BJORT Paper Analysis and Extension

E/Ea

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Contents

Chapter 1

BJORT

In this chapter we carefully go through the paper Directional derivatives and higher order chain rules for abelian functor calculus [BJO+18]. Unless stated otherwise, composition will always by in applicative order to match with [BJO+18].

1.1.0 Cross Effects for Functors

Throughout let $\mathcal{B}$ be a category with a basepoint (i.e. an initial and terminal object), and finite coproducts $\vee$. Let $\mathcal{A}$ denote an abelian category with zero object 0 and biproducts $\oplus$.

Definition 1.1.1 (Cross Effects) We define the **nth cross effect** of a functor $F : \mathcal{B} \to \mathcal{A}$ implicitly, and recursively, as the n-variable functor $\mathrm{cr}_n(F) : \mathcal{B}^n \to \mathcal{A}$ such that:

$$F(X) \cong F(\star) \oplus \mathrm{cr}_1(F)(X)$$

$$\mathrm{cr}_1(F)(X_1 \vee X_2) \cong \mathrm{cr}_1(F)(X_1) \oplus \mathrm{cr}_1(F)(X_2) \oplus \mathrm{cr}_2 F(X_1, X_2)$$

and in general

$$\begin{aligned}\mathrm{cr}_{n-1}(F)(X_1 \vee X_2, X_3, ..., X_n) \cong \mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n) &\oplus \mathrm{cr}_{n-1}(F)(X_2, ..., X_n) \\ &\oplus \mathrm{cr}_n(F)(X_1, ..., X_n)\end{aligned}$$

We prove that this defines a family of functors which are symmetric through a series of lemmas:

Lemma 1.1.2 For all $X, Y \in \mathcal{B}_0$, the inclusion $X \hookrightarrow X \vee Y$ is a split monomorphism.

Proof. Let ι_X and ι_Y denote the coproduct inclusions. Let $\hat{!} : Y \xrightarrow{!} \star \xrightarrow{¡} X$ denote the unique map ¡∘!. Then by the unniversal property of the coproduct we obtain a map

$$X \vee Y \xrightarrow{\langle 1_X | \hat{!} \rangle} X$$

such that $\left\langle 1_X \middle| \hat{!} \right\rangle \circ \iota_X = 1_X$, so ι_X is a split monomorphism, or in other words a section. ■

Throughout these notes $\hat{!}$ will denote the unique map which factors through the basepoint (note that by the universal property this is independent of the choice of basepoint). Additionally, we let $\left\langle 1_X \middle| \hat{!} \right\rangle$ denote the splitting for the inclusion $\iota_X : X \hookrightarrow X \vee Y$ in $\mathcal{B}$.

Lemma 1.1.3 Let $A, B, C \in \mathcal{A}_0$. Then $A \oplus B \cong A \oplus C$ if and only if $B \cong C$.

Proof. The reverse direction follows by functoriality of $A \oplus -$ in $\mathcal{A}$. For the forward direction suppose $A \oplus B \cong A \oplus C$ with isomorphism ψ. Then the claim is that the composite map $\pi_C \circ \psi \circ \iota_B$ is an isomorphism with inverse $\pi_B \circ \psi^{-1} \circ \iota_C$. Indeed, from the axioms of a biproduct we have

$$\pi_B \circ \psi^{-1} \circ \iota_C \circ \pi_C \circ \psi \circ \iota_B = 1_B, \quad \pi_C \circ \psi \circ \iota_B \circ \pi_B \circ \psi^{-1} \circ \iota_C = 1_C$$

as desired. ■

We now aim to make explicit this definition of the cross-effect functor, proceeding inductively. We can realize $\mathrm{cr}_1(F)(X)$ as the kernel of $F(!)$ in the following split short exact sequence.

$$\begin{tikzcd} \mathrm{cr}_1(F)(X) \arrow[r, tail, "ker"] & F(X) \arrow[r, shift left, "F(!)"] & F(\star) \arrow[l, shift left, "F(¡)"] \end{tikzcd}$$

We choose a representative kernel for each such $X \in \mathcal{B}_0$. In particular, if $F(\star) = 0$ (i.e. F is reduced), we choose $\mathrm{cr}_1(F)(X) := F(X)$. Note that by $F(¡)$ this map splits, with left splitting given by the universal property of the kernel in the diagram

$$\begin{tikzcd} \mathrm{cr}_1(F)(X) \arrow[r, "ker"] & F(X) \arrow[r, shift left, "F(!)"] & F(\star) \arrow[l, shift left, "F(¡)"] \\ F(X) \arrow[u, dashed, "r_{F,1}"] \arrow[ur, "1-F(\hat{!})"'] & & \end{tikzcd}$$

where $1 = 1_{F(X)}$ in the diagram. For simplicity of notation we write 1 for all identities moving forward, with the object of the identity given by context.

Then, given $X \xrightarrow{f} Y \in \mathcal{B}_1$, we obtain a unique map $\mathrm{cr}_1(F)(f)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{cr}_1(F)(X) \arrow[r, tail, "ker"] \arrow[d, dashed, "\mathrm{cr}_1(F)(f)"'] & F(X) \arrow[r, "F(!)"] \arrow[d, dashed, "F(f)"] & F(\star) \arrow[d, equal] \\
\mathrm{cr}_1(F)(Y) \arrow[r, tail, "ker"'] & F(Y) \arrow[r, "F(!)"] & F(\star)
\end{tikzcd}$$

by the universal property of the kernel, where uniqueness ensures that this defines a functor. Additionally, observe that

$$ker \circ r_{F,1} \circ F(f) \circ ker = (1 - F(\hat{!})) \circ F(f) \circ ker = F(f) \circ ker$$

using the definition of $\hat{!} =$ ¡∘! and the kernel. Uniqueness implies

$$\mathrm{cr}_1(F)(f) = r_{F,1} \circ F(f) \circ s_{F,1} \tag{1.1}$$

where we denote the kernel, which is also the inclusion into F of its first cross-effect, by $s_{F,1}$.

We show functoriality of the remaining $\mathrm{cr}_n(F)$ by induction. Suppose $\mathrm{cr}_{n-1}(F)$ is functorial and symmetric in each component, and we show that so is $\mathrm{cr}_n(F)$. We define $\mathrm{cr}_n(F)(X_1, ..., X_n)$ for $X_1, ..., X_n \in \mathcal{B}_0$ as the kernel

$$\begin{tikzcd}
\mathrm{cr}_n(F)(X_1, ..., X_n) \arrow[d, tail, "ker"] \\
\mathrm{cr}_{n-1}(F)(X_1 \vee X_2, X_3, ..., X_n) \arrow[d, shift left, "\left\langle \mathrm{cr}_{n-1}(F)(\langle 1_{X_1}|\hat{!}\rangle, 1_{X_3}, ..., 1_{X_n}), \mathrm{cr}_{n-1}(F)(\langle \hat{!}|1_{X_2}\rangle, 1_{X_3}, ..., 1_{X_n}) \right\rangle"] \\
\mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n) \oplus \mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n) \arrow[u, shift left, "\left\langle \mathrm{cr}_{n-1}(F)(\iota_{X_1}, 1_{X_3}, ..., 1_{X_n}) | \mathrm{cr}_{n-1}(F)(\iota_{X_2}, 1_{X_3}, ..., 1_{X_n}) \right\rangle"]
\end{tikzcd}$$

Again we choose representatives for the kernel. Let $\iota_i = \mathrm{cr}_{n-1}(F)(\iota_{X_i}, 1_{X_3}, ..., 1_{X_n})$ and $\pi_i = \mathrm{cr}_{n-1}(F)(\langle 1_{X_1}|\hat{!}\rangle, 1_{X_3}, ..., 1_{X_n})$. This SES is split. It is sufficient to show that $\langle \iota_1|\iota_2\rangle\langle \pi_1, \pi_2\rangle = 1$. Using the universal property of the coproduct it is sufficient to show $\iota_L\langle \iota_1|\iota_2\rangle\langle \pi_1, \pi_2\rangle = \iota_L$ and $\iota_R\langle \iota_1|\iota_2\rangle\langle \pi_1, \pi_2\rangle = \iota_R$ for the left and right inclusions. By symmetry it is sufficient to show this for just the left inclusion. First, using the definition of a map out of a coproduct

$$\iota_L\langle \iota_1|\iota_2\rangle\langle \pi_1, \pi_2\rangle = \iota_1\langle \pi_1, \pi_2\rangle$$

In order to show that this is equal to ι_L it is sufficient to show that post-composition with π_L yields the identity while post-composition with π_R yields zero by uniqueness of the map into a product and the structure of a biproduct. Observe that

$$\begin{aligned}
\iota_1\langle \pi_1, \pi_2\rangle\pi_L = \iota_1\pi_1 &= \mathrm{cr}_{n-1}(F)(\iota_{X_1}\langle 1_{X_1}|!\rangle, 1_{X_3}, ..., 1_{X_n}) \\
&= \mathrm{cr}_{n-1}(F)(1_{X_1}, 1_{X_3}, ..., 1_{X_n}) = 1_{\mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n)}
\end{aligned}$$

and

$$\iota_1\langle \pi_1, \pi_2\rangle\pi_R = \iota_1\pi_2 = \mathrm{cr}_{n-1}(F)(\iota_{X_1}\langle \hat{!}|1_{X_2}\rangle, 1_{X_3}, ..., 1_{X_n}) = \mathrm{cr}_{n-1}(F)(\hat{!}, 1_{X_3}, ..., 1_{X_n})$$

using functoriality of $\mathrm{cr}_{n-1}(F)$ from the inductive hypothesis. To show this second map is zero, it is sufficient to show that $\mathrm{cr}_{n-1}(F)(\star, X_3, ..., X_n) \cong 0$ for all $n \geq 2$.

Lemma 1.1.4 Let $n \in \mathbb{N}$ and $X_2, ..., X_n \in \mathcal{B}_0$. Then $\mathrm{cr}_n(F)(\star, X_2, ..., X_n) \cong 0$.

Proof. We proceed by induction on n using the implicit definition. If $n = 1$ then we have

$$F(\star) \cong F(\star) \oplus \mathrm{cr}_1(F)(\star)$$

By Lemma 1.1.3 $\mathrm{cr}_1(F)(\star) \cong 0$.
Now, suppose the claim holds for some $n - 1 \geq 1$. We have the direct sum decomposition:

$$\begin{aligned} \mathrm{cr}_{n-1}(F)(\star \vee X_2, X_3, ..., X_n) &\cong \mathrm{cr}_{n-1}(F)(\star, X_3, ..., X_n) \oplus \mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n) \\ &\oplus \mathrm{cr}_n(F)(\star, X_2, ..., X_n) \\ &\cong \mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n) \oplus \mathrm{cr}_n(F)(\star, X_2, ..., X_n) \end{aligned}$$

where the last isomorphism follows by the induction hypothesis and the fact that the 0 object is the monoidal unit for the biproduct. Then, since $\mathrm{cr}_{n-1}(F)(\star \vee X_2, X_3, ..., X_n) \cong \mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n)$, we are reduced to the base case, so again by Lemma 1.1.3 $\mathrm{cr}_n(F)(\star, X_2, ..., X_n) \cong 0$. ∎

Using Lemma 1.1.4 we then obtain the desired $\iota_1\pi_2 = 0$, so by uniqueness $\iota_L\langle\iota_1|\iota_2\rangle\langle\pi_1, \pi_2\rangle = \iota_L$, and by a symmetric argument for ι_R we obtain by uniqueness that the composite of the maps is the identity, so the SES splits. Additionally, the left splitting is given by

$$\begin{tikzcd} \mathrm{cr}_n(F)(X_1, ..., X_n) \arrow[r, "ker"] & \mathrm{cr}_{n-1}(F)(X_1 \vee X_2, \overline{X}) \arrow[r, bend left, "{\langle \mathrm{cr}_{n-1}(F)(\langle 1|\hat{!}\rangle, 1), \mathrm{cr}_{n-1}(F)(\langle \hat{!}|1\rangle, 1)\rangle}"] & \mathrm{cr}_{n-1}(F)(X_1, \overline{X}) \oplus \mathrm{cr}_{n-1}(F)(X_2, \overline{X}) \arrow[l, bend left, "{\langle \mathrm{cr}_{n-1}(F)(\iota_{X_1}, 1) | \mathrm{cr}_{n-1}(F)(\iota_{X_2}, 1)\rangle}"] \\ \mathrm{cr}_{n-1}(F)(X_1 \vee X_2, \overline{X}) \arrow[u, dashed, "r_{F,n}"] \arrow[ur, "{1 - \Delta(\mathrm{cr}_{n-1}(F)(1 \vee \hat{!}, 1) \oplus \mathrm{cr}_{n-1}(F)(\hat{!} \vee 1, 1))\nabla}"'] & & \end{tikzcd}$$

where $\Delta : B \to B \oplus B$ is the diagonal map, $\nabla : B \oplus B \to B$ is the codiagonal map, $\overline{X} = (X_3, ..., X_n)$, and $\langle 1|\hat{!}\rangle\iota_{X_1} = 1 \vee \hat{!}$ while $\langle \hat{!}|1\rangle\iota_{X_2} = \hat{!} \vee 1$.

It remains to show $\mathrm{cr}_n(F)$ is functorial in each component. We define $\mathrm{cr}_n(F)$ on a collection of $f_1, ..., f_n : X_i \to Y_i$ maps as the unique map making the diagram commute:

$$\begin{tikzcd} \mathrm{cr}_n(F)(X_1, ..., X_n) \arrow[r, "ker"] \arrow[d, dashed, "{\mathrm{cr}_n(F)(f_1, ..., f_n)}"'] & \mathrm{cr}_{n-1}(F)(X_1 \vee X_2, \overline{X}) \arrow[r, shift left, "{\langle \pi_1, \pi_2 \rangle}"] \arrow[d, "{\mathrm{cr}_{n-1}(F)(f_1 \vee f_2, \overline{f})}"'] & \mathrm{cr}_{n-1}(F)(X_1, \overline{X}) \oplus \mathrm{cr}_{n-1}(F)(X_2, \overline{X}) \arrow[l, shift left, "{\langle \iota_1 | \iota_2 \rangle}"] \arrow[d, "{\mathrm{cr}_{n-1}(F)(f_1, \overline{f}) \oplus \mathrm{cr}_{n-1}(F)(f_2, \overline{f})}"] \\ \mathrm{cr}_n(F)(Y_1, ..., Y_n) \arrow[r, "ker"'] & \mathrm{cr}_{n-1}(F)(Y_1 \vee Y_2, \overline{Y}) \arrow[r, shift left, "{\langle \pi_1, \pi_2 \rangle}"] & \mathrm{cr}_{n-1}(F)(Y_1, \overline{Y}) \oplus \mathrm{cr}_{n-1}(F)(Y_2, \overline{Y}) \arrow[l, shift left, "{\langle \iota_1 | \iota_2 \rangle}"] \end{tikzcd}$$

Functoriality follows from the inductive hypothesis and the uniqueness of the map between the kernels, which also implies identities are sent to identities. Additionally, as in the case of $n = 1$, by uniqueness the map is equal to

$$\mathrm{cr}_n(F)(f_1, ..., f_n) = r_{F,n} \circ \mathrm{cr}_{n-1}(F)(f_1 \vee f_2, f_3, ..., f_n) \circ s_{F,n} \tag{1.2}$$

To show that $\mathrm{cr}_n(F)(X_1, ..., X_n)$ is symmetric in each argument we proceed by induction. Let $\sigma \in \Sigma_n$ be a permutation on n letters. By the inductive hypothesis and the implicit definition we obtain

$$\mathrm{cr}_n(F)(X_{\sigma(1)}, ..., X_{\sigma(n)}) \oplus A \cong \mathrm{cr}_n(F)(X_1, ..., X_n) \oplus A$$

for $A \cong \mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n) \oplus \mathrm{cr}_{n-1}(F)(X_2, ..., X_n)$. Thus

$$\mathrm{cr}_n(F)(X_{\sigma(1)}, ..., X_{\sigma(n)}) \cong \mathrm{cr}_n(F)(X_1, ..., X_n) \tag{1.3}$$

by Lemma 1.1.3. These isomorphisms can be realized as unique maps given by the universal property of kernels. In particular, as we will soon show, these maps are the components of the natural transformation $\alpha : F \circ \sigma \Rightarrow F$ under cr_n.

> **Definition 1.1.5** We write $\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$ for the category of **strictly multi-reduced** functors from $\mathcal{B}^n$ to $\mathcal{A}$, i.e. those F such that for $F(X_1, ..., X_n) \cong 0$ if $X_i \cong \star$ for some i.

We have shown now that the cross-effect gives an object map $\mathrm{cr}_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$. It remains to show that this assignment is functorial. To this end let $\alpha : F \Rightarrow G$ be a natural transformation. We define $\mathrm{cr}_n(\alpha)$ inductively. For the case of $n = 1$ we let $\mathrm{cr}_1(\alpha)_X$ be the unique map making

$$\begin{tikzcd}
\mathrm{cr}_1 F(X) \arrow[r, tail] \arrow[d, dashed, "\mathrm{cr}_1(\alpha)_X"'] & F(X) \arrow[r, "F(!)"] \arrow[d, "\alpha_X"'] & F(\star) \arrow[d, "\alpha_\star"] \\
\mathrm{cr}_1 G(X) \arrow[r, tail] & G(X) \arrow[r, "G(!)"'] & G(\star)
\end{tikzcd}$$

commute. As in the case of the maps themselves, we observe that by uniqueness we have the formula

$$\mathrm{cr}_1(\alpha)_X = r_{G,1} \circ \alpha_X \circ s_{F,1} \tag{1.4}$$

Hence, to show naturality of $\mathrm{cr}_1(\alpha)$ it is sufficient to show naturality of $r_{G,1}$ and $s_{F,1}$. All of these naturalities follow from a general result on limits in Section 1.A.4. Since limits in a functor category are computed componentwise, it follows that the $s_{F,n}$ and $r_{F,n}$ bundle to form natural transformations. Additionally, $\mathrm{cr}_1(\alpha)$ is precisely the map induced by the limit for the map of diagrams

$$\begin{tikzcd}
F \arrow[r, Rightarrow] \arrow[d, "\alpha"'] & \mathrm{ev}_\star \circ F \arrow[d, "\mathrm{ev}_\star(\alpha)"] \\
G \arrow[r, Rightarrow] & \mathrm{ev}_\star \circ G
\end{tikzcd}$$

Inductively, suppose cr_{n-1} is functorial. Then by Lemma 1.A.41 and the inductive hypothesis, for $\alpha : F \Rightarrow G$, $\mathrm{cr}_n(\alpha)$ is the limit map induced by the map of diagrams

$$\begin{tikzcd}
(\vee_{i=1}^2 \times 1) \circ \mathrm{cr}_{n-1}(F) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\vee_{i=1}^2 \times 1)\circ \mathrm{cr}_{n-1}(\alpha)"'] & ((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(F)) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(F)) \arrow[d, "((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(\alpha)) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(\alpha))"] \\
(\vee_{i=1}^2 \times 1) \circ \mathrm{cr}_{n-1}(G) \arrow[r, shift left] \arrow[r, shift right] & ((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(G)) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(G))
\end{tikzcd}$$

where $\hat{\pi}_i : \mathcal{B}^n \to \mathcal{B}^{n-1}$ is the functor which skips the ith argument, and $\vee_{i=1}^n$ is the functor given in Lemma below.

Lemma 1.1.6 We have a functor $\vee_{i=1}^n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}^n, \mathcal{A})$ given by $\vee_{i=1}^n(G)(X_1, ..., X_n) = G(\vee_{i=1}^n X_i)$ and $\vee_{i=1}^n(G)(f_1, ..., f_n) = G(\vee_{i=1}^n f_i)$ on objects and by $\vee_{i=1}^n(\eta)_{X_1,...,X_n} = \eta_{\vee_{i=1}^n X_i}$ on arrows.

Proof. To prove $\vee_{i=1}^n$ is a functor we first show it is well-defined. Let $\eta : F \Rightarrow G$ be a natural transformation between single variable functors, and let $(f_i)_{i=1}^n : (X_1, ..., X_n) \to (Y_1, ..., Y_n)$ and $(g_i)_{i=1}^n : (Y_1, ..., Y_n) \to (Z_1, ..., Z_n)$ be maps in $\mathcal{B}^n$.
First, by uniqueness in the definition of $\vee_{i=1}^n f_i$ note that $\vee_{i=1}^n g_i \circ \vee_{i=1}^n f_i = \vee_{i=1}^n (g_i \circ f_i)$. Additionally, $\vee_{i=1}^n 1_{X_i} = 1_{\vee_{i=1}^n X_i}$. Combined with functoriality of F, we have that $\vee_{i=1}^n(F)$ is indeed a functor.
Next, naturality of $\vee_{i=1}^n(\eta)$ equates to the following diagram commuting

$$\begin{tikzcd}
F(\vee_{i=1}^n X_i) \arrow[r, "F(\vee_{i=1}^n f_i)"] \arrow[d, "\eta_{\vee_{i=1}^n X_i}"'] & F(\vee_{i=1}^n Y_i) \arrow[d, "\eta_{\vee_{i=1}^n Y_i}"] \\
G(\vee_{i=1}^n X_i) \arrow[r, "G(\vee_{i=1}^n f_i)"'] & G(\vee_{i=1}^n Y_i)
\end{tikzcd}$$

which follows from the naturality of η. Finally, if $\gamma : G \Rightarrow H$ is another natural transformation,

$$\vee_{i=1}^n(\gamma)_{X_1,...,X_n} \circ \vee_{i=1}^n(\eta)_{X_1,...,X_n} = \gamma(\vee_{i=1}^n X_i) \circ \eta(\vee_{i=1}^n X_i) = (\gamma \circ \eta)(\vee_{i=1}^n X_i) = \vee_{i=1}^n(\gamma \circ \eta)_{X_1,...,X_n}$$

by definition of composition of natural transformations, and

$$\vee_{i=1}^n(1_F)_{X_1,...,X_n} = 1_F(\vee_{i=1}^n X_i) = 1_{F(\vee_{i=1}^n X_i)} = 1_{\vee_{i=1}^n(F)(X_1,...,X_n)}$$

This finishes the proof that $\vee_{i=1}^n$ is a functor. ■

Additionally, as in the previous cases, using the uniqueness of the components of $\mathrm{cr}_n(\alpha)$ we can give the formula

$$\mathrm{cr}_n(\alpha) = r_{G,n} \circ ((\vee_{i=1}^2 \times 1)(\mathrm{cr}_{n-1}(\alpha))) \circ s_{F,n} \tag{1.5}$$

Therefore

$$\mathrm{cr}_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A}) \tag{1.6}$$

is indeed a functor between functor categories.

Remark:

We can define the cross-effect functors for $\mathcal{A}$ non-abelian, requiring simply that $\mathcal{A}$ has pullbacks and equilizers. To do this let $\mathcal{A}$ be such a category, and let $F : \mathcal{B} \to \mathcal{A}$. We consider the diagram

$$\begin{tikzcd}
F(X \vee Y) \arrow[r, "F(1_X \vee !)"] \arrow[d, "F(! \vee 1_Y)"'] & F(X) \arrow[d, "F(!)"] \\
F(Y) \arrow[r, "F(!)"'] & F(\star)
\end{tikzcd}$$

We remove the first vertex and take a homotopy limit:

$$\begin{tikzcd}
holim_{P_0(2)} F(\vee) \arrow[rr, bend left, to=1-3] \arrow[dr, bend right] & F(X \vee Y) \arrow[l, dashed, "\gamma"'] \arrow[r, "F(1_X \vee !)"] \arrow[d, "F(! \vee 1_Y)"'] & F(X) \arrow[d, "F(!)"] \\
& F(Y) \arrow[r, "F(!)"'] & F(\star)
\end{tikzcd}$$

We define the second cross effect of F to be

$$\mathrm{cr}_2(F) := \mathrm{hofib}\gamma$$

In the case of $n = 3$ we obtain a cubical diagram:

$$\begin{tikzcd}
F(X_1 \vee X_2 \vee X_3) \arrow[rr] \arrow[dr] \arrow[dd] & & F(X_2 \vee X_3) \arrow[dr] \arrow[dd, dashed] & \\
& F(X_1 \vee X_3) \arrow[rr] \arrow[dd] & & F(X_3) \arrow[dd] \\
F(X_1 \vee X_2) \arrow[rr, dashed] \arrow[dr] & & F(X_2) \arrow[dr] & \\
& F(X_1) \arrow[rr] & & F(\star)
\end{tikzcd}$$

The diagram can be labeled by $\mathcal{P}(\{1,2,3\})$, where the subset of $\{1,2,3\}$ corresponds to the complement of the indices on a particular node. Let $\chi(S)$ for $S \in \mathcal{P}(\{1,2,3\})$ denote the pullback for the subdiagram consisting on nodes labeled by subsets containing S. Then we define

$$\mathrm{cr}_3 F(X_1, X_2, X_3) := \mathrm{hofib}\gamma$$

where fiber indicates the pullback along zero.

Cross-effects preserve multi-reduced.

Lemma 1.1.7 Let $H : \mathcal{B}^n \to \mathcal{A}$ be a multi-reduced functor. Then $\mathrm{cr}_1(H) : \mathcal{B}^n \to \mathcal{A}$ is multi-reduced.

Proof. Since $\mathrm{cr}_1(H)$ is the kernel of the map from H to $H(0) = 0$, we must have that $\mathrm{cr}_1(H)$ is zero when H is zero, and so $\mathrm{cr}_1(H)$ is strictly multi-reduced. ■

Lemma 1.1.8 For any $n \geq 1$ and any $F : \mathcal{B} \to \mathcal{A}$, $\mathrm{cr}_n(s_{F,1}) : \mathrm{cr}_n(\mathrm{cr}_1(F)) \to \mathrm{cr}_n(F)$ is an isomorphism.

