

“Lecture notes” for Lie theory spring 2024

In these notes I will write down a plan for the course in Lie theory that is being held at Campus Førde for the robotics group and mathematicians on campus. The course is initially heavily inspired by the compendium of Brian Hall “An Elementary Introduction to Groups and Representations» that is open on arXiv, <https://arxiv.org/abs/math-ph/0005032>.

First session

We aim to go through some basics of abstract algebra that are required for further exploration of Lie theory.

What is a Lie group?

Let us start in medias res.

Definition (Lie Group):

A Lie group is a differentiable manifold G that also is a group such that the group operation

$$* : G \times G \rightarrow G,$$

and its inverse $g \rightarrow g^{-1}$ is differentiable.

So what is a group then?

This first session will mostly be about groups. So what is this strange and useful object?

Definition (group):

A group is a set G and an operation

$$* : G \times G \rightarrow G,$$

such that the following holds:

1. Associativity: For $g, h, i \in G$ we have $(g * h) * i = g * (h * i)$.
2. Identity: There exists an identity element $e \in G$ such that $g * e = e * g = g$ for all $g \in G$.
3. Inverse: To each element $g \in G$ there exists an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Note that the three requirements for $(G, *)$ to be a group are called the **group axioms**. It is also implied with from $* : G \times G \rightarrow G$, that the group is closed under the operation, i.e., if $g \in G$ and $h \in G$, then $g * h \in G$.

Extra definition (Abelian group) (Yes, after the Norwegian mathematician Niels Henrik Abel)

If a group is commutative under the operation, i.e., $g * h = h * g$ for all $g, h \in G$, then the group is called an abelian group.

About the $*$

I will eventually forget to write the operation symbol $*$. Therefore, we will make the usual convention that if two letters are written next to each other it is implied that there has happened an operation between them, $g * h = gh$.

Some examples of groups

1. The **trivial** group: The set with only one element, e , is a group where the group operation is defined by $ee = e$.

“Proof that the trivial group is a group”:

Let us call the group $T = \{e\}$, and check that all of the axioms are satisfied. First off all, the group is closed under the operation as $ee = e \in T$.

Associativity: $(ee)e = ee = e(ee)$.

Identity: $ee = e$ for all $e \in T$ as only e is in T .

Inverse: e is the inverse of e , therefore all elements of T has an inverse,

2. The integers under addition, $(\mathbb{Z}, +)$: The set of integers form a group with addition $+$ as the group operation.

In-between exercise – Talk to your neighbor and convince yourself (prove) that the trivial group is a group. In other words, check that the group axioms are satisfied and that the group is closed under the group operation.

3. The real numbers $(\mathbb{R}, +)$ and real-valued vectors $(\mathbb{R}^n, +)$ under addition.
4. Nonzero real numbers under multiplication (\mathbb{R}^+, \cdot) .
5. Non-zero Complex numbers under multiplication (\mathbb{C}^+, \cdot) .
6. Complex numbers of absolute value one under multiplication S^1 .
7. Invertible matrices under matrix multiplication $GL(n, \mathbb{R})$, this group is called the *general linear group*.
8. The set of matrices with determinant one is a group under matrix multiplication, $SL(n, \mathbb{R})$. This is called the *special linear group*.
9. Integers modulo n , \mathbb{Z}_n .
10. Permutation group. The set of one-to-one maps from $\{1, 2, \dots, n\}$ onto itself is a group under function composition (not important for us, but quite fun).

Exercise break:

Split the participants into groups of two that will together prove that selections of the groups above are groups. Determine which of the groups are abelian.

Show that \mathbb{Z} not is a group under multiplication.

Is \mathbb{R}^+ a group under the operation $a * b = \sqrt{ab}$.

Properties of groups:

1. The identity in a group is unique.
Proof: Let G be a group and assume that there exist two elements $e \in G$ and $f \in G$ such that $eg = g = fg$ for all $g \in G$. We then necessarily have that $e = ef = f$.
2. Each element in a group has a unique inverse.
Exercise.
3. In groups it is sufficient with $gh = e$ to be sure that h is the unique inverse of g .
Proof: Let $g, h \in G$ be such that $gh = e$. We can now multiply both sides by the inverse of g , $g^{-1}(gh) = g^{-1}e$. Then we have (by associativity and multiplication with identity) that $h = g^{-1}$.

4. The inverse of the inverse is the element itself, $(g^{-1})^{-1} = g$.

Exercise.

