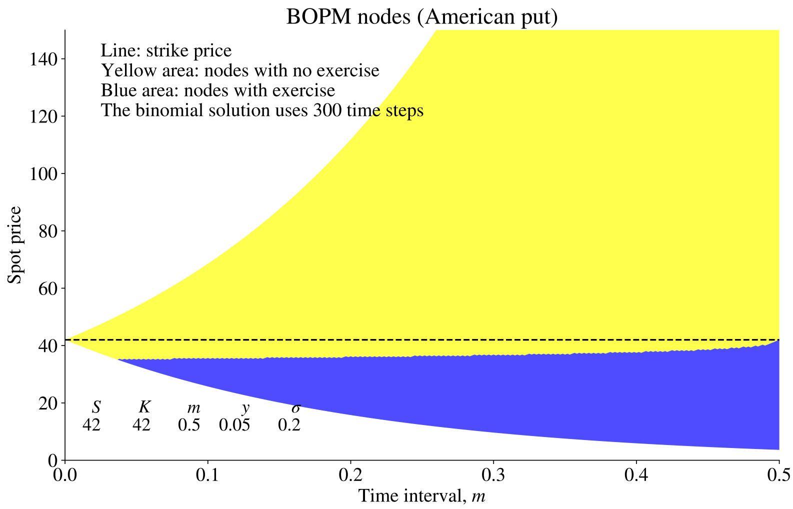


Figure 20.19: Numerical solution of an American put price

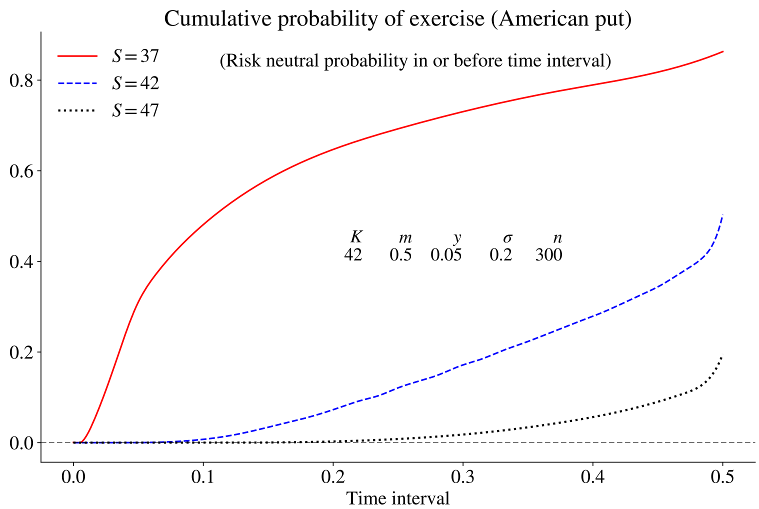


Figure 20.20: Probability of (early) exercise of American put option

Chapter 21

The Black-Scholes Model

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23 and Hull (2009) 13 and 17

Additional references: McDonald (2014) 15–16 and Cox, Ross, and Rubinstein (1979)

21.1 The Black-Scholes Model

Remark 21.1 *(On the notation) The notation is kept short. The current period is assumed to be $t = 0$ and the derivative expires in $t = m$. The current price of the underlying is denoted S (rather than S_0), the forward price according to a contract agreed on now and expiring in $t = m$ is F (rather than $F_0(m)$) and the continuously compounded interest between $t = 0$ and $t = m$ is y (rather than $y_0(m)$). However, to avoid confusion, the price of the underlying asset at expiration is denoted S_m. Otherwise, time subscripts are only introduced when strictly needed.*

21.1.1 The Basic Black-Scholes Model without Dividends

Assume that the change over a short interval (between t and $t + h$) in the log asset price is an iid process

$$\ln S_{t+h} - \ln S_t = \mu h + \varepsilon_{t+h}, \text{ with } \varepsilon_{t+h} \sim iid\ N(0, \sigma^2 h). \tag{21.1}$$

This clearly means that $\ln S_{t+h}$ is a random walk with drift: just add $\ln S_t$ to both sides of the equation. (Also, notice that $\ln S_{t+h} - \ln S_t = \ln S_{t+h}/S_t$, which is also used below.) This implies that, based on the information in period 0, the logarithm of the stock price in period m, S_m, is normally distributed

$$\ln S_m \sim N(\ln S + \mu m, \sigma^2 m), \tag{21.2}$$

where S is the current ($t = 0$) asset price. See Figure 21.1 for an illustration.

Proof. (of (21.2)) Notice that (21.1) implies that

$$\ln S_1 = \ln S_0 + \mu + \varepsilon_1 \text{ and}$$
$$\ln S_2 = \ln S_1 + \mu + \varepsilon_2 = (\ln S_0 + \mu + \varepsilon_1) + \mu + \varepsilon_2.$$

Since $\mathrm{E}_0\, \varepsilon_1 = 0$ and $\mathrm{E}_0\, \varepsilon_2 = 0$, the conditional means are $\mathrm{E}_0 \ln S_1 = \ln S_0 + \mu$ and $\mathrm{E}_0 \ln S_2 = \ln S_0 + 2\mu$. The conditional variances are just the variances of the forecast errors (the ε part), so $\mathrm{Var}_0(\ln S_1) = \mathrm{Var}(\varepsilon_1) = \sigma^2$ and $\mathrm{Var}_0(\ln S_2) = \mathrm{Var}(\varepsilon_1) + \mathrm{Var}(\varepsilon_2) = 2\sigma^2$. Finally, $\ln S_1$ and $\ln S_2$ are linear functions of the (independent) normally distributed $(\varepsilon_1, \varepsilon_2)$, and therefore also normally distributed. ■

If we take the proper limit as the time interval h goes towards zero, then we have a Brownian motion for the log asset price ($d \ln S_t = \mu dt + \sigma dW_t$, where dW_t are the increments to a Wiener process).

A hedging/no arbitrage argument similar to the binomial model then leads to the Black-Scholes formula for the price of a European call option (on an asset without dividends)

$$C = S\Phi(d_1) - e^{-ym} K \Phi(d_2), \text{ where} \tag{21.3}$$
$$d_1 = \frac{\ln(S/K) + (y + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.4}$$

In this formula, $\Phi(d)$ denotes the probability of $x \leq d$ when x has an $N(0,1)$ distribution (that is, the distribution function value at d).

The Black-Scholes call option price is increasing in the asset price, volatility, time to maturity and the interest rate, but decreasing in the strike price. See Figure 21.2. It is also straightforward to show (see Appendix) that when $\sigma = 0$ then $C = \max(S - e^{-ym}K, 0)$, and when $m = 0$ then $C = \max(S - K, 0)$.

Example 21.2 *(Call option price) With* $(S, K, y, m, \sigma) = (42, 42, 0.05, 0.5, 0.2)$, *(21.3)–(21.4) give* $C = 2.893$.

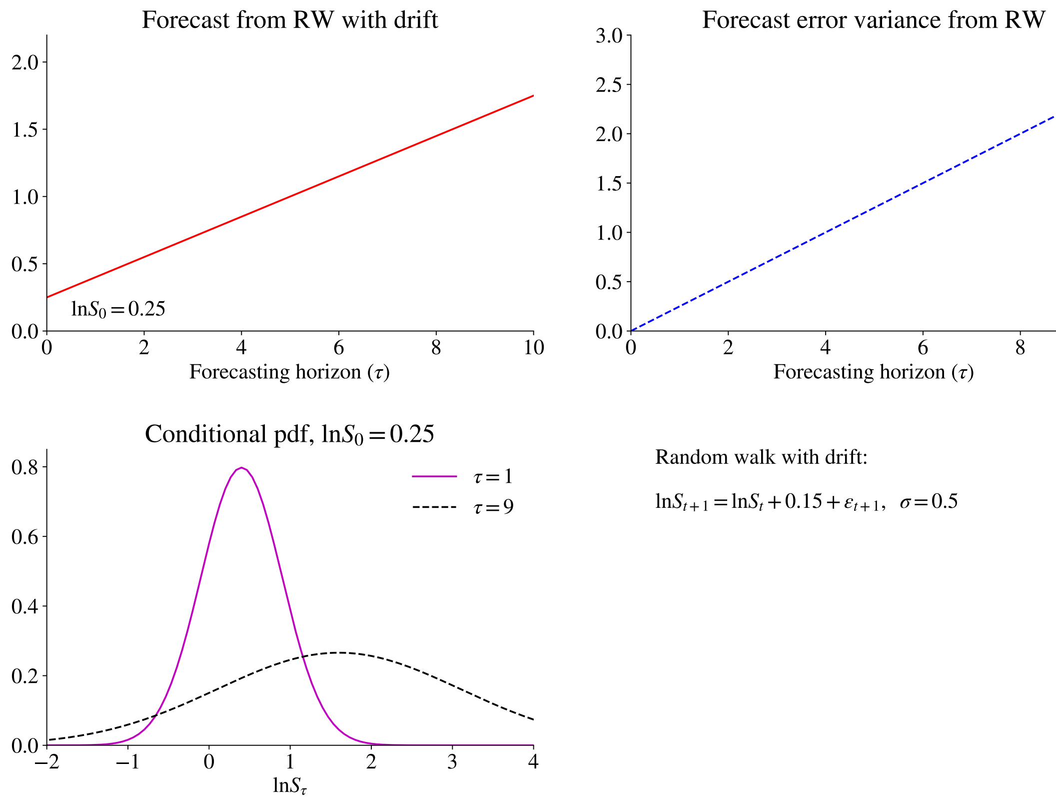


Figure 21.1: Conditional distribution from random walk with drift

21.1.2 The Black-Scholes Model with Dividends

Consider a European option on an underlying asset that pays (continuous or discrete) dividends before expiration. Then, the Black-Scholes formula is not correct. The basic reason is that the current price of the underlying embeds all future dividends, but the option will miss out on those dividends that are paid before the expiration.

To handle this, we could apply the BS formula to a forward contract on the underlying (expiring on the same day as the option) instead. Let a prepaid forward contract (present value of forward price, $e^{-ym}F$), play the role of the underlying asset in (21.1). Notice that the forward also misses out on the dividends until expiration. This is gives the BS formula (21.3)–(21.4) but with $e^{-ym}F$ substituted for S

$$C = e^{-ym}F\Phi(d_1) - e^{-ym}K\Phi(d_2), \text{ where} \tag{21.5}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.6}$$

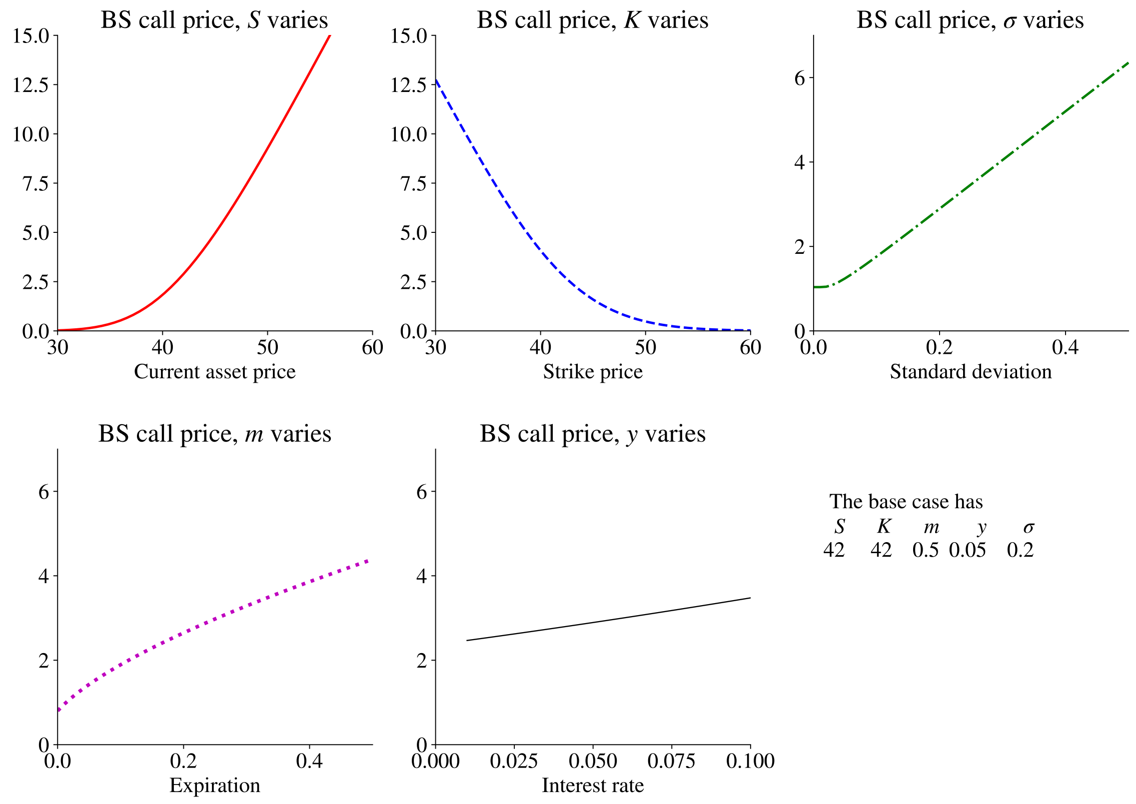


Figure 21.2: Call option price, Black-Scholes model

This is *Black's model* which has many applications.

For instance, for an asset with a continuous dividend rate of δ, the forward-spot parity says $F = Se^{(y-\delta)m}$. In this case (21.5)–(21.6) can also be written

$$C = e^{-\delta m} S \Phi(d_1) - e^{-ym} K \Phi(d_2), \text{ where} \tag{21.7}$$

$$d_1 = \frac{\ln(S/K) + (y - \delta + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.8}$$

When the asset is a currency (read: foreign money market account) and δ is the foreign interest rate, then this is the "Garman-Kolhagen" formula.

Using the put-call parity formula in (21.5), the pricing formula for a put option can be written

$$P = e^{-ym} K \Phi(-d_2) - e^{-\delta m} S \Phi(-d_1), \tag{21.9}$$

where d_1 and d_2 are defined in (21.8).

Example 21.3 *(Put price) Using the same parameters as in Example 21.2 and* $\delta = 0$*, we get* $P = 1.856$*. Instead, with* $\delta = 0.05$*, we get* $P = 2.309$*.*

Proof. (of (21.9)) Recall that the put-call parity for an asset with continuous dividends is $C - P = Se^{-\delta m} - e^{-my}K$. Use in (21.5) to get

$$P = e^{-\delta m}S[\Phi(d_1) - 1] - e^{-ym}K[\Phi(d_2) - 1].$$

Since $\Phi(d) + \Phi(-d) = 1$, this can be written as (21.9). ■

21.2 Convergence of the BOPM to Black-Scholes

21.2.1 The Main Result

This section demonstrates that the option price from the binomial option pricing model (BOPM) converges (as we take more, but shorter time steps to reach a fixed time to expiration m) to the price from the Black-Scholes model. See Figures 21.3–21.4 for an illustration of how the parameters (p, u, d) from the CRR approach and the resulting option price converge as the number of time steps increases.

21.2.2 The Risk Neutral Distribution in Black-Scholes

We know that the risk neutral pricing of a European call option is

$$C = e^{-ym}\,\mathrm{E}^* \max(0, S_m - K), \tag{21.10}$$

where E^* denotes the expectation according to the risk neutral distribution. This provides an alternative way to calculate the fair price of the option. We can rewrite as

$$C = e^{-ym}\int_0^\infty \max(0, S_m - K)f^*(S_m)dS_m, \tag{21.11}$$

where $f^*(S_m)$ is the risk neutral density function of the asset price at expiration (S_m). (Below K the value of the integrand is zero.)