Proof. Note that $s_{F,1,\star} : \mathrm{cr}_1(F)(\star) \to F(\star)$ is the kernel of $F(!) : F(\star) \to F(\star)$, which is the identity. Thus, by the characterization of limits of Functors in Section 1.A.4 we have a natural isomorphism 0_1 with components $0_{1,F} : \mathrm{cr}_1(F)(\star) \to 0$. Next, we also have $s_{\mathrm{cr}_1(F),1} : \mathrm{cr}_1(\mathrm{cr}_1(F)) \to \mathrm{cr}_1(F)$ which is the kernel of $\mathrm{cr}_1(F)(!) : \mathrm{cr}_1(F) \to \mathrm{cr}_1(F)(\star)$. Composing with the isomorphism $0_{1,F}$ shows that $s_{\mathrm{cr}_1(F),1} : \mathrm{cr}_1(\mathrm{cr}_1(F)) \to \mathrm{cr}_1(F)$ is an isomorphism.

We now proceed by induction on n. Suppose $\mathrm{cr}_n(s_{F,1}) : \mathrm{cr}_n(\mathrm{cr}_1(F)) \to \mathrm{cr}_n(F)$ for some $n \geq 1$. Then by definition of the cross-effect we have the commutative diagram of a map between equalizers

$$\begin{tikzcd}
\mathrm{cr}_n(\mathrm{cr}_1(F)) \arrow[r] \arrow[d, "\mathrm{cr}_n(s_{F,1})"'] & (\vee_{i=1}^2 \times 1) \circ \mathrm{cr}_{n-1}(\mathrm{cr}_1(F)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\vee_{i=1}^2 \times 1) \circ \mathrm{cr}_{n-1}(s_{F,1})"'] & ((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(\mathrm{cr}_1(F)) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(\mathrm{cr}_1(F))) \arrow[d, "((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(s_{F,1})) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(s_{F,1}))"'] \\
\mathrm{cr}_n(F) \arrow[r] & (\vee_{i=1}^2 \times 1) \circ \mathrm{cr}_{n-1}(F) \arrow[r, shift left] \arrow[r, shift right] & ((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}(F) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}(F))
\end{tikzcd}$$

By the inductive hypothesis the middle and right vertical map are isomorphisms. Since each of the rows are exact and the left maps are monomorphisms, being kernel maps, we can add zeros to the left and use the 5-lemma to conclude that $\mathrm{cr}_n(s_{F,1})$ is an isomorphism, as desired. ■

These properties demonstrate that the inclusion $\mathrm{Fun}_*(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$ admits a right adjoint, namely the first cross-effect functor cr_1. In other words, $\mathrm{Fun}_*(\mathcal{B}, \mathcal{A})$ is a coreflective subcategory of $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$. Since the left adjoint is full and faithful (being the inclusion of a full subcategory), the unit of this adjunction is an isomorphism $\eta_F : F \Rightarrow \mathrm{cr}_1\iota(F)$, which also re-affirms that $\mathrm{cr}_1 F \cong \mathrm{cr}_1^2 F$. In fact, by our choice of kernels, η_F has identities as components.

Proof. To demonstrate the adjunction we show the co-universal property where $\epsilon_F = s_{F,1}$ is the monic inclusion $\mathrm{cr}_1(F)(X) \rightarrowtail F(X)$. Since the $s_{F,1}$ are natural in X and F by Lemma 1.A.42 we need only show the co-universal property.
To show the co-universal property we take a natural transformation $\alpha : \iota F \to G$, for F a

strictly reduced functor. This generates a commutative diagram

$$\begin{tikzcd}
\mathrm{cr}_1 G(X) \arrow[r, tail, "(\epsilon_F)_X"] & G(X) \arrow[r, "G(!)"] & G(\star) \\
F(X) \arrow[u, dashed, "\hat{\alpha}_X"] \arrow[ur, "\alpha_X"] \arrow[r, "F(!)"'] & F(\star) \cong 0 \arrow[ur, "\alpha_\star"] &
\end{tikzcd}$$

where $\hat{\alpha}_X$ is the unique map from the universal property of the kernel. Thus by Lemma 1.A.42 $\hat{\alpha}$ is natural. If β was another natural transformation making the first diagram commute, then $\beta_X = \hat{\alpha}_X$ for all X, by uniqueness of the map to the kernel. In other words we would have $\beta = \hat{\alpha}$, so that $\hat{\alpha}$ is unique, proving the co-universal property. ■

The counit then is the natural inclusion $\epsilon_F : \iota \mathrm{cr}_1(F) \Rightarrow F$. We can extend this to an adjunction for cr_n. First, consider G, a reduced functor, and $X_1, ..., X_n \in \mathcal{B}_0$. I claim that $\mathrm{cr}_n(G)(X_1, ..., X_n)$ is a direct summand of $G(\vee_{i=1}^n X_i)$. Indeed, we have the inclusion ι_G given by the composite

$$\mathrm{cr}_n(G)(X_1, ..., X_n) \xrightarrow{\vee_{i=1}^n (s_{G,1}) \circ \cdots \circ s_{G,n}} G(\vee_{i=1}^n X_i)$$

and the projection π_G given by the composite

$$G(\vee_{i=1}^n X_i) \xrightarrow{r_{G,n} \circ \cdots \circ \vee_{i=1}^n (r_{G,1})} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

In particular, from our previous work ι_G and π_G are composites of natural transformations, and hence themselves are natural. In particular, we have the natural transformations

Lemma 1.1.9 The components ι_G defined above constitute a natural transformation $\iota : \mathrm{cr}_n \Rightarrow \vee_{i=1}^n$.

and

Lemma 1.1.10 The components π_G defined above constitute a natural transformation $\pi : \vee_{i=1}^n \Rightarrow \mathrm{cr}_n$.

Additionally, for $F \in \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$ we will let i denote the composite

$$F(X_1, ..., X_n) \xrightarrow{F(i_1, ..., i_n)} \Delta^*(F)(\vee_{i=1}^n X_i) \xrightarrow{\pi_{\Delta^*(F)}} \mathrm{cr}_n(\Delta^*(F))(X_1, ..., X_n)$$

The map $F(i_1, ..., i_n)$ is natural in both F and the X_i, so that i is also a natural transformation $1_{\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})} \Rightarrow \mathrm{cr}_n \circ \Delta^*$.

Lemma 1.1.11 We have a natural transformation $\bar{i} : 1_{\mathrm{Fun}(\mathcal{B}^n, \mathcal{A})} \Rightarrow \vee_{i=1}^n \circ \Delta^*$ which restricts to a natural transformations between functors on $\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$

Proof. Let $\alpha : F \to G$ be a map of functors and let $f_1 : X_1 \to Y_1, ..., f_n : X_n \to Y_n$ be a collection of maps in $\mathcal{B}$. Naturality of $\bar{i}_F$ in the X_i is given by the commutative diagram

$$\begin{tikzcd}
F(X_1, ..., X_n) \arrow[r, "F(i_1,...,i_n)"] \arrow[d, "F(f_1,...,f_n)"'] & \Delta^*(F)(\vee_{i=1}^n X_i) \arrow[d, "\Delta^*(F)(\vee_{i=1}^n f_i)"] \\
F(Y_1, ..., Y_n) \arrow[r, "F(i_1,...,i_n)"'] & \Delta^*(F)(\vee_{i=1}^n Y_i)
\end{tikzcd}$$

which commutes by definition of $\vee_{i=1}^n f_i$ and the functoriality of F. On the other hand, naturality of $\bar{i}$ itself is given by the diagram

$$\begin{tikzcd}
F(X_1, ..., X_n) \arrow[r, "F(i_1,...,i_n)"] \arrow[d, "\alpha_{X_1,...,X_n}"'] & \Delta^*(F)(\vee_{i=1}^n X_i) \arrow[d, "\Delta^*(\alpha)_{\vee_{i=1}^n X_i}"] \\
G(X_1, ..., X_n) \arrow[r, "G(i_1,...,i_n)"'] & \Delta^*(G)(\vee_{i=1}^n X_i)
\end{tikzcd}$$

which commutes by naturality of α. ■

Another important natural transformation we require is given by the $+$ operation on disjoint unions which gives the unique map sending a disjoint union of a single object to itself with all inclusions the identity.

Lemma 1.1.12 We have a natural transformation $+ : \vee_{i=1}^n \Rightarrow 1_{\mathcal{B}}$.

Proof. Let $f : X \to Y$ be a map in $\mathcal{B}$. Then naturality equates to the commutivity of

$$\begin{tikzcd}
\vee_{i=1}^n X \arrow[r, "\vee_{i=1}^n f"] \arrow[d, "+"'] & \vee_{i=1}^n Y \arrow[d, "+"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

However, the lower composite is precisely the unique map for each inclusion into Y given by f, while the upper composite has as inclusions the composite $X \xrightarrow{f} Y \rightarrowtail \vee_{i=1}^n Y \xrightarrow{+} Y$ which by definition also equals f. Thus the diagram commutes by uniqueness of the map out of a coproduct. ■

Next we argue that the isomorphisms in Definition 1.1.1 can be upgraded to natural isomorphisms.

Lemma 1.1.13 We have natural isomorphisms

$$1_{\mathrm{Fun}(\mathcal{B},\mathcal{A})} \cong \mathrm{ev}_\star \oplus \mathrm{cr}_1 \tag{1.7}$$

$$\vee_{i=1}^{2} \circ \mathrm{cr}_1 \cong ((\pi_1)^* \circ \mathrm{cr}_1) \oplus ((\pi_2)^* \circ \mathrm{cr}_1) \oplus \mathrm{cr}_2 \tag{1.8}$$

and in general

$$(\vee_{i=1}^{2} \times 1_{\mathcal{B}^{n-2}}) \circ \mathrm{cr}_{n-1} \cong ((\hat{\pi}_2)^* \circ \mathrm{cr}_{n-1}) \oplus ((\hat{\pi}_1)^* \circ \mathrm{cr}_{n-1}) \oplus \mathrm{cr}_n \tag{1.9}$$

Indeed, these natural isomorphisms follow from the fact that the inclusion and retractions formed natural transformations, and hence the sequence of functors defining the cross-effect split.

We can apply these natural isomorphisms inductively to obtain the following isomorphism of functors given in [JM03].

Theorem 1.1.14 For any $n \in \mathbb{N}$, we have a natural isomorphism

$$\vee_{i=1}^{n} \cong \mathrm{ev}_\star \oplus \left(\bigoplus_{m=1}^{n} \left(\bigoplus_{j_1 < \cdots < j_m =: \overline{j}} \pi_{\overline{j}}^* \circ \mathrm{cr}_m \right) \right)$$

where $\pi_{\overline{j}} : \mathcal{B}^n \to \mathcal{B}^m$ projects onto the components $j_1 < \cdots < j_m =: \overline{j}$.

Proof. If $n = 1$ this isomorphism is precisely Equation (1.7). Inductively suppose this isomorphism exists for some $n-1$, $n \geq 2$. Let φ_{n-1} denote this isomorphism. Recall the functor $(\vee_{i=1}^{2} \times 1_{\mathcal{B}^{n-2}}) : \mathrm{Fun}(\mathcal{B}^{n-1}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}^n, \mathcal{A})$, and observe that $\vee_{i=1}^{n} \cong (\vee_{i=1}^{2} \times 1_{\mathcal{B}^{n-1}}) \circ \vee_{i=1}^{n-1}$ by the universal property of the coproduct. Then applying φ_{n-1}, the isomorphisms in Lemma 1.1.13, as well as isomorphisms associated with re-ordering the direct sum we obtain the desired natural isomorphism. ■

To help with future computations we describe the composite $\iota_G \circ \pi_G$ for a functor G a bit more explicitly. For this remark we emphasize the order of the projection and inclusion by writing $\iota_{G,n}$ and $\pi_{G,n}$.

Remark:

Projection Formulas We construct a formula for the composite $\iota_{G,n} \circ \pi_{G,n}$ by induction on n. In the case of $n = 1$ $\iota_{G,1} = s_{G,1}$ and $\pi_{G,1} = r_{G,1}$, so by construction of the retraction

$$\iota_{G,1} \circ \pi_{G,1} = 1 - G(\hat{!}) \tag{1.10}$$

In the case of $n = 2$, $\iota_{G,2} = \vee_{i=1}^{2}(s_{G,1}) \circ s_{G,2}$ and $\pi_{G,2} = r_{G,2} \circ \vee_{i=1}^{2}(r_{G,1})$. Then using Equation 1.1 the composite is given by

$$\begin{aligned} \iota_{G,2} \circ \pi_{G,2} &= \vee_{i=1}^{2}(s_{G,1}) \circ s_{G,2} \circ r_{G,2} \circ \vee_{i=1}^{2}(r_{G,1}) \\ &= \vee_{i=1}^{2}(s_{G,1}) \circ (1 - (\mathrm{cr}_1(G)(1 \vee \hat{!}) + \mathrm{cr}_1(G)(\hat{!} \vee 1))) \circ \vee_{i=1}^{2}(r_{G,1}) \end{aligned}$$

$$
\begin{aligned}
&= \vee_{i=1}^2(1 - G(\hat{!})) - \vee_{i=1}^2(1 - G(\hat{!})) \circ G(1 \vee \hat{!}) \circ \vee_{i=1}^2(1 - G(\hat{!})) \\
&- \vee_{i=1}^2(1 - G(\hat{!})) \circ G(\hat{!} \vee 1) \circ \vee_{i=1}^2(1 - G(\hat{!})) \\
&= \vee_{i=1}^2(1 - G(\hat{!})) - (G(1 \vee \hat{!}) - G(\hat{!} \vee \hat{!})) \circ \vee_{i=1}^2(1 - G(\hat{!})) \\
&- (G(\hat{!} \vee 1) - G(\hat{!} \vee \hat{!})) \circ \vee_{i=1}^2(1 - G(\hat{!})) \\
&= \vee_{i=1}^2(1 - G(\hat{!})) - (G(1 \vee \hat{!}) - G(\hat{!} \vee \hat{!})) - (G(\hat{!} \vee 1) - G(\hat{!} \vee \hat{!})) \\
&= 1 - G(1 \vee \hat{!}) - G(\hat{!} \vee 1) + G(\hat{!} \vee \hat{!})
\end{aligned}
$$

Therefore, our formula for $n = 2$ becomes

$$
\iota_{G,2} \circ \pi_{G,2} = 1 - G(1 \vee \hat{!}) - G(\hat{!} \vee 1) + G(\hat{!} \vee \hat{!}) \tag{1.11}
$$

Now, suppose that for some $n \geq 2$ we have the formula

$$
\iota_{G,n} \circ \pi_{G,n} = 1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}} \hat{!})
$$

where the sum is over sequences of distinct integers from 1 to n of lengths ≥ 1, the $k_{\bar{i}}$ are integers, and where $\vee_{\bar{i}}\hat{!}$ has $\hat{!}$ in each entry i_j, and identities in all other entries. Then our formula for $\iota_{G,n+1}$ for the $n+1$ case can be written as $(\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n}) \circ s_{G,n+1}$, while $\pi_{G,n+1} = r_{G,n+1} \circ (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\pi_{G,n})$. Note by Equation 1.2 applied inductively, $\mathrm{cr}_n(G)(f_1, ..., f_n) = \pi_{G,n} \circ G(\vee_{i=1}^n f_i) \circ \iota_{G,n}$. Then by our inductive hypothesis we can compute

$$
\begin{aligned}
\iota_{G,n+1} \circ \pi_{G,n+1} &= (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n}) \circ s_{G,n+1} \circ r_{G,n+1} \circ (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\pi_{G,n}) \\
&= (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n}) \circ (1 - (\mathrm{cr}_{n+1}(G)(1 \vee \hat{!}, 1) + \mathrm{cr}_{n+1}(G)(\hat{!} \vee 1, 1))) \\
&\circ (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\pi_{G,n}) \\
&= \left(\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}}\right) \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}} \hat{!})\right) \\
&- (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n} \circ \pi_{G,n}) \circ G(1 \vee \hat{!} \vee 1_{n-1}) \circ (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n} \circ \pi_{G,n}) \\
&- (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n} \circ \pi_{G,n}) \circ G(\hat{!} \vee 1 \vee 1_{n-1}) \circ (\vee_{i=1}^2 \times 1_{\mathcal{B}^{n-2}})(\iota_{G,n} \circ \pi_{G,n}) \\
&= \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}'} \hat{!})\right) \\
&- \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}'} \hat{!})\right) \circ G(1 \vee \hat{!} \vee 1_{n-1}) \circ \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}'} \hat{!})\right) \\
&- \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}'} \hat{!})\right) \circ G(\hat{!} \vee 1 \vee 1_{n-1}) \circ \left(1 + \sum_{\bar{i}} k_{\bar{i}} G(\vee_{\bar{i}'} \hat{!})\right)
\end{aligned}
$$

where $\bar{i}'$ is obtained from $\bar{i}$ by either shifting up all degrees by 1 if $1 \notin \bar{i}$, or by shifting up all degrees by 1 and adding 1 to $\bar{i}$ if it does contain 1. Observe that the second composites will consist of integer combinations of maps of the form $G(\vee_{\bar{i}} \hat{!})$ where $\bar{i}$ is non-empty. Thus, $\iota_{G,n+1} \circ \pi_{G,n+1}$ is of the desired form.

These natural isomorphisms will prove valuable for proving that cr_n is the right adjoint in an adjunction between categories of reduced functors.

Proposition 1.1.15 The n-th cross effect is a right adjoint to the diagonal functor $\Delta^* : \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}, \mathcal{A})$.

Proof. We demonstrate the adjunction by showing the co-universal property. Pictorially this can be represented by:

$$\begin{tikzcd}
\mathrm{cr}_n(G) & \Delta^*(\mathrm{cr}_n(G)) \arrow[r, "\epsilon_G"] & G \\
F \arrow[u, dashed, "\hat{\alpha}"] & \Delta^*(F) \arrow[u, dashed] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Here $\epsilon_G = G(+) \circ \iota_G$. Given such an α, we let $\hat{\alpha}$ be given by the composite

$$F(X_1, ..., X_n) \xrightarrow{i} \mathrm{cr}_n(\Delta^*(F))(X_1, ..., X_n) \xrightarrow{\mathrm{cr}_n(\alpha)_{X_1,...,X_n}} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

First we show the components of this proposed map make the diagram commute. Using naturality of ι we can re-write this composite as

$$F(X, ..., X) \xrightarrow{i} \mathrm{cr}_n(\Delta^*(F))(X, ..., X) \xrightarrow{\iota_{\Delta^*(F)}} \Delta^*(F)(\vee_{i=1}^n X) \xrightarrow{\alpha_{\vee_{i=1}^n X}} G(\vee_{i=1}^n X) \xrightarrow{G(+)} G(X)$$

Then using naturality of α we obtain

$$F(X, ..., X) \xrightarrow{i} \mathrm{cr}_n \Delta^*(F)(X, ..., X) \xrightarrow{\iota_{\Delta^*(F)}} \Delta^*(F)(\vee_{i=1}^n X) \xrightarrow{\Delta^*(F)(+)} \Delta^*(F)(X) \xrightarrow{\alpha_X} G(X)$$

It remains to show $\Delta^*(F)(+) \circ \iota_{\Delta^*(F)} \circ i = 1_{\Delta^*(F)(X)}$. However, $i = \pi_{\Delta^*(F)} \circ F(i_1, ..., i_n)$, and from our previous remark $\iota_{\Delta^*(F)} \circ \pi_{\Delta^*(F)}$ is equal to $1_{\Delta^*(F)(\vee_{i=1}^n X)}$ plus terms which involve at least one $\hat{!}$. Composing any term which involves $\hat{!}$ with $\Delta^*(F)(+)$ will result in the zero map since F is reduced. Thus, the composite becomes

$$\Delta^*(F)(+) \circ F(i_1, ..., i_n) = 1_{\Delta^*(F)(X)}$$

Next we show that $\hat{\alpha}$ is natural. However, this follows immediately from Lemma 1.1.10 and Lemma 1.1.11, so $\hat{\alpha}$ is a composite of natural transformations.

Finally, it remains to show uniqueness of $\hat{\alpha}$. It is sufficient to show that if $\beta : F \Rightarrow \mathrm{cr}_n(G)$, then $\widehat{\epsilon_G \circ \Delta^*(\beta)} = \beta$, or in other words the composite

$$F(X_1, ..., X_n) \xrightarrow{i} \mathrm{cr}_n(\Delta^*(F))(X_1, ..., X_n) \xrightarrow{\mathrm{cr}_n(\Delta^*(\beta))} \mathrm{cr}_n(\Delta^*(\mathrm{cr}_n(G)))(X_1, ..., X_n)$$

$$\xrightarrow{\mathrm{cr}_n(\Delta^*(\iota_G))} \mathrm{cr}_n(\Delta^*(\vee_{i=1}^n(G)))(X_1, ..., X_n)$$
$$\xrightarrow{\mathrm{cr}_n(G(+))} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

equals β. Using naturality of the projection and the inclusions into $\vee_{i=1}^n X_i$, this composite can be written as

$$F(X_1, ..., X_n) \xrightarrow{\beta} \mathrm{cr}_n(G)(X_1, ..., X_n) \xrightarrow{\mathrm{cr}_n(G)(i_1,...,i_n)} \Delta^*(\mathrm{cr}_n(G))(\vee_{i=1}^n X_i)$$
$$\xrightarrow{\pi_{\Delta^*(\mathrm{cr}_n(G))}} \mathrm{cr}_n(\Delta^*(\mathrm{cr}_n(G)))(X_1, ..., X_n)$$
$$\xrightarrow{\mathrm{cr}_n(\Delta^*(\iota_G))} \mathrm{cr}_n(\Delta^*(\vee_{i=1}^n(G)))(X_1, ..., X_n)$$
$$\xrightarrow{\mathrm{cr}_n(G(+))} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

It remains to show that the composite after β is the identity. By naturality of the projection twice this composite becomes

$$\mathrm{cr}_n(G)(X_1, ..., X_n) \xrightarrow{\mathrm{cr}_n(G)(i_1,...,i_n)} \Delta^*(\mathrm{cr}_n(G))(\vee_{i=1}^n X_i) \xrightarrow{\Delta^*(\iota_G)} \Delta^*(\vee_{i=1}^n(G))(\vee_{i=1}^n X_i)$$
$$\xrightarrow{G(+)} G(\vee_{i=1}^n X_i)$$
$$\xrightarrow{\pi_G} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

Next, using the naturality of ι we obtain

$$\mathrm{cr}_n(G)(X_1, ..., X_n) \xrightarrow{\iota_G} \vee_{i=1}^n(G)(X_1, ..., X_n) \xrightarrow{\vee_{i=1}^n(G)(i_1,...,i_n)} \Delta^*(\vee_{i=1}^n(G))(\vee_{i=1}^n X_i)$$
$$\xrightarrow{G(+)} G(\vee_{i=1}^n X_i)$$
$$\xrightarrow{\pi_G} \mathrm{cr}_n(G)(X_1, ..., X_n)$$

Now, the middle two arrows compose to give the identity, while we also have that $\pi_G \circ \iota_G$ is the identity, completing the proof.

■

Composing with the adjunction $\mathrm{inc} \dashv \mathrm{cr}_1$ and using Lemma 1.1.8 we obtain an adjunction

$$\mathrm{Fun}(\mathcal{B}, \mathcal{A}) \underset{\mathrm{cr}_n \circ \mathrm{cr}_1}{\overset{\Delta^*}{\underset{\perp}{\leftrightarrows}}} \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$$

We use this adjunction to define a family of comonads.

Definition 1.1.16 For each $n \in \mathbb{N}$, we have a comonad $C_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$ given by $C_n := \Delta^* \circ \mathrm{cr}_n \circ \mathrm{cr}_1$. The counit of the comonad is given by the composite

$$C_n(G)(X) = \Delta^*(\mathrm{cr}_n(\mathrm{cr}_1(G)))(X) \xrightarrow{\Delta^*(\iota_{\mathrm{cr}_1(G)})} \mathrm{cr}_1(G)(\vee_{i=1}^n X) \xrightarrow{\mathrm{cr}_1(G)(+)} \mathrm{cr}_1(G)(X) \xrightarrow{s_{G,1,X}} G(X)$$

while the comultiplication is given by the composite

$$\Delta^*(\mathrm{cr}_n(\mathrm{cr}_1(G)))(X) \xrightarrow{\mathrm{cr}_n(\mathrm{cr}_1(G))(i_1,\ldots,i_n)} \Delta^*(\mathrm{cr}_n(\mathrm{cr}_1(G)))(\vee_{i=1}^n X) \xrightarrow{\pi_{\Delta^*(\mathrm{cr}_n(\mathrm{cr}_1(G)))}} C_n(C_n(G))(X)$$

1.1.0.1 Contracting Homotopies

In this section we aim to show that the contracting homotopy in the following lemma is natural in A.

Lemma 1.1.17 Let $\mathcal{A} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{B}$ define an adjunction between abelian categories inducing a comonad $C = LR$ on $\mathcal{A}$ with counit $\epsilon : LR \Rightarrow \mathrm{id}$. Then for each $A \in \mathcal{A}_0$ the chain complex in $\mathcal{B}$ with differentials defined to be the alternating sums $\sum_{i\geq 0}^{k}(-1)^i R(LR)^i\epsilon$ admits a contracting homotopy.