Subgroups

Definition (subgroup):

A subgroup H of a group G is a subset such that H is itself a group under the same operation as G . One only needs to check the following conditions:

1. The identity is in H .
2. If $h \in H$ then $h^{-1} \in H$.
3. H is closed, i.e., if $h_1, h_2 \in H$ then $h_1 h_2 \in H$.

Examples:

1. \mathbb{Z} under addition is a subgroup of \mathbb{R} under addition.
2. S^1 is a subgroup of \mathbb{C}^+
3. $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

Exercise: Show that this is true.

Maps between groups (homomorphisms)

We will now consider what happens when we assign elements of groups to each other using a special type of maps called a **homomorphism**.

Definition (homomorphisms):

Let G and H be groups. A map $\phi : G \rightarrow H$ is called a homomorphism if $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$.

Note that the group operations inside ϕ and outside ϕ are not (necessarily) the same as they can be operations on different groups.

If a homomorphism is bijective, it is called an **isomorphism**. If there exist an isomorphism between two groups, they are called **isomorphic**. Two groups that are isomorphic somehow act the same as groups, although they are not strictly the same.

An isomorphism of a group with itself is called an **automorphism**.

Proposition (Important fact in everyday normal guy's language):

Identities and inverses are preserved through homomorphisms. I.e., $\phi(e_g) = e_h$ and $\phi(g^{-1}) = (\phi(g))^{-1}$.

Proof. Exercise

Definition (Kernel):

The kernel of a homomorphism $\phi: G \rightarrow H$ is the subset $\ker(\phi) \subseteq G$ such that $\phi(g) = e_h$.

The kernel of a homomorphism is a subgroup of G . (This can easily be verified).

Proposition: If a kernel of a homomorphism only includes the identity element, then the homomorphism is injective (i.e., no two elements are sent to the same element by the homomorphism).

Proof. Assume that $\ker(\phi) = e_g$ for $\phi: G \rightarrow H$. Let now $g_1, g_2 \in G$ be such that $\phi(g_1) = \phi(g_2)$. We then have $e_h = \phi(g_1)\phi(g_1)^{-1} = \phi(g_2)\phi(g_1)^{-1} = \phi(g_2g_1^{-1})$, i.e., $g_2g_1^{-1} \in \ker(\phi)$. Since $\ker(\phi) = e_g$ we have that $g_2g_1^{-1} = e_g$ and by the uniqueness of the inverse we have that $g_1 = g_2$.

(René Des) **Cartesian products:**

Let G and H be two groups. Then the Cartesian product $G \times H$ with the product $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$ is itself a group.

Both G and H are isomorphic to subgroups of $G \times H$ by fixing one element in the opposite group.

If there is more time left

We look at exercises 2, 5, 8 (what about $SL(n, \mathbb{R})$?) and 13 from the compendium.

Second session – Matrix Lie groups

In this session we will define what a matrix Lie group is (it is in some way easier to define that going through the whole process of manifolds), then we will see some examples and non-examples, look at lie group homomorphisms and finally state that all matrix Lie groups also are Lie groups (a not completely trivial result).

Definition (Matrix Lie group)

A matrix Lie group is a subgroup H of $GL(n, \mathbb{C})$ with the property that any convergent sequence in H either converges to a matrix in H or the matrix is not invertible (not in $GL(n, \mathbb{C})$).

Note 1: Convergence here, means elementwise convergence with the usual definition of limits (ϵ/δ).

Note 2: Saying that a subgroup H of $GL(n, \mathbb{C})$ is a matrix Lie group is the same as saying that H is closed in $GL(n, \mathbb{C})$. It needs however not be closed in the set of all matrices, although many are.

Note 3: To be able to talk about topological properties of Matrix Lie Groups (open/closed sets, continuous functions, differentiability) we can identify the space of $n \times n$ matrices with the space \mathbb{C}^{n^2} and its usual topology (defines as one would imagine, extrapolated from 1, 2 and 3 dimensional real space).

A counter-example

A subgroup of $GL(n, \mathbb{C})$ that is not a matrix Lie group is the group of all invertible matrices with rational entries. The set is clearly a subgroup (multiplication and addition of rational numbers are still rational), but every irrational number can be written as a limit of rational numbers (a math fact), hence there are sequences of that converge in $GL(n, \mathbb{C})$ but not to elements of the subgroup. This counterexample can also be traced back to the fact that subgroups of rational numbers are not Lie groups as they cannot be manifolds (not locally similar enough to real numbers).