For the Black-Scholes model, the (physical) normal distribution for the log asset price (21.2) implies that the risk neutral distribution of $\ln S_m$ is

$$\ln S_m \sim^* N(\ln S + ym - \sigma^2 m/2, \sigma^2 m), \tag{21.12}$$

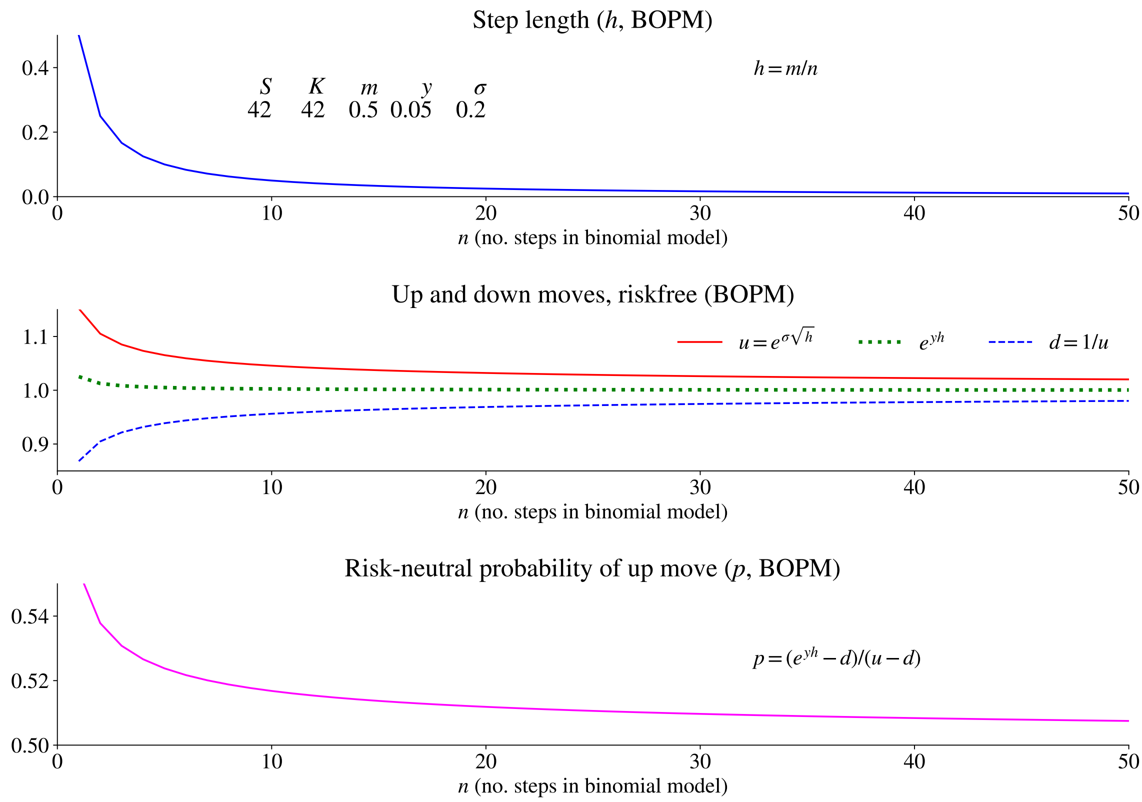


Figure 21.3: Convergence of the parameters in a binomial model

where S is the current asset price. This risk neutral distribution typically has a lower mean than the physical distribution (certainly if the expected return is higher than the risk free rate), but the same variance. Calculating (21.11) by using (21.12) for $f^*(S_m)$ gives the Black-Scholes formula (21.3). Proving this is just a matter of calculating the integral. (See Appendix 21.5 for a proof.)

We can alternaively calculate (21.11) by numerical integration to verify that we get the same value as from the Black-Scholes formula. See Figure 21.5 for an illustration.

21.2.3 The Risk Neutral Binomial Distribution

In the binomial option pricing model (BOPM), the risk neutral binomial process for the asset price gives the following binomial process for the *log returns* (changes of the log

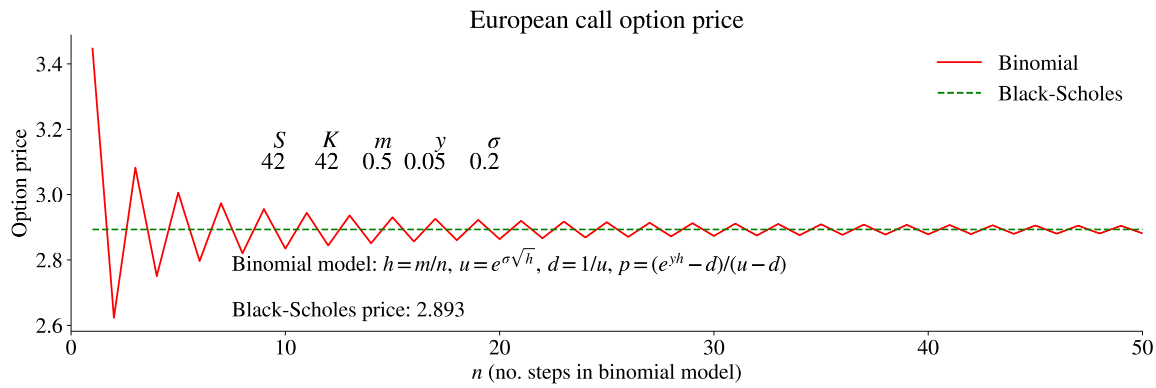


Figure 21.4: Convergence of the binomial price to the Black-Scholes price

asset price)

$$r_{t+h} = \ln(S_{t+h}/S_t) = \begin{cases} \ln u & \text{with probability } p \\ \ln d & \text{with probability } 1-p. \end{cases} \tag{21.13}$$

(In the risk neutral binomial tree $S_{t+h} = S_t u$ with probability p and $S_t d$ with probability $1-p$. This implies (21.13).) The parameters u, d and p all depend on the time step length h in such a way that we match the mean and variance of the price series. In fact, they are chosen so that the mean and variance of r_{t+h} are (at least in the limit) proportional to h.

Clearly, the binomial tree means that we reach $\ln S_m$ by starting at S_0 and adding n steps of the kind in (21.13)

$$\ln S_m = \ln S_0 + \ln(S_h/S_0) + \ldots + \ln(S_{nh}/S_{(n-1)h}) \tag{21.14}$$

$$= \ln S_0 + \textstyle\sum_{i=1}^{n} r_i, \tag{21.15}$$

where r_i is the log return between $(i-1)h$ and ih.

I demonstrate the convergence of this to the Black-Scholes risk neutral distribution (21.12) in two steps: first, that the binomial distribution converges to a normal distribution; and second that both distributions have the same mean and variance in the limit.

21.2.4 The Central Limit Theorem at Work

The Black-Scholes model is based on normally distributed changes of log prices. In the binomial model, the log price changes can only take two values, but the sum of many such changes will converge to a normally distributed variable as the number of time steps

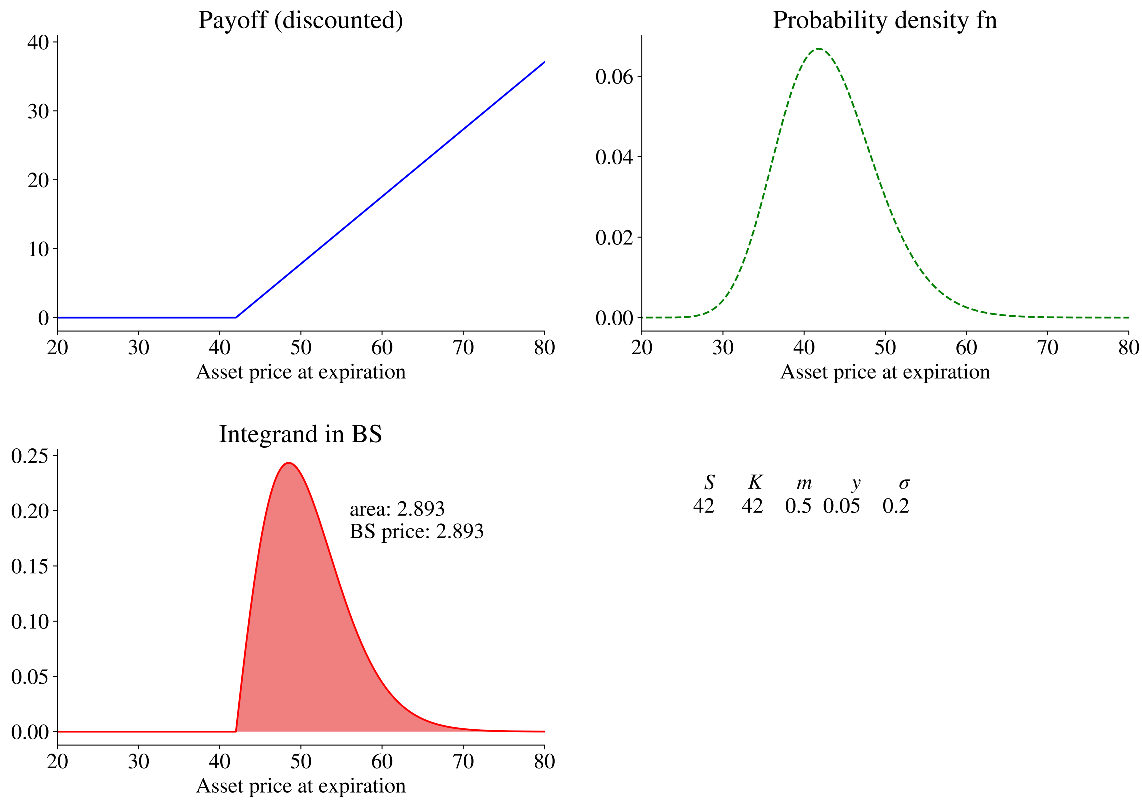


Figure 21.5: Numerical integration to the BS call price

increases. This may seem counter intuitive since central limit theorems apply to samples averages (times the square root of the sample size), not to sums. However, the rescaling of the log price changes as the number of time steps increases implies that the sum is effectively a (scaled) sample average—so a CLT indeed applies.

See Figure 21.6 for an example of how the distribution converges. Notice that the figure shows the density functions for the *log* asset price (at expiration). Also, notice that the discrete distribution from the binomial model is illustrated by bars centered on the outcome—and that the bars are normalised to have an area of one.

Proposition 21.4 *If u, d and p in the binomial process (21.13) are such that the mean and variance of $\ln S_{t+h} - \ln S_t$ are proportional to h, then the distribution of $\sum_{i=1}^{n} r_i$ converges to a normal distribution as the number of time steps n increases, keeping the maturity m constant (so $h = m/n$).*

Proof. (*of Proposition 21.4) The binomial model (21.13)–(21.15) means that we can write return $r_i = \varepsilon_i \sqrt{h} + \mu h$, where ε_i is an iid zero mean random variable with variance

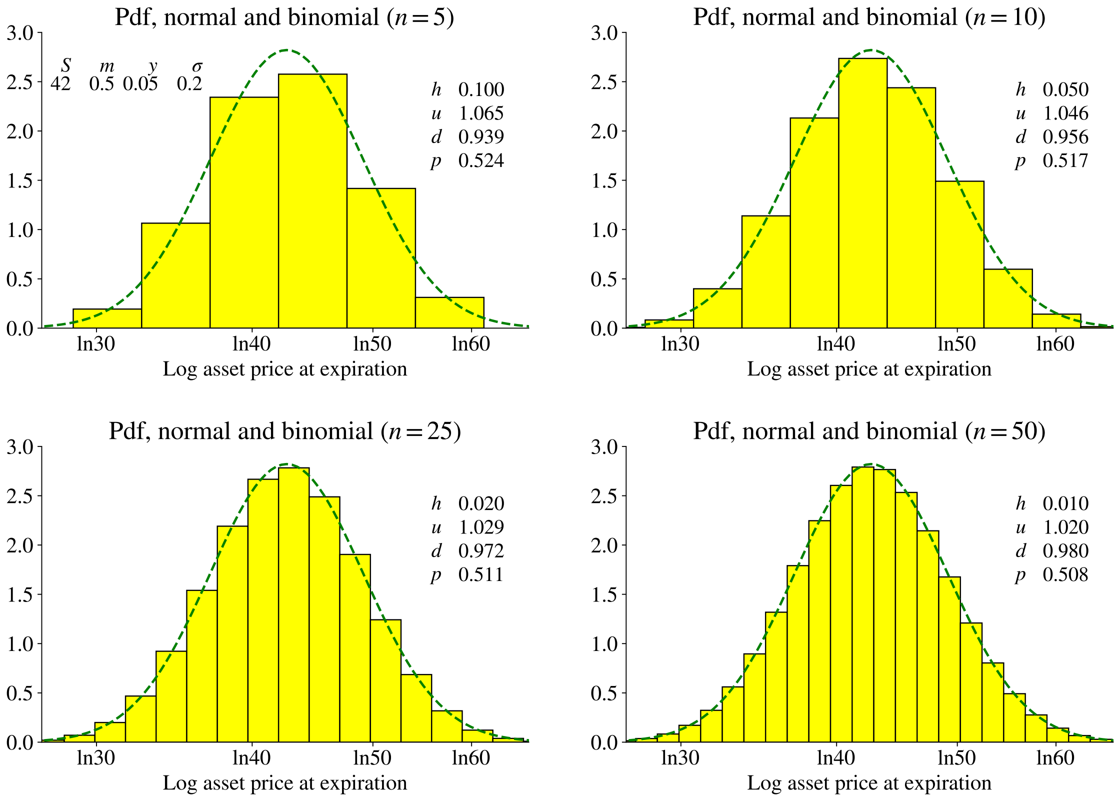


Figure 21.6: Convergence of the binomial model to the Black-Scholes model. The figure shows results for the log assset price. The distribution of from the binomial distribution is scaled so the area of the bars equals one.