Contracting Homotopy Proof. Define $s_k = \eta_{R(LR)^k A}$ using the unit $\eta : \mathrm{id} \Rightarrow RL$ of the adjunction.
We first show that the described data defines a chain complex. Observe for $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\left(\sum_{i\geq 0}^{n}(-1)^i R(LR)^i\epsilon_{(LR)^{n-i}A}\right) \circ \left(\sum_{i\geq 0}^{n+1}(-1)^i R(LR)^i\epsilon_{(LR)^{n+1-i}A}\right)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}R(LR)^i\epsilon_{(LR)^{n-i}A} \circ R(LR)^j\epsilon_{(LR)^{n+1-j}A}\\
&= \sum_{i=0}^{n}\sum_{i<j}^{n+1}(-1)^{i+j}R(LR)^i\epsilon_{(LR)^{n-i}A} \circ R(LR)^j\epsilon_{(LR)^{n+1-j}A}\\
&+ \sum_{i=0}^{n}\sum_{j\leq i}(-1)^{i+j}R(LR)^i\epsilon_{(LR)^{n-i}A} \circ R(LR)^j\epsilon_{(LR)^{n+1-j}A}\\
&= \sum_{i=0}^{n}\sum_{i<j}^{n+1}(-1)^{i+j}R(LR)^i(\epsilon_{(LR)^{n-i}A} \circ (LR)^{j-i}\epsilon_{(LR)^{n+1-j}A})\\
&+ \sum_{i=0}^{n}\sum_{j\leq i}(-1)^{i+j}R(LR)^j((LR)^{i-j}\epsilon_{(LR)^{n-i}A} \circ \epsilon_{(LR)^{n+1-j}A})\\
&= \sum_{i=0}^{n}\sum_{i\leq k}^{n}(-1)^{i+k+1}R(LR)^i(\epsilon_{(LR)^{n-i}A} \circ (LR)^{k+1-i}\epsilon_{(LR)^{n-k}A}) \qquad \text{(substituting } k = j-1\text{)}\\
&+ \sum_{i=0}^{n}\sum_{j\leq i}(-1)^{i+j}R(LR)^j((LR)^{i-j}\epsilon_{(LR)^{n-i}A} \circ \epsilon_{(LR)^{n+1-j}A})
\end{aligned}$$

$$= \sum_{i=0}^{n} \sum_{i \le k}^{n} (-1)^{i+k+1} R(LR)^i((LR)^{k-i}\epsilon_{(LR)^{n-k}A} \circ \epsilon_{(LR)^{n+1-i}A}) \qquad \text{(by naturality of } \epsilon\text{)}$$

$$+ \sum_{i=0}^{n} \sum_{j \le i} (-1)^{i+j} R(LR)^j((LR)^{i-j}\epsilon_{(LR)^{n-i}A} \circ \epsilon_{(LR)^{n+1-j}A})$$

$$= -\sum_{i=0}^{n} \sum_{i \le k}^{n} (-1)^{i+k} R(LR)^i((LR)^{k-i}\epsilon_{(LR)^{n-k}A} \circ \epsilon_{(LR)^{n+1-i}A})$$

$$+ \sum_{i=0}^{n} \sum_{j \le i} (-1)^{i+j} R(LR)^j((LR)^{i-j}\epsilon_{(LR)^{n-i}A} \circ \epsilon_{(LR)^{n+1-j}A})$$

$$= -\sum_{k=0}^{n} \sum_{i \le k} (-1)^{i+k} R(LR)^i((LR)^{k-i}\epsilon_{(LR)^{n-k}A} \circ \epsilon_{(LR)^{n+1-i}A})$$

(switching the order of summation)

$$+ \sum_{i=0}^{n} \sum_{j \le i} (-1)^{i+j} R(LR)^j((LR)^{i-j}\epsilon_{(LR)^{n-i}A} \circ \epsilon_{(LR)^{n+1-j}A})$$

$$= 0$$

so the maps are differentials of a complex.
Next we show that s_k, as defined, is a contracting homotopy for our chain complex. This is equivalent to saying that $s_{k-1} \circ \partial_{k-1} + \partial_k \circ s_k = 1_{R(LR)^k A}$, where $\partial_k : R(LR)^{k+1}A \to R(LR)^k A$ is our differential defined above. Then observe that

$$\begin{aligned} s_{k-1} \circ \partial_{k-1} + \partial_k \circ s_k &= \sum_{i=0}^{k-1} (-1)^i \eta_{R(LR)^{k-1}A} \circ R(LR)^i \epsilon_{(LR)^{k-1-i}A} \\ &+ \sum_{i=0}^{k} (-1)^i R(LR)^i \epsilon_{(LR)^{k-i}A} \circ \eta_{R(LR)^k A} \\ &= 1_{R(LR)^k A} + \sum_{i=0}^{k-1} (-1)^i \eta_{R(LR)^{k-1}A} \circ R(LR)^i \epsilon_{(LR)^{k-1-i}A} \\ &+ \sum_{i=1}^{k} (-1)^i R(LR)^i \epsilon_{(LR)^{k-i}A} \circ \eta_{R(LR)^k A} \quad \text{(using the triangle identities)} \end{aligned}$$

It remains to show the extra sum is zero. After re-indexing the first sum it becomes:

$$-\sum_{i=1}^{k} (-1)^i \eta_{R(LR)^{k-1}A} \circ R(LR)^{i-1} \epsilon_{(LR)^{k-i}A} + \sum_{i=1}^{k} (-1)^i R(LR)^i \epsilon_{(LR)^{k-i}A} \circ \eta_{R(LR)^k A}$$

which is zero by naturality of η. ■

Additional to the result of this lemma, we claim that the contracting chain homotopy yields a natural transformation $s_k : R(LR)^k \Rightarrow R(LR)^{k+1}$, as $\eta_{R(LR)^k}$ is natural.

Finally, we have the following proposition.

Proposition 1.1.18 For each $n \geq 1$, the functors $\mathrm{cr}_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$ and $C_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$ are exact.

Proof. Since the functor categories are abelian, showing exactness is equivalent to showing that the functors preserve short exact sequences. The proof for cr_n follows by the 3×3 lemma and induction on n. After the proof for cr_n the result for C_n follows immediately. ■

1.1.0.2 Properties of the Cross-Effect

1.2.0 A categorical context for abelian functor calculus

Classically, abelian functor calculus deals with functors to abelian categories. In order to discuss universal properties "up to homotopy" we replace abelian categories by some type of homotopical categories where weak universal properties replace strict ones.

We denote the category of chain complexes of an abelian category $\mathcal{A}$ concentrated in non-negative degrees by $\mathsf{Ch}\mathcal{A}$. In this section we will construct a category which has as arrows maps from abelian categories to categories of chains. An important theorem for this construction is the Dold-Kan Equivalence, which is reviewed in Appendix 1.A.2.

1.2.1 Pointwise versus Natural Equivalences

> Remark:
>
> If we choose pointwise isomorphisms for the functors in the definition of AbCat, then composition in $\mathsf{AbCat}_{\mathsf{Ch}}$ will not be well-defined since we require that for any equivalent functors, G and H, and any simplicial object $\hat{A}$ with codomain equal to the domain of G and H, $G \circ \hat{A} \cong H \circ \hat{A}$ as simplicial objects.

Since the polynomial and linearization functors of [JM04] are only defined up to quasi-isomorphism in certain viewpoints, in order for them to be well-defined we must pass to the homotopy category $\mathsf{HoAbCat}_{\mathsf{Ch}}$. First, in this section we let composition in $\mathsf{AbCat}_{\mathsf{Ch}}$ be defined for $G : \mathcal{B} \to \mathsf{Ch}(\mathcal{C})$ and $F : \mathcal{A} \to \mathsf{Ch}(\mathcal{B})$ by

$$G \lhd F := N_{\mathcal{C}} \Delta_{\mathcal{C}} (\Gamma_{\mathcal{C}})_* G_* \Gamma_{\mathcal{B}} F$$

where $\Delta_{\mathcal{C}} : (\mathcal{C}^{\Delta^{op}})^{\Delta^{op}} \to \mathcal{C}^{\Delta^{op}}$ is the diagonal functor. It remains to show that this does define a categorical structure on $\mathsf{AbCat}_{\mathsf{Ch}}$, which we check through the following list of conditions:

1. Let $F : \mathcal{A} \to \mathsf{Ch}(\mathcal{B})$ be a functor. The identity is given by $\deg_0$. Indeed observe that

   $$[\deg_0^{\mathcal{B}} \lhd F] = [N_{\mathcal{B}} \Delta_{\mathcal{B}} (\Gamma_{\mathcal{B}})_* (\deg_0^{\mathcal{B}})_* \Gamma_{\mathcal{B}} F]$$

   However, by Lemma 1.A.28 $\Gamma_{\mathcal{B}} \circ \deg_0^{\mathcal{B}} \cong \iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^{\Delta^{op}}$ is the constant functor. It follows that $\Delta_{\mathcal{B}}(\Gamma_{\mathcal{B}} \circ \deg_0^{\mathcal{B}} \circ \Gamma_{\mathcal{B}} F) = \Gamma_{\mathcal{B}} F$, so

   $$[\deg_0^{\mathcal{B}} \lhd F] = [N_{\mathcal{B}} \Delta_{\mathcal{B}} (\Gamma_{\mathcal{B}})_* (\deg_0^{\mathcal{B}})_* \Gamma_{\mathcal{B}} F] = [N_{\mathcal{B}} \Gamma_{\mathcal{B}} F] = [F]$$

   On the other hand,

   $$[F \lhd \deg_0^{\mathcal{A}}] = [N_{\mathcal{B}} \Delta_{\mathcal{B}} (\Gamma_{\mathcal{B}})_* F_* \Gamma_{\mathcal{A}} \deg_0^{\mathcal{A}}] = [N_{\mathcal{B}} \Delta_{\mathcal{B}} (\Gamma_{\mathcal{B}})_* F_* \iota_{\mathcal{A}}]$$

Observe that $F_*\iota_{\mathcal{A}} = \iota_{\mathsf{Ch}(\mathcal{B})} \circ F$. Similarly, $(\Gamma_{\mathcal{B}})_*\iota_{\mathsf{Ch}(\mathcal{B})} = \iota_{\mathcal{B}^{\Delta^{op}}} \circ \Gamma_{\mathcal{B}}$. Finally,

$$\Delta_{\mathcal{B}}(\iota_{\mathcal{B}^{\Delta^{op}}} \circ \Gamma_{\mathcal{B}} \circ F) = \Gamma_{\mathcal{B}} \circ F$$

so

$$[F \lhd \deg_0^{\mathcal{A}}] = [N_{\mathcal{B}}\Delta_{\mathcal{B}}(\Gamma_{\mathcal{B}})_* F_* \Gamma_{\mathcal{A}} \deg_0^{\mathcal{A}}] = [N_{\mathcal{B}}\Gamma_{\mathcal{B}}F] = [F]$$

2. It remains to show composition is associative, so consider $F : \mathcal{A} \to \mathsf{Ch}(\mathcal{B}), H : \mathcal{B} \to \mathsf{Ch}(\mathcal{C}), G : \mathcal{C} \to \mathsf{Ch}(\mathcal{D})$. Then we compute:

$$\begin{aligned}[(G \lhd H) \lhd F] &= [N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*(G \lhd H)_*\Gamma_{\mathcal{B}}F] \\ &= [N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*(N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*G_*\Gamma_{\mathcal{C}}H)_*\Gamma_{\mathcal{B}}F] \\ &= [N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*G_*)_*(\Gamma_{\mathcal{C}})_*H_*\Gamma_{\mathcal{B}}F]\end{aligned}$$

while

$$\begin{aligned}[G \lhd (H \lhd F)] &= [G \lhd (N_{\mathcal{C}}\Delta_{\mathcal{C}}(\Gamma_{\mathcal{C}})_*H_*\Gamma_{\mathcal{B}}F)] \\ &= [N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*G_*\Gamma_{\mathcal{C}}(N_{\mathcal{C}}\Delta_{\mathcal{C}}(\Gamma_{\mathcal{C}})_*H_*\Gamma_{\mathcal{B}}F)] \\ &= [N_{\mathcal{D}}\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*G_*\Delta_{\mathcal{C}}(\Gamma_{\mathcal{C}})_*H_*\Gamma_{\mathcal{B}}F]\end{aligned}$$

Hence, it is sufficient to show that $[(\Gamma_{\mathcal{D}})_*G_*\Delta_{\mathcal{C}}] = [(\Delta_{\mathcal{D}}(\Gamma_{\mathcal{D}})_*G_*)_*]$. Recall that $(\Gamma_{\mathcal{D}})_*G_* : \mathcal{C}^{\Delta^{op}} \to \mathsf{Ch}(\mathcal{D})^{\Delta^{op}} \to (\mathcal{D}^{\Delta^{op}})^{\Delta^{op}}$ and $((\Gamma_{\mathcal{D}})_*G_*)_* : (\mathcal{C}^{\Delta^{op}})^{\Delta^{op}} \to ((\mathcal{D}^{\Delta^{op}})^{\Delta^{op}})^{\Delta^{op}}$. Additionally, $(\Delta_{\mathcal{D}})_* = (-)^{\Delta^{op}}\Delta_{\mathcal{D}} = \Delta_{\mathcal{D}^{\Delta^{op}}}$, so this equality is exactly naturality of $\Delta_{(-)}$, which is shown in Section 1.A.1.

Therefore, this composition provides the structure of a 1-category for $\mathsf{AbCat}_{\mathsf{Ch}}$.

Next we start describing the homotopy structure on this category.

Definition 1.2.1 Two functors $H, G : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ are said to be **pointwise chain homotopy equivalent** if the chain complexes $H(X)$ and $G(X)$ are chain homotopy equivalent in $\mathsf{Ch}(\mathcal{A})$ for each $X \in \mathcal{B}_0$, and **naturally chain homotopy equivalent** if the chain homotopies are natural in X.
Explicitly, naturally chain homotopy equivalent means that for each $X \in \mathcal{B}_0$ we have natural transformations $h : H \Rightarrow G, g : G \Rightarrow H$, together with homotopies $s_X : h_X \circ g_X \simeq 1_{G(X)}$ and $r_X : g_X \circ h_X \simeq 1_{H(X)}$ which are natural in the sense that for each n, $s_n : (-)_n \circ G \Rightarrow (-)_{n+1} \circ G$ and $r_n : (-)_n \circ H \Rightarrow (-)_{n+1} \circ H$ are natural transformations.

Both of these notions define equivalence relations on the category $\mathsf{AbCat}_{\mathsf{Ch}}$. Indeed, any functor is naturally chain homotopy equivalent to itself through identity natural transformations and zero homotopies, and a natural chain homotopy equivalence from H to G is precisely the same as a natural chain homotopy equivalence from G to H. It remains to show that if $H \simeq_{ChN} G \simeq_{ChN} F$, then $H \simeq_{ChN} F$. Let (h, g, s, r) and (g', f, s', r') be the quadruples witnessing the natural chain homotopy equivalence. Then we define a new quadruple by $(g' \circ h, g \circ f, g' \circ s \circ f + s', g \circ r' \circ h + r)$ which has all natural components since the composition of natural transformations is natural, and $+$ is functorial in an abelian category. To see that this does indeed define a chain homotopy equivalence observe that denoting the chain maps by ∂, and suppressing subscripts, we compute

$$\begin{aligned}\partial(g' \circ s \circ f + s') + (g' \circ s \circ f + s')\partial &= \partial g' s f + \partial s' + g' s f \partial + s' \partial \\ &= g' \partial s f + g' s \partial f + \partial s' + s' \partial \\ &= g'(\partial s + s \partial) f + \partial s' + s' \partial \\ &\quad \text{(using the fact } g' \text{ and } f \text{ are chain maps)} \\ &= g'(hg - 1_G) f + (g' f - 1_F) \\ &\quad \text{(by definition of the homotopies } s \text{ and } s') \\ &= g' h g f - 1_F\end{aligned}$$

Finally, it remains to show that this equivalence relation is well-defined for isomorphism classes of functors. It is sufficient to show that if $[H] = [H']$ and H is (naturally) chain homotopy equivalent to G, then so is H'. Let (h, g, s, r) witness the (natural) chain homotopy, and let $\alpha : H \to H'$ be a natural isomorphism witnessing the equivalence. Then I claim that the quadruple $(h\alpha^{-1}, \alpha g, s, \alpha[+1] r \alpha^{-1})$ is a (natural) chain homotopy, where $\alpha[+1]$ is the induced natural isomorphism between $H[+1]$ and $H'[+1]$.

First, observe that this definition preserves naturality, since α is natural, so it is sufficient in both cases to demonstrate that the components define a chain homotopy. Then for $X \in \mathcal{B}_0$ we can compute

$$\partial s_X + s_X \partial = h_X g_X - 1_{G(X)} = (h_X \alpha_X^{-1})(\alpha_X g_X) - 1_{G(X)}$$

while for a given n (to make the computation more tractable)

$$\begin{aligned}&\partial_{n+1}((\alpha_X[+1])_n (r_X)_n (\alpha_X^{-1})_n) + ((\alpha_X[+1])_{n-1} (r_X)_{n-1} (\alpha_X^{-1})_{n-1}) \partial_n \\ &\quad = \partial_{n+1}((\alpha_X)_{n+1} (r_X)_n (\alpha_X^{-1})_n) + ((\alpha_X)_n (r_X)_{n-1} (\alpha_X^{-1})_{n-1}) \partial_n \\ &\quad = (\alpha_X)_n \partial_{n+1} (r_X)_n (\alpha_X^{-1})_n + (\alpha_X)_n (r_X)_{n-1} \partial_n (\alpha_X^{-1})_n \\ &\quad = (\alpha_X)_n (\partial_{n+1} (r_X)_n + (r_X)_{n-1} \partial_n)(\alpha_X^{-1})_n \\ &\quad = (\alpha_X)_n ((g_X)_n (h_X)_n - (1_{H(X)})_n)(\alpha_X^{-1})_n \\ &\quad = (\alpha_X)_n (g_X)_n (h_X)_n (\alpha_X^{-1})_n - (1_{H(X)})_n\end{aligned}$$

so that the homotopies indeed hold. Note that these proofs thus far are independent of the composition we have chosen for $\mathsf{AbCat}_{\mathsf{Ch}}$. In particular, in the case of natural chain homotopies the results follow from the fact that they correspond to chain homotopies in a category of chain complexes by the work in Section 1.A.1.2. In order to define a homotopy category from these equivalence relations we must now show that they are in fact congruence relations. We proceed only for the natural chain homotopies.

Lemma 1.2.2 If $G, H : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ are naturally chain homotopy equivalent functors, then for any pair of functors $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $K : \mathcal{D} \to \mathsf{Ch}(\mathcal{C})$, the composites

$$F \lhd G \lhd K, F \lhd H \lhd K : \mathcal{D} \to \mathsf{Ch}(\mathcal{A})$$

are naturally chain homotopy equivalent.

Proof. We separate this proof into two parts. Let G, H be naturally chain homotopy equivalent, and let F and K be functors as in the question. Let $(h, g, s, r) : H \simeq_{\mathsf{Ch}} G$ be a natural chain homotopy between our original functors. We begin the construction in the case of pre-composition by K:

[$- \lhd K$] From the definition of $\lhd$, the composites $H \lhd K$ and $G \lhd K$ are given by

$$H \lhd K = N_{\mathcal{B}}\Delta_{\mathcal{B}}(\Gamma_{\mathcal{B}})_* H_* \Gamma_{\mathcal{C}} K \quad \text{and} \quad G \lhd K = N_{\mathcal{B}}\Delta_{\mathcal{B}}(\Gamma_{\mathcal{B}})_* G_* \Gamma_{\mathcal{C}} K$$

First, note that since the homotopy is natural $H \lhd K \simeq G \lhd K$ by the results in Sections 1.A.1.1 and 1.A.2.

[$F \lhd -$] The composites $F \lhd H$ and $F \lhd G$ are given by

$$F \lhd H = N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H \quad \text{and} \quad F \lhd G = N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} G$$

Using the results in Sections 1.A.2 and 1.A.1.1 we have the chain of implications

$$\begin{aligned} H \simeq_{\mathsf{Ch}} G &\implies \Gamma_{\mathcal{B}} H \simeq_{Ho} \Gamma_{\mathcal{B}} G \\ &\implies F_* \Gamma_{\mathcal{B}} H \simeq_{Ho} F_* \Gamma_{\mathcal{B}} G \\ &\implies (\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H \simeq_{Ho} (\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} G \\ &\implies \Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H \simeq_{Ho} \Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} G \\ &\implies N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H \simeq_{\mathsf{Ch}} N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} G \end{aligned}$$

where each implication preserves naturality as well.

■

We now seek to upgrade the category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ in [BJO+18] to natural chain homotopy equivalences.

Definition 1.2.3 There exists a (large) category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ consisting of the following data:

- Objects are abelian categories
- Morphisms $\mathcal{B} \rightsquigarrow \mathcal{A}$ are natural chain homotopy equivalence classes of functors

$\mathcal{B} \to \mathsf{Ch}(\mathcal{A})$

- Composition of maps $\mathcal{C} \rightsquigarrow \mathcal{B}$ and $\mathcal{B} \rightsquigarrow \mathcal{A}$, corresponding to functors $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ and $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is defined by the equivalence class of the composite $F \lhd G$

This definition exists since the natural chain homotopy equivalences form a congruence relation on $\mathsf{AbCat}_{\mathsf{Ch}}$.

1.3.0 The Taylor tower in abelian functor calculus

With the appropriate categorical language built up, we can begin constructing Taylor towers, as in [JM04]. This process involves defining certain polynomial functors of **degree** n with natural transformations which go down by one in degree. In [BJO+18] many definitions are given in terms of pointwise chain homotopies. In order to introduce two-dimensional structures on the cat $\mathsf{HoAbCat}_{\mathsf{Ch}}$ we aim to upgrade these to natural homotopies. We distinguish where these changes are made by changing the text color to red for occurrences of natural which aren't originally in the text.

Definition 1.3.1 A functor $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is **degree** n if $\mathrm{cr}_{n+1}(F) : \mathcal{B}^{n+1} \to \mathsf{Ch}(\mathcal{A})$ is **contractible**, i.e., natural chain homotopy equivalent to zero.

Note that $\mathsf{cr}_k(F) \simeq_{\mathsf{Ch}} 0$ implies $\mathsf{cr}_\ell(F) \simeq_{\mathsf{Ch}} 0$ for any $\ell > k$. Indeed, since $G \oplus F \simeq_{\mathsf{Ch}} 0$ implies $F \simeq_{\mathsf{Ch}} 0$ from Section 1.A.1.2, this follows from the inductive definition. Consequently, functors of degree k are also of degree ℓ for $\ell > k$.

Proof. Let F be degree n, so $\mathrm{cr}_{n+1}(F) \simeq_{\mathsf{Ch}} 0$. We show $\mathrm{cr}_{n+k}(F) \simeq_{\mathsf{Ch}} 0$ for all $k \geq 1$ by induction. The base case is by assumption, so suppose $\mathrm{cr}_{n+k}(F) \simeq_{\mathsf{Ch}} 0$ for some $k \geq 1$. Then we have the isomorphism

$$\begin{aligned}\mathrm{cr}_{n+k}(F)(X_1 \vee X_2, X_3, \cdots, X_n) \cong\ & \mathrm{cr}_{n+k}(F)(X_1, X_3, ..., X_n) \oplus \mathrm{cr}_{n+k}(F)(X_2, ..., X_n) \\ & \oplus \mathrm{cr}_{n+k+1}(F)(X_1, ..., X_n)\end{aligned}$$

which is natural in the X_i and F. Since all cr_{n+k} terms are naturally contractible, Lemmas 1.4.4 and 1.4.5 imply that

$$\mathrm{cr}_{n+k+1}(F) \simeq_{\mathsf{Ch}} 0$$

as desired. ∎

From [JM04, Defn 2.4], chain complexes can be constructed from a pair of a comonad and an object in the category on which the comonad acts.

Lemma 1.3.2 Let $C : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$ be a comonad on a functor category where $\mathcal{A}$ is an abelian category (so in particular $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$ is abelian). Then C induces a functor

$$C^{\mathsf{Ch}} : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$$

Proof. Let $\epsilon : C \to 1$ be the counit of the comonad. Then for $F : \mathcal{B} \to \mathcal{A}$, define $C^{\mathsf{Ch}}(F)$ to be the chain

$$\cdots \to C^3(F) \xrightarrow{\epsilon_{C^2(F)} - C\epsilon_{C(F)} + C^2\epsilon_F} C^2(F) \xrightarrow{\epsilon_{C(F)} - C\epsilon_F} C(F) \xrightarrow{\epsilon_F} F$$

where the kth differential is defined by the alternating sum $\sum_{i=0}^{k-1}(-1)^i C^i \epsilon_{C^{(k-i)}}$. Note these differentials are indeed natural, and hence this defines a sequence of functors and natural transformations. To see that this sequence forms a chain complex observe that

$$\begin{aligned}
&\left(\sum_{i=0}^{n}(-1)^i C^i \epsilon_{C^{n-i}(F)}\right) \circ \left(\sum_{i=0}^{n+1}(-1)^i C^i \epsilon_{C^{n+1-i}(F)}\right) \\
&\qquad = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j} C^i \epsilon_{C^{n-i}(F)} \circ C^j \epsilon_{C^{n+1-j}(F)} \\
&\qquad = \sum_{i=0}^{n}\sum_{i<j}^{n+1}(-1)^{i+j} C^i \epsilon_{C^{n-i}(F)} \circ C^j \epsilon_{C^{n+1-j}(F)} \\
&\qquad + \sum_{i=0}^{n}\sum_{i\geq j}(-1)^{i+j} C^i \epsilon_{C^{n-i}(F)} \circ C^j \epsilon_{C^{n+1-j}(F)} \\
&\qquad = \sum_{i=0}^{n}\sum_{i<j}^{n+1}(-1)^{i+j} C^i \left(\epsilon_{C^{n-i}(F)} \circ C^{j-i} \epsilon_{C^{n+1-j}(F)}\right) \\
&\qquad + \sum_{i=0}^{n}\sum_{i\geq j}(-1)^{i+j} C^j \left(C^{i-j}\epsilon_{C^{n-i}(F)} \circ \epsilon_{C^{n+1-j}(F)}\right) \\
&\qquad = \sum_{i=0}^{n}\sum_{i\leq k}^{n}(-1)^{i+k+1} C^i \left(\epsilon_{C^{n-i}(F)} \circ C^{k+1-i} \epsilon_{C^{n-k}(F)}\right) \\
&\qquad\qquad \text{(Substituting } k = j-1\text{)} \\
&\qquad + \sum_{i=0}^{n}\sum_{i\geq j}(-1)^{i+j} C^j \left(C^{i-j}\epsilon_{C^{n-i}(F)} \circ \epsilon_{C^{n+1-j}(F)}\right) \\
&\qquad = \sum_{i=0}^{n}\sum_{i\leq k}^{n}(-1)^{i+k+1} C^i \left(C^{k-i}\epsilon_{C^{n-k}(F)} \circ \epsilon_{C^{n+1-i}(F)}\right) \quad \text{(Naturality of } \epsilon\text{)} \\
&\qquad + \sum_{i=0}^{n}\sum_{i\geq j}(-1)^{i+j} C^j \left(C^{i-j}\epsilon_{C^{n-i}(F)} \circ \epsilon_{C^{n+1-j}(F)}\right) \\
&\qquad = -\sum_{k=0}^{n}\sum_{k\geq i}(-1)^{i+k} C^i \left(C^{k-i}\epsilon_{C^{n-k}(F)} \circ \epsilon_{C^{n+1-i}(F)}\right) \\
&\qquad\qquad \text{(Re-ordering the sum)} \\
&\qquad + \sum_{i=0}^{n}\sum_{i\geq j}(-1)^{i+j} C^j \left(C^{i-j}\epsilon_{C^{n-i}(F)} \circ \epsilon_{C^{n+1-j}(F)}\right) \\
&\qquad = 0
\end{aligned}$$

Next, let $\alpha : F \Rightarrow G$ be a natural transformation. Then $C^{\mathsf{Ch}}(\alpha) : C^{\mathsf{Ch}}(F) \to C^{\mathsf{Ch}}(G)$ is defined by $C^{\mathsf{Ch}}(\alpha)_n := C^n\alpha : C^nF \to C^nG$. To see that this is a chain map observe that for

$0 \leq i \leq n$ we have the commutative diagram

$$\begin{tikzcd} C^{n+1}(F) \arrow[r, "C^i\epsilon_{C^{n-i}(F)}"] \arrow[d, "C^{n+1}(\alpha)"'] & C^n(F) \arrow[d, "C^n(\alpha)"] \\ C^{n+1}(G) \arrow[r, "C^i\epsilon_{C^{n-i}(G)}"'] & C^n(G) \end{tikzcd}$$

which commutes by naturality of ϵ. Since composition is bilinear with respect to the group operation on hom sets in an abelian category we have that the $C^n(\alpha)$ form a chain map in $\mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$. Further, since C is a functor so is C^{Ch}, completing the proof. ■

Note that we can realize this functor as going to $\mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$. Indeed, $\mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A})) \cong \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$, as we show in Lemma 1.A.11. We can use this technique to define the polynomial approximations in the Taylor tower for a functor.

