“Lecture notes” for Lie theory spring 2024

In these notes I will write down a plan for the course in Lie theory that is being held at Campus Førde for the robotics group and mathematicians on campus. The course is initially heavily inspired by the compendium of Brian Hall “An Elementary Introduction to Groups and Representations» that is open on arXiv, https://arxiv.org/abs/math-ph/0005032.

First session

We aim to go through some basics of abstract algebra that are required for further exploration of Lie theory.

What is a Lie group?

Let us start in medias res.

Definition (Lie Group):

A Lie group is a differentiable manifold G that also is a group such that the group operation

$$* : G \times G \to G,$$

and its inverse $g \to g^{-1}$ is differentiable.

So what is a group then?

This first session will mostly be about groups. So what is this strange and useful object?

Definition (group):

A group is a set G and an operation

$$* : G \times G \to G,$$

such that the following holds:

1. Associativity: For $g, h, i \in G$ we have $(g * h) * i = g * (h * i)$.
2. Identity: There exists an identity element $e \in G$ such that $g * e = e * g = g$ for all $g \in G$.
3. Inverse: To each element $g \in G$ there exists an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Note that the three requirements for $(G,*)$ to be a group are called the **group axioms**. It is also implied with from $* : G \times G \to G$, that the group is closed under the operation, i.e., if $g \in G$ and $h \in G$, then $g * h \in G$.

Extra definition (Abelian group) (Yes, after the Norwegian mathematician Niels Henrik Abel)

If a group is commutative under the operation, i.e., $g * h = h * g$ for all $g, h \in G$, then the group is called an abelian group.

About the $*$

I will eventually forget to write the operation symbol $*$. Therefore, we will make the usual convention that if two letters are written next to each other it is implied that there has happened an operation between them, $g * h = gh$.

Some examples of groups

1. The **trivial** group: The set with only one element, e, is a group where the group operation is defined by $ee = e$.

"Proof that the trivial group is a group":

Let us call the group $T = \{e\}$, and check that all of the axioms are satisfied. First off all, the group is closed under the operation as $ee = e \in T$.

Associativity: $(ee)e = ee = e(ee)$.

Identity: $ee = e$ for all $e \in T$ as only e is in T.

Inverse: e is the inverse of e, therefore alle elements of T has an inverse,

2. The integers under addition, $(\mathbb{Z}, +)$: The set of integers form a group with addition $+$ as the group operation.

In-between exercise – Talk to your neighbor and convince yourself (prove) that the trivial group is a group. In other words, check that the group axioms are satisfied and that the group is closed under the group operation.

3. The real numbers $(\mathbb{R}, +)$ and real-valued vectors $(\mathbb{R}^n, +)$ under addition.
4. Nonzero real numbers under multiplication $(\mathbb{R}^+, \cdot)$.
5. Non-zero Complex numbers under multiplication $(\mathbb{C}^+, \cdot)$.
6. Complex numbers of absolute value one under multiplication S^1.
7. Invertible matrices under matrix multiplication $GL(n, \mathbb{R})$, this group is called the *general linear group.*
8. The set of matrices with determinant one is a group under matrix multiplication, $SL(n, \mathbb{R})$. This is called the *special linear group.*
9. Integers modulo n, $\mathbb{Z}_n$.
10. Permutation group. The set of one-to-one maps from $\{1,2, \dots, n\}$ onto itself is a group under function composition (not important for us, but quite fun).

Exercise break:

Split the participants into groups of two that will together prove that selections of the groups above are groups. Determine which of the groups are abelian.

Show that $\mathbb{Z}$ not is a group under multiplication.

Is $\mathbb{R}^+$ a group under the operation $a * b = \sqrt{ab}$.

Properties of groups:

1. The identity in a group is unique.
   Proof: Let G be a group and assume that there exist two elements $e \in G$ and $f \in G$ such that $eg = g = fg$ for all $g \in G$. We then necessarily have that $e = ef = f$.
2. Each element in a group has a unique inverse.
   Exercise.
3. In groups it is sufficient with $gh = e$ to be sure that h is the unique inverse of g.
   Proof: Let $g, h \in G$ be such that $gh = e$. We can now multiply both sides by the inverse of g, $g^{-1}(gh) = g^{-1}e$. Then we have (by associativity and multiplication with identity) that $h = g^{-1}$.

4. The inverse of the inverse is the element itself, $(g^{-1})^{-1} = g$.
   Exercise.

Subgroups

Definition (subgroup):

A subgroup H of a group G is a subset such that H is itself a group under the same operation as G. One only needs to check the following conditions:

1. The identity is in H.
2. If $h \in H$ then $h^{-1} \in H$.
3. H is closed, i.e., if $h_1, h_2 \in H$ then $h_1 h_2 \in H$.

Examples:

1. $\mathbb{Z}$ under addition is a subgroup of $\mathbb{R}$ under addition.
2. S^1 is a subgroup of $\mathbb{C}^+$
3. $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

Exercise: Show that this is true.