σ^2. Notice that $\mathrm{E}\, r_i = \mu h$ and $\mathrm{Var}(r_i) = \sigma^2 h$, so both moments are proportional to h. Write (21.15) as

$$\textstyle\sum_{i=1}^{n} r_i = \sqrt{h}\sum_{i=1}^{n} \varepsilon_i + nh\mu.$$

Since $h = m/n$, this can be written

$$\textstyle\sum_{i=1}^{n} r_i = \sqrt{m}\underbrace{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i}_{A} + \mu m.$$

The term A is $\sqrt{n}$ times the sample average of an iid random variable (ε_i) with $\mathrm{E}\, \varepsilon_i = 0$ and $\mathrm{Var}(\varepsilon_i) = \sigma^2 < \infty$. We can therefore apply the (Lindeberg-Lévy) central limit theorem to show that $A \overset{d}{\to} N(0, \sigma^2)$. The second term (μm) is just a constant. Together, we get that $\Sigma_{i=1}^{n} r_i \overset{d}{\to} N(\mu m, \sigma^2 m)$. ■

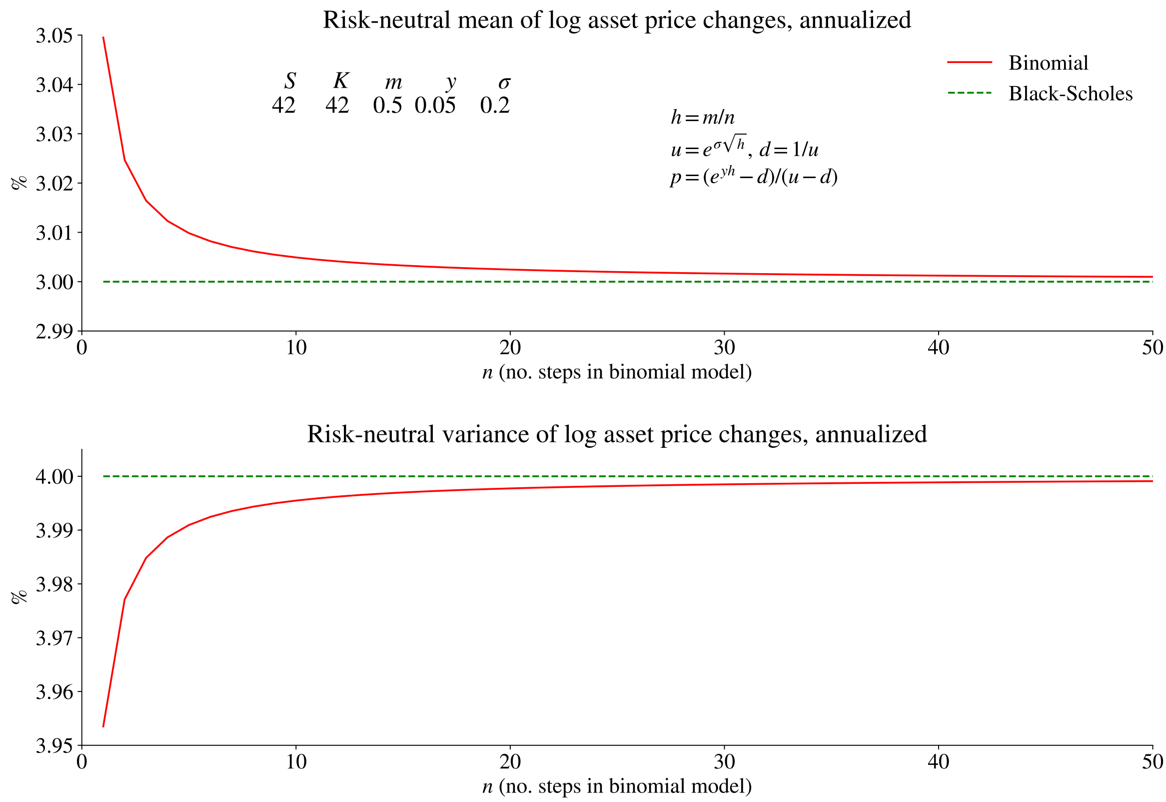


Figure 21.7: Convergence of the binomial mean and variance

21.2.5 Convergence of the Mean and Variance

This section demonstrates that the mean and variance of the binomial distribution converge to the same values as in the risk neutral distribution of the Black-Scholes model (21.12). See Figure 21.7 for an illustration.

Proposition 21.5 *(Moments of CRR steps) In the Cox, Ross, and Rubinstein (1979) tree, the parameters in (21.13) are*

$$\ln u = \sigma\sqrt{h}, \ln d = -\sigma\sqrt{h} \text{ and } p = (e^{yh} - d)/(u - d).$$

As $n \to \infty$, but $h = m/n$ we have (since the price changes are independent)

$$\mathrm{E}\textstyle\sum_{i=1}^{n} r_i = m(y - \sigma^2/2) \text{ and } \mathrm{Var}(\textstyle\sum_{i=1}^{n} r_i) = m\sigma^2.$$

This is the same as in the risk neutral distribution of the Black-Scholes model.

Proof. (*of Proposition 21.5) Recall that the mean and variance of r_i are $p \ln u +$

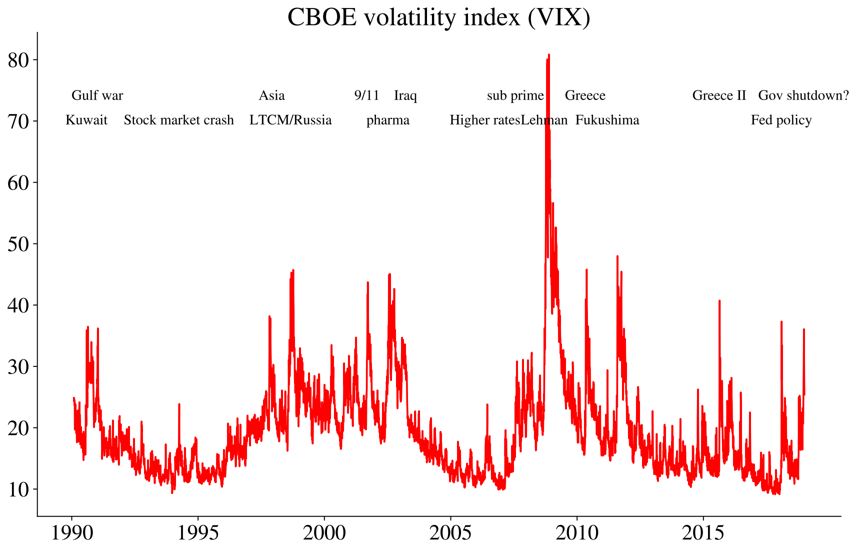


Figure 21.8: CBOE VIX, summary measure of implied volatilities (30 days) on US stock markets

$(1-p)\ln d$ and $p(1-p)(\ln u - \ln d)^2$ respectively. Since the terms in (21.15) are uncorrelated, the mean and the variance of the sum are $n\,\mathrm{E}\,r_i$ and $n\,\mathrm{Var}(r_i)$. Substitute for u, d and p and take the limits of as $n \to \infty$, but $h = m/n$. (This is straightforward, but slightly messy, calculus.) ■

21.3 Testing the BS Model

The Black-Scholes formula ((21.3)–(21.4) or (21.5)–(21.6)) contains only one unknown parameter: the variance $\sigma^2 m$ in the distribution of $\ln S_m$ (see 21.2). With data on the option price, spot and forward prices, the interest rate, and the strike price, we can solve for the standard deviation σ (see from Figure 21.2 that the option price and the volatility have a monotonic relation). You can also calculate σ from a put, since the put-call parity shows that a call and a put with the same strike price have the same implied volatility. The σ calculated in this way is called the *implied volatility*—and it is often used as an indicator of market uncertainty about the future asset price, S_m. It can be thought of as an annualized standard deviation. See Figure 21.8 for an empirical example.

Note that we can solve for one implied volatility for each available strike price. If the Black-Scholes formula is correct, that is, if the assumption in (21.1) is correct, then these volatilities should be the same across strike prices.

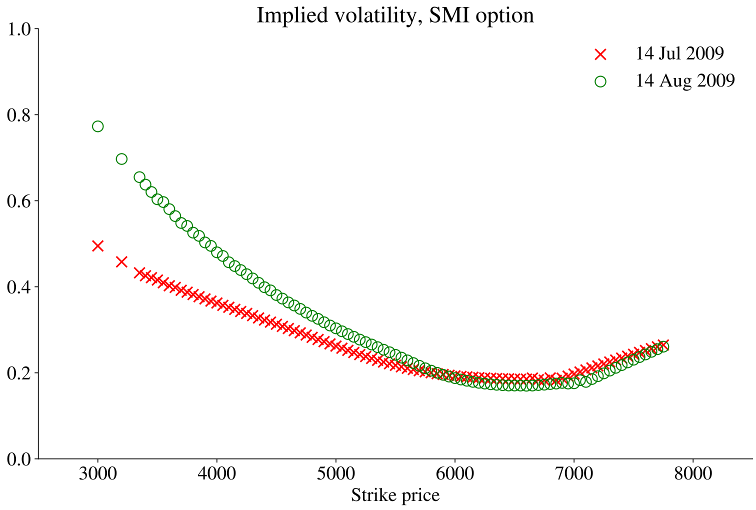


Figure 21.9: Implied volatilities of SMI options, selected dates

However, on currency markets, we often find a volatility "smile" (volatility is a U-shaped function of the strike price). One possible explanation is that the (perceived) distribution of the future asset price has relatively more probability mass in the tails ("fat tails") than a normal distribution has. On equity markets, we often find a volatility "smirk" instead, where the volatility is very high for very low strike prices. This is often interpreted as that investors are willing to pay a lot for put options that protect them from a dramatic fall in the stock price. One possible explanation is thus that the distribution has more probability mass than a normal distribution at very low stock prices (negative skewness). See Figures 21.9–21.10 for empirical examples.

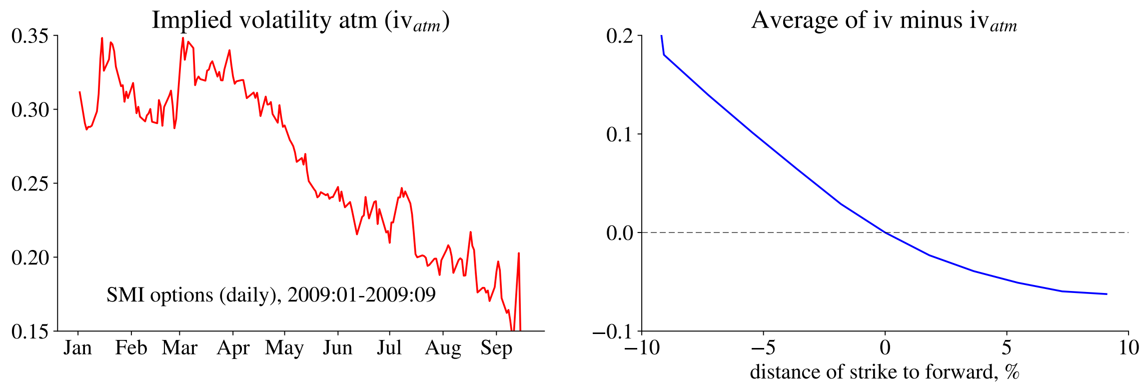


Figure 21.10: Implied volatilities over nine months

21.4 Appendix: More Details on the Black-Scholes Model*

21.4.1 Limits of the Black-Scholes Formula when $\sigma = 0$ or $m = 0$

Remark 21.6 *(Black-Scholes formula when $\sigma = 0$*) From (21.4) $\lim_{\sigma\to 0} d_1 = \lim_{\sigma\to 0} d_2 = \infty$ if $e^{ym}S \geq K$ and $-\infty$ otherwise. Therefore, $\lim_{\sigma\to 0} \Phi(d_1) = \lim_{\sigma\to 0} \Phi(d_2) = 1$ if $e^{ym}S \geq K$ and 0 otherwise. The Black-Scholes call option price at $\sigma = 0$ is therefore* $\max(S - e^{-ym}K, 0)$.

Remark 21.7 *(Call option price when $\sigma = 0$, version 2*) When the underlying asset is riskfree ($\sigma = 0$), then its return (denoted μ in (21.2)) must equal the riskfree rate y, so the value of the underlying asset is $e^{ym}S$ at expiration. The present value of the known call payoff is $e^{-ym}\max(e^{ym}S - K, 0)$, which is the same as in the previous remark.*

Remark 21.8 *(Black-Scholes formula when $m = 0$*) From (21.4) $\lim_{\sigma\to 0} d_1 = \lim_{\sigma\to 0} d_2 = \infty$ if $S \geq K$ and $-\infty$ otherwise. Therefore, $\lim_{\sigma\to 0} \Phi(d_1) = \lim_{\sigma\to 0} \Phi(d_2) = 1$ if $S \geq K$ and 0 otherwise. The Black-Scholes call option price at $m = 0$ is therefore* $\max(S - K, 0)$.

21.4.2 Calculating Black's model with Computer Code for the Black-Scholes Model

Remark 21.9 *(Coding Black's model with a forward price*) Suppose you have a computer code for the BS model (21.3)—(21.4) which takes the inputs (S, K, y, m, σ). To use that code for Black's model (21.5)–(21.6), substitute $(F, 0)$ for (S, y) and multiply the results by e^{-ym}.*

Remark 21.10 *(Coding the BS model with continuous dividends*) Suppose you have a computer code for the BS model (21.3)—(21.4) which takes the inputs* (S, K, y, m, σ). *To use that code for Black's model (21.5)–(21.6), substitute* $e^{-\delta m}S$ *for* S.

Remark 21.11 *(Practical hint: finding the dividend rate*) If you don't know what the dividend rate is, use the forward-spot parity,* $F = Se^{(y-\delta)m}$, *to calculate it as* $\delta = y - \ln(F/S)/m$.

21.5 Appendix: The Probabilities in the BOPM and Black-Scholes Model*

The price of a European (call or put) option calculated by the binomial model converges to the Black-Scholes price as the number of subintervals increases (keeping the time to expiration constant, so the subintervals become shorter). This is illustrated in Figure 21.4.

Both the binomial option pricing model (BOPM) and the Black-Scholes model imply that the call option price can be written as the discounted risk neutral expected payoff (21.10). We can clearly rewrite (21.10) as

$$
\begin{aligned}
C &= e^{-ym}\,\mathrm{E}^*(S_m - K|S_m > K)\,\Pr{}^*(S_m > K) && (21.16)\\
&= e^{-ym}\,\mathrm{E}^*(S_m|S_m > K)\,\Pr{}^*(S_m > K) - e^{-ym}K\,\Pr{}^*(S_m > K)\,. && (21.17)
\end{aligned}
$$

The first term is (the present value of) the risk neutral expected asset price conditional on exercise, times the risk neutral probability of exercise. The second term is (the present value of) the strike price times the risk neutral probability of exercise.

Example 21.12 *(Binomial model with $n = 2$) The price of a European call option is*

$$C = e^{-ym}\left[p^2\max(Suu - K, 0) + 2p(1-p)\max(Sud - K, 0) + (1-p)^2\max(Sdd - K, 0)\right].$$

Suppose we only exercise in the Suu node ($Suu > K$ but $Sud < K$). The call price can then be written

$$
\begin{aligned}
C &= e^{-ym}p^2(Suu - K)\\
&= e^{-ym}\underbrace{Suu}_{\mathrm{E}^*(S_m|S_m>K)}\underbrace{p^2}_{\Pr^*(Suu)} - e^{-ym}K\underbrace{p^2}_{\Pr^*(Suu)}.
\end{aligned}
$$

21.5.1 The Probabilities in the Black-Scholes Model

Remark 21.13 *(Properties of a lognormal distribution) Let $x \sim N(\mu, s^2)$ and define $k_0 = (\ln K - \mu)/s$. First, $\Pr(e^x > K) = \Phi(-k_0)$. Second, $\mathrm{E}(e^x|e^x > K) = e^{\mu+s^2/2}\Phi(s - k_0)/\Phi(-k_0)$. (To prove this, just integrate.)*

Proposition 21.14 *(Riskneutral probability of $S_m > K$) The $\Phi(d_2)$ term in the Black-Scholes formula (21.3)–(21.4) is the risk-neutral probability that $S_m > K$.*

Proposition 21.15 *($S\Phi(d_1)$ in Black-Scholes) The $S\Phi(d_1)$ term in the Black-Scholes formula (21.3)–(21.4) is (the present value of) the expected asset price conditional on exercise, times the probability of exercise, that is, the first term in (21.17).*

Proof. (of Proposition 21.14) The risk neutral probability of $\ln S_m$ is $N(\mu, s^2)$ with $\mu = \ln S + ym - \sigma^2 m/2$ and $s^2 = \sigma^2 m$. Use Remark 21.13 to calculate the probability $\Pr(S_m > K)$ as $\Phi(-k_0)$ where $k_0 = (\ln K - \mu)/s$. Clearly, $-k_0$ is the same as d_2 in (21.4). ∎

Proof. (of Proposition 21.15) Using Remark 21.13, the first term in (21.17), here denoted A, can be written

$$A = e^{-ym} e^{\mu + s^2/2} \Phi(s - k_0),$$

since the two $\Phi(-k_0)$ terms cancel. Since $\ln S_m$ is $N(\mu, s^2)$ with $\mu = \ln S + ym - \sigma^2 m/2$ and $s^2 = \sigma^2 m$. we get

$$\mu + s^2/2 = \ln S + ym, \text{ and}$$
$$s - k_0 = \sigma\sqrt{m} - \frac{\ln K - \left(\ln S + ym - \sigma^2 m/2\right)}{\sigma\sqrt{m}} = d_1,$$

where the last line follows from comparing with (21.4). We can therefore write A as $S\Phi(d_1)$, since the $e^{-ym}e^{ym}$ term cancels. This is the same as in the Black-Scholes formula. ∎

21.6 Appendix: Statistical Tables

	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018
-2.9	0.0019	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025
-2.8	0.0026	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034
-2.7	0.0035	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045
-2.6	0.0047	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060
-2.5	0.0062	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080
-2.4	0.0082	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104
-2.3	0.0107	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136
-2.2	0.0139	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174
-2.1	0.0179	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222
-2.0	0.0228	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281
-1.9	0.0287	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351
-1.8	0.0359	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436
-1.7	0.0446	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537
-1.6	0.0548	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655
-1.5	0.0668	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793
-1.4	0.0808	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951
-1.3	0.0968	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131
-1.2	0.1151	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335
-1.1	0.1357	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562
-1.0	0.1587	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814
-0.9	0.1841	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090
-0.8	0.2119	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389
-0.7	0.2420	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709
-0.6	0.2743	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050
-0.5	0.3085	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409
-0.4	0.3446	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783
-0.3	0.3821	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168
-0.2	0.4207	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562
-0.1	0.4602	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960

Table 21.1: Values of the standard normal distribution function at x where x is the sum of the values in the first column and the first row.