Definition 1.3.3 The **nth polynomial approximation** is the composite functor $P_n := (\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{Fun}^{\mathsf{ch}} \circ C^{\mathsf{Ch}}_{n+1} : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$.

Recall by Lemma 1.1.8 if $n = 0$, $C_1(F) = \mathrm{cr}_1(\mathrm{cr}_1(F)) = \mathrm{cr}_1(F)$ (by our choice in defining cr_1), and so $C_1^{\times k}(F) = \mathrm{cr}_1(F)$ for each $k \geq 1$. Further, the co-unit $\epsilon : \Delta^*(\mathrm{cr}_1(\mathrm{cr}_1(F)))(X) = \mathrm{cr}_1(F)(X) \to F(X)$ is the kernel map in $\mathrm{cr}_1(F) \to F(X) \to F(\star)$. In the case of ϵ_{C_1} the kernel map is the identity from our definition of cr_1, while by the definition of cr_1 on maps we have

$$\begin{tikzcd} \mathrm{cr}_1(F)(X) \arrow[r, equal] \arrow[d, "\mathrm{cr}_1(\epsilon)"'] & \mathrm{cr}_1(F)(X) \arrow[r, "!"] \arrow[d, "\epsilon"'] & 0 \arrow[d] \\ \mathrm{cr}_1(F)(X) \arrow[r, tail, "\epsilon"'] & F(X) \arrow[r] & F(\star) \end{tikzcd}$$

so $C_1(\epsilon)$ is the identity as well since ϵ is monic. Hence, we obtain the chain complex of functors

$$\cdots \xrightarrow{0} \mathrm{cr}_1(F) \xrightarrow{\mathrm{Id}} \mathrm{cr}_1(F) \xrightarrow{0} \mathrm{cr}_1(F) \xrightarrow{\epsilon} F$$

Since $F \cong \mathrm{cr}_1(F) \oplus F(\star)$, the chain complex defining $P_0(F)$ can be written as a direct sum of the two chain complexes

$$\cdots \xrightarrow{0} \mathrm{cr}_1(F) \xrightarrow{\mathrm{Id}} \mathrm{cr}_1(F) \xrightarrow{0} \mathrm{cr}_1(F) \xrightarrow{\mathrm{Id}} \mathrm{cr}_1(F)$$

and

$$\cdots \to 0 \to 0 \to 0 \to F(\star)$$

Note that the chain complex in the top line is contractible. Recall the totalization commutes with direct sums. The totalization of the second complex is isomorphic to $F(\star)$ itself. On the other hand, the totalization of the first complex, $C_\bullet$, has $C_0 = \mathrm{cr}_1(F)_0$, $C_1 = \mathrm{cr}_1(F)_1 \oplus \mathrm{cr}_1(F)_0$, and in general

$$C_n = \bigoplus_{i=0}^{n} \mathrm{cr}_1(F)_i$$

with differential $\partial_n : C_n \to C_{n-1}$ given by

$$\partial_n = (\delta_{n-i,even}\pi_{C_{i-1}} - \partial_i^{\mathrm{cr}_1(F)} \circ \pi_{C_i})_{1\leq i\leq n}$$

where $\delta_{n-i,even}$ is 1 when $2 \mid n-i$ and 0 else. Then $P_0(F)$ is the direct sum of these two sequences. However, since the first complex before totalization is contractible, we can model $P_0(F)(X) \cong F(\star)$.

We test the above computation, and the results to follow, using the example of $\deg_0^{\mathcal{A}} : \mathcal{A} \to \mathsf{Ch}(\mathcal{A})$

Example 1.3.1 () :

First, note that the chain complex $\deg_0^{\mathcal{A}}(0)$ is the zero complex. Since $\deg_0^{\mathcal{A}}$ is reduced we also have that $\mathrm{cr}_1(\deg_0^{\mathcal{A}}) = \deg_0^{\mathcal{A}}$. It follows that $P_0(\deg_0^{\mathcal{A}})_n = 1_{\mathcal{A}}$ for each $n \geq 0$, and $\partial_n = \delta_{n-1,even}1_{\mathcal{A}}$, or in other words

$$P_0(\deg_0^{\mathcal{A}}) := \cdots \xrightarrow{1_{1_{\mathcal{A}}}} 1_{\mathcal{A}} \xrightarrow{0} 1_{\mathcal{A}} \xrightarrow{1_{1_{\mathcal{A}}}} 1_{\mathcal{A}}$$

Note that this complex is contractible.

Using the isomorphisms in Section 1.A.1.2 we can now describe the cross-effect on functors into chain complexes explicitly. Next we show compatibility of this functor with the isomorphism $\mathrm{Fun}^{\mathsf{Ch}}$.

Lemma 1.3.4 For any pointed category $\mathcal{B}$ and abelian category $\mathcal{A}$, we have a natural isomorphism

$$\mathrm{cr}_n^{\mathcal{B},\mathsf{Ch}(A)} \cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}}) \circ (\mathrm{Fun}^{\mathsf{Ch}})^{-1}$$

Proof. Since the cross effect is additive we can apply Ch, so the claim is well-posed. Note that finite limits in functor categories between abelian categories are computed pointwise, up to natural isomorphism, and the same holds for chain complexes, from Sections 1.A.4 and 1.A.1.2. It follows that

$$\mathrm{cr}_n^{\mathcal{B},\mathsf{Ch}(A)} \cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}}) \circ (\mathrm{Fun}^{\mathsf{Ch}})^{-1}$$

∎

Note that in addition $\Delta^* \circ \mathrm{Fun}^{\mathsf{Ch}} = \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(\Delta^*)$, so we also have

$$C_n^{\mathcal{B},\mathsf{Ch}(A)} \circ \mathrm{Fun}^{\mathsf{Ch}} \cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(C_n^{\mathcal{B},\mathcal{A}})$$

Finally, we show that the C^{Ch} construction also commutes with the chain complex functor when C is additive.

Lemma 1.3.5 Let (C, ϵ, δ) be a comonad on $\mathcal{A}$ which is also an additive functor. Then we have the isomorphism

$$\mathsf{Ch}(C)^{\mathsf{Ch}} \cong \mathsf{Ch}(C^{\mathsf{Ch}})$$

Proof. Let $A_\bullet \in \mathsf{Ch}(\mathcal{A})$. Then both $\mathsf{Ch}(C)^{\mathsf{Ch}}(A_\bullet)$ and $\mathsf{Ch}(C^{\mathsf{Ch}})(A_\bullet)$ give the same double complex after the isomorphism $\mathsf{Ch}(\mathsf{Ch}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Ch}(\mathcal{A}))$ which swaps the rows and columns (i.e. transposition)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & C^2(A_{n+1}) \arrow[r, "C\epsilon_{A_{n+1}} - \epsilon_{CA_{n+1}}"] \arrow[d, "C^2(\partial^A_{n+1})"'] & C(A_{n+1}) \arrow[r, "\epsilon_{A_{n+1}}"] \arrow[d, "C(\partial^A_{n+1})"] & A_{n+1} \arrow[d, "\partial^A_{n+1}"] \\
\cdots \arrow[r] & C^2(A_n) \arrow[r, "C\epsilon_{A_n} - \epsilon_{CA_n}"] \arrow[d, "C^2(\partial^A_n)"'] & C(A_n) \arrow[r, "\epsilon_{A_n}"] \arrow[d, "C(\partial^A_n)"] & A_n \arrow[d, "\partial^A_n"] \\
\cdots \arrow[r] & C^2(A_{n-1}) \arrow[r, "C\epsilon_{A_{n-1}} - \epsilon_{CA_{n-1}}"'] \arrow[d] & C(A_{n-1}) \arrow[r, "\epsilon_{A_{n-1}}"'] \arrow[d] & A_{n-1} \arrow[d] \\
 & \vdots & \vdots & \vdots
\end{tikzcd}$$

■

These results and Lemma 1.A.19 allow us to formalize preservation of chain homotopies for our approximation functors. Due to Proposition 1.1.18 we also obtain nice properties for the functors $P_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$.

Proposition 1.3.6 For any $n \geq 0$,

(i) $P_n : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$ is exact

(ii) P_n preserves natural chain homotopies, chain homotopy equivalences, and contractibility.

Proof. It is sufficient to prove (i) for short exact sequences. Let $0 \to F \to G \to H \to 0$ be a SES of functors in $\mathsf{AbCat}_{\mathsf{Ch}}$. By Proposition 1.1.18 we obtain a SES of bicomplexes in the definition of the nth polynomial approximation. Since totalization is exact we obtain a SES $0 \to P_n(F) \to P_n(G) \to P_n(H) \to 0$.

For (ii), observe that from our work in Sections 1.A.1.2 and 1.A.4

$$(\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{Fun}^{\mathsf{Ch}} \circ C^{\mathsf{Ch}}_{n+1} \cong (\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{Fun}^{\mathsf{Ch}} \circ \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(C^{\mathsf{Ch}}_{n+1}) \circ (\mathrm{Fun}^{\mathsf{Ch}})^{-1}$$

$$\cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathrm{Tot}_{\mathrm{Fun}(\mathcal{B},\mathcal{A})} \circ \mathsf{Ch}(C_{n+1})^{\mathsf{Ch}} \circ (\mathrm{Fun}^{\mathsf{Ch}})^{-1}$$

Then since $\mathsf{Ch}(C_{n+1})^{\mathsf{Ch}} = \mathsf{Ch}(C^{\mathsf{Ch}}_{n+1})$ preserves chain homotopies and by [dJ05, 12.18] $\mathrm{Tot}_{\mathrm{Fun}(\mathcal{B},\mathcal{A})}$ also preserves chain homotopies, it follows by Lemma 1.A.13 that P_n preserves natural chain homotopies. ■

For each $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$, the functor $P_n(F)$ comes equipped with a natural transformation $p_n : F \to P_n(F)$ defined by inclusion into the degree zero part of the chain complex $P_n(F)$. Explicitly, can define $p_n : \mathbb{1} \Rightarrow P_n$ as done by Jason Parker:

Remark:

First we define a natural transformation $i : (\deg^{\mathsf{Ch}(\mathcal{A})})_* \Rightarrow \mathrm{Fun}^{\mathsf{Ch}} \circ C^{\mathsf{Ch}}_{n+1} : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}^2(\mathcal{A}))$, and then we define $p_n := (\mathrm{Tot}_{\mathcal{A}})_* \circ i$ along with Lemma 1.3.7.
For $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ we define $i_F : \deg^{\mathsf{Ch}(\mathcal{A})} \circ F \Rightarrow \mathrm{Fun}^{\mathsf{Ch}} \circ C^{\mathsf{Ch}}_{n+1}(F) : \mathcal{B} \Rightarrow \mathsf{Ch}^2(\mathcal{A})$ where for each $B \in \mathcal{B}_0$, we define $i_{F,B} : \deg^{\mathsf{Ch}(\mathcal{A})}(FB) \to (\mathrm{Fun}^{\mathsf{Ch}}(C^{\mathsf{Ch}}_{n+1}(F)))B$ in $\mathsf{Ch}^2(\mathcal{A})$ by saying for all $m \geq 0$,

$$(i_{F,B})_m = \begin{cases} 0 & m > 0 \\ 1_{FB} : FB \to FB & m = 0 \end{cases}$$

since $\mathrm{Fun}^{\mathsf{Ch}}(C^{\mathsf{Ch}}_{n+1}(F)))(B)_0 = C^0_{n+1}(F)(B) = F(B)$. These form appropriate chain maps which are natural in B and F as all components are either zero or identities. Then explicitly,

$$(p_n)_{F,B,m} = F(B)_m \to \bigoplus_{p+q=m} C^p_{n+1}(F)(B)_q$$

is the inclusion into the direct summand $F(B)_m$.

Lemma 1.3.7 We have a natural isomorphism

$$(\mathrm{Tot}_{\mathcal{A}})_* \circ \deg^{\mathsf{Ch}(\mathcal{A})} \cong \mathbb{1}_{\mathsf{Ch}(\mathcal{A})}$$

Proof. Let $A \in \mathsf{Ch}(\mathcal{A})$. Then for $n \geq 0$

$$(\mathrm{Tot}_{\mathcal{A}})_* \circ \deg^{\mathsf{Ch}(\mathcal{A})}(A)_n = \bigoplus_{i+j=n} \deg^{\mathsf{Ch}((\mathcal{A})}(A)_i)_j \cong A_n$$

since all other terms are zero. This isomorphism is given uniquely by the universal property of the biproduct and zero map, so induces the desired isomorphism in the statement of the Lemma. ■

In order to show some basic properties of the approximation functor we first prove the following Lemma related to the behaviour of the cross effect functor.

Lemma 1.3.8 Let $\mathcal{B}$ be a pointed category and let $\mathcal{A}, \mathcal{C}$ be abelian categories. Then if $F : \mathcal{A} \to \mathcal{C}$ is an exact functor we have a natural isomorphism

$$\mathrm{cr}_n^{\mathcal{B},\mathcal{C}} \circ F_* \cong F_* \circ \mathrm{cr}_n^{\mathcal{B},\mathcal{A}}$$

where $\mathrm{cr}_n^{\mathcal{B},-} : \mathrm{Fun}(\mathcal{B}, -) \to \mathrm{Fun}_*(\mathcal{B}^n, -)$ specifies the codomain category.

Actually, this statement is true whenever F preserves direct sums. However we will only use it for exact functors.

Proof. We will prove this by induction. For the base case on objects let $G : \mathcal{B} \to \mathcal{A}$ be a functor. Then (by the definition of the cross-effect) $G(X) \cong G(*) \oplus \mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(X)$ for every object X of $\mathcal{B}$. Applying F to this equality and using that F preserves direct sums, we obtain that

$$F \circ G(X) \cong F(G(X) \cong G(*) \oplus \mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(X)) \cong F(G(*)) \oplus F(\mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(X))$$

Applying the definition of the cross-effect to this we obtain that

$$\mathrm{cr}_n(F \circ G)(X) \cong F(\mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(X)).$$

As $\mathrm{cr}_n(F \circ G)$ send a morphisms f of $\mathcal{B}$ to the unique induced map into the limit that is the cross-effect and $F(\mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(f)$ gives one such map, $\mathrm{cr}_n^{\mathcal{B},\mathcal{C}} \circ F_*(G) \cong F_* \circ \mathrm{cr}_n^{\mathcal{B},\mathcal{A}}(G)$ as functors $\mathcal{B} \to \mathcal{C}$.
For the base case on morphisms, let $\varphi : G \Rightarrow G'$ be a natural transformation between functors $G, G' : \mathcal{B} \to \mathcal{A}$. Then the component φ_X corresponds to $\varphi_* \oplus \mathrm{cr}_1^{\mathcal{B},\mathcal{A}} \varphi_X$ in the sense that

$$\begin{tikzcd}
G(X) \arrow[rr, "\varphi_X"] \arrow[d, "\cong"'] & & G'(X) \arrow[d, "\cong"] \\
G(*) \oplus \mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G(X) \arrow[rr, "\varphi_* \oplus (\mathrm{cr}_1^{\mathcal{B},\mathcal{A}} \varphi)_X"] & & G'(*) \oplus \mathrm{cr}_1^{\mathcal{B},\mathcal{A}} G'(X)
\end{tikzcd}$$

commutes (this is how the cross-effect is defined on morphisms). Applying F to this we can read off that $F(\varphi)_X$ corresponds to $F(\varphi_*) \oplus F(\mathrm{cr}_1^{\mathcal{B},\mathcal{A}}(\varphi)_X)$, so by the definition of the cross-effect $\mathrm{cr}_1^{\mathcal{B},\mathcal{C}}(F \circ \varphi) \cong F(\mathrm{cr}_1^{\mathcal{B},\mathcal{A}}(\varphi))$.
For the inductive step, let the statement be true for $\mathrm{cr}_n - 1$ (as it will be analogous to the base case we will only sketch this part). Then the definition of the cross-effect tells us

$$\mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}} G(X_1 \vee X_2, X_3, ...) = \mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}} G(X_1, X_3, ...) \oplus \mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}} G(X_1, X_3, ...) \oplus \mathrm{cr}_n^{\mathcal{B},\mathcal{A}} G(X_1, X_2, X_3, ...).$$

Applying F and the inductive hypothesis, we obtain

$$\mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{C}}(F \circ G)(X_1 \vee X_2, X_3, ...) = \mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{C}}(F \circ G)(X_1, X_3, ...) \oplus \mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{C}}(F \circ G)(X_1, X_3, ...) \oplus F(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}} G(X_1, X_2, X_3, ...)$$

from which we can see (by the definition of the cross-effect) that

$$\mathrm{cr}_n^{\mathcal{B},\mathcal{C}}(F \circ G)(X_1, X_2, X_3, ...) = F(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}} G(X_1, X_2, X_3, ...))$$

Again the uniqueness of the induced map into a limit gives us that $\mathrm{cr}_n^{\mathcal{B},\mathcal{C}}(F \circ G) = F \circ \mathrm{cr}_n^{\mathcal{B},\mathcal{A}} G$. In order to do the induction step on morphisms, let $\varphi : G \to G'$ be a natural transformation. Then, by the definition of the cross-effect on morphisms, $(\mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}}\varphi)_{X_1 \vee X_2,\dots}$ corresponds to $(\mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}}\varphi)_{X_1,X_3\dots} \oplus (\mathrm{cr}_{n-1}^{\mathcal{B},\mathcal{A}}\varphi)_{X_2,X_3\dots} \oplus (\mathrm{cr}_{n}^{\mathcal{B},\mathcal{A}}\varphi)_{X_1,X_2,\dots}$. Applying F and using the inductive hypothesis, we can read off that

$$(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}} F(\varphi))_{X_1,X_2,\dots} = F(\mathrm{cr}_n^{\mathcal{B},\mathcal{A}}\varphi)_{X_1,X_2,\dots}$$

which proves the inductive step for morphisms. ■

The basic properties of this approximation are given in the following proposition.

Proposition 1.3.9 For $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$,

(i) The functor $P_n(F)$ is degree n

(ii) If F is degree n, then the map $p_n : F \to P_n(F)$ is a chain homotopy equivalence (natural)

(iii) The pair $(P_n(F), p_n : F \to P_n(F))$ is universal up to chain homotopy equivalence with respect to degree n functors receiving natural transformations from F.

Finally, one last result we will require that is used in the proof of (iii) is the following computation for a functor $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$.

Remark:

By construction $p_{n,P_n(F)}$ is the inclusion of $P_n(F)$ into $P_n(P_n(F))$ via the totalization after inclusion into the degree zero part of the bicomplex defining $P_n(P_n(F))$. On the other hand, applying P_n to $p_{n,F}$ **TBC**

Proof of Proposition 1.3.9. Let $F_k : \mathcal{B} \to \mathcal{A}$ be the kth degree component of $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ (under the isomorphism $\mathrm{Fun}^{\mathsf{Ch}}$). To prove (i) we show $\mathrm{cr}_{n+1}(P_n(F))$ is contractible (i.e. naturally contractible). By Lemma 1.3.8

$$\mathrm{cr}_{n+1}^{\mathcal{B},\mathsf{Ch}(\mathcal{A})} \circ (\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{Fun}^{\mathsf{Ch}} \circ C_{n+1}^{\mathsf{Ch}} \cong (\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{cr}_{n+1}^{\mathcal{B},\mathsf{Ch}^2(\mathcal{A})} \circ \mathrm{Fun}^{\mathsf{Ch}} \circ C_{n+1}^{\mathsf{Ch}}$$

Since Tot preserves natural homotopies it is sufficient to show that the cross effect for the bicomplex $\mathrm{cr}_{n+1}^{\mathcal{B},\mathsf{Ch}^2(\mathcal{A})} \circ \mathrm{Fun}^{\mathsf{Ch}} \circ C_{n+1}^{\mathsf{Ch}}(F)$ defining $P_n(F)$ is contractible. By Lemma 1.3.4 this is equivalent to showing $\mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(\mathrm{cr}_{n+1}^{\mathcal{B},\mathsf{Ch}(\mathcal{A})}) \circ C_{n+1}^{\mathsf{Ch}}(F)$ is contractible, or since the naturally homotopies on either side of the isomorphism agree, it is sufficient to show $\mathsf{Ch}(\mathrm{cr}_{n+1}^{\mathcal{B},\mathsf{Ch}(\mathcal{A})}) \circ C_{n+1}^{\mathsf{Ch}}(F) \in \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})))$ is contractible. In particular, we can consider the corresponding double complex $\mathsf{Ch}^2(\mathrm{Fun}(\mathcal{B},\mathcal{A}))$, so we can apply the results in Appendix 1.6. Then the kth row of this bicomplex is given by

$$\cdots \to \mathrm{cr}_{n+1}^{\mathcal{B},\mathcal{A}} C_{n+1}^{\times 2}(F_k) \xrightarrow{\mathrm{cr}_{n+1}^{\mathcal{B},\mathcal{A}}(\epsilon C_{n+1} - C_{n+1}\epsilon)} \mathrm{cr}_{n+1}^{\mathcal{B},\mathcal{A}} C_{n+1}(F_k) \xrightarrow{\mathrm{cr}_{n+1}^{\mathcal{B},\mathcal{A}}\epsilon} \mathrm{cr}_{n+1}^{\mathcal{B},\mathcal{A}}(F_k)$$

By Lemma 1.1.17 we have a family of horizontal contractions for each row after applying cr_{n+1}, denoted $s^{k,h}$. Setting the vertical contractions, s^v, to be zero, we obtain a natural contraction for the bicomplex, so under the totalization we obtain a natural contraction for the chain complex $\mathrm{cr}_{n+1}(P_n(F))$, as desired.

For (ii) let F be of degree n, so $\mathrm{cr}_{n+1}(F)$ is naturally contractible. Recall the kth column of the bicomplex defining $P_n(F)$ is $C_{n+1}^{\times k}(F) = (\Delta^*\mathrm{cr}_{n+1})^k(F)$. The map $p_n : F \to P_n(F)$ is the natural inclusion of the 0th column into the totalization. Note that $C_{n+1}^{\times k}(F)$ is contractible for each $k \geq 1$ since $C_{n+1} \cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathsf{Ch}(C_n) \circ (\mathrm{Fun}^{\mathsf{Ch}})^{-1}$ preserves chain homotopies and F is of degree n. Under the isomorphisms in Section 1.A.1.2, p_n becomes the inclusion of $F_\bullet$ into $\mathrm{Tot}_{\mathsf{Fun}(\mathcal{B},\mathcal{A})}(C_{n+1}^{\mathsf{Ch}}(F))$, so by Corollary 1.6.6 we obtain a chain homotopy equivalence, which becomes a natural chain homotopy equivalence under the desired isomorphisms.