Maps between groups (homomorphisms)

We will now consider what happens when we assign elements of groups to each other using a special type of maps called a **homomorphism**.

Definition (homomorphisms):

Let G and H be groups. A map $\phi : G \to H$ is called a homomorphism if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. Note that the group operations inside ϕ and outside ϕ are not (necessarily) the same as they can be operations on different groups.

If a homomorphism is bijective, it is called an **isomorphism**. If there exist an isomorphism between two groups, they are called **isomorphic**. Two groups that are isomorphic somehow act the same as groups, although they are not strictly the same.

An isomorphism of a group with itself is called an **automorphism**.

Proposition (Important fact in everyday normal guy's language):

Identities and inverses are preserved through homomorphisms. I.e., $\phi(e_g) = e_h$ and $\phi(g^{-1}) = (\phi(g))^{-1}$.

Proof. Exercise

Definition (Kernel):

The kernel of a homomorphism $\phi: G \to H$ is the subset $\ker(\phi) \subseteq G$ such that $\phi(g) = e_h$.

The kernel of a homomorphism is a subgroup of G. (This can easily be verified).

Proposition: If a kernel of a homomorphism only includes the identity element, then the homomorphism is injective (i.e., no two elements are sent to the same element by the homomorphism).

Proof. Assume that $\ker(\phi) = e_g$ for $\phi: G \to H$. Let now $g_1, g_2 \in G$ be such that $\phi(g_1) = \phi(g_2)$. We then have $e_h = \phi(g_1)\phi(g_1)^{-1} = \phi(g_2)\phi(g_1)^{-1} = \phi(g_2 g_1^{-1})$, i.e., $g_2 g_1^{-1} \in \ker(\phi)$. Since $\ker(\phi) = e_g$ we have that $g_2 g_1^{-1} = e_g$ and by the uniqueness of the inverse we have that $g_1 = g_2$.

(René Des)**Cartesian products:**

Let G and H be two groups. Then the Cartesian product $G \times H$ with the product $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ is itself a group.

Both G and H are isomorphic to subgroups of $G \times H$ by fixing one element in the opposite group.

If there is more time left

We look at exercises 2, 5, 8 (what about $SL(n, \mathbb{R})$?) and 13 from the compendium.

Some extra material for later in relation to manifolds and the unit circle

The unit circle (for example described through the complex numbers)

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is a group under multiplication $\cdot$. Let us check that the closedness and the axioms for being a group are satisfied:

First, remember that all elements in S^1 can be described by an angle through Euler's formula

$$z \in S^1 \Rightarrow \exists \theta \in \mathbb{R} \text{ such that } z = e^{i\pi\theta}.$$

Now, given three arbitrary elements of S^1, $z_1 = e^{i\pi\theta_1}$, $z_2 = e^{i\pi\theta_2}$ and $z_3 = e^{i\pi\theta_3}$, we can first multiply two of them together to check that the group is closed under multiplication

$$z_1 \cdot z_2 = e^{i\pi\theta_1} \cdot e^{i\pi\theta_2} = e^{i\pi(\theta_1+\theta_2)} \in S^1.$$

Then we have the three axioms:

1. Associativity:

$$(z_1 \cdot z_2) \cdot z_3 = \left(e^{i\pi\theta_1} \cdot e^{i\pi\theta_2}\right) \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2)} \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2+\theta_3)} = e^{i\pi\theta_1} \cdot \left(e^{i\pi\theta_2} \cdot e^{i\pi\theta_3}\right) = z_1 \cdot (z_2 \cdot z_3).$$

2. Identity: $z = 1 = e^0$.
3. Inverse: Given $z = e^{i\pi\theta}$, the inverse is given by $z^{-1} = e^{-i\pi\theta}$.

And what is a differentiable (smooth) manifold?

That is a bit more work to define, but we will manage.

A manifold is a (topological) space that locally is equivalent (homeomorphic) to Euclidean space ($\mathbb{R}^n$). Many fancy words, but the important property is that around every point on the manifold there should exist a neighborhood and an invertible continuous map (whose inverse is also continuous) from that neighborhood to a subset of $\mathbb{R}^n$.

Sidenote on topological spaces

A topological space is a set equipped with (rules for how to define) open sets (closed sets are complements of the open sets). In that regard S^1 is a topological space where all sets of the form $(z_1, z_2) \coloneqq \left\{z = e^{i\pi\theta} \in S^1 \mid \theta \in (\theta_1, \theta_2)\right\}$ (and unions of them) are open sets.

The unit circle S^1 is a manifold

We can easily check that S^1 is a manifold. We already saw that it is a topological space. Now we just need to see that it is locally homeomorphic to Euclidean space. Given a point $z \in S^1$, we can always choose a small neighborhood (a small open set as defined above, important that the angles are less than π apart) around the point $(z_1, z_2) = \left(e^{i\pi\theta_1}, e^{i\pi\theta_2}\right)$.