	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Table 21.2: Values of the standard normal distribution function at x where x is the sum of the values in the first column and the first row.

Chapter 22

Hedging Options

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23 and Hull (2009) 13 and 17

Additional references: McDonald (2014) 15–16 and Cox, Ross, and Rubinstein (1979)

22.1 Hedging an Option

This section discusses how we can hedge an option. The setting is that we have written (sold, issued) such an option, but we do not want to carry the risk.

A first order approximation suggests that the change (here indicated by d) in the call price due to a change in the underlying price is

$$dC \approx \frac{\partial C}{\partial S} dS, \tag{22.1}$$

where $\partial C_t / \partial S$ is positive, see Figure 22.1.

Remark 22.1 *(dX notation) Warning: this section uses dX to indicate a change in variable X, mostly since Δ has another, and well established, interpretation in the option literature.*

22.2 An Approximate Hedge

22.2.1 Basic Setup

Consider a portfolio with v of the underlying asset (the hedging portfolio) and short one call option. The value of the overall position is

$$V = vS + M - C, \tag{22.2}$$

where M is money on a short-term money market account. The idea is now to find v so that vS and C are equally sensitive to changes in S.

For now, we focus on movements of the price of the underlying, disregarding, for instance, movements in volatility and the value of the money market account. Use (22.1) to approximate the change (indicated by d) of the value of the overall portfolio as

$$\begin{aligned} dV &\approx v dS - \frac{\partial C}{\partial S} dS \\ &\approx 0 \text{ if } v = \frac{\partial C}{\partial S} = \Delta, \end{aligned} \tag{22.3}$$

where the second line uses Δ as a symbol for the derivative $\partial C / \partial S$ (as is standard in the option literature). This approach is therefore called a *delta hedge*. Clearly, the delta is likely to change from period to period, so the portfolio needs to be (frequently) rebalanced.

In practice, the overall portfolio includes a position in a short-term money market account to make the initial portfolio value zero, so

$$M_0 = C_0 - \Delta_0 S_0. \tag{22.4}$$

This position is typically negative for a call option, and it will surely be if you use the Black-Scholes model to calculate Δ. This means that we finance the purchase of the underlying asset with the proceeds from the selling the option and from borrowing. See Remark 22.2 for details on how the portfolio value changes over time.

Remark 22.2 *(Overall portfolio value over several subperiods*) Start by creating a hedge portfolio with a zero initial value as in (22.4). In $t + h$ (say, after one day so $h = 1/365$), this portfolio is worth (this is the marking-to-market)*

$$V_{t+h} = \Delta_t (D_{t+h} + S_{t+h}) + M_t e^{y_t h} - C_{t+h},$$

where the underlying pays a dividend ($D_{t+h} = 0$ if no dividends), the prices are measured after dividends and y_t is the interest rate. In $t + h$ we need Δ_{t+h} units of the underlying asset (value $\Delta_{t+h} S_{t+h}$). Since we already own Δ_t of the underlying asset, this means that we must withdraw an additional $(\Delta_{t+h} - \Delta_t) S_{t+h}$ from the money market account. On that account, we have since last period $M_t e^{y_t h} + \Delta_t D_{t+h}$ (old holdings with interest plus the dividends we received in cash), so our holdings in $t + h$ (after having rebalanced *the holdings) is*

$$M_{t+h} = M_t e^{y_t h} + \Delta_t D_{t+h} - (\Delta_{t+h} - \Delta_t) S_{t+h}.$$

The value of the overall portfolio in $t+2h$ (marking-to-market) is computed as in the first equation, but with subscripts advanced one period. See Figure 22.3 for an illustration. In that figure, "m-to-m" stands for the marking-to-market stage (first equation in this remark) and "rebalancing" for the stage after rebalancing the portfolio (second equation in this remark).

22.2.2 Deltas from the Black-Scholes Model

The results in the following remark will be useful later on.

Remark 22.3 *(The "Greeks") The Black-Scholes formula for an asset with continuous dividends (δ) is*

$$\begin{aligned} C &= e^{-\delta m} S \Phi\left(d_1\right) - e^{-ym} K \Phi\left(d_2\right), \text{ where} \\ d_1 &= \frac{\ln(S/K) + (y - \delta + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \end{aligned}$$

(Warning: d_1 and d_2 indicate the usual terms in the Black-Scholes formula. Do not confuse with the d used to indicate a change.) The derivatives are

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial S} = e^{-\delta m} \Phi\left(d_1\right) \\ \Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta m} \phi\left(d_1\right)}{S\sigma\sqrt{m}} \\ \theta &= \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial m} = \delta S e^{-\delta m} \Phi(d_1) - yKe^{-ym}\Phi(d_2) - \frac{1}{2\sqrt{m}} e^{-\delta m} S \phi(d_1)\sigma \\ \text{(vega)} &= \frac{\partial C}{\partial \sigma} = S e^{-\delta m} \phi\left(d_1\right)\sqrt{m} \\ \rho &= \frac{\partial C}{\partial y} = mKe^{-ym}\Phi(d_2), \end{aligned}$$

where $\phi()$ is the standard normal probability density function (the derivative of $\Phi()$). See Figures 22.1–22.2. It is also useful to notice that the sensitivity to a forward price ($F = Se^{(y-\delta)m}$) is

$$\frac{\partial C}{\partial F} = e^{-ym} \Phi\left(d_1\right).$$

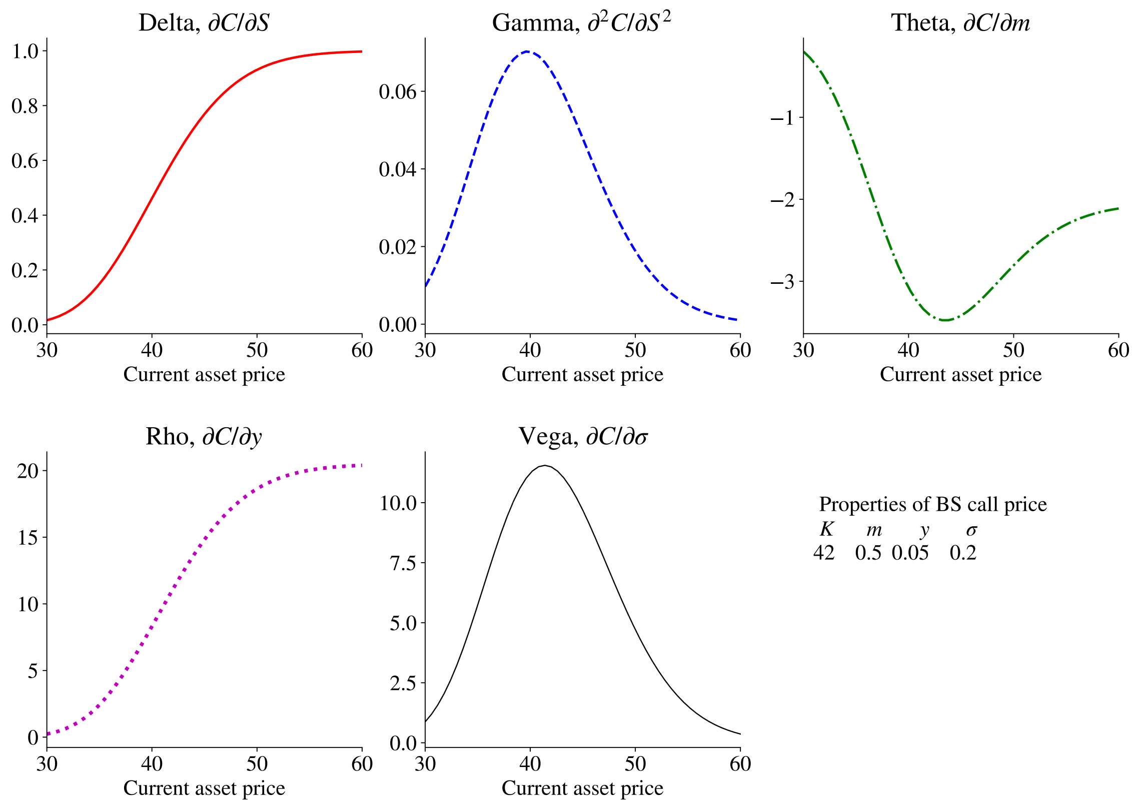


Figure 22.1: The Greeks in the Black-Scholes model as a function of the asset price

See Figures 22.1–22.2 for an illustration of how Δ (and the other "Greeks") depend on the strike and underlying price (according to the Black-Scholes model). In particular, notice that $0 \leq \Delta \leq 1$ and that Δ is increasing in the price of the underlying asset. Intuitively, an option that is deep out of the money will not be very sensitive to the asset price—since the chance of exercising is so low. Conversely, an option that is deep in the money moves in tandem with the asset price, since it will almost for sure be exercised.

Remark 22.4 *(Hedging a put option) Instead, if you want to hedge a put option, replace C by P in (22.1) and notice that the delta of a put option is*

$$\frac{\partial P}{\partial S} = e^{-\delta m}[\Phi\left(d_1\right) - 1] = -e^{-\delta m}\Phi\left(-d_1\right),$$

which is negative. Warning: d_1 indicates the usual term in the Black-Scholes formula. Do do confuse with the d used to indicate a change. (This result follows from the put-call parity and by using the symmetry of the normal distribution. To see this, notice that the put-call parity says $P = C - Se^{-\delta m} + e^{-my}K$, so $\partial P/\partial S = \partial C/\partial S - e^{-\delta m}$.)

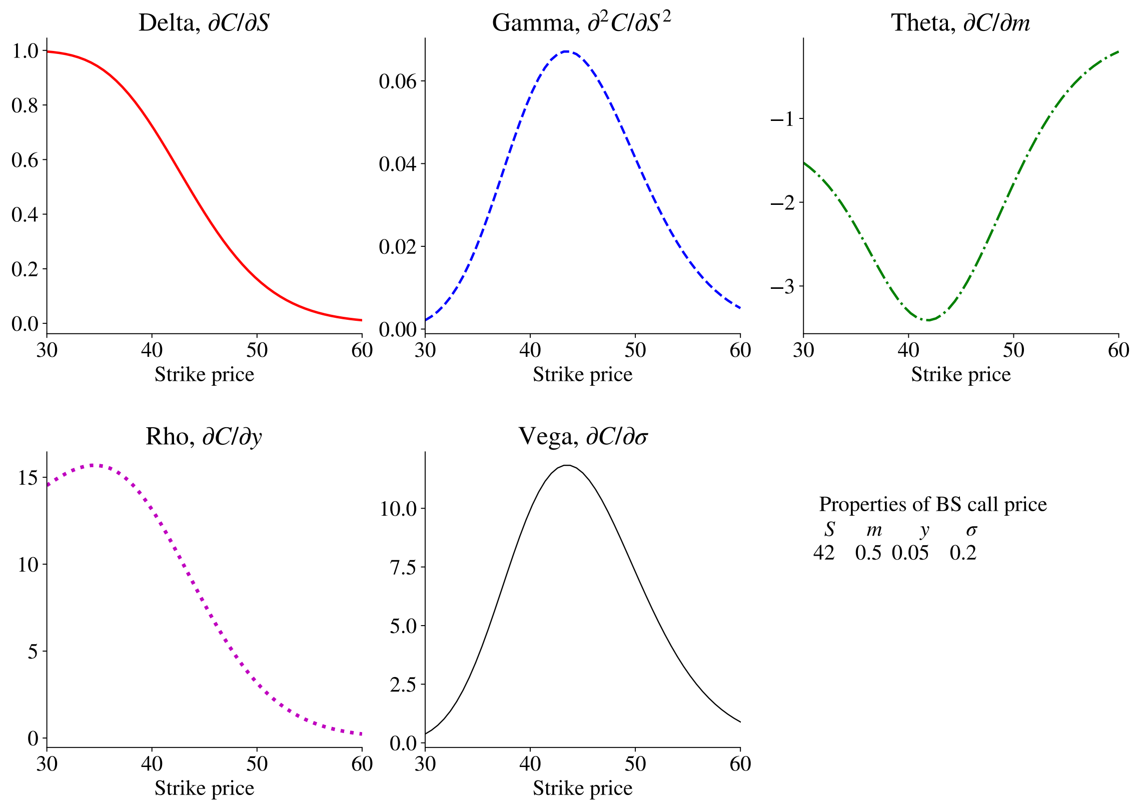


Figure 22.2: The Greeks in the Black-Scholes model as a function of the strike price

Example 22.5 *(Deltas) With* $(S, K, y, m, \sigma) = (42, 42, 0.05, 0.5, 0.2)$ *and* $\delta = 0$*, we have* $\partial C/\partial S = 0.60$ *and* $\partial P/\partial S = -0.40$*. The difference is clearly equal to one (since* $\delta = 0$*).*

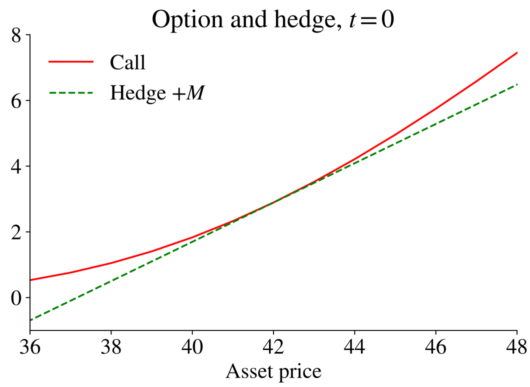
Remark 22.6 *(Initial position when hedging a call option*) Using Remark 22.3 in (22.4) gives*

$$M_0 = -e^{-ym} K \Phi(d_2),$$

which is negative.

See Figure 22.3 for how the hedging portfolio approximates the call option price (as well for numbers for the positions in the hedge). In that figure, “m-to-m” stands for the marking-to-market stage and “rebalancing”for the stage after rebalancing the portfolio.