To show (iii) let $\tau : F \to G$ be a natural transformation transformation where G is a functor of degree n. By naturality of p_n in the functor F we have a commutative diagram

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[d, "p_{n,F}"'] & G \arrow[d, "p_{n,G}"] \\ P_n(F) \arrow[r, "P_n(\tau)"'] & P_n(G) \end{tikzcd}$$

where the right hand $p_{n,G}$ is a natural chain homotopy equivalence by (ii). Let $s_{n,G}$ denote a natural chain homotopy inverse of $p_{n,G}$. Setting $\tau^\# = s_{n,G} \circ P_n(\tau)$ we have that

$$\tau^\# \circ p_{n,F} = s_{n,G} \circ P_n(\tau) \circ p_{n,F} = s_{n,G} \circ p_{n,G} \circ \tau \simeq_{\mathsf{Ch}} \tau$$

This shows τ factors through $p_n : F \to P_n(F)$ up to natural chain homotopy equivalence. To show uniqueness suppose $\sigma : P_n(F) \to G$ is another map such that τ is naturally chain homotopy equivalent to $\sigma \circ p_{n,F}$. Then by naturality of the p_n, we have a commuting diagram

$$\begin{tikzcd} F \arrow[r, "p_{n,F}"] \arrow[d, "p_{n,F}"'] & P_n(F) \arrow[r, "\sigma"] \arrow[d, "p_{n,P_n(F)}"] & G \arrow[d, "p_{n,G}"] \\ P_n(F) \arrow[r, "P_n p_{n,F}"'] & P_n(P_n(F)) \arrow[r, "P_n(\sigma)"'] & P_n(G) \end{tikzcd}$$

where $p_{n,P_n(F)}$ and $p_{n,G}$ are natural chain homotopy equivalences by (ii).
By Lemma **REF** we have that $P_n(p_{n,F})$ is a natural chain homotopy equivalence with inverse $S_{n,F}$. Observe that

$$\begin{aligned} \sigma &\simeq_{\mathsf{Ch}} s_{n,G} \circ p_{n,G} \circ \sigma \\ &= s_{n,G} \circ P_n(\sigma) \circ p_{n,P_n(F)} \end{aligned}$$

and

$$P_n(\sigma) \simeq_{\mathsf{Ch}} P_n(\sigma) \circ P_n(p_{n,F}) \circ S_{n,F} \simeq_{\mathsf{Ch}} P_n(\tau) \circ S_{n,F}$$

It follows that

$$\sigma \simeq_{\mathsf{Ch}} s_{n,G} \circ P_n(\tau) \circ S_{n,F} \circ p_{n,P_n(F)}$$

Since this is independent of σ, in particular we have that $\sigma \simeq_{\mathsf{Ch}} \tau^\#$. ■

Note that the universal property in bullet (iii) of Proposition 1.3.9, along with the fact that degrees of a functor are characterized by their minimal value, we obtain a factorization (up to homotopy in general)

$$\begin{tikzcd}
 & & F \arrow[dll] \arrow[dl, "p_{n+1}"'] \arrow[d, "p_n"] \arrow[dr, "p_{n-1}"] \arrow[drrr, "p_0"] & & & \\
\cdots \arrow[r] & P_{n+1}(F) \arrow[r, "q_{n+1}"'] & P_n(F) \arrow[r, "q_n"'] & P_{n-1}(F) \arrow[r] & \cdots \arrow[r, "q_1"'] & P_0(F)
\end{tikzcd}$$

which is known as the **algebraic Taylor tower** of the functor F, where the q_n are determined uniquely by the universal property of the $P_n(F)$. We can also realize the $q_n : P_n(F) \to P_{n-1}(F)$ as being induced by a natural transformation $\rho_n : C_{n+1} \Rightarrow C_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A})$, with components given by

$$C_{n+1}(F)(X) = \Delta^*\mathrm{cr}_{n+1}(\mathrm{cr}_1(F))(X) \rightarrowtail \mathrm{cr}_n(F)(X \oplus X, X, ..., X) \xrightarrow{\mathrm{cr}_n(F)(+, 1_X, ..., 1_X)} \Delta^*\mathrm{cr}_n(\mathrm{cr}_1(F))(X) = C_n(F)(X$$

Proof of construction of q_n. Let $\rho_n : C_{n+1} \Rightarrow C_n$ be the natural transformation described above. Then we have natural transformations $\rho_n([k]) : C_{n+1}^{\times(k+1)} \Rightarrow C_n^{\times(k+1)}$ given by

$$\rho_n([k]) = C_n^{\times k}\rho_n \circ C_n^{\times(k-1)}(\rho_n)_{C_{n+1}} \circ \cdots \circ C_n(\rho_n)_{C_{n+1}^{\times(k-1)}} \circ (\rho_n)_{C_{n+1}^{\times k}}$$

These natural transformations define a natural map of the bicomplexes defining P_n and P_{n-1}. Applying totalization we obtain the natural transformation $q_n : P_n \Rightarrow P_{n-1}$ described. Since the map of bicomplexes defining q_n is the identity on the zeroth column given by the identity, it follows that the desired triangle commutes. ■

Example 1.3.2 () :

Note that for $\deg_0^{\mathcal{A}}$, since $\mathrm{cr}_1(\deg_0^{\mathcal{A}}) = \deg_0^{\mathcal{A}}$ is the identity concentrated in degree zero, by the inductive definition $\mathrm{cr}_n(\deg_0^{\mathcal{A}}) \cong 0$ for $n \geq 2$. Then the bicomplex defining $P_n(\deg_0^{\mathcal{A}})$ has $\deg_0^{\mathcal{A}}$ as the 0th column, with all other columns zero. This implies that $P_n(\deg_0^{\mathcal{A}}) = \deg_0^{\mathcal{A}}$ for $n \geq 1$.
The map $q_n(\deg_0^{\mathcal{A}}) : P_n(\deg_0^{\mathcal{A}}) \to P_{n-1}(\deg_0^{\mathcal{A}})$ for $n \geq 2$ is the identity. For $n = 1$ the ρ_1 defining q_1 has 0th component the identity and kth component the zero map for $k \geq 1$. It follows that $q_1(\deg_0^{\mathcal{A}}) : P_1(\deg_0^{\mathcal{A}}) \to P_0(\deg_0^{\mathcal{A}})$ is the natural inclusion.

1.4.0 Linear Approximations

In this section we define the linear approximation of a functor $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ as a functor $D_1(F)$ in $\mathsf{AbCat}_{\mathsf{Ch}}$. This construction will coincie, up to homotopy, with the homotopy fiber of the map $q_1 : P_1(F) \to P_0(F)$. All of the properties we will describe for $D_1(F)$ are developed only up to natural chain homotopy equivalence.

Definition 1.4.1 The **linearization** of $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is the functor $D_1(F) : \mathcal{B} \to \mathcal{A}$ given as the totalization of the explicit chain complex of chain complexes $(D_1(F)_\bullet, \partial_\bullet)$ where:

$$D_1(F)_k := \begin{cases} C_2^{\times k}(F) & k \geq 1 \\ \mathrm{cr}_1(F) & k = 0 \\ 0 & \text{else} \end{cases}$$

and the differential $\partial_1 : D_1(F)_1 \to D_1(F)_0$ is given by ρ_1, while for $k \geq 2$, $\partial_k : D_1(F)_k \to D_1(F)_{k-1}$ is given by the alternating sum $\sum_{i=0}^{k-1}(-1)^i C_2^{\times i} \epsilon_{C_2^{\times(k-1-i)}}$. Note $\rho_1 = \epsilon$ since $C_2(F) \cong C_2(\mathrm{cr}_1(F))$.

Recall that $P_0(F)$ is naturally chain homotopy equivalent to $F(0)$. Using this model the induced map $q_1 : P_1(F) \to P_0(F) \to F(0)$ is a degree-wise epimorphism, and hence a fibration in the standard model structure on non-negatively graded chain complexes.

Proof of Epimorphism. Note by our explicit construction of q_1, $q_1 \circ p_1 : F \to F(0)$ is exactly the projection for the isomorphism $F \cong F(0) \oplus \mathrm{cr}_1(F)$. But this is epi as a map of chain complexes in each degree, so q_1 must also be epi. ■

This implies that the kernel of q_1 is a model of the homotopy fiber, when this structure makes sense. Note that our definition of $D_1(F)$ coincides with $P_1(\mathrm{cr}_1(F))$ due to Lemma 1.1.8, which is exactly the kernel of $q_1 : P_1(F) \to F(0)$.

To start understanding the linear approximation we consider an example:

Example 1.4.1 () :

We consider the affine functor $F(X) = A \oplus X$. Recall $\mathrm{cr}_2(F) \cong 0$ and $\mathrm{cr}_1(F) \cong \mathrm{Id}$. This implies that $D_1(F)$ is chain homotopy equivalent to Id in degree 0.

We begin by inspecting properties of D_1. First we obtain immediate results since $D_1 \cong P_1 \circ \mathrm{cr}_1$.

Proposition 1.4.2

(i) $D_1 : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$ is exact

(ii) D_1 preserves (natural) chain homotopies, chain homotopy equivalences, and contractibility.

Proof. These results follow from Propositions 1.3.6 and 1.1.18 using the isomorphism $D_1 \cong P_1 \circ \mathrm{cr}_1$. ■

Relaxing some of our previous constraints to the level of natural chain homotopy equivalence, we define what it means for a functor to be linear.

Definition 1.4.3 A functor $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is said to be **linear** if it is degree one and weakly reduced, so $F(0)$ is naturally contractible.

This definition is equivalent to a characterization in terms of finite direct sums. In order to establish this equivalence we must first prove results on how direct sums interact with natural chain homotopy equivalence. These results are a generalization of Lemma 1.1.3.

Lemma 1.4.4 Let $F, G, H : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ be functors and let $f, g : F \Rightarrow G$ be chain homotopic maps. Then $f \oplus 1_H$ is chain homotopic to $g \oplus 1_H$.

Proof. Note that under the isomorphism $\mathrm{Fun}^{\mathsf{Ch}} : \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{A}))$, it is sufficient to show for $F_\bullet, G_\bullet, H_\bullet \in \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$, with $f, g : F_\bullet \Rightarrow G_\bullet$, and let for each $n \in \mathbb{Z}$, $s_n : F_n \Rightarrow G_{n+1}$ denote the component natural transformations for the homotopy. Note that $- \oplus H_\bullet : \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A})) \to \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$ is a functor. We consider $s_n \oplus 0 : F_n \oplus H_n \Rightarrow G_{n+1} \oplus H_{n+1}$. Then

$$\begin{aligned}(\partial^G_{n+1} \oplus \partial^H_{n+1}) \circ (s_n \oplus 0) + (s_{n-1} \oplus 0) \circ (\partial^F_n \oplus \partial^H_n) &= (\partial^G_{n+1} \circ s_n + s_{n-1} \circ \partial^F_n) \oplus 0 \\ &= (f_n - g_n) \oplus (1_{H_n} - 1_{H_n}) \\ &= f_n \oplus 1_{H_n} - g_n \oplus 1_{H_n}\end{aligned}$$

as desired. ■

We also have the converse result.

Lemma 1.4.5 Let $F, G, H : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ be functors and let $f, g : F \Rightarrow G$ be maps such that $f \oplus 1_H$ and $g \oplus 1_H$ are chain homotopic maps. Then f is chain homotopic to g.

Proof. Once again we pass to $F_\bullet, G_\bullet, H_\bullet \in \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$ with $f, g : F_\bullet \Rightarrow G_\bullet$. Let

$$\begin{pmatrix} s_n^{1,1} & s_n^{1,2} \\ s_n^{2,1} & s_n^{2,2} \end{pmatrix} : F_n \oplus H_n \to G_{n+1} \oplus H_{n+1}$$

be the homotopy witnessing $f \oplus 1_H$ homotopic to $g \oplus 1_H$. Then the homotopy condition takes the form

$$\begin{pmatrix} \partial^G_{n+1} s^{1,1}_n + s^{1,1}_{n-1}\partial^G_n & \partial^G_{n+1} s^{1,2}_n + s^{1,2}_{n-1}\partial^H_n \\ \partial^H_{n+1} s^{2,1}_n + s^{2,1}_{n-1}\partial^G_n & \partial^H_{n+1} s^{2,2}_n + s^{2,2}_{n-1}\partial^H_n \end{pmatrix} = \begin{pmatrix} f_n - g_n & 0 \\ 0 & 0 \end{pmatrix}$$

It follows that $s^{1,1}_n : F_n \to G_{n+1}$ forms a homotopy from f to g. ■

As an immediate corollary of Lemma 1.4.4 and Lemma 1.4.5 we have that chain homotopy equivalences are preserved by direct sum and satisfy the cancellative property.

We can now prove the previously asserted equivalence.

Proposition 1.4.6 A functor $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is linear if and only if the natural map

$$F(X) \oplus F(Y) \hookrightarrow F(X \oplus Y)$$

is a natural chain homotopy equivalence.

Proof. First suppose $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is linear. Then we have that $\mathrm{cr}_2(F) \simeq_{\mathsf{Ch}} 0$ and $F(0) \simeq_{\mathsf{Ch}} 0$. Recall from the inductive definition of the cross effect that

$$\mathrm{cr}_1(F)(X \oplus Y) \cong \mathrm{cr}_1(F)(X) \oplus \mathrm{cr}_1(F)(Y) \oplus \mathrm{cr}_2(F)(X, Y)$$

naturally in X and Y. Then by Lemma 1.4.4 we have that

$$\mathrm{cr}_1(F)(X) \oplus \mathrm{cr}_1(F)(Y) \oplus \mathrm{cr}_2(F)(X, Y) \simeq_{\mathsf{Ch}} \mathrm{cr}_1(F)(X) \oplus \mathrm{cr}_1(F)(Y) \oplus 0 \cong \mathrm{cr}_1(F)(X) \oplus \mathrm{cr}_1(F)(Y)$$

so $\mathrm{cr}_1(F)$ preserves direct sums up to natural chain homotopy equivalence. Then since $F(0) \simeq_{\mathsf{Ch}} 0$ and $F \cong \mathrm{cr}_1(F) \oplus F(0)$, it follows that F also preserves direct sums up to natural chain homotopy equivalence.

Conversely, if F preserves direct sums up to natural chain homotopy equivalence we have the identity

$$F(0) \cong F(0 \oplus 0) \simeq_{\mathsf{Ch}} F(0) \oplus F(0)$$

so by Lemma 1.4.5 it follows that $F(0) \simeq_{\mathsf{Ch}} 0$. Similarly, using the inductive formula for $\mathrm{cr}_2(F)$ and Lemma 1.4.5 we have that $\mathrm{cr}_2(F)$ is naturally contractible, so F is degree 1. ■

Note that since D_1 is an exact functor it is linear in the sense that it preserves the zero object and direct sums of functors up to natural isomorphism. Additionally, $D_1(F)$ is also a linear functor for any F.

Lemma 1.4.7 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$. Then $D_1(F) : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ is strictly reduced and linear.

Proof. Recall $D_1(F) \cong P_1(\mathrm{cr}_1(F))$. Then as $\mathrm{cr}_1(F)$ is strictly reduced and each $C_2(\mathrm{cr}_1(F))$ is strictly reduced, it follows that $P_1(\mathrm{cr}_1(F))$ is strictly reduced.

By Proposition 1.3.9 we have that $P_1(\mathrm{cr}_1(F))$ is degree 1, which completes the proof. ■

The next main result we wish to show is that D_1 preserves composition up to naturally chain homotopy, upgrading the original pointwise chain homotopy results in [BJO+18]. The primary work and lemmas for this result are contained in Section 1.7. In particular, we will prove the following result in Section 1.7.

Proposition 1.4.8 If $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ are composible functors in $\mathsf{AbCat}_{\mathsf{Ch}}$ such that G is reduced, Then

$$D_1(F \lhd G) \simeq_{\mathsf{Ch}} D_1(F) \lhd D_1(G)$$

In this section we will extend this result to non-reduced G. First, by Lemma 1.1.8 we have that $\mathrm{cr}_1(\mathrm{cr}_1(F)) \cong \mathrm{cr}_1(F)$, so as $D_1(F) \cong P_1(\mathrm{cr}_1(F))$ and post-composition preserves natural isomorphisms we obtain that $D_1(F) \cong D_1(\mathrm{cr}_1(F))$. Before extending the proposition we first prove some preliminary results on the relationship between the direct sum operation on functors and our composition, $\lhd$, in $\mathsf{AbCat}_{\mathsf{Ch}}$.

Lemma 1.4.9 Let $F, G : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $H : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$. Then there is a natural isomorphism

$$(F \oplus G) \lhd H \cong (F \lhd H) \oplus (G \lhd H)$$

Proof. First, note that since Γ preserves direct sums, $\Gamma_{\mathcal{A}} \circ (F \oplus G) \cong (\Gamma_{\mathcal{A}} \circ F) \oplus (\Gamma_{\mathcal{A}} \circ G)$. Then by definition of the direct sum of functors

$$((\Gamma_{\mathcal{A}} \circ F) \oplus (\Gamma_{\mathcal{A}} \circ G))_* \Gamma_{\mathcal{B}} H = (\Gamma_{\mathcal{A}} \circ F)_* \Gamma_{\mathcal{B}} H \oplus (\Gamma_{\mathcal{A}} \circ G)_* \Gamma_{\mathcal{B}} H$$

Finally, since $\Delta_{\mathcal{A}}$ and $N_{\mathcal{A}}$ also preserve finite limits, we have that

$$(F \oplus G) \lhd H \cong (F \lhd H) \oplus (G \lhd H)$$

as desired. ■

As a simple corollary we find that post-composing by a 0 functor is zero.

Corollary 1.4.10 Let $H : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ and let $0 : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ be a zero functor. Then $0 \lhd H$ is a zero functor.

Proof. Note that by Lemma 1.4.9 we have that $0 \lhd H \cong (0 \lhd H) \oplus (0 \lhd H)$, so from general properties of Abelian categories we have that $0 \lhd H \cong 0$. ■

Lemma 1.4.11 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $H : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$. If F or H are constant functors, then so is $F \lhd H$.

Proof. First, suppose H is a constant functor associated to $B_\bullet \in \mathsf{Ch}(\mathcal{B})$. Then for any $C \in \mathcal{C}_0$ we have that

$$(F \lhd H)(C) = N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H(C) = N_{\mathcal{A}}(\Delta_{\mathcal{A}}((\Gamma_{\mathcal{A}})_* \circ F_* \circ \Gamma_{\mathcal{B}}(B_\bullet)))$$

with maps sent to identities. Similarly, if F is constant determined by $A_\bullet \in \mathsf{Ch}(\mathcal{B})$, then

$$\begin{aligned}(F \lhd H)(C) &= N_{\mathcal{A}}\Delta_{\mathcal{A}}(\Gamma_{\mathcal{A}})_* F_* \Gamma_{\mathcal{B}} H(C) \\ &= N_{\mathcal{A}}(\Delta_{\mathcal{A}}((\Gamma_{\mathcal{A}})_* \circ F_* \circ \Gamma_{\mathcal{B}}(B_\bullet))) \\ &= N_{\mathcal{A}}(\Delta_{\mathcal{A}}((\Gamma_{\mathcal{A}})_* \circ \Delta_{A_\bullet})) \\ &= N_{\mathcal{A}}\Gamma_{\mathcal{A}}(A_\bullet) \\ &\cong A_\bullet\end{aligned}$$

using the Dold-Kan equivalence, where $\Delta_{A_\bullet} : \Delta^{op} \to \mathsf{Ch}(\mathcal{A})$ is the constant simplicial object at $A_\bullet$. ■

Lemma 1.4.12 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $H_1, ..., H_n : \mathcal{C} \to \mathcal{B}$. Then

$$\begin{aligned}\mathrm{cr}_{n-1}(F)(H_1 \oplus H_2, H_3, ..., H_n) \cong \mathrm{cr}_{n-1}(F)(H_1, H_3, ..., H_n) &\oplus \mathrm{cr}_{n-1}(F)(H_2, ..., H_n) \\ &\oplus \mathrm{cr}_n(F)(H_1, ..., H_n)\end{aligned}$$

Proof. Note that we have functors $H_1 \times \cdots \times H_n : \mathcal{C}^n \to \mathcal{B}^n$. Then the desired isomorphism is obtained from the isomorphism in Equation (1.9) by whiskering by $H_1 \times \cdots \times H_n$. ■

Lemma 1.4.13 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $H_1, ..., H_n : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$. Then

$$\begin{aligned}\mathrm{cr}_{n-1}(F) \lhd (H_1 \oplus H_2, H_3, ..., H_n) \cong \mathrm{cr}_{n-1}(F) \lhd (H_1, H_3, ..., H_n) &\oplus \mathrm{cr}_{n-1}(F) \lhd (H_2, ..., H_n) \\ &\oplus \mathrm{cr}_n(F) \lhd (H_1, ..., H_n)\end{aligned}$$

Proof. Since $\Gamma_{\mathcal{B}}$ preserves finite limits and colimits, we have that **TBD**

$$\mathrm{cr}_{n-1}(F)_* \Gamma_{\mathcal{B}^{n-1}}(H_1 \oplus H_2, H_3, ..., H_n) \cong$$

■

In order to extend the proposition we require one additional result.

Lemma 1.4.14 If $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ are composible functors, then

$$\mathrm{cr}_1(F \lhd G)(X) \cong (\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G))(X) \oplus (\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G)(X)))$$

Proof. First, observe that $\mathrm{cr}_1(F \lhd G) \oplus (F \lhd G)(0) \cong F \lhd G$ by construction of the cross-effect. Next, using the isomorphism again but now on F and G individually,

$$F \lhd G \cong F \lhd (G(0) \oplus \mathrm{cr}_1(G)) \cong (F(0) \oplus \mathrm{cr}_1(F)) \lhd (G(0) \oplus \mathrm{cr}_1(G))$$

From our Lemmas 1.4.9 and 1.4.11 we have that

$$(F(0) \oplus \mathrm{cr}_1(F)) \lhd (G(0) \oplus \mathrm{cr}_1(G)) \cong F(0) \oplus (\mathrm{cr}_1(F) \lhd (G(0) \oplus \mathrm{cr}_1(G)))$$

Next applying Lemma 1.4.13 we obtain that

$$\mathrm{cr}_1(F \lhd G) \oplus (F \lhd G)(0) \cong F(0) \oplus (\mathrm{cr}_1(F) \lhd G(0)) \oplus (\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G)) \oplus (\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G)))$$

Expanding $(F \lhd G)(0) \cong F(0) \oplus \mathrm{cr}_1(F) \lhd G(0)$ and using the cancellative property of the direct sum, we obtain the desired isomorphism. ∎

We now extend Proposition 1.4.8.

Proposition 1.4.15 If $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$, then there is a natural chain homotopy equivalence

$$D_1(F \lhd G) \simeq_{\mathsf{Ch}} (D_1(F) \lhd D_1(G)) \oplus D_1((\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G))))$$

Proof. Using Lemma 1.4.14 we have that

$$D_1(F \lhd G) \cong D_1(\mathrm{cr}_1(F \lhd G)) \cong D_1((\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G)) \oplus (\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G))))$$

Using the linearity of D_1 we have that

$$\begin{aligned} &D_1((\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G)) \oplus (\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G)))) \\ &\cong D_1(\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G)) \oplus D_1(\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G))) \end{aligned}$$

Since $\mathrm{cr}_1(G)$ is reduced, we can apply Proposition 1.4.8 and Lemma 1.4.4 we have a natural chain homotopy equivalence

$$\begin{aligned} &D_1(\mathrm{cr}_1(F) \lhd \mathrm{cr}_1(G)) \oplus D_1(\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G))) \\ &\simeq_{\mathsf{Ch}} (D_1(\mathrm{cr}_1(F)) \lhd D_1(\mathrm{cr}_1(G))) \oplus D_1(\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G))) \end{aligned}$$

Applying the isomorphism $D_1 \circ \mathrm{cr}_1 \cong D_1$ we have that

$$D_1(F \lhd G) \cong D_1(\mathrm{cr}_1(F \lhd G)) \simeq_{\mathsf{Ch}} (D_1(F) \lhd D_1(G)) \oplus D_1(\mathrm{cr}_2(F) \lhd (G(0) \oplus \mathrm{cr}_1(G)))$$

as desired. ∎

We now move on to multilinearization of multivariable functors, $F : \mathcal{B}^n \to \mathsf{Ch}(\mathcal{A})$. We introduce the same convention as in [BJO+18, Conv 5.11] where for $X_1, ..., X_{i-1}, X_{i+1}, ..., X_n \in \mathcal{B}$ fixed we define $F_i : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ to be the functor given by

$$F_i(Y) := F(X_1, ..., X_{i-1}, Y, X_{i+1}, ..., X_n)$$

In general we define the following.

Definition 1.4.16 Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ be a set of distinct increasing indices. Let $\pi_{i_1\times\cdots\times i_k} : \mathcal{B}^n \to \mathcal{B}^k$ and let $\iota_{i_1\times\cdots\times i_k} : \mathcal{B}^{n-k} \times \mathrm{Fun}(\mathcal{B}^n, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}^k, \mathcal{A})$ be the functor defined by

$$\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, F)((B_{i_\ell})_{1\leq \ell\leq k}) := F((B_i)_{1\leq i\leq n})$$

with natural action on maps. We define a functor $D_1^{i_1\times\cdots\times i_k} : \mathrm{Fun}(\mathcal{B}^n, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}^n, \mathcal{A})$ by

$$D_1^{i_1\times\cdots\times i_k}(F)((B_i)_{1\leq i\leq n}) := D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, F))((B_{i_\ell})_{1\leq \ell\leq k})$$

where on maps for $f_i : B_i \to B_i'$, we have that

$$D_1^{i_1\times\cdots\times i_k}(F)((f_i)_{1\leq i\leq n}) := D_1(\iota_{i_1\times\cdots\times i_k}((f_j)_{j\neq i_\ell}, F))_{(B'_{i_\ell})_{1\leq \ell\leq k}} \circ D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, F))((f_{i_\ell})_{1\leq \ell\leq k})$$

which is functorial by definition of composition of natural transformations and functoriality of D_1, F, and $\iota_{i_1\times\cdots\times i_k}$. Now, if $\alpha : F \Rightarrow G$ is a natural transformation, we define

$$D_1^{i_1\times\cdots\times i_k}(\alpha)_{(B_i)_{1\leq i\leq n}} := D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, \alpha))_{(B_{i_\ell})_{1\leq \ell\leq k}}$$

Note that we can also apply this multilinearization procedure sequentially by applying D_1^i terms. When we are multilinearing over all variables in this way we write

$$D_1^{(n)}(F) := D_1^n \circ \cdots D_1^1(F)$$

We proceed to prove a number of lemmas on the behaviour of these multilinearization operations.