Exercise:

The subgroup of $GL(n, \mathbb{C})$,

$$H = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-ita} \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

is not a matrix Lie group although it is a Lie group. **Hint:** ($-I$ is not in this group, can you make a limit of elements in this group that converges to I ?)

Examples

Here follows a list of examples of matrix Lie groups.

1. The general linear groups: $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.

Exercise: Prove that these are matrix Lie groups.

2. The special Linear groups: $SL(n, \mathbb{R})$, and $SL(n, \mathbb{C})$.

Proof: Both groups are subgroups of the general linear groups. Moreover, the determinant is a continuous function, hence the limit of matrices with unit determinant also needs to have unit determinant.

3. The orthogonal and special orthogonal groups, $O(n)$, and $SO(n)$. These are subgroups of $GL(n, \mathbb{R})$.

Reminder: Orthogonal matrices are those with orthonormal columns. The orthogonal group consist of those. The special orthonormal group is the subgroup of the orthonormal group with determinant equal to one. The proof that these are matrix Lie groups are the same as for the special linear group, i.e., the conditions to be in these groups are continuous.

Exercise 6 in hall for classification.

4. The unitary and special unitary groups, $U(n)$, and $SU(n)$. These are the complex versions of $O(n)$ and $SO(n)$.

Note: There is one very important distinction between the orthogonal and the unitary groups. The determinant of matrices in the unitary group has complex modulus equal to one, i.e., $|\det A| = 1$, while the matrices in the orthogonal group has determinants with “real modulus” (absolute value) equal to one. Absolute value equal to one means 1 or -1, while complex modulus equal to one governs the whole unit circle. Therefore the “size” difference between $U(n)$ and $SU(n)$ is greater than that of $O(n)$ and $SO(n)$. In fact, while considered as manifolds $U(n)$ is of one dimension more than $SU(n)$, while $O(n)$ and $SO(n)$ are of the same dimension.

Exercise 8 in Hall.

5. The groups \mathbb{R}^* , \mathbb{C}^* , S^1 , \mathbb{R} , and \mathbb{R}^n can all be shown to be isomorphic to matrix Lie groups.

Exercise: Prove it! Hint: The groups they are isomorphic to have all been defined above.

6. The group of nonzero quaternions \mathbb{H} .

It is isomorphic to the subgroup of $GL(2, \mathbb{C})$ with matrices of the form $\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ with isomorphism

$$\phi(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Exercise: Should we try to show that the group of nonzero quaternions indeed is a group? Furthermore, that the isomorphism actually is an isomorphism?

7. The group of unit quaternions are isomorphic to $SU(2)$.

The same isomorphism as the above does the trick. In other words, if we understand the matrix Lie group $SU(2)$ we can also understand the group of unit quaternions that can act on three-dimensional space to produce any rotation.

8. The Euclidean and the special Euclidean groups $E(n)$, $SE(n)$. The Euclidean group is the group of all distance preserving bijections of \mathbb{R}^n to itself. It also turns out that any element in $E(n)$ is of the form

$$f(\vec{v}) = R\vec{v} + \vec{t},$$

where $R \in O(n)$ and $\vec{t} \in \mathbb{R}^n$. For this reason, the Euclidean group is also often called the group of Rigid transformations. It does however not preserve orientations, and for that reason the special Euclidean group might be even more often considered. It is denoted $SE(n)$, and often called the group of rigid body motions (or just the group of rigid motions) and is the subgroup of $E(n)$ with $R \in SO(n)$.

Exercise:

1. Show that $E(n)$ is a group with function composition as the operation.
2. $E(n)$ is not, directly, a matrix Lie group as it is not a subgroup of $GL(n, \mathbb{C})$. Find a suitable subgroup of $GL(n+1, \mathbb{C})$ and an isomorphism showing that $E(n)$ (and thereby also $SE(n)$) is isomorphic to a matrix Lie group, you then also need to argue that this subgroup of $GL(n+1, \mathbb{C})$ has the closedness property that is required to be a matrix Lie group.

Secret hint: consider matrices of the form

$$\begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix}.$$

Definition (Lie group homomorphism)

A homomorphism between two matrix Lie groups G and H is a **Lie group homomorphism** if the homomorphism is continuous. If the homomorphism additionally is an isomorphism and the inverse map is continuous then it is called a **Lie group isomorphism**.

Theorem: All matrix Lie groups are Lie groups.