Example 22.7 *(Delta hedging) Using the same parameters as in Example 22.5 and* $\delta = 0$*, Figure 22.3 illustrates the initial positions (day 0), and two snap shots of the day after*



K	m	y	σ
42	0.5	0.05	0.2

Portfolio value:
$V = \Delta S + M - C$

Call option: see above

Underlying: 42 on day 0,
but 43 on day 1

	S	Δ	ΔS	M	$-C$	V
Day 0	42.00	0.60	25.10	-22.21	-2.89	0.00
Day 1 (m-to-m)	43.00	0.60	25.70	-22.22	-3.51	-0.02
Day 1 (rebal)	43.00	0.66	28.40	-24.91	-3.51	-0.02

Figure 22.3: Delta hedging over time

(day 1: after marking to market, day 1: after rebalancing). On day, the overall portfolio includes $\Delta = 0.6$ *of the underlying asset (at a value of* $0.6 \times 42 = 25.10$*),* -1 *of the option (at the value* -2.89*) and the balance on a money market account (*$-25.10 + 2.89 = -22.21$*) so the total portfolio is worth zero. This clearly means that the investor has borrowed.*

Finally, see Figure 22.4 for an example of how a delta hedge works on real data.

Remark 22.8 *(*Hedging with a forward contract) Consider using a forward contract as hedging instrument. Recall that* $W_t = e^{-ym}(F_t - F_\tau)$ *is the value of an old forward contract (written in* $\tau < t$*). The hedge portfolio is* $V = vW + M - C$*. This portfolio is almost stable if* $v = e^{ym}\partial C/\partial F$ *(see Remark 22.3 for an expression). To see this, notice that* $dV = vdW - dC \approx ve^{-ym}dF - \frac{\partial C_t}{\partial F}dF$*.*

22.2.3 Deltas from Other Models

The derivative in (22.3) could also be computed from some other option pricing model, for instance, the binomial model.

The basic approach is straightforward: consider two different values of the underlying asset (S_a and S_b), use the model to compute the option price at each of them (get $C(S_a)$ and $C(S_b)$) and approximate the derivative with a finite difference ratio: $[C(S_a) - C(S_b)]/(S_a - S_b)$.

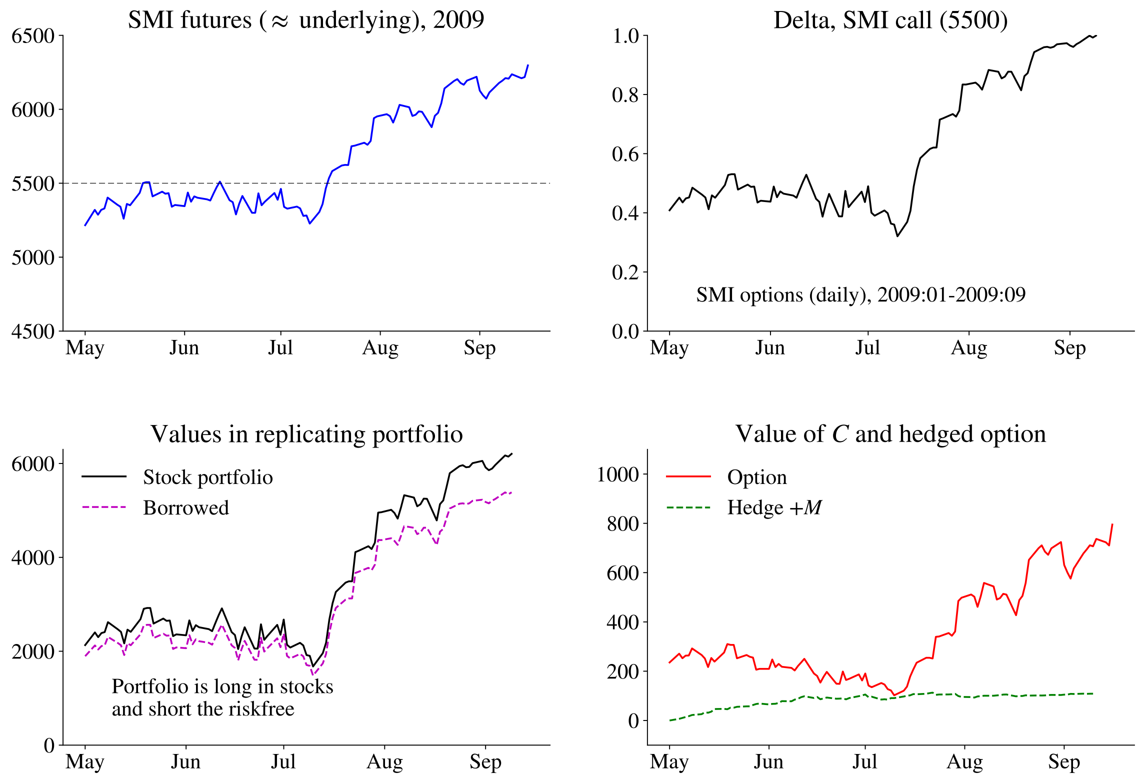


Figure 22.4: Delta hedging an SMI call option

In particular, the binomial model has the advantage that it allows us to handle also American style options. See Figure 22.5 and notice that the deltas of American puts tend to be more negative than for European puts, especially at low prices of the underlying.

22.3 Higher-Order Hedging*

22.3.1 Delta-Gamma Hedging*

Delta hedging can be imprecise if the price of the underlying asset changes a llot. We can improve the precision by using a second-order Taylor approximation of the option price

$$dC \approx \varDelta dS + \frac{1}{2}\varGamma\,(dS)^2\,, \text{ where } \varDelta = \frac{\partial C}{\partial S} \text{ and } \varGamma = \frac{\partial^2 C}{\partial S^2}. \tag{22.5}$$

The $\varDelta$ and $\varGamma$ of the Black-Scholes model are given in Remark 22.3; see Figures 22.1–22.2 for illustrations.

To hedge, consider a portfolio (denoted V) with v of the underlying asset, w of another

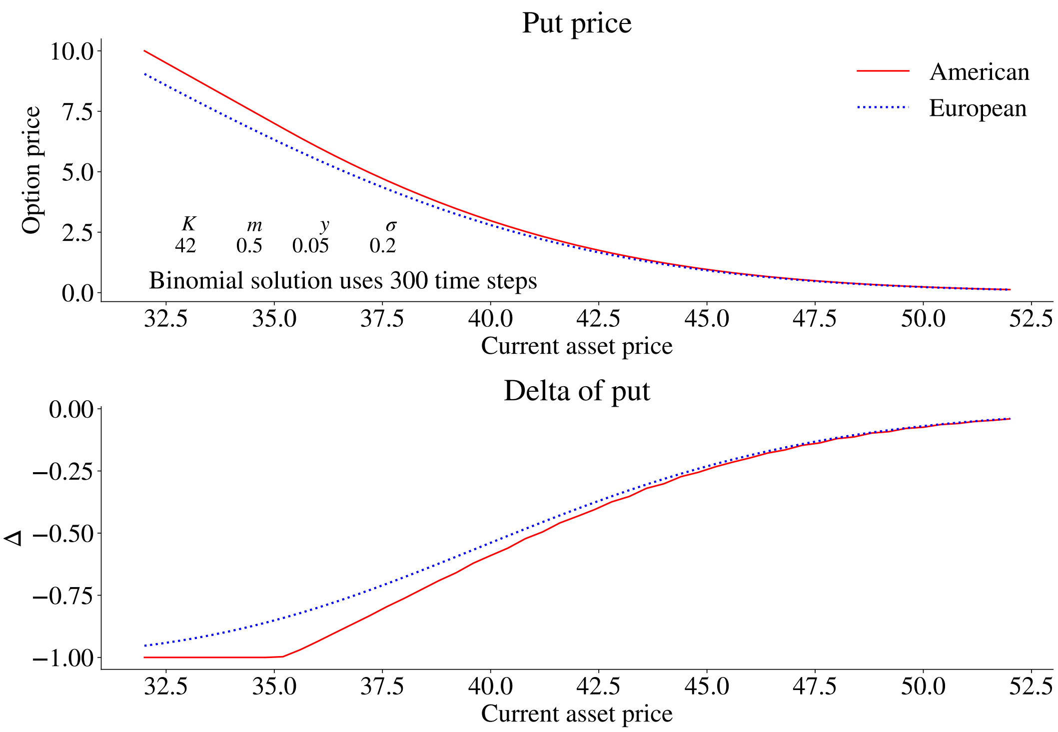


Figure 22.5: The deltas of American and European puts

option (or another asset) with a price denoted C^* and then short one option (with price C)

$$V = vS + wC^* - C. \tag{22.6}$$

A second-order Taylor approximation like (22.5) of the value of portfolio V gives

$$\begin{aligned} dV &= vdS + w[\Delta^* dS + \frac{1}{2}\Gamma^* (dS)^2] - [\Delta dS + \frac{1}{2}\Gamma (dS)^2] \\ &= (v + w\Delta^* - \Delta)dS + (w\Gamma^* - \Gamma)\frac{1}{2}(dS)^2, \end{aligned} \tag{22.7}$$

where Δ^* and Γ^* are the delta and gamma of C^*.

By setting

$$w = \Gamma/\Gamma^*, \text{ and} \tag{22.8}$$

$$v = \Delta - (\Gamma/\Gamma^*)\,\Delta^*, \tag{22.9}$$

we get $dV \approx 0$.

Example 22.9 *(Delta-gamma hedging) Suppose* $(\Delta, \Gamma) = (0.5, 0.07)$ *and* $(\Delta^*, \Gamma^*) =$

$(0.3, 0.03)$, *which requires* $w = 2.33$ *and* $v = -0.2$. *Clearly, this is quite different from a delta hedge (which has* $v = 0.5$ *and* $w = 0$*). Here, the lower sensitivity (gamma) of the second option to the quadratic term means that the hedge portfolio includes a lot of the second option. As a consequence, it becomes overexposed to the linear term, which is compensated for by a short position in the underlying asset.*

22.3.2 Delta-Vega Hedging*

The volatility of financial markets seems to change over time. To account for that, a first-order Taylor approximation of the call option price with respect to *both* the underlying and volatility is

$$dC \approx \Delta dS + \frac{\partial C}{\partial \sigma} d\sigma, \text{ where } \Delta = \frac{\partial C}{\partial S}, \tag{22.10}$$

and where $\partial C / \partial \sigma$ is the "vega" of the option. The Δ and vega of the Black-Scholes model are given in Remark 22.3, but notice that this model is inconsistent with time-variation in volatility—so it can only be used as an approximation.

Consider hedging by holding the following portfolio

$$V = vS + wC^* - C, \tag{22.11}$$

where C^* is the price of some other option (or asset). A first-order Taylor approximation like (22.10) of the value of portfolio V gives

$$\begin{aligned} dV &= vdS + w(\Delta^* dS + \frac{\partial C^*}{\partial \sigma} d\sigma) - (\Delta dS + \frac{\partial C}{\partial \sigma} d\sigma) \\ &= (v + w\Delta^* - \Delta) dS + (w \frac{\partial C^*}{\partial \sigma} - \frac{\partial C}{\partial \sigma}) d\sigma, \end{aligned} \tag{22.12}$$

where Δ^* and $\partial C^* / \partial \sigma$ are the delta and vega of C^*.

By setting

$$w = \frac{\partial C}{\partial \sigma} / \frac{\partial C^*}{\partial \sigma}, \text{ and} \tag{22.13}$$

$$v = \Delta - w\Delta^*, \tag{22.14}$$

we get $dV \approx 0$. For instance, if the C^* asset is directly linked to VIX, then $\Delta^* = 0$ and $\partial C^* / \partial \sigma = 1$.

22.4 Hedging in the Binomial Model*

The binomial model can be used to calculate the derivatives used in the hedging above. If the binomial model is correct, then this should actually provide an *exact hedge*—not just an approximation as in (22.3).

To see that, recall that in any node (ij, where i is time step i and j indicates different values of the underlying asset) of the binomial model, we can replicate the derivative by the portfolio

$$\begin{aligned}
&\Delta_{ij} S_{ij} \text{ in the underlying asset, and} \\
&C_{ij} - \Delta_{ij} S_{ij} \text{ on a money market account, where} \\
\Delta_{ij} &= \frac{C_{u,ij} - C_{d,ij}}{S_{ij}\,(u-d)}.
\end{aligned} \tag{22.15}$$

where $(\Delta_{ij}, S_{ij}, C_{ij})$ are the values in the *current node* (time step i,) and $(C_{u,ij}, C_{d,ij})$ are the values of the derivative in the *next time step* (depending on whether the underlying moves up to $S_{ij}u$ or down to $S_{ij}d$). This means that we hold Δ_{ij} stocks (each of which costs S_{ij}) and the rest on a money market account.

Notice that Δ_{ij} is just the number of underlying assets that is needed to replicate the derivative. However, the right hand side of (22.15) shows that it actually is a finite difference ratio—essentially measuring how the derivative price reacts to changes in the underlying, that is, the analogue to $\partial C/\partial S$. Also notice that the amount on the money market account is the same as in (22.4).

See Figure 22.8 for an example of how the hedge portfolio is structured at each node. It is straightforward to show that the value of this portfolio (in the next time step) is the same as the value of the call option in Figure 22.7.

Example 22.10 *(Replicating portfolio in the binomial model). In the initial node in Figure 22.8, we buy* 0.561 *underlying assets (*$\Delta = 0.561$*) and borrow* 5.392 *on the money market. If the underlying then moves up to 10.5, then this portfolio is worth* $0.561 \times 10.5 - 5.392$ *(since the interest rate is zero), that is,* 0.5*. This is the same as the value of the call option in 22.7.*

Theoretically, the portfolio (22.15) should provide a perfect replication of the derivative irrespective of whether the underlying asset moves up or down. However, if the binomial model is just an approximation (the most likely case), then the hedge will also be approximate.

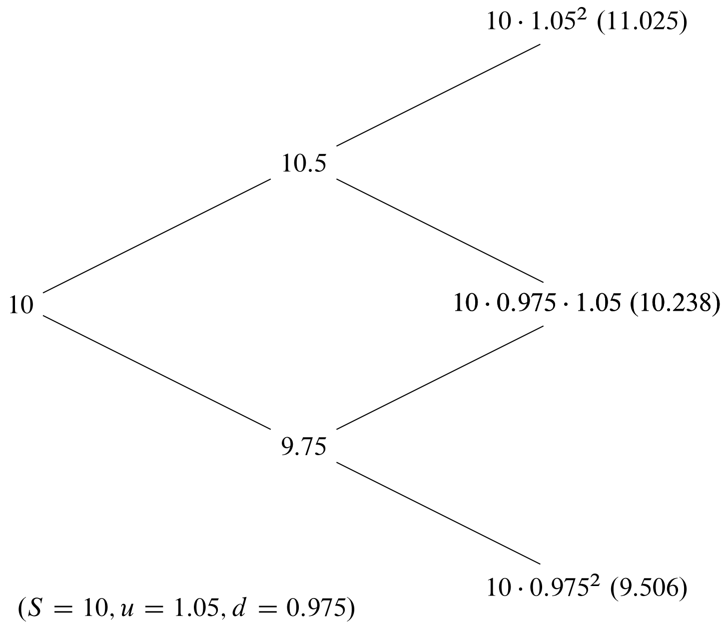


Figure 22.6: Numerical example of a binomial tree for underlying asset ($n = 2$)

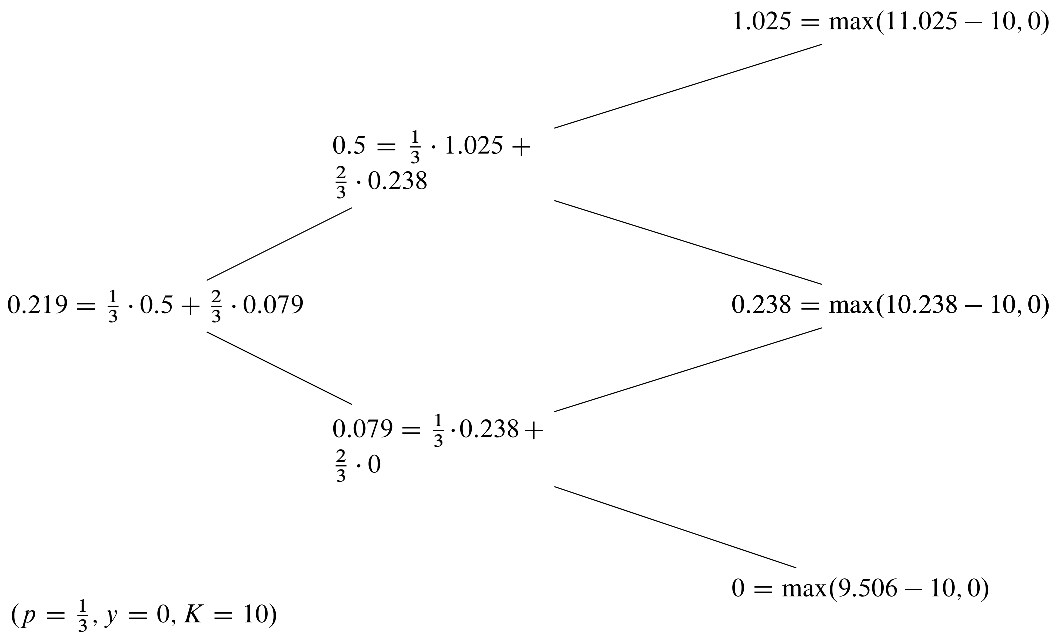


Figure 22.7: Numerical example of binomial tree for European call option ($n = 2$), zero interest rate. The underlying is described in Figure 22.6.