Lemma 1.4.17 Let $H : \mathcal{B}^n \to \mathsf{Ch}(\mathcal{A})$ be a strictly multi-reduced functor. Then for any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $D_1^{i_1\times\cdots\times i_k}(H)$ is contractible.

Proof. Let $B_1, ..., B_n \in \mathcal{B}$ and let $C_{i_1}, ..., C_{i_k} \in \mathcal{B}$. For $1 \leq i \leq n$ let A_i' be $B_{i_j} \oplus C_{i_j}$ if there exists $1 \leq j \leq k$ such that $i = i_j$, and B_i otherwise. Then we have that

$$D_1^{i_1\times\cdots\times i_k}(H)((A_i')_{1\leq i\leq n}) = D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, H))((B_{i_j} \oplus C_{i,j})_{1\leq j\leq k})$$

But by Lemma 1.4.7 D_1 of a functor is linear, so we have a natural chain homotopy

$$D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, H))((B_{i_j} \oplus C_{i_j})_{1\leq j\leq k}) \simeq_{\mathsf{Ch}} D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i_\ell}, H))((B_{i_j})_{1\leq j\leq k}) \oplus D_1(\iota_{i_1\times\cdots\times i_k}((B_j)_{j\neq i}$$

Note that in the special case where $B_{i_j} = 0$ for $1 < j \leq k$ and $C_{i_1} = 0$, the natural chain homotopy equivalence implies that $D_1^{i_1 \times \cdots \times i_k}(H)$ is contractible since D_1 of a multireduced functor is multireduced by Lemma 1.1.7. ■

An important form of Lemma 1.4.17 is the following.

Corollary 1.4.18 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $H : \mathcal{B}^n \to \mathsf{Ch}(\mathcal{A})$ be functors such that $F = H \circ \Delta$, where H is strictly multi-reduced. Then $D_1(F)$ is contractible.

Proof. By Proposition 1.4.8 we have that $D_1(F)$ is naturally chain homotopy equivalent to $D_1(H) \lhd D_1(\Delta)$ since Δ is strictly reduced. But $D_1(H) = D_1^{1 \times \cdots \times n}(H)$, so by Lemma 1.4.17 $D_1(H)$ is naturally contractible. Further, since Δ is degree 1 we have that Δ is naturally chain homotopy equivalent to $P_1(\Delta) \cong P_1(\mathrm{cr}_1(\Delta)) \cong D_1(\Delta)$ by Proposition 1.3.9.

Since $\lhd$ preserves natural chain homotopy equivalences we have that

$$D_1(H) \lhd D_1(\Delta) \simeq_{\mathsf{Ch}} D_1(H) \lhd \Delta \simeq_{\mathsf{Ch}} 0 \lhd \Delta \cong 0$$

where the last isomorphism is by Corollary 1.4.10. ■

Next we will provide an isomorphism for sequential linearization, analogous to the commutivity of partial differentiation from calculus when dealing with smooth functions.

Lemma 1.4.19 For any $F : \mathcal{B}^n \to \mathsf{Ch}(\mathcal{A})$ and any $1 \leq i, j \leq n$ we have a natural isomorphism

$$D_1^i(D_1^j(F)) \cong D_1^j(D_1^i(F))$$

Proof. **TBD** ■

Next, we show that linearization preserves strictly linear functors up to isomorphism.

Proposition 1.4.20 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ be a functor that is strictly linear. Then $D_1(F) \cong F$.

Proof. Since F is strictly linear $\mathrm{cr}_2(F) \cong 0$ and $\mathrm{cr}_1(F) \cong F$. Thus, the bicomplex defining $D_1(F)$ contains F in the zeroth column and zeros elsewhere, so its totalization is isomorphic to F. ■

As a simple consequence we have that the projection functor $\pi_i : \mathcal{B}^n \to \mathcal{B}$ is its own linearization, which is to say $D_1(\pi_i) \cong \pi_i$.

Corollary 1.4.21 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$. Then $D_1(F \circ \pi_1)(B_1, B_2) \cong D_1(F)(B_1)$ for all $B_1, B_2 \in \mathcal{B}$.

Proof. By the kernel definition of the cross-effect we have that $\mathrm{cr}_1(F \circ \pi_1) \circ \iota_{B_2} \cong \mathrm{cr}_1(F)$ for any $B_2 \in \mathcal{B}$. Similarly, from our kernel description of the second cross-effect $\mathrm{cr}_2(F) \cong \mathrm{cr}_2(F \circ \pi_1) \circ (\iota_{B_2} \times \iota_{B_1})$. Then since these are natural isomorphisms we conclude that $D_1(F \circ \pi_1) \circ \iota_{B_2} \cong D_1(F)$ for all $B_2 \in \mathcal{B}$. ■

Finally, using this result we can obtain a commutativity between functors into products and linearization.

Lemma 1.4.22 Let $F : \mathcal{C} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ and consider $\langle F, G \rangle : \mathcal{C} \to \mathsf{Ch}(\mathcal{A}) \times \mathsf{Ch}(\mathcal{B})$. Under the isomorphism $\mathsf{Ch}(\mathcal{A}) \times \mathsf{Ch}(\mathcal{B}) \cong \mathsf{Ch}(\mathcal{A} \times \mathcal{B})$ we obtain an isomorphisms

$$D_1(\langle F, G \rangle) \cong \langle D_1(F), D_1(G) \rangle$$

Proof. Since limits are computed componentwise in the product category $\mathcal{A} \times \mathcal{B}$ we have that $\mathrm{cr}_n(\langle F, G \rangle) \cong \langle \mathrm{cr}_n(F), \mathrm{cr}_n(G) \rangle$, from which the result follows since the bicomplex defining D_1 is expressed in terms of cross-effects. ■

1.5.0 The first directional derivative

With the linearization results in the previous section we can now introduce directional differentiation for functors. We follow the procedure used to define the directional derivative in [BJO+18], and then proceed to show that we obtain a cartesian differential structure on our homotopy category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ before extending results to include 2-categorical structure.

Definition 1.5.1 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{B})$ and let $B, V \in \mathcal{B}$, where we consider V to act analogously to a direction vector in a Banach space. We define a functor $\nabla F : \mathcal{B} \times \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ by

$$\nabla F(V; X) := D_1(F(X \oplus -))(V)$$

We can also define the directional derivative of $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ using the limit formulation of the directional derivative. These two constructions are equivalent.

Lemma 1.5.2 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$. Then there is an isomorphism

$$D_1^V(\ker(F(X \oplus V) \xrightarrow{F(\pi_X)} F(X))) \cong D_1(F(X \oplus -))(V)$$

as well as an isomorphism

$$D_1^V(\ker(F(X \oplus V) \xrightarrow{F(\pi_X)} F(X))) \cong D_1(F)(V) \oplus D_1^V(\mathrm{cr}_2(F))(X, V)$$

Proof. First, note that the kernel described in the Lemma is exactly our definition of the first cross-effect of $F(X \oplus -)$, so we obtain an isomorphism

$$D_1^V(\ker(F(X \oplus V) \xrightarrow{\pi_X} F(X))) \cong D_1^V(\mathrm{cr}_1(F(X \oplus -)))(V)$$

An alternate perspective on $\ker(F(X \oplus V) \xrightarrow{F(\pi_X)} F(X))$ uses the fact that $F(X) \xrightarrow{F(\iota_X)} F(X \oplus V)$ is a section for the projection. Now, from the isomorphisms

$$\begin{aligned} F(X \oplus V) &\cong F(0) \oplus \mathrm{cr}_1(F)(X \oplus V) \\ &\cong F(0) \oplus \mathrm{cr}_1(F)(X) \oplus \mathrm{cr}_1(F)(V) \oplus \mathrm{cr}_2(F)(X, V) \\ &\cong F(X) \oplus \mathrm{cr}_1(F)(V) \oplus \mathrm{cr}_2(F)(X, V) \end{aligned}$$

it follows that $\ker(F(X \oplus V) \xrightarrow{F} (X))$ is isomorphic to $\mathrm{cr}_1(F)(V) \oplus \mathrm{cr}_2(F)(X, V)$. Then, since D_1 is strictly linear with respect to functors by Proposition 1.4.2, we obtain an isomorphism

$$\begin{aligned} D_1^V(\ker(F(X \oplus V) \xrightarrow{\pi_X} F(X))) &\cong D_1^V(\mathrm{cr}_1(F)(V) \oplus \mathrm{cr}_2(F)(X, V)) \\ &\cong D_1^V(\mathrm{cr}_1(F))(V) \oplus D_1^V(\mathrm{cr}_2(F))(X, V) \\ &\cong D_1^V(F)(V) \oplus D_1^V(\mathrm{cr}_2(F))(X, V) \end{aligned}$$

as desired. ■

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Note that since D_1 preserves natural chain homotopy equivalences, so does ∇, so for $F \simeq_{\mathsf{Ch}} G$, $\nabla F \simeq_{\mathsf{Ch}} \nabla G$.

We now will state the theorem that with this operation $\mathsf{HoAbCat}_{\mathsf{Ch}}$ becomes a cartesian differential category before proving each piece of the theorem through a sequence of lemmas.

Theorem 1.5.3 The category $\mathsf{AbCat}_{\mathsf{Ch}}$ together with ∇ satosfies the following properties for functors $F, G : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$, $H : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$, and $K : \mathcal{B} \to \mathsf{Ch}(\mathcal{D})$:

(i) ∇ is linear in the sense that

$$\nabla(F \oplus G) \cong \nabla(F) \oplus \nabla(G)$$

(ii) ∇F is linear in the direction variable, which is to say ∇F is strictly reduced and degree 1, so

$$\nabla F(V \oplus W; X) \simeq_{\mathsf{Ch}} \nabla F(V; X) \oplus \nabla F(W; X)$$

and $\nabla F(0; X) \cong 0$.

(iii) The directional derivative of the degree zero functor $\deg_0^{\mathcal{A}} : \mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ is the projection onto the direction, which is to say

$$\nabla \deg_0^{\mathcal{A}}(V; X) \cong V$$

(iv) We have an isomorphism

$$\nabla\langle F, K\rangle(V; X) \cong \langle \nabla F(V; X), \nabla G(V; X)\rangle$$

(v) There is a natural chain homotopy equivalence

$$\nabla(F \lhd H)(V; X) \simeq_{\mathsf{Ch}} \nabla F \lhd (\nabla G(V; X); G(X))$$

(vi) There is an isomorphism

$$\nabla(\nabla F)((Z; 0); (0; X)) \cong \nabla F(Z; X)$$

(vii) There is a natural chain homotopy equivalence

$$\nabla(\nabla F)((Z; W); (V; X)) \simeq_{\mathsf{Ch}} \nabla(\nabla F)((Z; V); (W; X))$$

As a result of Theorem 1.5.3 we conclude that the homotopy category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ is a cartesian differential category. We now proceed to the proof of Theorem 1.5.3 in segments.

1.6.0 A General Bicomplex Retraction

In order to construct certain explicit chain homotopy equivalences in the text we require a criteria for when total complexes of certain bicomplexes are chain homotopy equivalent. Throughout we consider $A_{\bullet,\bullet}$ to be denote a first-quadrant bicomplex. This is sufficient for our case since all our bicomplexes are constructed from chain complexes concentrated in non-negative degrees. As in [BJO$^+$18] we proceed with bicomplexes having anti-commuting squares. To apply this to the work elsewhere all that must be done is the replacement of $d_h : A_{p,q} \to A_{p,q-1}$ by $(-1)^p d_h$.

Definition 1.6.1 We say a morphism $\iota : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$ **admits a row-wise strong deformation retraction** if for all $p \geq 0$ there exists a map $f_{p,\bullet} : B_{p,\bullet} \to A_{p,\bullet}$ such that

(i) $f_{p,\bullet} \circ \iota_{p,\bullet} = 1_{A_{p,\bullet}}$

(ii) there exist morphisms $s_h : B_{p,q} \to B_{p,q+1}$ such that $d_h s + s d_h = 1 - \iota_{p,q} f_{p,q}$ and $s_h \circ \iota_{p,q} = 0$ (i.e. we have a strong chain homotopy equivalence between $A_{p,\bullet}$ and $B_{p,\bullet}$)

Throughout this section we will denote the horizontal differentials of a bicomplex by $d_h : A_{p,q} \to A_{p,q-1}$ and the vertical differentials by $d_v : B_{p,q} \to B_{p-1,q}$. Although our maps in definition 1.6.1 are given only for $p, q \geq 0$, they can easily be extended to all p, q by setting ones with negative indices equal to zero. We record some commutativity equalities for use in the proofs to follow

$$d_h^2 = 0 \;\; d_v^2 = 0 \;\; d_h d_v + d_v d_h = 0 \;\; d_h s_h + s_h d_h = 1 - \iota \circ f \;\; s_h \circ \iota = 0$$
$$f \circ \iota = 1 \;\; f \circ d_h = d_h \circ f \;\; \iota \circ d_h = d_h \circ \iota \;\; \iota \circ d_v = d_v \circ \iota$$

We begin with the following lemma. (**Note:** Juxtaposition in the following lemma still denotes functional compositional ordering for the sake of preserving space).

Lemma 1.6.2 For any $k \geq 0$ we have the following equalities:

(i) $d_v f(-d_v s_h)^k + d_h f(-d_v s_h)^{k+1} = f(-d_v s_h)^k d_v + f(-d_v s_h)^{k+1} d_h$

(ii) $d_v s_h(-d_v s_h)^k + d_h s_h(-d_v s_h)^{k+1} = -\iota f(-d_v s_h)^{k+1} - s_h d_h(-d_v s_h)^{k+1}$

(iii) $s_h(-d_v s_h)^{k+1} d_h + s_h(-d_v s_h)^k d_v = -(-s_h d_v)^{k+1} d_h s_h$

(iv) $s_h d_h(-d_v s_h)^{k+1} = -(-s_h d_v)^{k+1} d_h s_h$

Proof. We will prove each formula by induction.

(i) If $k = 0$ we want to show

$$d_v f + d_h f(-d_v s_h) = f d_v + f(-d_v s_h) d_h$$

Using our relations

$$\begin{aligned} d_v f + d_h f(-d_v s_h) &= d_v f - f d_h d_v s_h \\ &= d_v f + f d_v d_h s_h \\ &= d_v f + f d_v (1 - \iota f - s_h d_h) \\ &= d_v f + f d_v - f d_v \iota f - f d_v s_h d_h \\ &= d_v f + f d_v - d_v f + f(-d_v s_h) d_h \\ &= f d_v + f(-d_v s_h) d_h \end{aligned}$$

as desired. Suppose now that the claim holds for some $k \geq 0$. Then

$$\begin{aligned} d_v f(-d_v s_h)^{k+1} + d_h f(-d_v s_h)^{k+2} &= [f(-d_v s_h)^k d_v + f(-d_v s_h)^{k+1} d_h](-d_v s_h) \\ &= f(-d_v s_h)^k d_v(-d_v s_h) + f(-d_v s_h)^{k+1} d_h(-d_v s_h) \\ &= f(-d_v s_h)^{k+1} d_v(1 - s_h d_h - \iota f) \\ &= f(-d_v s_h)^{k+1} d_v - f(-d_v s_h)^{k+1} d_v s_h d_h - f(-d_v s_h)^{k+1} d_v \iota f \\ &= f(-d_v s_h)^{k+1} d_v - f(-d_v s_h)^{k+2} d_h - f(-d_v s_h)^{k+1} \iota d_v f \\ &= f(-d_v s_h)^{k+1} d_v - f(-d_v s_h)^{k+2} d_h \end{aligned}$$

as desired.

(ii) We can immediately compute

$$\begin{aligned} d_v s_h(-d_v s_h)^k + d_h s_h(-d_v s_h)^{k+1} &= [d_v s_h + d_h s_h(-d_v s_h)](-d_v s_h)^k \\ &= [d_v s_h + (1 - s_h d_h - \iota f)(-d_v s_h)](-d_v s_h)^k \\ &= [-\iota f(-d_v s_h) - s_h d_h(-d_v s_h)](-d_v s_h)^k \\ &= -\iota f(-d_v s_h)^{k+1} - s_h d_h(-d_v s_h)^{k+1} \end{aligned}$$

as desired.

(iii) If $k = 0$ we compute

$$\begin{aligned} s_h(-d_v s_h) d_h + s_h d_v &= -s_h d_v(1 - d_h s_h - \iota f) + s_h d_v \\ &= s_h d_v d_h s_h + s_h d_v \iota f \\ &= -(-s_h d_v) d_h s_h + s_h \iota d_v f \\ &= -(-s_h d_v) d_h s_h \end{aligned}$$

Now if the claim holds for $k \geq 0$ we can compute

$$\begin{aligned} -(-s_h d_v)^{k+2} d_h s_h &= (-s_h d_v)[s_h(-d_v s_h)^{k+1} d_h + s_h(-d_v s_h)^k d_v] \\ &= s_h(-d_v s_h)^{k+2} d_h + s_h(-d_v s_h)^{k+1} d_v \end{aligned}$$

as desired.

(iv) If $k = 0$ we observe that

$$s_h d_h(-d_v s_h) = s_h d_v d_h s_h = -(-s_h d_v) d_h s_h$$

If the claim holds for some $k \geq 0$, then we can compute

$$\begin{aligned} s_h d_h(-d_v s_h)^{k+2} &= -(-s_h d_v)^{k+1} d_h s_h(-d_v s_h) \\ &= (-s_h d_v)^{k+1}(1 - s_h d_h - \iota f) d_v s_h \\ &= (-s_h d_v)^{k+1} d_v s_h - (-s_h d_v)^{k+1} s_h d_h d_v s_h - (-s_h d_v)^{k+1} \iota f d_v s_h \\ &= -(-s_h d_v)^{k+2} d_h s_h \end{aligned}$$

as desired since $d_v^2 = 0$, $d_v \iota = \iota d_v$, and $s_h \iota = 0$.

■

We recall that for first-quadrant bicomplexes the totalization at each degree is a finite direct sum, so its differentials can be described by finite matrices. Explicitly the nth differential $\mathrm{Tot}(A_{\bullet,\bullet})_n \to \mathrm{Tot}(A_{\bullet,\bullet})_{n-1}$ is given by the matrix

$$\begin{pmatrix} d_v & d_h & 0 & \cdots & \cdots & 0 \\ 0 & d_v & d_h & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_v & d_h & 0 \\ 0 & \cdots & \cdots & 0 & d_v & d_h \end{pmatrix}$$

Proposition 1.6.3 Let $\iota : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$ be a map of bicomplexes that admits a row-wise strong deformation retraction. Then the induced morphism of total complexes $\mathrm{Tot}(\iota) : \mathrm{Tot}(A_{\bullet,\bullet})_\bullet \to \mathrm{Tot}(B_{\bullet,\bullet})_\bullet$ admits a retraction $\rho : \mathrm{Tot}(B_{\bullet,\bullet})_\bullet \to \mathrm{Tot}(A_{\bullet,\bullet})_\bullet$ defined in degree n by the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} f & 0 & \cdots & \cdots & 0 & 0 & \\ f(-d_v s_h) & f & 0 & \ddots & \ddots & 0 & \\ f(-d_v s_h)^2 & f(-d_v s_h) & f & 0 & \ddots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ f(-d_v s_h)^n & f(-d_v s_h)^{n-1} & \cdots & \cdots & f(-d_v s_h) & f & \end{pmatrix}$$

Proof. First, to see that ρ is a chain map fix $n \geq 1$. Then $\partial_n \rho_n$ and $\rho_{n-1}\partial_n$ are $n \times (n+1)$ matrices with i, j component given by

$$(\partial_n \rho_n)_{i,j} = \begin{cases} 0 & i+1 < j \\ d_h f & i+1 = j \\ d_h f(-d_v s_h) + d_v f & i = j \\ d_h f(-d_v s_h)^{i-j+1} + d_v f(-d_v s_h)^{i-j} & i > j \end{cases}$$

while

$$(\rho_{n-1}\partial_n)_{i,j} = \begin{cases} 0 & i+1<j \\ fd_h & i+1=j \\ fd_v & i=j=1 \\ f(-d_vs_h)d_h + fd_v & i=j\neq 1 \\ f(-d_vs_h)^{i-j+1}d_v + f(-d_vs_h)^{i-j}d_h & i>j \end{cases}$$

We have equality for $i > j$ by equation (i) of Lemma 1.6.2, equality for $i+1 < j$ vacuously, equality for $i+1 = j$ since f is a chain map with respect to the horizontal differentials, equality for $i = j \neq 1$ is also from equation (i) of Lemma 1.6.2, and $i = j = 1$ is equation (i) and the fact that $A_{n+1,-1} = 0$ as the bicomplex is concentrated in non-negative degree.

To show ρ is a retraction it remains to show $\rho_n\iota_n = 1$. Observe that for i, j,

$$(\rho_n\iota_n)_{i,j} = \begin{cases} 0 & i>j \\ f\circ\iota & i=j \\ f(-d_vs_h)^{j-i}\circ\iota & i<j \end{cases}$$

But since f is part of a row-wise strong deformation retraction, $f \circ \iota = 1$, and $s_h \circ \iota = 0$, so the matrix is the identity. ■

It remains to show that ρ is associated with a deformation retraction for ι. In other words, we want to show that $\iota \circ \rho$ is chain homotopic to the identity.

Proposition 1.6.4 The composite map $\iota\circ\rho : \mathrm{Tot}(B)_\bullet \to \mathrm{Tot}(B)_\bullet$ is chain homotopic to the identity via the chain homotopy $\sigma : \mathrm{Tot}(B)_\bullet \to \mathrm{Tot}(B)_{\bullet+1}$ defined in degree n by the $n+2 \times n+1$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 \\ s_h & 0 & \ddots & \ddots & \ddots & 0 \\ s_h(-d_vs_h) & s_h & 0 & \ddots & \ddots & \vdots \\ s_h(-d_vs_h)^2 & s_h(-d_vs_h) & s_h & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ s_h(-d_vs_h)^{n-1} & \cdots & \cdots & s_h(-d_vs_h) & s_h & 0 \\ s_h(-d_vs_h)^n & s_h(-d_vs_h)^{n-1} & \cdots & \cdots & s_h(-d_vs_h) & s_h \end{pmatrix}$$

Proof. Let $\partial_n : \mathrm{Tot}(B)_n \to \mathrm{Tot}(B)_{n-1}$ be the total complex differential. Observe that that

$1 - \iota \circ \rho$ is the matrix

$$\begin{pmatrix} 1-\iota f & 0 & \cdots & \cdots & 0 \\ -\iota f(-d_v s_h) & 1-\iota f & 0 & \cdots & 0 \\ -\iota f(-d_v s_h)^2 & -\iota f(-d_v s_h) & 1-\iota f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\iota f(-d_v s_h)^n & \cdots & -\iota f(-d_v s_h)^2 & -\iota f(-d_v s_h) & 1-\iota f \end{pmatrix}$$

On the other hand, we can compute for $1 \leq i, j \leq n+1$

$$(\partial_{n+1}\sigma_n)_{i,j} = \begin{cases} 0 & i < j \\ d_h s_h & i = j \\ d_v s_h(-d_v s_h)^{i-j-1} + d_h s_h(-d_v s_h)^{i-j} & i > j \end{cases}$$

and

$$(\sigma_{n-1}\partial_n)_{i,j} = \begin{cases} 0 & i < j \\ s_h d_h & i = j \\ s_h(-d_v s_h)^{i-j} d_h + s_h(-d_v s_h)^{i-j-1} d_v & i > j \end{cases}$$

Adding these together we observe that the case of $i = j$ gives equality since $d_h s_h + s_h d_h = 1 - \iota f$. On the other hand, for $i > j$ we can use the relations in Lemma 1.6.2

$$\begin{aligned} d_v s_h(-d_v s_h)^{i-j-1} + d_h s_h(-d_v s_h)^{i-j} &+ s_h(-d_v s_h)^{i-j} d_h + s_h(-d_v s_h)^{i-j-1} d_v \\ &= -\iota f(-d_v s_h)^{i-j} - s_h d_h(-d_v s_h)^{i-j} \\ &+ s_h(-d_v s_h)^{i-j} d_h + s_h(-d_v s_h)^{i-j-1} d_v \qquad \text{(by (ii))} \\ &= -\iota f(-d_v s_h)^{i-j} - s_h d_h(-d_v s_h)^{i-j} - (-s_h d_v)^{i-j} d_h s_h \\ &\qquad \text{(by (iii))} \\ &= -\iota f(-d_v s_h)^{i-j} - s_h d_h(-d_v s_h)^{i-j} + s_h d_h(-d_v s_h)^{i-j} \\ &\qquad \text{(by (iv))} \\ &= -\iota f(-d_v s_h)^{i-j} \end{aligned}$$

which is precisely the i, j entry of $1 - \iota\rho$, completing the proof. ∎

Together these lemmas prove the following result:

Theorem 1.6.5 Let $\iota : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$ be a morphism of first-quadrant bicomplexes that admit a row-wise strong deformation retraction. Then ι induces a chain homotopy equivalence of total complexes $\mathrm{Tot}(A_{\bullet,\bullet}) \to \mathrm{Tot}(B_{\bullet,\bullet})$.

As a quick corollary we obtain the following sufficient condition for a chain homotopy equivalence between the degree 0 inclusion of a chain complex into a first quadrant bicomplex with contractible rows and its totalization.

Corollary 1.6.6 Let $A_{\bullet,\bullet}$ be a first-quadrant bicomplex so that every row except the zeroth row $A_{0,\bullet}$ is contractible. Then the natural inclusion $A_{0,\bullet} \hookrightarrow \mathrm{Tot}(A)_\bullet$ is a chain homotopy equivalence.