Third session

The third session will be a proper exploration of rotations in 2D ($S^1, SO(2)$), rotations in 3D quaternions, $SU(2)$, $S^3, SO(3)$, and of the rigid body motions.

Fourth session – The exponential map

We are now going to turn our attention towards what is known as the Lie algebra, which is a linear space (a vector space) that comes as a pair with any Lie group. The way that we are going to define this might potentially be a bit different as to how we are going to think about it, but it will probably still be quite useful to see it done in this way. At least it helped be, in comparison to pure handwaving¹.

¹ Handwaving is mathematical terminology for speaking/writing about things while not writing them down with proper mathematical notation and proofs. Unfortunately, the definition of being handwavy is itself handwavy, and it might vary significantly from mathematician to mathematician what is considered to be handwavy and what is considered to be rigorous (the opposite). Moreover, me mentioning handwaving above is pure nonsense as most mathematician will call this entire note for handwaving, and there is no way we are going to do Lie algebras rigorously.

The goal here will be to define the Lie algebra as the space that maps into a matrix Lie group through the exponential map. Therefore, we begin with the exponential map, and a small trip down memory lane. Recall the definition of the Euler number

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2,718281828.$$

A crazy number with some properties that are found to be useful in all of mathematics. Naturally, in Lie theory as well. One of these crazy properties is that treating the Euler number as the base of exponentiation as a function lead to the exponential function which is defined for real and complex numbers as

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The exponential function can be made on any set of objects that can be operated (multiplied) together. Specifically, we can do it with matrices X :

$$\exp(X) = \sum_n \frac{X^n}{n!}.$$

It is also customary to write $e^X = \exp(X)$, but I am a bit sceptic to use that notation as it feels like all rules of powers should hold, and that is not entirely true. Therefore, in this text, I will only use $\exp(X)$ and it should be understood as the series presented above.

[Some extra material for later in relation to manifolds and the unit circle](#)

The unit circle (for example described through the complex numbers)

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is a group under multiplication \cdot . Let us check that the closedness and the axioms for being a group are satisfied:

First, remember that all elements in S^1 can be described by an angle through Euler's formula

$$z \in S^1 \Rightarrow \exists \theta \in \mathbb{R} \text{ such that } z = e^{i\pi\theta}.$$

Now, given three arbitrary elements of S^1 , $z_1 = e^{i\pi\theta_1}$, $z_2 = e^{i\pi\theta_2}$ and $z_3 = e^{i\pi\theta_3}$, we can first multiply two of them together to check that the group is closed under multiplication

$$z_1 \cdot z_2 = e^{i\pi\theta_1} \cdot e^{i\pi\theta_2} = e^{i\pi(\theta_1+\theta_2)} \in S^1.$$

Then we have the three axioms:

1. Associativity:

$$(z_1 \cdot z_2) \cdot z_3 = (e^{i\pi\theta_1} \cdot e^{i\pi\theta_2}) \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2)} \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2+\theta_3)} = e^{i\pi\theta_1} \cdot (e^{i\pi\theta_2} \cdot e^{i\pi\theta_3}) \\ = z_1 \cdot (z_2 \cdot z_3).$$

2. Identity: $z = 1 = e^0$.

3. Inverse: Given $z = e^{i\pi\theta}$, the inverse is given by $z^{-1} = e^{-i\pi\theta}$.

And what is a differentiable (smooth) manifold?

That is a bit more work to define, but we will manage.

A manifold is a (topological) space that locally is equivalent (homeomorphic) to Euclidean space (\mathbb{R}^n). Many fancy words, but the important property is that around every point on the manifold there should exist a neighborhood and an invertible continuous map (whose inverse is also continuous) from that neighborhood to a subset of \mathbb{R}^n .

Sidenote on topological spaces

A topological space is a set equipped with (rules for how to define) open sets (closed sets are complements of the open sets). In that regard S^1 is a topological space where all sets of the form $(z_1, z_2) := \{z = e^{i\pi\theta} \in S^1 \mid \theta \in (\theta_1, \theta_2)\}$ (and unions of them) are open sets.

The unit circle S^1 is a manifold

We can easily check that S^1 is a manifold. We already saw that it is a topological space. Now we just need to see that it is locally homeomorphic to Euclidean space. Given a point $z \in S^1$, we can always choose a small neighborhood (a small open set as defined above, important that the angles are less than π apart) around the point $(z_1, z_2) = (e^{i\pi\theta_1}, e^{i\pi\theta_2})$.