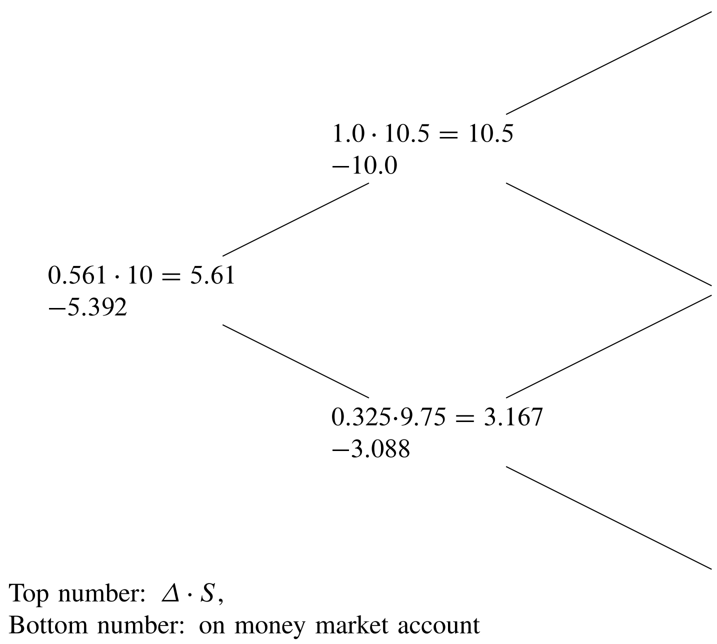


Figure 22.8: Numerical example of how to replicate a European call option ($n = 2$) in a binomial tree, zero interest rate. The underlying and call option are described in Figures 22.6 –22.7.

Chapter 23

FX and Interest Rate Options

Main references: Hull (2009) 15 and Wystup (2006)

23.1 FX Options: Put or Call?

Buying one currency entails selling another. It should therefore come as no surprise that a call option on a currency is also a put option on the other currency. To be precise, the option prices are related according to

$$C_d(\text{strike} = K) = S_t K P_f(\text{strike} = 1/K). \tag{23.1}$$

On the left hand side, C_d is the domestic price of a call option on the foreign currency—with the strike price (K) is expressed in the domestic currency. On the right hand side, S_t is the current exchange rate (domestic price of one unit of the foreign currency), and P_f is the foreign price of a put option on the domestic currency—with the strike price ($1/K$).

In particular, we can rewrite the expression as

$$P_f(\text{strike} = 1/K) = \frac{C_d(\text{strike} = K)}{S_t K}, \tag{23.2}$$

which is the price (measured in foreign currency) of the put on the domestic currency.

Example 23.1 *Let* $C_d = £0.01$ *for an option on US dollars and the strike price is £0.6 (to get one dollar). If the current exchange rate is £0.58 (per dollar), then the dollar price of a put option on GBP with a strike price of* $1/0.6$ *dollars per GBP is* $0.01/(0.58 \times 0.6) =$ \$0.0287.

Remark 23.2 *(Option price quoted in which currency?) In practice, it is important to consider which currency the option price is quoted in. For instance, most options involv-*

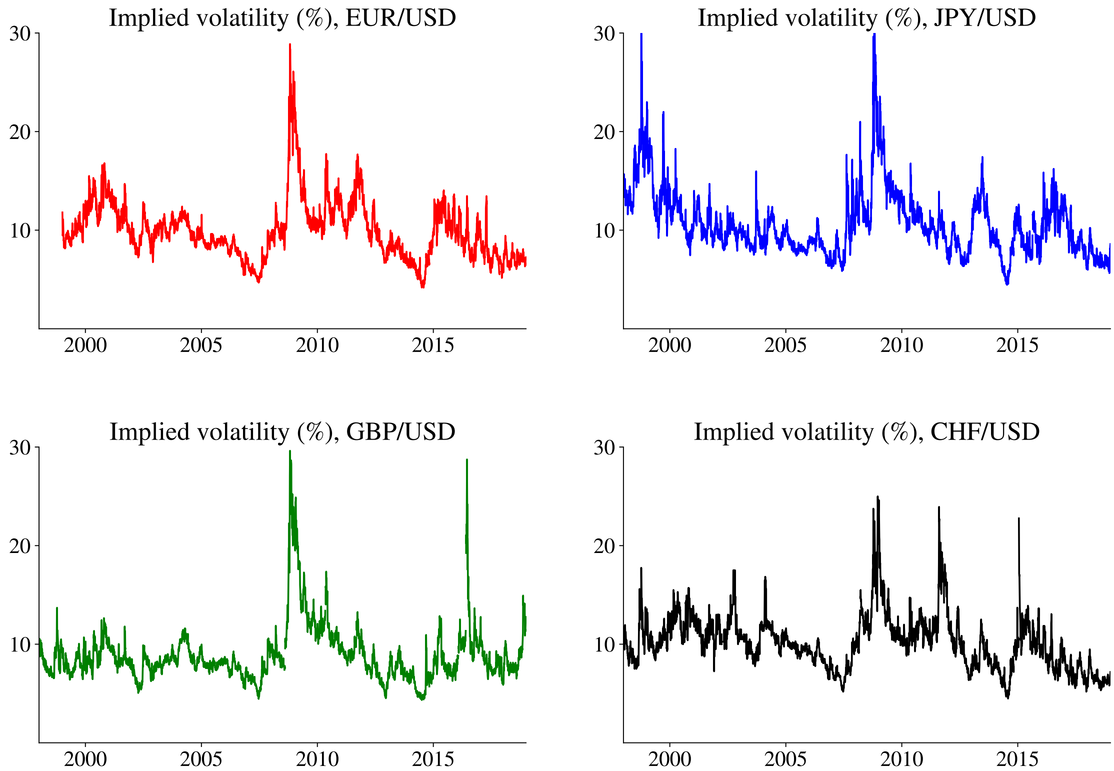


Figure 23.1: Implied volatility (atm) from different 1-month FX options

ing the USD have option prices quoted in USD, while most options involving the EUR (unless also the USD is involved) have prices quoted in EUR.

Proof. (of (23.1)) The payoff of a call option (denominated in the domestic currency) on foreign currency with strike price K is

$$\max(0, S_{t+m} - K),$$

where K is the strike price and S_{t+m} is the exchange rate at expiration—both expressed as the domestic price of one unit of foreign currency (for instance, GBP 0.6 per USD). The payoff is clearly expressed in the domestic currency. In contrast, the payoff of a put option (denominated in the foreign currency) on the domestic currency (with strike price $1/K$) has the payoff

$$\max(0, 1/K - 1/S_{t+m}),$$

which is clearly expressed in the foreign currency. Notice that both options are exercised when $S_{t+m} > K$. In fact, these options are identical, except for a scaling factor and the currency denomination. To see that, consider buying K of the foreign denominated options and then convert the payoff to the domestic currency (multiply by S_{t+m})

$$S_{t+m}K \max(0, 1/K - 1/S_{t+m}) = \max(0, S_{t+m} - K),$$

which is clearly the same as for the first option. For that reason, buying K of the foreign currency denominated put options should have the same price (when measured in domestic currency—multiply by S_t) as the domestically denominated call option. ■

23.2 FX Option Portfolios

Options on the FX (exchange rate) markets are often sold (on the OTC market) as special portfolios (consisting of straddles, risk-reversals and strangles) and quoted in terms of the implied volatilities. Apart from these conventions, options on exchange rates are no different from options on other assets (but, remember that currencies carry "dividends" since holding a currency in practice means holding a money market account in that currency).

23.2.1 Summary of the Black-Scholes Model

This section summarizes some important facts about the Black-Scholes model needed below. See separate lecture notes for details.

For an asset with a continuous dividend rate of δ, the forward-spot parity says $F = Se^{(y-\delta)m}$. In this case the Black-Scholes formula can be written

$$C = e^{-\delta m} S\Phi(d_1) - e^{-ym} K\Phi(d_2), \text{ where} \tag{23.3}$$

$$d_1 = \frac{\ln(S/K) + (y - \delta + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{23.4}$$

When the asset is a currency (read: foreign money market account) and δ is the foreign interest rate, then this is the "Garman-Kolhagen" formula. The sensitivity to the underlying asset price is

$$\frac{\partial C}{\partial S} = e^{-\delta m}\Phi(d_1), \tag{23.5}$$

where d_1 is given by (23.4). Without dividends, just set $\delta = 0$. From the put-call parity, it is clear that the sensitivity of a put option is

$$\frac{\partial P}{\partial S} = e^{-\delta m}[\Phi(d_1) - 1] = -e^{-\delta m}\Phi(-d_1), \tag{23.6}$$

which is negative. The second equality follows from the symmetry of the normal distribution.)

Using $F = Se^{(y-\delta)m}$ in (23.3)–(23.4), we get *Black's model* for an option on a forward contract:

$$C = e^{-ym} F \Phi(d_1) - e^{-ym} K \Phi(d_2), \text{ where} \tag{23.7}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{23.8}$$

For an asset with a continuous dividend, this d_1 is the same as in (23.4). The sensitivity of the call option price to the forward price is

$$\frac{\partial C}{\partial F} = e^{-ym} \Phi(d_1), \tag{23.9}$$

where d_1 is given by (23.8). Similarly, the sensitivity of a put option is

$$\frac{\partial P}{\partial F} = e^{-ym}[\Phi(d_1) - 1] = -e^{-ym} \Phi(-d_1). \tag{23.10}$$

23.2.2 Straddles

A *delta-neutral straddle*, that is, a long position in a call and also in a put. To make it delta-neutral (with respect to the spot), we need

$$\frac{\partial C}{\partial S} + \frac{\partial P}{\partial S} = 0, \tag{23.11}$$

which from (23.5)–(23.6) gives (with d_1 defined by (23.8) or equivalently (23.4))

$$d_1 = 0, \text{ that is, } K_{atm} = Se^{(y-\delta)m + \sigma_{atm}^2 m/2}. \tag{23.12}$$

This straddle is typically quoted in terms of the implied volatility (σ_{atm}) of an option at K_{atm}. A higher value of the straddle indicates more overall uncertainty. See Figure 23.2 for illustrations of the profits of different option portfolios. Also, see Figure 23.1 for an empirical illustration of how the implied volatilities for different FX options have developed.

Remark 23.3 *(Deltas with respect to the forward price*) The market convention is that developed market currencies with time to expiration up to a year are quoted in deltas with respect to the spot price, while all other FX options are quoted in deltas with respect to the forward price. If we use the convention that the deltas are with respect to the forward* *price then K_{atm} is the same as in (23.12). Both conventions are used. (The forward deltas are more common for options with long time expiration and for emerging market currencies.)*

Proof. (of (23.12)) If we use spot deltas, then (23.5)–(23.6) give

$$\frac{\partial C}{\partial S} + \frac{\partial P}{\partial S} = e^{-\delta m}\Phi(d_1) - e^{-\delta m}\Phi(-d_1) = 0,$$

which requires $d_1 = 0$. With d_1 defined by (23.4) we have

$$\ln K = \ln S + (y - \delta + \sigma^2/2)m = \ln F + (\sigma^2/2)m$$

If we instead use forward deltas, use (23.9)–(23.10) and set to zero

$$\frac{\partial C}{\partial F} + \frac{\partial P}{\partial F} = e^{ym}\left[e^{-ym}\Phi(d_1) - e^{-ym}\Phi(-d_1)\right] = 0,$$

which still requires $d_1 = 0$ (and d_1 is the same in (23.8) and (23.4)). ■

23.2.3 Risk Reversals

A *25-delta risk reversal* is a portfolio of one call option with a strike price K_2 such that the delta is 0.25 and short one put option with a strike price K_1 such that the delta is -0.25. (Other values of the deltas are also used.) Both options are out of the money so the strike price for the put is lower than the forward price, which in turn is lower than the strike price of the call ($K_1 < F < K_2$). The risk reversal is typically quoted as the difference of the two implied volatilities

$$rr = \sigma_2 - \sigma_1, \tag{23.13}$$

where σ_2 and σ_1 are the implied volatilities of the options with strike prices K_2 and K_1 respectively (notice that, by the put-call parity, a put and a call with the same strike price have the same implied volatility). A higher value of the risk reversal indicates beliefs of an increase in the underlying—so it captures skewness of the exchange rate distribution.

23.2.4 Strangles and Butterflies

A *25-delta strangle* has a long position in the 25-delta call and also in the 25-delta put. A *25-delta butterfly* is a portfolio that is long one 25-delta straddle and short one delta-neutral straddle. It is typically quoted as the average implied volatility of the K_2 and K_1

options (call and put, respectively) minus the at-the-money volatility

$$bf = \frac{\sigma_2 + \sigma_1}{2} - \sigma_{atm}. \tag{23.14}$$

An increase in bf signals a belief in fatter tails, so it captures kurtosis. Notice that a proportional increase of all volatilities does not change bf (it is "vega" neutral).

With the quotes on the risk reversal (23.13) and the butterfly (23.14), we can solve for the implied volatilities σ_1 and σ_2 as

$$\begin{aligned}\sigma_1 &= bf + \sigma_{atm} - rr/2 \\ \sigma_2 &= bf + \sigma_{atm} + rr/2.\end{aligned} \tag{23.15}$$

23.2.5 Finding the Strike Prices of the Risk Reversals, Strangles and Butterflies

It is straightforward to invert the formulas for the deltas to derive what the strike prices are. If we use the convention that the deltas are with respect to the spot price, then by setting $\partial C/\partial S = \Delta$ (say, $\Delta = 0.25$) in (23.5) to derive the strike price K_2 and $\partial P/\partial S = -\Delta$ in (23.6) to derive the strike price K_1 we get the following strike prices

$$\begin{aligned}K_1 &= Se^{(y-\delta)m}\exp[\sigma_1\sqrt{m}\Phi^{-1}(e^{\delta m}\Delta) + m\sigma_1^2/2] \\ K_2 &= Se^{(y-\delta)m}\exp[-\sigma_2\sqrt{m}\Phi^{-1}(e^{\delta m}\Delta) + m\sigma_2^2/2],\end{aligned} \tag{23.16}$$

Clearly, by changing to $\Delta = 0.10$, we get the strikes for a 10-delta risk reversal.