Proof. Let $\iota : \deg_0(A_{0,\bullet}) \to A_{\bullet,\bullet}$ denote the inclusion in degree 0. Let $f_{p,\bullet} : A_{p,\bullet} \to \deg_0(A_{0,\bullet})_{p,\bullet}$ denote the chain complex map for the contraction for $p > 0$ (which is the zero map), the identity for $p = 0$, and zero for $p < 0$. Let $s_h : A_{p,q} \to A_{p,q+1}$ denote the contraction for $p \geq 1$, and 0 for $p \leq 0$, so $d_h s_h + s_h d_h = 1 - \iota f$ and $s_h \iota = 0$ since ι is zero for $p > 0$ and s_h is zero for $p \leq 0$. By Theorem 1.6.5 there exists a chain homotopy equivalence $\iota : A_{0,\bullet} \hookrightarrow \mathrm{Tot}(A)_\bullet$ where the retraction $\rho : \mathrm{Tot}(A)_\bullet \to A_{0,\bullet}$ is the $1 \times (n+1)$ is given By

$$\begin{pmatrix} f(-d_v s_h)^n & f(-d_v s_h)^{n-1} & \cdots & f(-d_v s_h) & f \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} (-d_v s_h)^n & (-d_v s_h)^{n-1} & \cdots & (-d_v s_h) & 1 \end{pmatrix}$$

since f is the identity on $A_{0,\bullet}$. ■

1.7.0 Quasi-isomorphism for Composition

In this section we will prove Proposition 1.4.8 using the work in Section 1.6. To this goal let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ be composable functors with G reduced. We first observe the following lemma which will allow us to reduce to the case that F is also reduced.

Lemma 1.7.1 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ be composable functors with G reduced. Then $\mathrm{cr}_1(F \lhd G) \cong \mathrm{cr}_1(F) \lhd G$.

Proof. Observe that using our construction of the cross-effect we have isomorphisms

$$F \lhd G \cong (F \lhd G)(0) \oplus \mathrm{cr}_1(F \lhd G)$$

and by Lemma 1.4.9 and Lemma 1.4.11

$$F \lhd G \cong (\mathrm{cr}_1(F) \oplus F(0)) \lhd G \cong (\mathrm{cr}_1(F) \lhd G) \oplus (F(0) \lhd G) \cong (\mathrm{cr}_1(F) \lhd G) \oplus F(0)$$

Further, $(F \lhd G)(0) \cong F(0)$ since G is reduced and $\Gamma_{\mathcal{B}}$ preserves zero objects. Thus taking direct sum complements we obtain the desired isomorphism. ■

We can use this lemma to begin reducing our goal.

Corollary 1.7.2 Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ and $G : \mathcal{C} \to \mathsf{Ch}(\mathcal{B})$ be composable functors with G reduced. Then

$$D_1(F \lhd G) \cong D_1(\mathrm{cr}_1(F) \lhd G)$$

Proof. Recall that $D_1 \cong D_1 \circ \mathrm{cr}_1$ since the first cross-effect is idempotent. Then by Lemma 1.7.1

$$D_1(F \lhd G) \cong D_1(\mathrm{cr}_1(F \lhd G)) \cong D_1(\mathrm{cr}_1(F) \lhd G)$$

as desired. ■

Note that since $D_1(F) \lhd D_1(G) \cong D_1(\mathrm{cr}_1(F)) \lhd D_1(G)$, Corollary 1.7.2 implies that it is sufficient to prove Proposition 1.4.8 when both functors are reduced. Note that Lemma 1.A.17 restricts to a functor $\mathsf{Ch} : \mathrm{Fun}_*(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}_*(\mathsf{Ch}(\mathcal{A}), \mathsf{Ch}(\mathcal{C}))$.

<u>Remark:</u>

Let $F : \mathcal{A} \to \mathsf{Ch}(\mathcal{B})$ be a strictly reduced functor. We define a comparison map $\mathrm{sw}_F : \Gamma_{\mathsf{Ch}(\mathcal{B})} \circ \mathsf{Ch}(F) \to F_* \circ \Gamma_{\mathcal{A}}$ at $A_\bullet \in \mathsf{Ch}(\mathcal{A})$ and $n \in \mathbb{N}$,

$$\mathrm{sw}_{F,A_\bullet,n} : \bigoplus_{[n] \twoheadrightarrow [k]} F(A_k) \to F\left(\bigoplus_{[n] \twoheadrightarrow [k]} A_k\right)$$

given by the universal coproduct property of the biproduct. This is evidently natural in n, $A_\bullet$, and F. Then we have that $(\Gamma_{\mathcal{B}})_*\mathrm{sw}_F$ is given by

$$(\Gamma_{\mathcal{B}})_*\mathrm{sw}_{F,A_\bullet,n} : \Gamma_{\mathcal{B}}\left(\bigoplus_{[n]\twoheadrightarrow[k]} F(A_k)\right) \to \Gamma_{\mathcal{B}} \circ F\left(\bigoplus_{[n]\twoheadrightarrow[k]} A_k\right)$$

which at m is

$$((\Gamma_{\mathcal{B}})_*\mathrm{sw}_{F,A_\bullet,n})_m : \bigoplus_{[m]\twoheadrightarrow[\ell]}\left(\bigoplus_{[n]\twoheadrightarrow[k]} F(A_k)_\ell\right) \to \bigoplus_{[m]\twoheadrightarrow[\ell]} F\left(\bigoplus_{[n]\twoheadrightarrow[k]} A_k\right)_\ell$$

Taking the diagonal we obtain at $A_\bullet$ and n the map

$$\bigoplus_{[n]\twoheadrightarrow[\ell]}\left(\bigoplus_{[n]\twoheadrightarrow[k]} F(A_k)_\ell\right) \to \bigoplus_{[n]\twoheadrightarrow[\ell]} F\left(\bigoplus_{[n]\twoheadrightarrow[k]} A_k\right)_\ell$$

We now show some chain homotopy equivalence results for linear functors. First we investigate how linear functors act on sums of maps.

Remark:

Let $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ be a linear functor, so by Proposition 1.4.6 we have a natural chain homotopy equivalence

$$F \oplus F \simeq_{\mathsf{Ch}} F(- \oplus -)$$

induced by the natural inclusion $F \oplus F \hookrightarrow F(- \oplus -)$. Now consider maps $f, g : B \to B'$ in $\mathcal{B}$. Then the sum $f + g$ can be represented as the composite

$$B \xrightarrow{\langle 1_B, 1_B \rangle} B \oplus B \xrightarrow{f \oplus g} B' \oplus B' \xrightarrow{\langle 1_{B'} | 1_{B'} \rangle} B'$$

Now we can apply F to this **TBC**

Lemma 1.7.3 Let $F : \mathcal{A} \to \mathsf{Ch}(\mathcal{B})$ be linear. Then $F_*\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathsf{Ch}(\mathcal{B})}\mathsf{Ch}(F)$ are naturally chain homotopy equivalent.

Proof. Note that by assumption F is linear and strictly reduced. Then by Proposition 1.4.6, for each $A_\bullet \in \mathsf{Ch}(\mathcal{A})$ and each n we have a natural chain homotopy equivalence

$$F\left(\bigoplus_{[n]\twoheadrightarrow[k]} A_k\right) \simeq_{\mathsf{Ch}} \bigoplus_{[n]\twoheadrightarrow[k]} F(A_k)$$

which is natural in the A_k, and hence in $A_\bullet$. We want to enhance these to a natural chain homotopy $F_*\Gamma_{\mathcal{A}} \simeq_{\mathsf{Ch}} \Gamma_{\mathsf{Ch}(\mathcal{B})}\mathsf{Ch}(F)$. To this end let $\alpha_n, \beta_n, s^n, r^n$ denote such a natural chain homotopy for each n. ■

1.A.0 Appendices

1.A.1 Simplicial Object 2-Monad

In this section we attempt to construct a (pseudo)monad on $\mathsf{2Ab}$ corresponding to simplicial objects. The goal is that this (pseudo)monad is easier to construct than the chain complex pseudomonad, and that via conjugation by the Dold-Kan equivalence, we can obtain the chain complex pseudomonad, at least up to a suitable equivalence.

We define $(-)^{\Delta^{op}} : \mathsf{2Ab} \to \mathsf{2Ab}$ as a pseudofunctor as follows:

1. On 0-cells, $(-)^{\Delta^{op}}$ sends an abelian category $\mathcal{A}$ to its category of simplicial objects $\mathcal{A}^{\Delta^{op}}$
2. Given abelian categories $\mathcal{A}, \mathcal{B}$, we have a functor $(-)^{\Delta^{op}} : [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}^{\Delta^{op}}, \mathcal{B}^{\Delta^{op}}]$ given as follows:
   (a) A functor $F : \mathcal{A} \to \mathcal{B}$ is sent to its push-forward $F_* : \mathcal{A}^{\Delta^{op}} \to \mathcal{B}^{\Delta^{op}}$ defined by post-composition
   (b) A natural transformation $\gamma : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ is sent to a natural transformation $\gamma^{\Delta^{op}} : F_* \Rightarrow G_*$ such that for $X \in \mathcal{A}_0^{\Delta^{op}}$,
   $$\gamma_X^{\Delta^{op}} : F \circ X \Rightarrow G \circ X := \gamma_X$$
3. We observe $m(F, G) := 1_{(G \circ F)_*} : G_* \circ F_* \Rightarrow (G \circ F)_*$ is our comparison 2-cell
4. For each abelian category $\mathcal{A}$, an invertible 2-cell $i := 1_{1_{\mathcal{A}^{\Delta^{op}}}} : 1_{\mathcal{A}^{\Delta^{op}}} \Rightarrow (1_{\mathcal{A}})_*$ which is an identity.

The psuedofunctor comes with the following monad data:

1. A pseudonatural transformation $\eta : 1_{\mathsf{2Ab}} \Rightarrow (-)^{\Delta^{op}}$ given by the following data:
   (a) For each abelian category $\mathcal{A}$, a functor $\eta_{\mathcal{A}} : \mathcal{A} \Rightarrow \mathcal{A}^{\Delta^{op}}$ given by the diagonal functor, sending an object A to the constant functor for A with identities on arrows.
   (b) For each functor $F : \mathcal{A} \to \mathcal{B}$ a natural transformation $\eta_F : \eta_{\mathcal{B}} \circ F \Rightarrow F_* \circ \eta_{\mathcal{A}}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\eta_{\mathcal{B}}"] \arrow[dl, phantom, "\swarrow \eta_F"] \\ \mathcal{A}^{\Delta^{op}} \arrow[r, "F_*"'] & \mathcal{B}^{\Delta^{op}} \end{tikzcd}$$

   which is the identity, since the square commutes

2. A pseudonatural transformation $m : (-)^{\Delta^{op}} \circ (-)^{\Delta^{op}} \Rightarrow (-)^{\Delta^{op}}$ given by the following data:

   (a) For every abelian category $\mathcal{A}$, a functor $m_{\mathcal{A}} : (\mathcal{A}^{\Delta^{op}})^{\Delta^{op}} \to \mathcal{A}^{\Delta^{op}}$. For $A \in (\mathcal{A}^{\Delta^{op}})^{\Delta^{op}}$

   $$m_{\mathcal{A}}(A)([n]) := A([n])([n])$$

   and for $\alpha : [n] \to [m]$ we set

   $$m_{\mathcal{A}}(A)(\alpha) : A([m])([m]) \to A([n])([n]) := A([n])(\alpha) \circ A(\alpha)_{[m]} = A(\alpha)_{[n]} \circ A([m])(\alpha)$$

   by naturality of $A(\alpha)$. Given a map of simplicial objects $\beta : A \Rightarrow B$ in $(\mathcal{A}^{\Delta^{op}})^{\Delta^{op}}$, we set

   $$m_{\mathcal{A}}(\beta) : m_{\mathcal{A}}(A) \to m_{\mathcal{A}}(B)$$

   with $[n]$th component given by

   $$m_{\mathcal{A}}(\beta)_{[n]} : A([n])([n]) \to B([n])([n]) := (\beta_{[n]})_{[n]}$$

   (b) For each functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, a natural transformation $m_F : m_{\mathcal{B}} \circ (F_*)_* \Rightarrow F_* \circ m_{\mathcal{A}}$

   $$\begin{tikzcd}
(\mathcal{A}^{\Delta^{op}})^{\Delta^{op}} \arrow[r, "(F_*)_*"] \arrow[d, "m_{\mathcal{A}}"'] & (\mathcal{B}^{\Delta^{op}})^{\Delta^{op}} \arrow[d, "m_{\mathcal{B}}"] \arrow[dl, phantom, "\Swarrow m_F"] \\
\mathcal{A}^{\Delta^{op}} \arrow[r, "F_*"'] & \mathcal{B}^{\Delta^{op}}
\end{tikzcd}$$

   which is the identity since the square commutes. Indeed, for each $A \in (\mathcal{A}^{\Delta^{op}})^{\Delta^{op}}$, and each $[n] \in \mathsf{Ob}(\Delta)$

   $$m_{\mathcal{B}} \circ (F_*)_*(A)([n]) = m_{\mathcal{B}}(F_* \circ A)([n]) = F(A([n])([n])) = F_* \circ m_{\mathcal{A}}(A)([n])$$

   while for $\alpha : [m] \to [n]$

   $$\begin{aligned} m_{\mathcal{B}} \circ (F_*)_*(A)(\alpha) &= m_{\mathcal{B}}(F_* \circ A)(\alpha) \\ &= F(A([n])(\alpha) \circ A(\alpha)_{[m]}) \\ &= F_* \circ m_{\mathcal{A}}(A)(\alpha) \end{aligned}$$

   Further, for $\alpha : A \to A'$ in $(\mathcal{A}^{\Delta^{op}})^{\Delta^{op}}$, and $[n] \in \mathsf{Ob}(\Delta)$,

   $$\begin{aligned} m_{\mathcal{B}} \circ (F_*)_*(\alpha)_{[n]} &= m_{\mathcal{B}}(F_*\alpha)_{[n]} \\ &= F(\alpha_{[n]})_{[n]}) \\ &= F(m_{\mathcal{A}}(\alpha)_{[n]}) \\ &= F_* \circ m_{\mathcal{A}}(\alpha)_{[n]} \end{aligned}$$

   Thus the functors along each edge are equal, so the comparison cell is the identity.

3. An invertible modification $\mu : m \circ (-)^{\Delta^{op}} m \Rrightarrow m \circ m_{(-)^{\Delta^{op}}}$ given by the following data:

(a) For each abelian category $\mathcal{A}$, a natural transformation $\mu_{\mathcal{A}} : m_{\mathcal{A}} \circ (-)^{\Delta^{op}} m_{\mathcal{A}} \Rightarrow m_{\mathcal{A}} \circ m_{\mathcal{A}^{\Delta^{op}}}$ which has identity components since for a simplicial object $A \in ((\mathcal{A}^{\Delta^{op}})^{\Delta^{op}})^{\Delta^{op}}$

$$\begin{aligned} m_{\mathcal{A}}((m_{\mathcal{A}})_* A)([n]) &= (m_{\mathcal{A}} \circ A)([n])([n]) \\ &= m_{\mathcal{A}}(A([n]))([n]) \\ &= A([n])([n])([n]) \\ &= m_{\mathcal{A}^{\Delta^{op}}}(A)([n])([n]) \\ &= (m_{\mathcal{A}} \circ m_{\mathcal{A}^{\Delta^{op}}}(A))([n]) \end{aligned}$$

and for $\alpha : [m] \to [n]$,

$$\begin{aligned} m_{\mathcal{A}}((m_{\mathcal{A}})_* A)(\alpha) &= (m_{\mathcal{A}} \circ A)([n])(\alpha) \circ (m_{\mathcal{A}} \circ A)(\alpha)_{[m]} \\ &= m_{\mathcal{A}}(A([n]))(\alpha) \circ m_{\mathcal{A}}(A(\alpha))_{[m]} \\ &= A([n])([n])(\alpha) \circ A([n])(\alpha)_{[m]} \circ (A(\alpha)_{[m]})_{[m]} \\ &= A([n])([n])(\alpha) \circ (A([n])(\alpha) \circ A(\alpha)_{[m]})_{[m]} \\ &= m_{\mathcal{A}^{\Delta^{op}}}(A)([n])(\alpha) \circ m_{\mathcal{A}^{\Delta^{op}}}(A)(\alpha)_{[m]} \\ &= (m_{\mathcal{A}}(m_{\mathcal{A}^{\Delta^{op}}}(A)))(\alpha) \end{aligned}$$

4. An invertible modification $\lambda : m \circ \eta_{(-)^{\Delta^{op}}} \Rrightarrow 1_{(-)^{\Delta^{op}}}$ given by the following data:

(a) For each abelian category $\mathcal{A}$, a natural transformation $\lambda_{\mathcal{A}} : m_{\mathcal{A}} \circ \eta_{\mathcal{A}^{\Delta^{op}}} \Rightarrow 1_{\mathcal{A}^{\Delta^{op}}}$ which is given by identities since for a simplicial object A

$$m_{\mathcal{A}}(\eta_{\mathcal{A}^{\Delta^{op}}}(A))([n]) = \eta_{\mathcal{A}^{\Delta^{op}}}(A)([n])([n]) = A([n])$$

and for $\alpha : [m] \to [n]$,

$$m_{\mathcal{A}}(\eta_{\mathcal{A}^{\Delta^{op}}}(A))(\alpha) = \eta_{\mathcal{A}^{\Delta^{op}}}(A)([n])(\alpha) \circ \eta_{\mathcal{A}^{\Delta^{op}}}(A)(\alpha)_{[m]} = A(\alpha) \circ (1_A)_{[m]} = A(\alpha)$$

5. An invertible modification $\rho : m \circ (-)^{\Delta^{op}} \eta \Rrightarrow 1_{(-)^{\Delta^{op}}}$ given by the following data:

(a) For each abelian category $\mathcal{A}$, a natural transformation $\rho_{\mathcal{A}} : m_{\mathcal{A}} \circ (-)^{\Delta^{op}} \eta_{\mathcal{A}} \Rightarrow 1_{\mathcal{A}^{\Delta^{op}}}$ which is also given by identities since for a simplicial object A

$$m_{\mathcal{A}}((-)^{\Delta^{op}} \eta_{\mathcal{A}}(A))([n]) = (\eta_{\mathcal{A}} \circ A)([n])([n]) = \eta_{\mathcal{A}}(A([n]))([n]) = A([n])$$

and for $\alpha : [m] \to [n]$,

$$m_{\mathcal{A}}((-)^{\Delta^{op}} \eta_{\mathcal{A}}(A))(\alpha) = (\eta_{\mathcal{A}} \circ A)([n])(\alpha) \circ (\eta_{\mathcal{A}} \circ A)(\alpha)_{[m]} = 1_{A([n])} \circ \eta_{\mathcal{A}}(A(\alpha))_{[m]} = A(\alpha)$$

Since all the higher comparison cells are identities, it follows that all coherence diagrams commute automatically, and in particular, the simplicial objects functor is a strict 2-monad on the (large) 2-category of abelian categories, $\mathsf{2Ab}$.

1.A.1.1 Simplicial Homotopies

Homotopies in categories $\mathcal{C}^{\Delta^{op}}$ will be important in our analysis with the Dold-Kan Equivalence. This requires the consideration how to form products with simplicial sets in $\mathcal{C}^{\Delta^{op}}$, which we can obtain from [dJ05, Defn 14.13.1].

> **Definition 1.A.1** Let $\mathcal{C}$ be a category with finite coproducts and let $X \in \mathcal{C}^{\Delta^{op}}$. If $U \in \mathsf{Set}^{\Delta^{op}}$ is a finite, non-empty, simplicial set, we define the product $X \times U$ to be the simplicial object with nth component
>
> $$(X \times U)_n := \coprod_{u \in U_n} X_n$$
>
> such that for any map $\varphi : [m] \to [n]$, $(X \times U)(\varphi) : \coprod_{u \in U_n} X_n \to \coprod_{u' \in U_m} X_m$ is defined by
>
> $$(X \times U)(\varphi) \circ \iota_u = \iota_{U(\varphi)(u)} \circ X(\varphi)$$
>
> Given maps $f : X \Rightarrow Y$ and $g : U \Rightarrow V$ of simplicial objects and simplicial sets, respectively, we obtain a map of simplicial objects $f \times g : X \times U \to Y \times V$ given on components by
>
> $$(f \times g)_n : \coprod_{u \in U_n} X_n \to \coprod_{v \in V_n} Y_n, \quad (f \times g)_n \circ \iota_u = \iota_{g_n(u)} \circ f_n$$

We can now define simplicial homotopies. Let $\Delta^n := Hom_{\Delta}(-, [n])$ be the standard n-simplex as a simplicial set. Recall that Δ^0 is a singleton in each component, while

$$(\Delta^1)_n = \{\alpha_0^n, ..., \alpha_{n+1}^n\}, \quad \alpha_i^n(j) = \begin{cases} 0 & j < i \\ 1 & j \geq i \end{cases}$$

By Yoneda we can identify these maps with natural isomorphisms, so in particular we have $\alpha_0^0 : \Delta^0 \Rightarrow \Delta^1$ and $\alpha_1^0 : \Delta^0 \Rightarrow \Delta^1$ corresponding to sending 0 to 1 and sending 0 to 0, respectively (note the flip). We will write $e_0 := \alpha_1^0$ and $e_1 := \alpha_0^0$. Noting that for any simplicial object $U \in \mathcal{C}^{\Delta^{op}}$ $U \times \Delta^0 \cong U$, we obtain $e_0, e_1 : U \Rightarrow U \times \Delta^1$. This is sufficient to define simplicial homotopies [dJ05, Defn 14.26.1].

> **Definition 1.A.2** Let $X, Y \in \mathcal{C}^{\Delta^{op}}$ be simplicial objects in a category with finite coproducts, and let $f, g : X \Rightarrow Y$ be simplicial maps. Then a **simplicial homotopy** between f and g is a simplicial map $h : X \times \Delta^1 \Rightarrow Y$ making the following diagram

commute

$$\begin{tikzcd} X \arrow[d, "e_0"'] \arrow[dr, "f"] & \\ X \times \Delta^1 \arrow[r, dashed, "h"] & Y \\ X \arrow[u, "e_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

When $\mathcal{C}$ is an abelian category this defines an additive equivalence relation on the simplicial maps $X \Rightarrow Y$ [Wei94]. Otherwise, we say f and g are simplicially homotopic if there is a sequence $f = f_0, f_1, ..., f_n = g$ of maps such that there is a simplicial homotopy from f_i to f_{i+1} or from f_{i+1} to f_i for each $i < n$.

We can extend this definition to functors valued in simplicial objects.

Definition 1.A.3 Let $F, G : \mathcal{B} \to \mathcal{A}^{\Delta^{op}}$ be functors valued in simplicial objects. We say F and G are **pointwise homotopy equivalent** if for each $B \in \mathcal{B}$, we have a simplicial homotopy equivalence $(f_B : F(B) \to G(B), g_B : G(B) \to F(B), h_B : F(B) \times \Delta^1 \to F(B), h'_B : G(B) \times \Delta^1 \to G(B))$. We say F and G are **naturally homotopy equivalent** if we have natural transformations $(f : F \Rightarrow G, g : G \to F, h : F \times \Delta^1 \Rightarrow F, h' : G \times \Delta^1 \Rightarrow G)$ which comprise homotopy equivalences at each $B \in \mathcal{B}$.

We also have a completely combinatorial description of simplicial homotopies which is equivalent when working with simplicial objects in finitely cocomplete categories [Wei94]. In particular, a simplicial homotopy between $f, g : X \to Y$ is a family of maps $h_i^n : X_n \to Y_{n+1}$ for $n \in \mathbb{N}$ and $0 \le i \le n$, such that $Y_{d_0^{n+1}} h_0^n = f_n$, $Y_{d_{n+1}^{n+1}} h_n^n = g_n$, and

$$Y(d_i^{n+1})h_j^n = \begin{cases} h_{j-1}^{n-1} X(d_i^n) & i < j \\ Y(d_i^{n+1})h_{i-1}^n & i = j \neq 0 \\ h_j^{n-1} X(d_{i-1}^n) & i > j+1 \end{cases}$$

$$Y(s_i^{n+1})h_j^n = \begin{cases} h_{j+1}^{n+1} X(s_i^n) & i \le j \\ h_j^{n+1} X(s_{i-1}^n) & i > j \end{cases}$$

First we show that these homotopies are preserved by biproducts in abelian categories.

Lemma 1.A.4 Let $A, B, C, D \in \mathcal{C}^{\Delta^{op}}$ for $\mathcal{C}$ an abelian category, and let $f, g : A \to B$ and $h : C \to D$ be simplicial maps. Then f and g are simplicially homotopic if and only if $f \oplus h$ is simplicially homotopic to $g \oplus h$.

Proof. First, suppose that f and g are simplicially homotopic by maps $h_i^n : A_n \to B_{n+1}$ for $n \in \mathbb{N}$ and $0 \le i \le n$. Then I claim $h_i^n \oplus 0 : A_n \oplus C_n \to B_{n+1} \oplus D_{n+1}$ gives a homotopy from $f \oplus h$ to $g \oplus h$. Indeed since $\oplus$ is functorial the simplicial identities still hold in the first variable, and they also hold in the second variable vacuously since the composite with zero by any map is zero.