See Figure 23.3 for how the strike prices are calculated.

Example 23.4 *(σ_{atm}, rr and bf on 1 April 2005, 1-month EUR/GBP) For this particular date and contract σ_{atm} was 4.83%, the 25 delta risk reversal was 0.18% and the 25 delta strangle (really, a 25 delta butterfly) was 0.15%. (See Wystup (2006), tables 1.7–9.) This gives*

$$\begin{aligned}\sigma_1 &= 0.15 + 4.83 - 0.18/2 = 4.89 \\ \sigma_2 &= 0.15 + 4.83 + 0.18/2 = 5.07.\end{aligned}$$

The spot exchange rate was 0.6859 (the price of one EUR, in terms of GBP) and the 1-month interest rates were 4.87 in the UK and 2.10 in the euro zone, so the forward rate was $F = 0.6859 \times \exp[(0.0487 - 0.0210)/12] \approx 0.6875$. This gives $K_1 = 0.6811$, $K_{atm} = 0.6876$ and $K_2 = 0.6941$.

Remark 23.5 *(Deltas with respect to the forward price*) If we use the convention that the deltas are with respect to the forward price then K_1 and K_1 are similar to (23.16), but where Δ is substituted for $e^{\delta m}\Delta$.*

Proof. (of (23.16)) If we use the spot delta, then set (23.5) equal to 0.25

$$0.25 = e^{-\delta m}\Phi\,(d_1)\,, \text{ so we need } d_1 = \Phi^{-1}(e^{\delta m}0.25)$$

With d_1 given by (23.4) we get

$$\ln K_2 = \ln F + (\sigma^2/2)m - \sigma\sqrt{m}\Phi^{-1}(e^{\delta m}0.25),$$

since $\ln F = \ln S + (y - \delta)m$. Instead, if we use the forward delta from using (23.9)

$$\begin{aligned} 0.25 &= e^{ym}\frac{\partial C_t}{\partial F} = \Phi\,(d_1)\,, \text{ so} \\ d_1 &= \Phi^{-1}(0.25). \end{aligned}$$

With d_1 given by (23.8) we get

$$\ln K_2 = \ln F + (\sigma^2/2)m - \sigma\sqrt{m}\Phi^{-1}(0.25).$$

The calculations for the strike prices K_1 for the put are similar. ■

23.2.6 Another Type of Quotation: Implied Volatility for Different Deltas*

Another way to quote FX option prices is to list the implied volatility for different strike prices (instead of the portfolios discussed above)—but where the strike prices are expressed as deltas. For instance, $\Delta = (-0.25, 0, 0.25)$. Often, these are labelled "$25\Delta P$", atm, and "$25\Delta C$", where $25\Delta P$ stands for the strike price where a put has a delta of -0.25, atm stands for the strike price at the money, and $25\Delta C$ is the strike price where a call has a delta of 0.25.

Typically, the atm strike price is as in (23.12), while the "$25\Delta P$" strike price is calculated as K_1 in (23.16) by setting $\Delta = 0.25$ and the "$25\Delta C$" strike price is calculated as K_2 in (23.16) by setting $\Delta = 0.25$. Other deltas are similar.

Remark 23.6 *(Premium-adjusted deltas) When the option price is quoted in the foreign currency, then the deltas reported do not correspond to (23.12) and (23.16). See Wystup (2006) for more details.*

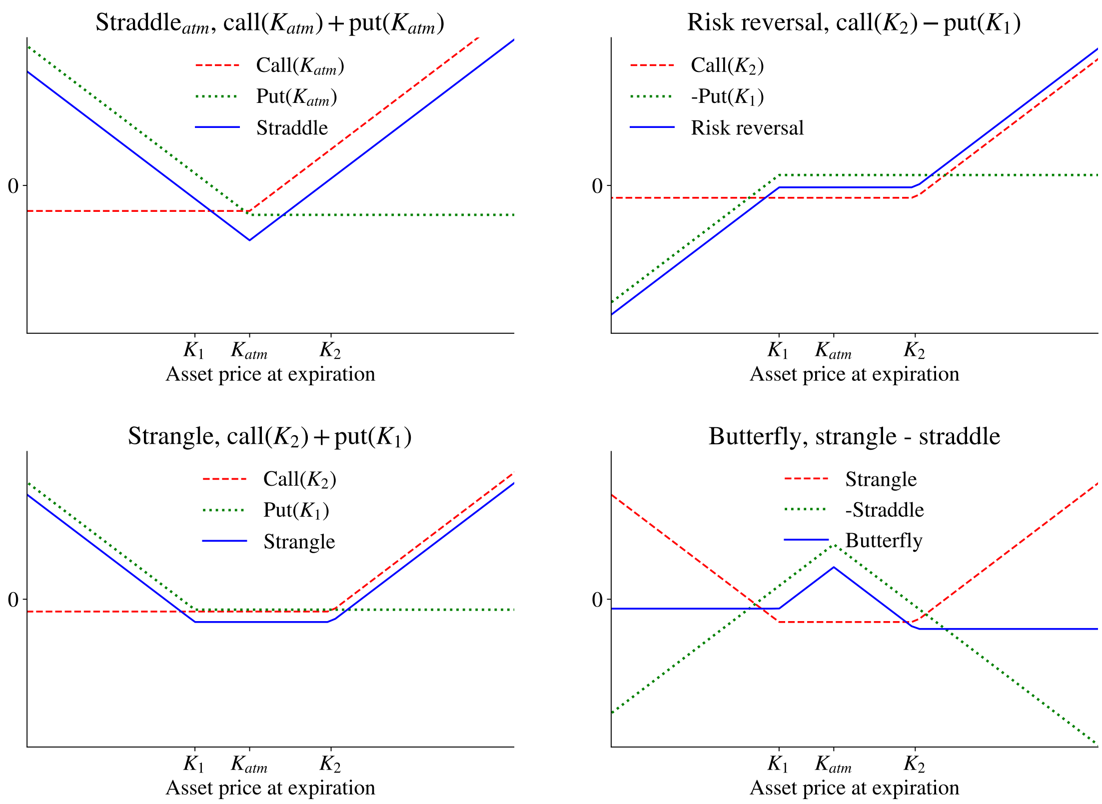


Figure 23.2: Profits diagrams for FX option portfolios

23.3 Options on Interest Rates: Caps and Floors

Options on bonds are basically no different from options on equity, although bonds typically pay "dividends" (the coupons). For instance, a call option on a bond gives the right to buy the bond (at the expiration of the option) at the strike price.

Options on interest rates are also very similar, but often have a more complicated structure. A *caplet* is a call option that protects against higher interest rates (typically a floating 3-month market rate or similar). Let Z_{t+s} be the (annualized) market interest rate for a loan between $t+s$ and $t+s+m$ and let Z_K be the (annualized) cap rate. The payoff in $t+s+m$ (notice: paid at the end of the borrowing period) is

$$\max[0, m(Z_{t+s} - Z_K)]. \tag{23.17}$$

The second term is the interest rate cost for a loan (with a face value of unity) between $t+s$ and $t+s+m$ according to the market rate minus the same cost according to the

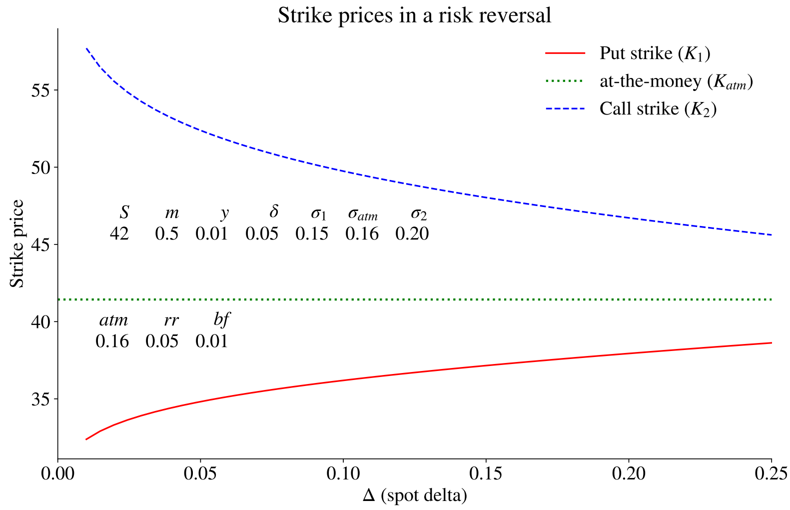


Figure 23.3: Strike prices in a risk reversal

cap rate. Clearly, buying such an option is a way to make sure that interest rate paid on a loan will not exceed the cap rate. If settled at $t+s$ the payoff is just the discounted value

$$\frac{\max[0, m(Z_{t+s} - Z_K)]}{1 + mZ_{t+s}}. \tag{23.18}$$

The payoff in (23.18) can be rewritten as

$$(1 + mZ_K) \max\left(0, \frac{1}{1 + mZ_K} - B_{t+s}(m)\right) \tag{23.19}$$

Notice that the max() term defines the payoff of a put option on an m-period bond in $t+s$ (whose value turns out to be $B_{t+s}(m) = 1/(1 + mZ_{t+s})$)—with a strike price of of $1/(1 + mZ_K)$. The caplet is therefore proportional to a put option on a bond.

Proof. (of (23.19)) Multiply and divide (23.18) by $(1 + mZ_K)$ and rearrange

$$(1 + mZ_K) \max\left[0, \frac{mZ_{t+s} - mZ_K}{(1 + mZ_{t+s})(1 + mZ_K)}\right]$$
$$= (1 + mZ_K) \max\left(0, \frac{1}{1 + mZ_K} - \frac{1}{1 + mZ_{t+s}}\right).$$

Notice that $B_{t+s}(m) = 1/(1 + mZ_{t+s})$. ■

We can apply the Black's formula (23.7)–(23.8) to price the caplet by assuming that a forward contract on either Z_{t+s} or (somewhat less often) B_{t+s} has a lognormal distribution. (These two assumptions are not compatible, since the latter is the same as assuming that $1 + mZ_{t+s}$ has a lognormal distribution.)

Remark 23.7 *(Simple interest rates) If Z is a simple interest rates, then of a zero-coupon bond that gives unity at maturity is*

$$B(m) = \frac{1}{1 + mZ(m)}, \text{ or } Z(m) = \frac{1/B(m) - 1}{m}.$$

A simple forward rate for the period s to s + m periods in the future is defined as

$$Z^f(s, s+m) = \frac{1}{m}\left[\frac{B(s)}{B(s+m)} - 1\right].$$

A forward rate (determined t) for the future investment period $t+s$ to $t+s+m$, denoted Z^f, clearly coincides with the market rate in $t+s$. We can therefore apply Black's formula to the underlying mZ^f by assuming that it is lognormally distributed—and using the strike "price" mZ_K. However, we need to discount by $\exp[-(s+m)y]$ instead of $\exp(-sy)$ since the payoff (23.17) is paid in $t+s+m$ (not in $t+s$). The value of this caplet is therefore

$$\text{Caplet}(s, m; \sigma, Z_K) = me^{-(s+m)y}[Z^f \Phi(d_1) - Z_K \Phi(d_2)], \text{ where} \tag{23.20}$$

$$d_1 = \frac{\ln(Z^f/Z_K) + (\sigma^2/2)s}{\sigma\sqrt{s}} \text{ and } d_2 = d_1 - \sigma\sqrt{s}, \tag{23.21}$$

where σ is the (annualized) volatility of the log forward rate.

An *interest rate cap* is a portfolio of different caplets which protects the owner over several *tenors* (subperiods). Typically, the first caplet is deleted (as there is no uncertainty about what the short rate is today) and the last payment is done on the maturity date n. Therefore, the tenors are $[m, 2m]$, $[2m, 3m]$ and so forth until the last one which is $[n-m, n]$ so there are $n/m-1$ caplets. (The start/end of a tenor is called a reset/settlement date.) For instance, a 1-year cap on the 3-month Libor consists of 3 caplets. See Figure 23.4 for an illustration. (The cap could also be scheduled to start at a later date.)

If we apply the same volatility to all caplets ("flat volatilities"), then the price of a cap (according to the Black-Scholes model) starting now and ending in n, is

$$\text{Cap}(n, m; \sigma, Z_K) = \sum\nolimits_{i=1}^{n/m-1} \text{Caplet}(im, m; \sigma, Z_K). \tag{23.22}$$

Caps are often quoted in terms of the implied volatility (σ) that solves this equation—meaning that there is one implied volatility per cap contract, but it may differ across cap

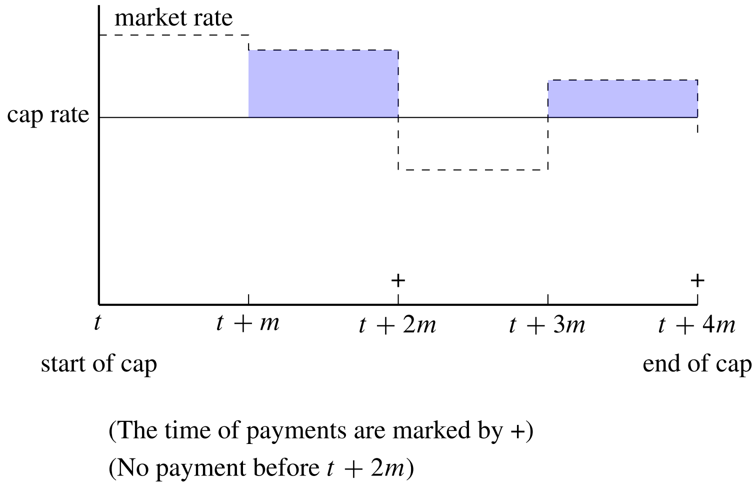


Figure 23.4: Interest rate cap

rates ("strike prices") and maturities. (If the cap is scheduled to start S periods ahead, instead of now, then im should be replaced by $S + im$.)

Example 23.8 *(1-year Cap starting now, 3-month tenors) Let* $n = 1$ *(1-year cap) and* $m = 1/4$ *(3-month tenors). The payoffs are based on the difference between the 3-month Libor and the cap rate at the beginning of the tenors* $(1/4, 2/4, 3/4)$*, but are paid one quarter later. Equation (23.22) is therefore*

$$Cap(1, 1/4; \sigma, Z_K) = Caplet(1/4, 1/4; \sigma, Z_K) + Caplet(2/4, 1/4; \sigma, Z_K) + Caplet(3/4, 1/4; \sigma, Z_K).$$

Floorlets and *floors* are similar to caplets and caps, except that they pay off when the interest goes below the cap rate.

Chapter 24

Trading Volatility

Reference: Gatheral (2006) and McDonald (2014) 29

More advanced material is denoted by a star (*). It is not required reading.

24.1 The Purpose of Trading Volatility

By using option portfolios (for instance, straddles) it is possible to create a position that is a bet on volatility—and is (in principle) not sensitive to the direction of change of the underlying. See Figure 24.1 for an illustration.

Volatility, as an asset class, has some interesting features. In particular, returns on the underlying asset and volatility are typically negatively correlated: very negative returns are typically accompanied by increases in future actual volatility as well as beliefs about higher future volatility (as priced into options). See Figure 24.2 for an illustration, where changes in the VIX are taken to proxy the one-day holding return on a straddle.

There are several ways of trading volatility: straddles (and other option portfolios), futures (and options) on the VIX, as well as volatility (and variance) swaps.

24.2 VIX and VIX Futures

The VIX is an index of volatility, calculated from 1-month options on S&P 500. It used to be calculated as an average of implied volatilities, but since 2003 the calculation is more complicated (the old series is now called VXO). It can be shown (although it is a bit tricky) that the VIX is a very good approximation to the square root of the variance swap rate (see below) for a 30-day contract. There are also futures contracts on VIX with payoff

$$\text{VIX futures payoff}_{t+m} = VIX_{t+m} - \text{futures price}_t. \tag{24.1}$$

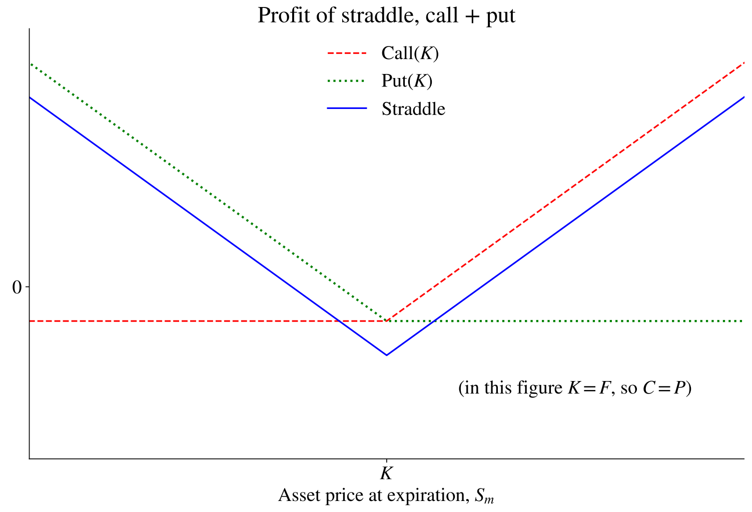


Figure 24.1: Profit of straddle

Notice that VIX_{t+m} is really a guess of what the volatility will be during the month after $t + m$, so the futures contract pays off when the expected volatility (in $t + m$) is higher than what was thought in t.

See Figures 24.3–24.4 for an empirical illustration. Notice that the futures prices indicate that volatility is mean reverting: high VIX levels are associated with negative spreads (the futures is lower than the current VIX). This indicates that market participants believe that volatility will settle down.

Remark 24.1 *(Calculation of VIX) Let F be the forward price, $\Delta K_i = (K_{i+1} - K_{i-1})/2$ and let K_0 denote the first strike price below F. Then, the VIX is calculated as*

$$VIX^2 = \frac{2}{m}\exp(ym)\sum_{K_i \le K_0}\frac{\Delta K_i}{K_i^2}P(K_i) + \frac{2}{m}\exp(ym)\sum_{K_i > K_0}\frac{\Delta K_i}{K_i^2}C(K_i) - \frac{1}{m}(F/K_0 - 1)^2,$$

where m is the time to expiration (around 1/12), y the interest rate, $P()$ the put price and $C()$ the call price.

24.3 Variance and Volatility Swaps

Instead of investing in straddles, it is also possible to invest in *variance swaps*. Such a contract has a zero price in inception (in t) and the payoff at expiration (in $t + m$) is

$$\text{Variance swap payoff}_{t+m} = \text{realized variance}_{t+m} - \text{variance swap rate}_t, \qquad (24.2)$$

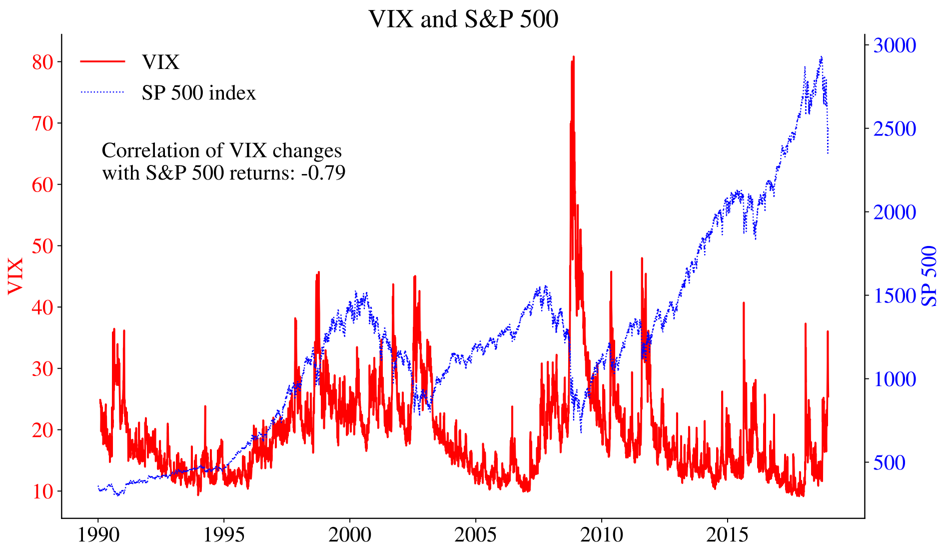


Figure 24.2: S&P 500 and VIX

where the variance swap rate (also called the strike or forward price for) is agreed on at inception (t) and the realized volatility is just the sample variance for the swap period. Both rates are typically annualized, for instance, if data is daily and includes only trading days, then the variance is multiplied by 252 or so (as a proxy for the number of trading days per year).

A *volatility swap* is similar, except that the payoff it is expressed as the difference between the standard deviations instead of the variances

$$\text{Volatility swap payoff}_{t+m} = \sqrt{\text{realized variance}_{t+m}} - \text{volatility swap rate}_t, \qquad (24.3)$$

If we use daily data to calculate the realized variance from t until the expiration(RV_{t+m}), then

$$RV_{t+m} = \frac{252}{m}\textstyle\sum_{s=1}^{m} R_{t+s}^2, \qquad (24.4)$$

where R_{t+s} is the net return on day $t+s$. (This formula assumes that the mean return is zero—which is typically a good approximation for high frequency data. In some cases, the average is taken only over $m-1$ days.)

Notice that both variance and volatility swaps pay off if actual (realized) volatility between t and $t+m$ is higher than expected in t. In contrast, the futures on the VIX pays off when the expected volatility (in $t+m$) is higher than what was thought in t. In a way, we can think of the VIX futures as a futures on a volatility swap (between $t+m$ and a

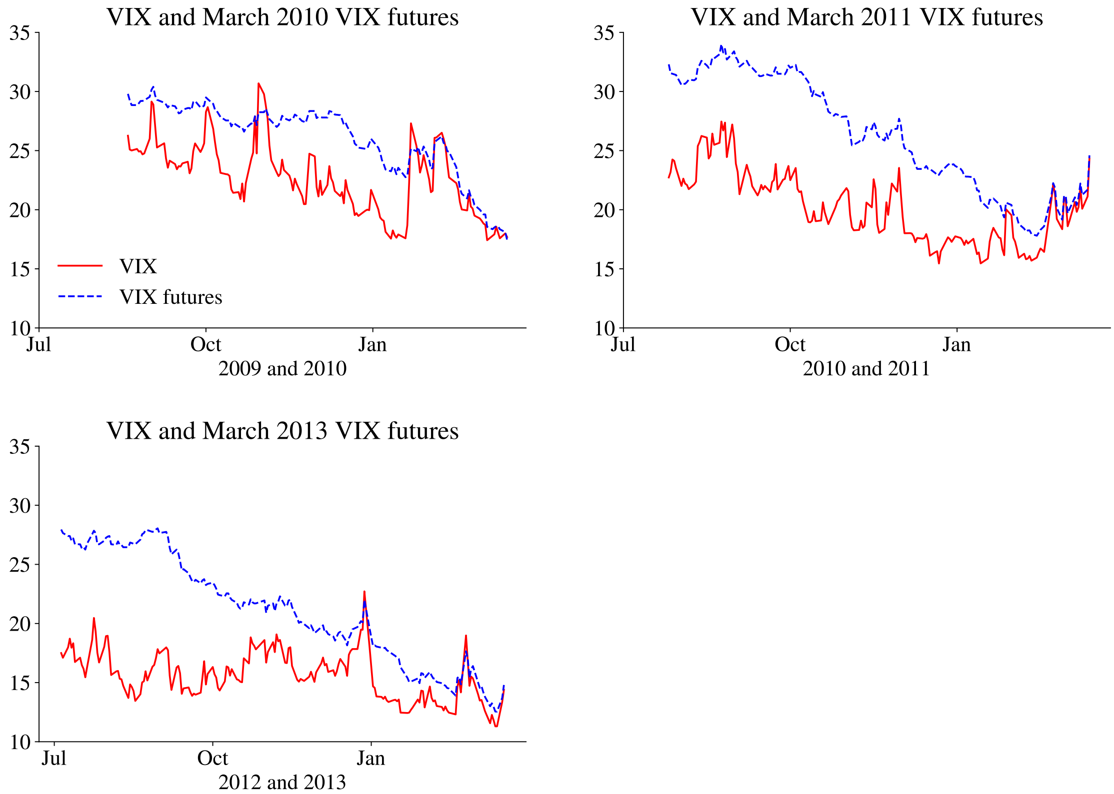


Figure 24.3: VIX and futures contract on VIX

month later).

Since VIX^2 is a good approximation of variance swap rate for a 30-day contract, the return can be approximated as

$$\text{Return of a variance swap}_{t+m} = (RV_{t+m} - VIX_t^2)/VIX_t^2. \tag{24.5}$$

Figures 24.5–24.6 illustrate the properties for the VIX and realized volatility of the S&P 500. It is clear that the return of a variance swap (with expiration of 30 days) would have been negative on average. (Notice: variance swaps were not traded for the early part of the sample in the figure.) The excess return (over a riskfree rate) would, of course, have been even more negative. This suggests that selling variance swaps (which has been the specialty of some hedge funds) might be a good deal—except that it will incur some occasional really large losses (the return distribution has positive skewness). Presumably, buyers of the variance swaps think that this negative average return is a reasonable price to pay for the "hedging" properties of the contracts—although the data does not suggest

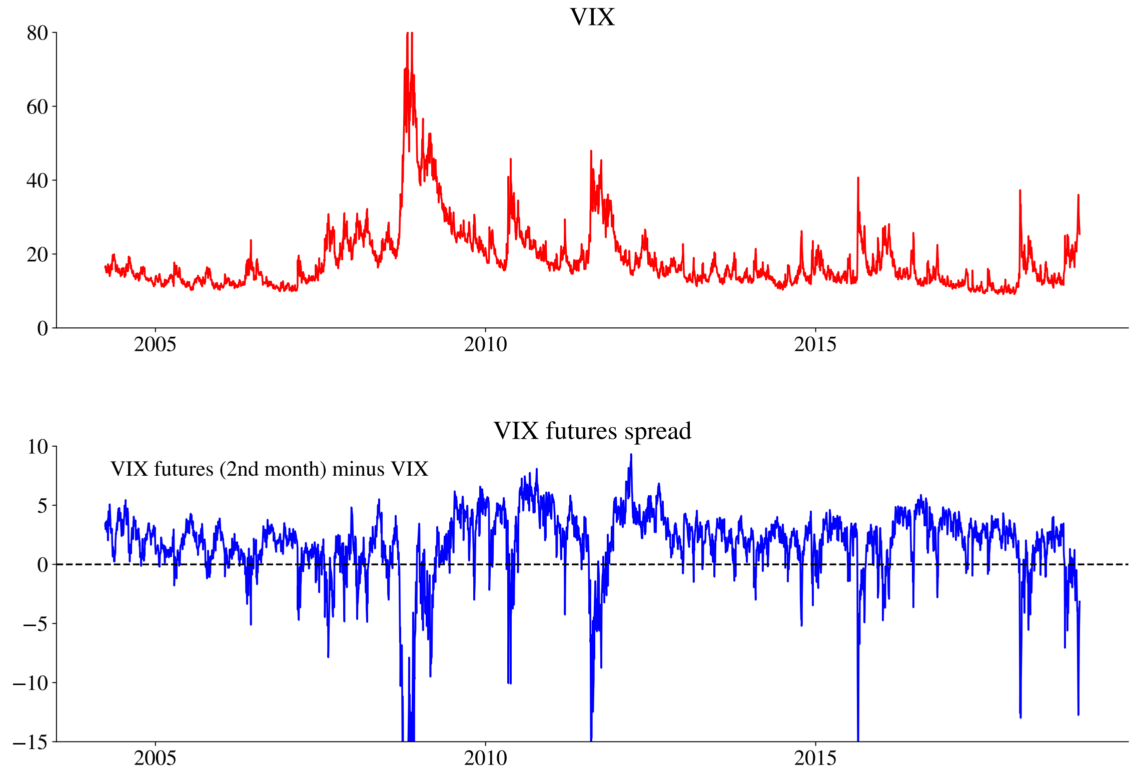


Figure 24.4: VIX futures spread

a very strong negative correlation with S&P 500 returns.

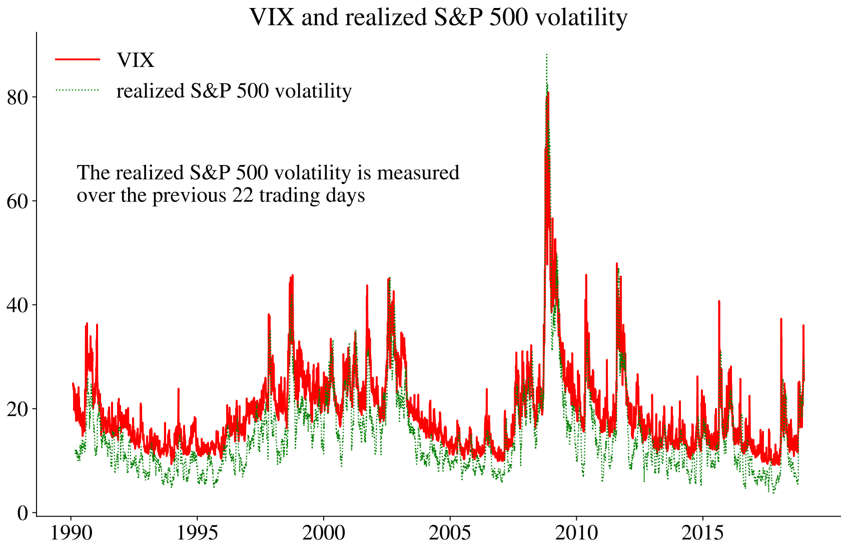


Figure 24.5: VIX and realized volatility (variance)

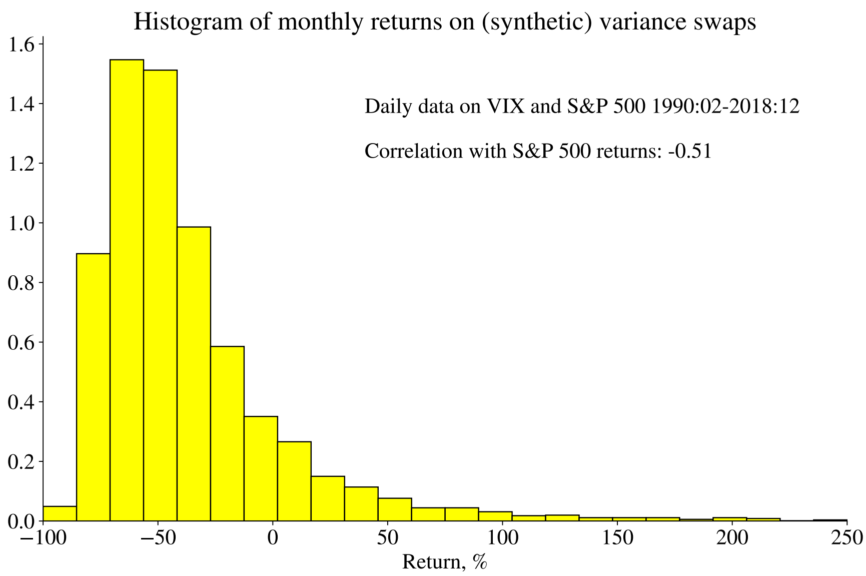


Figure 24.6: Distribution of return from investing in variance swaps

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