Conversely, suppose we have a homotopy

$$A_n \oplus C_n \xrightarrow{\begin{pmatrix} h^{1,1}_{n,i} & h^{1,2}_{n,i} \\ h^{2,1}_{n,i} & h^{2,2}_{n,i} \end{pmatrix}} B_{n+1} \oplus D_{n+1}$$

from $f \oplus h$ to $g \oplus h$. The homotopy conditions then become the equalities

$$\begin{pmatrix} B(d_i^{n+1})h^{1,1}_{n,j} & B(d_i^{n+1})h^{1,2}_{n,j} \\ D(d_i^{n+1})h^{2,1}_{n,j} & D(d_i^{n+1})h^{2,2}_{n,j} \end{pmatrix} = \begin{pmatrix} h^{1,1}_{n-1,j-1}A(d_i^n) & h^{1,2}_{n-1,j-1}C(d_i^n) \\ h^{2,1}_{n-1,j-1}A(d_i^n) & h^{2,2}_{n-1,j-1}C(d_i^n) \end{pmatrix} \qquad (i < j)$$

$$\begin{pmatrix} B(d_i^{n+1})h^{1,1}_{n,j} & B(d_i^{n+1})h^{1,2}_{n,j} \\ D(d_i^{n+1})h^{2,1}_{n,j} & D(d_i^{n+1})h^{2,2}_{n,j} \end{pmatrix} = \begin{pmatrix} B(d_i^{n+1})h^{1,1}_{n,i-1} & B(d_i^{n+1})h^{1,2}_{n,i-1} \\ D(d_i^{n+1})h^{2,1}_{n,i-1} & D(d_i^{n+1})h^{2,2}_{n,i-1} \end{pmatrix} \qquad (i = j \neq 0)$$

$$\begin{pmatrix} B(d_i^{n+1})h^{1,1}_{n,j} & B(d_i^{n+1})h^{1,2}_{n,j} \\ D(d_i^{n+1})h^{2,1}_{n,j} & D(d_i^{n+1})h^{2,2}_{n,j} \end{pmatrix} = \begin{pmatrix} h^{1,1}_{n-1,j}A(d_{i-1}^n) & h^{1,2}_{n-1,j}C(d_{i-1}^n) \\ h^{2,1}_{n-1,j}A(d_{i-1}^n) & h^{2,2}_{n-1,j}C(d_{i-1}^n) \end{pmatrix} \qquad (i > j+1)$$

$$\begin{pmatrix} B(s_i^{n+1})h^{1,1}_{n,j} & B(s_i^{n+1})h^{1,2}_{n,j} \\ D(s_i^{n+1})h^{2,1}_{n,j} & D(s_i^{n+1})h^{2,2}_{n,j} \end{pmatrix} = \begin{pmatrix} h^{1,1}_{n+1,j+1}A(s_i^n) & h^{1,2}_{n+1,j+1}C(s_i^n) \\ h^{2,1}_{n+1,j+1}A(s_i^n) & h^{2,2}_{n+1,j+1}C(s_i^n) \end{pmatrix} \qquad (i \leq j)$$

$$\begin{pmatrix} B(s_i^{n+1})h^{1,1}_{n,j} & B(s_i^{n+1})h^{1,2}_{n,j} \\ D(s_i^{n+1})h^{2,1}_{n,j} & D(s_i^{n+1})h^{2,2}_{n,j} \end{pmatrix} = \begin{pmatrix} h^{1,1}_{n+1,j}A(s_{i-1}^n) & h^{1,2}_{n+1,j}C(s_{i-1}^n) \\ h^{2,1}_{n+1,j}A(s_{i-1}^n) & h^{2,2}_{n+1,j}C(s_{i-1}^n) \end{pmatrix} \qquad (i > j)$$

It follows that $h^{1,1}_{n,i}$ give a simplicial homotopy from f to g, as desired. ■

We now show how these equivalences behave under composition as well as some behaviour of the category of functors into simplicial objects.

Lemma 1.A.5 If $\mathcal{A}, \mathcal{B}$ are categories, there exists an isomorphism of categories

$$\mathrm{Fun}(\mathcal{A}, \mathcal{B}^{\Delta^{op}}) \cong \mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\Delta^{op}}$$

Proof. We define a natural isomorphism $\gamma : \mathrm{Fun}(\mathcal{A}, \mathcal{B}^{\Delta^{op}}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\Delta^{op}}$ on an object $F : \mathcal{A} \to \mathcal{B}^{\Delta^{op}}$ by

$$\gamma(F)_n(A) := F(A)_n,$$

For each n $\gamma(F)_n$ is a functor so for $\alpha : [m] \to [n]$ we need to show a natural transformation $\gamma(F)_\alpha : \gamma(F)_n \to \gamma(F)_m$ which we define to be

$$(\gamma(F)_\alpha)_A := F(A)_\alpha$$

This is functorial since each $F(A)$ is a functor. If $f : A \to B$, then since $F(f)$ is a natural transformation, we also have the commuting square

$$\begin{tikzcd} F(A)_n \arrow[r, "F(f)_n"] \arrow[d, "F(A)_\alpha"'] & F(B)_n \arrow[d, "F(B)_\alpha"] \\ F(A)_m \arrow[r, "F(f)_m"'] & F(B)_m \end{tikzcd}$$

It follows that $\gamma(F)_\alpha$ is natural. Thus γ is a well-defined functor. Similarly, γ has an inverse $\tau : \mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\Delta^{op}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{B}^{\Delta^{op}})$ given by $\tau(F)(A)_n := F_n(A)$. ■

Note that under this isomorphism, natural simplicial homotopy is simplicial homotopy in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\Delta^{op}}$.

Lemma 1.A.6 We have a functor $\mathrm{Fun}(-,-)^{\Delta^{op}} : \mathsf{Cat}^{op} \times \mathsf{Cat} \to \mathsf{Cat}$, and for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{Cat}$ where $\mathcal{C}$ has finite coproducts, and any functor $F : \mathcal{A} \to \mathcal{B}$, $\mathrm{Fun}(F, \mathcal{C})^{\Delta^{op}} : \mathrm{Fun}(\mathcal{B}, \mathcal{C})^{\Delta^{op}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})^{\Delta^{op}}$ preserves simplicial homotopies. Similarly, if $\mathcal{B}$ and $\mathcal{C}$ both have finite colimits, and $F : \mathcal{B} \to \mathcal{C}$, then $\mathrm{Fun}(\mathcal{A}, F)$ preserves simplicial homotopies.

Proof. The functor $\mathrm{Fun}(-,-)^{\Delta^{op}}$ is simply the composite of $\mathrm{Fun}(-,-)$ with the 2-monad $^{\Delta^{op}}$. Now let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{Cat}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Then suppose $G, H \in \mathrm{Fun}(\mathcal{B}, \mathcal{C})^{\Delta^{op}}$ and suppose $g, h : G \to H$ are simplicially homotopic. Then there exists $f : G \times \Delta^1 \to H$ such that $f \circ e_0 = g$ and $f \circ e_1 = h$. Then $\mathrm{Fun}(F, \mathcal{C})^{\Delta^{op}}$ is defined by sending $K \in \mathrm{Fun}(\mathcal{B}, \mathcal{C})^{\Delta^{op}}$ K_F, where $(K_F)_n := K_n \circ F$. Observe that

$$((G \times \Delta_1)_F)_n = (\coprod_{u \in \Delta^1_n} G_n) \circ F = \coprod_{u \in \Delta^1_n} (G_n \circ F) = ((G \circ F) \times \Delta^1)_n$$

so $(G \times \Delta^1)_F = (G \circ F) \times \Delta^1$. Additionally, $((e_i)_F)_n = ((e_i)_n)_F$, which is exactly $(e_i)_n : (G \circ F)_n \to ((G \circ F) \times \Delta^1)_n$. Thus $\mathrm{Fun}(F, \mathcal{C})^{\Delta^{op}}$ preserves simplicial homotopies.

Conversely, if $\mathcal{B}$ and $\mathcal{C}$ are finitely cocomplete and $F : \mathcal{B} \to \mathcal{C}$, we can use the combinatorial description of homotopies. In this case, $\mathrm{Fun}(\mathcal{A}, F)$ is defined by ${}_F H_n := F \circ H_n$ on objects and ${}_F h_n := F h_n$ on maps. Let $f, g : H \to K$ be simplicially homotopic maps in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\Delta^{op}}$, by a simplicial homotopy $h^n_i : H_n \to K_{n+1}$. Then $F h^n_i : F \circ H_n \to F \circ K_{n+1}$ defines a simplicial homotopy between ${}_F f$ and ${}_F g$, as desired. ■

Lemma 1.A.7 Let $F, G : \mathcal{C} \to (\mathcal{B}^{\Delta^{op}})^{\Delta^{op}}$ and let $f, g : F \to G$ be naturally simplicially homotopic. Then $\Delta_{\mathcal{B}} f, \Delta_{\mathcal{B}} g : \Delta_{\mathcal{B}} \circ F \to \Delta_{\mathcal{B}} \circ G$ are naturally simplicially homotopic.

Proof. Let $h : F \times \Delta^1 \to G$ be a simplicial homotopy between f and g. Then

$$\Delta_{\mathcal{B}}(F \times \Delta^1)(C)_n = \coprod_{u \in \Delta^1_n} F(C)_{n,n} = (\Delta_{\mathcal{B}} F \times \Delta^1)(C)_n$$

In particular, $\Delta_{\mathcal{B}} h$ defines a simplicial homotopy from $\Delta_{\mathcal{B}} f$ to $\Delta_{\mathcal{B}} g$. ■

Lemma 1.A.8 Let $F, G : \mathcal{C} \to \mathcal{B}^{\Delta^{op}}$ and let $f, g : F \to G$ be naturally simplicially homotopic. Then $\Delta^{op}(f), \Delta^{op}(g) : \Delta^{op}(F) \to \Delta^{op}(G)$ are naturally simplicially homotopic.

Proof. Let $h : F \times \Delta^1 \to G$ be a simplicial homotopy between f and g. Then observe that

$$\Delta^{op}(F \times \Delta^1)(C)_n = F(C_n) \times \Delta^1 = \Delta^{op}(F)(C)_n \times \Delta^1$$

Then $\Delta^{op}(h) : \Delta^{op}(F) \times \Delta^1 \to \Delta^{op}(G)$, and since Δ^{op} is a strict 2-functor it preserves composition so $\Delta^{op}(h)$ is a homotopy from $\Delta^{op}(f)$ to $\Delta^{op}(g)$. ■

Next, we construct simplicial objects out comonads on a category.

Proposition 1.A.9 Let (C, ϵ, δ) be a comonad on a category $\mathcal{C}$. Then there exists a functor

$$C^{\bullet+1} : \mathcal{C} \to \mathcal{C}^{\Delta^{op}}$$

Proof. Let $A \in \mathcal{C}_0$. Then we define the simplicial object $C^{\bullet+1}(A)$ by

$$C^{\bullet+1}(A)_n := C^{n+1}(A)$$

while for the face operators $d_i^n : [n-1] \to [n]$ and degeneracy operators $s_i^n : [n+1] \to [n]$ we define

$$C^{\bullet+1}(A)_{d_i^n} := C^i \epsilon_{C^{n-i}(A)} : C^{n+1}(A) \to C^n(A), \quad C^{\bullet+1}(A)_{s_i^n} := C^i \delta_{C^{n-i}(A)} : C^n(A) \to C^{n+1}(A)$$

It remains to show the simplicial identities for these maps:

$$\begin{aligned}
C^i \epsilon_{C^{n-i}(A)} \circ C^j \epsilon_{C^{n+1-j}(A)} &= C^i(\epsilon_{C^{n-i}(A)} \circ C^{j-i} \epsilon_{C^{n+1-j}(A)}) \\
&= C^i(C^{j-1-i} \epsilon_{C^{n+1-j}(A)} \circ \epsilon_{C^{n+1-i}(A)}) = C^{j-1} \epsilon_{C^{n-(j-1)}(A)} \circ C^i \epsilon_{C^{n+1-i}(A)} \\
&\qquad (i < j) \\
C^i \delta_{C^{n+1-i}(A)} \circ C^j \delta_{C^{n-j}(A)} &= C^j(C^{i-j} \delta_{C^{n+1-i}(A)} \circ \delta_{C^{n-j}(A)}) \\
&= C^j(\delta_{C^{n+1-j}(A)} \circ C^{i-1-j} \delta_{C^{n+1-i}(A)}) = C^j \delta_{C^{n+1-j}(A)} \circ C^{i-1} \delta_{C^{n-(i-1)}(A)} \\
&\qquad (i > j) \\
C^i \epsilon_{C^{n+1-i}(A)} \circ C^j \delta_{C^{n-j}(A)} &= 1_{A_n} \qquad (i \in \{j, j+1\}) \\
C^i \epsilon_{C^{n+1-i}(A)} \circ C^j \delta_{C^{n-j}(A)} &= C^i(\epsilon_{C^{n+1-i}(A)} \circ C^{j-i} \delta_{C^{n-j}(A)}) \\
&= C^i(C^{j-i-1} \delta_{C^{n-j}(A)} \circ \epsilon_{C^{n-i}(A)}) = C^{j-1} \delta_{C^{n-1-(j-1)}(A)} \circ C^i \epsilon_{C^{n-i}(A)} \\
&\qquad (i < j) \\
C^i \epsilon_{C^{n+1-i}(A)} \circ C^j \delta_{C^{n-j}(A)} &= C^j(C^{i-j} \epsilon_{C^{n+1-i}(A)} \circ \delta_{C^{n-j}(A)}) \\
&= C^j(\delta_{C C^{n-1-j}(A)} \circ C^{i-1-j} \epsilon_{C^{n+1-i}(A)}) = C^j \delta_{C^{n-1-j}(A)} \circ C^{i-1} \epsilon_{C^{n-(i-1)}(A)} \\
&\qquad (i > j)
\end{aligned}$$

where we have used the naturality of ϵ and δ, as well as the monad identities. Finally, if $f : A \to A'$ in $\mathcal{C}$ we define $C^{\bullet+1}(f)_n := C^{n+1}(f)$. This gives a map of simplicial sets since the face and degeneracy operators are natural. Since C is a functor this assignment is functorial. ■

For any category we have the trivial comonad given by identities. All comonads then given simplicial objects which are augmented over the simplicial object for this comonad.

Proposition 1.A.10 Let (C, ϵ, δ) be a comonad on $\mathcal{C}$. Then there exists a natural transformation $C^{\bullet+1} \to 1_{\mathcal{C}}^{\bullet+1}$

Proof. We define $\rho : C^{\bullet+1} \to 1_{\mathcal{C}}^{\bullet+1}$ at $A \in \mathcal{C}_0$ and $n \in \mathbb{N}$ by

$$C^{n+1}(A) \xrightarrow{\epsilon_A \circ \epsilon_{C(A)} \circ \cdots \circ \epsilon_{C^n(A)}} A$$

Then for the image of the face operators we obtain the commuting rectangle

$$\begin{tikzcd}
C^{n+1}(A) \arrow[r, "\epsilon_{C^n(A)}"] \arrow[d, "C^i\epsilon_{C^{n-i}(A)}"'] & C^n(A) \arrow[r] \arrow[d, "C^{i-1}\epsilon_{C^{n-i}(A)}"] & \cdots \arrow[r] & C(A) \arrow[r, "\epsilon_A"] & A \arrow[d, equal] \\
C^n(A) \arrow[r, "\epsilon_{C^{n-1}(A)}"'] & C^{n-1}(A) \arrow[r] & \cdots \arrow[r] & C(A) \arrow[r] & A
\end{tikzcd}$$

by applying naturality i times. Similarly, applying naturality of ϵ i times to

$$\begin{tikzcd}
C^{n+1}(A) \arrow[r, "\epsilon_{C^n(A)}"] \arrow[d, "C^i\delta_{C^{n-i}(A)}"'] & C^n(A) \arrow[r] \arrow[d, "C^{i-1}\delta_{C^{n-i}(A)}"] & \cdots \arrow[r] & C(A) \arrow[r, "\epsilon_A"] & A \arrow[d, equal] \\
C^{n+2}(A) \arrow[r, "\epsilon_{C^{n+1}(A)}"'] & C^{n+1}(A) \arrow[r] & \cdots \arrow[r] & C(A) \arrow[r] & A
\end{tikzcd}$$

we obtain the diagram

$$\begin{tikzcd}
C^{n-i+1}(A) \arrow[r, "\epsilon_{C^{n-i}(A)}"] \arrow[d, "\delta_{C^{n-i}(A)}"'] \arrow[dr, equal] & C^{n-i}(A) \arrow[r] & \cdots \arrow[r] & A \arrow[d, equal] \\
C^{n-i+2}(A) \arrow[r, "\epsilon_{C^{n-i+1}(A)}"'] & C^{n-i+1}(A) \arrow[r] & \cdots \arrow[r] & A
\end{tikzcd}$$

which commutes. Thus ρ_A is a map of simplicial sets. For $f : A \to A'$, we have the commuting diagram

$$\begin{tikzcd}
C^{n+1}(A) \arrow[r, "\epsilon_{C^n(A)}"] \arrow[d, "C^{n+1}(f)"'] & C^n(A) \arrow[r] \arrow[d, "C^n(f)"] & \cdots \arrow[r] & C(A) \arrow[r, "\epsilon_A"] \arrow[d, "C(f)"] & A \arrow[d, "f"] \\
C^{n+1}(A') \arrow[r, "\epsilon_{C^n(A')}"'] & C^n(A') \arrow[r] & \cdots \arrow[r] & C(A') \arrow[r, "\epsilon_{A'}"'] & A'
\end{tikzcd}$$

so ρ is natural. ■

1.A.1.2 Chain Homotopies

In this section we expand on the behaviour of natural chain homotopies and the interaction between natural chain homotopies and direct sums. We begin by proving equivalent formulations of natural chain homotopies.

Lemma 1.A.11 For $\mathcal{A}$ an abelian category, we have an isomorphism of categories

$$\mathsf{Ch}(\mathrm{Fun}(\mathcal{B},\mathcal{A})) \cong \mathrm{Fun}(\mathcal{B},\mathsf{Ch}(\mathcal{A})) \tag{1.12}$$

Proof. Define a functor $\gamma : \mathsf{Ch}(\mathrm{Fun}(\mathcal{B},\mathcal{A})) \to \mathrm{Fun}(\mathcal{B},\mathsf{Ch}(\mathcal{A}))$ given on a chain complex of functors $F_\bullet$ by

$$\gamma(F_\bullet)(B)_n := F_n(B),\ \forall B \in \mathcal{B}$$

where the differentials are given by the natural transformation differentials in $F_\bullet$ evaluated at B. Given a map of chain complexes $\alpha_\bullet : F_\bullet \to G_\bullet$ we set

$$(\gamma(\alpha_\bullet)_B)_n := (\alpha_n)_B$$

This defines a chain map $\gamma(F_\bullet)(B) \to \gamma(G_\bullet)(B)$ since $\alpha_\bullet$ is a chain map of natural transformations, so all squares with differentials commute. Further, $\gamma(\alpha_\bullet)$ is natural in B since if $f : B \to B'$ is a map in $\mathcal{B}$, then in

$$\begin{tikzcd}
 & F_{n+1}(B') \arrow[rr, "\partial_{n+1}"] \arrow[dd, "(\alpha_{n+1})_{B'}"] & & F_n(B') \arrow[dd, "(\alpha_n)_{B'}"] \\
F_{n+1}(B) \arrow[ur, "F_{n+1}(f)"] \arrow[rr, "\partial_{n+1}"] \arrow[dd, "(\alpha_{n+1})_B"'] & & F_n(B) \arrow[ur, "F_n(f)"'] \arrow[dd, "(\alpha_n)_B"] & \\
 & G_{n+1}(B') \arrow[rr, "\partial_{n+1}"] & & G_n(B') \\
G_{n+1}(B) \arrow[ur, "G_{n+1}(f)"] \arrow[rr, "\partial_{n+1}"'] & & G_n(B) \arrow[ur, "G_n(f)"'] &
\end{tikzcd}$$

the front and back faces commute since $\alpha_\bullet$ is a chain map, the top and bottom faces commute by naturality of the boundary maps, and the side faces commute by naturality of the α_n. Since this definition is in terms of the components of $\alpha_\bullet$ it is inherently functorial.
Next we must witness an inverse $\rho : \mathrm{Fun}(\mathcal{B},\mathsf{Ch}(\mathcal{A})) \to \mathsf{Ch}(\mathrm{Fun}(\mathcal{B},\mathcal{A}))$ functor. Given $F : \mathcal{B} \to \mathsf{Ch}(\mathcal{A})$ we set $\rho(F)$ to have nth component $(-)_n \circ F$ and differential ∂_n given by components the nth differential of F evaluated at $B \in \mathcal{B}$. Naturality of the differential equates to the commutivity of

$$\begin{tikzcd}
F(B)_n \arrow[r, "\partial_n(B)"] \arrow[d, "F(f)_n"'] & F(B)_{n-1} \arrow[d, "F(f)_{n-1}"] \\
F(B')_n \arrow[r, "\partial_n(B')"'] & F(B')_{n-1}
\end{tikzcd}$$

for any $f : B \to B'$, which follows since $F(f)$ is a chain map. Next, if $\alpha : F \to G$ is a natural transformation between two such functors we set $\rho(\alpha)$ such that $\rho(\alpha)_n$ is the natural transformation defined by $(\rho(\alpha)_n)_B := (\alpha_B)_n$. Naturality and the chain condition follow by the commutivity of

$$\begin{tikzcd}
& F(B')_{n+1} \arrow[rr, "\partial_{n+1}(B')"] \arrow[dd, "(\alpha_{B'})_{n+1}"] & & F(B')_n \arrow[dd, "(\alpha_{B'})_n"] \\
F(B)_{n+1} \arrow[ur, "F(f)_{n+1}"] \arrow[rr, "\partial_{n+1}(B)"] \arrow[dd, "(\alpha_B)_{n+1}"'] & & F(B)_n \arrow[ur, "F(f)_n"'] \arrow[dd, "(\alpha_B)_n"] & \\
& G(B')_{n+1} \arrow[rr, "\partial_{n+1}(B')"] & & G(B')_n \\
G(B)_{n+1} \arrow[ur, "G(f)_{n+1}"] \arrow[rr, "\partial_{n+1}(B)"'] & & G(B)_n \arrow[ur, "G(f)_n"'] &
\end{tikzcd}$$

where the bottom and top faces are the fact $G(f)$ and $F(f)$ are chain maps, the front and back faces are the fact α_B is a chain map, and finally the side faces are naturality of α. Once again, since $\rho(\alpha)$ is defined in terms of the components of α the assignment is inherently functorial. Further, these operations are exactly inverse of each other as they correspond to swapping the element and natural number indices (in particular, on the other side of the Dold-Kan Equivalence this is simply the swap natural isomorphism on functors of two variables). ■

Moving forward we write $\mathrm{Fun}^{\mathsf{Ch}}$ for the isomorphism γ in the proof. We also have another description of this category:

Lemma 1.A.12 Let $\mathbb{Z}$ denote the category associated with the linear ordered set $(\mathbb{Z}, \geq)$. Let $\mathrm{Fun}_{\mathsf{Ch}}(\mathcal{B} \times \mathbb{Z}, \mathcal{A})$ be the sub-category such that $F(-, n+2 \leq n)$ is the zero map. Then under the adjunction $- \times \mathbb{Z} \dashv \mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ we have the isomorphism

$$\mathrm{Fun}_{\mathsf{Ch}}(\mathcal{B} \times \mathbb{Z}, \mathcal{A}) \cong \mathrm{Fun}(\mathcal{B}, \mathrm{Fun}_{\mathsf{Ch}}(\mathbb{Z}, \mathcal{A}))$$

and $\mathrm{Fun}_{\mathsf{Ch}}(\mathbb{Z}, \mathcal{A}) \cong \mathsf{Ch}(\mathcal{A})$.

Proof. It is sufficient to show that the isomorphism in the adjunction restricts to the isomorphism above. But this follows immediately by definition, so all that there is to show is $\mathrm{Fun}_{\mathsf{Ch}}(\mathbb{Z}, \mathcal{A}) \cong \mathsf{Ch}$. But this is also immediate by sending $A : \mathbb{Z} \to \mathcal{A}$ to $A_\bullet$, where $A_n = A(n)$ and $\partial_n^A = A(n \geq n-1)$, and vice-versa. ■

Lemma 1.A.13 Chain homotopies correspond to natural chain homotopies of functors under the isomorphism $\mathrm{Fun}^{\mathsf{Ch}} : \mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}(\mathcal{B}))$.

Proof. First, let $\alpha, \beta : F_\bullet \to G_\bullet$ be a map of chain complexes of functors in $\mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$. Then a chain homotopy from α to β is, for each $n \in \mathbb{Z}$, a natural transformation $s_n : F_n \Rightarrow G_{n+1}$ such that

$$\partial^G_{n+1} \circ s_n + s_{n-1} \circ \partial^G_n = \alpha_n - \beta_n$$

On the other hand, under $\mathrm{Fun}^{\mathsf{Ch}}$ α and β correspond to natural transformations between functors valued in chain complexes, F, G. By definition, a natural chain homotopy is then a family of natural transformations $s_n : (-)_n \circ F \to (-)_{n+1} \circ G$, where $(-)_n : \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$. But this is precisely the same data as the chain homotopy in $\mathsf{Ch}(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$. ■

Next, we also show an equivalent form of chain homotopies.

Lemma 1.A.14 Chain homotopies between maps $f, g : A_\bullet \to B_\bullet$ in $\mathsf{Ch}(\mathcal{A})$ are equivalent to chain maps $H : \mathrm{cyl}(-1_{A_\bullet}) \to B_\bullet$ such that the triangle

$$\begin{tikzcd} \mathrm{cyl}(-1_{A_\bullet}) \arrow[dr, "H"] & \\ A_\bullet \oplus A_\bullet \arrow[u, "q_1+q_2"] \arrow[r, "f+g"'] & B_\bullet \end{tikzcd}$$

commutes where q_1 is the inclusion in the top of the cylinder and q_2 is the inclusion in the bottom cylinder.

Proof. We begin with a triangle as in the statement of the Lemma where $\mathrm{cyl}(1_{A_\bullet})$ is the chain complex with nth degree term given by $A_{n-1} \oplus A_n \oplus A_n$ and chain map given by

$$A_n \oplus A_{n+1} \oplus A_{n+1} \xrightarrow{\begin{pmatrix} \partial^A_n & 0 & 0 \\ (-1)^n 1_{A_n} & \partial^A_{n+1} & 0 \\ (-1)^{n+1} 1_{A_n} & 0 & \partial^A_{n+1} \end{pmatrix}} A_{n-1} \oplus A_n \oplus A_n$$

Additionally, q_1 is given by

$$A_n \xrightarrow{\begin{pmatrix} 0 \\ 1_{A_n} \\ 0 \end{pmatrix}} A_{n-1} \oplus A_n \oplus A_n$$

and q_2 is given by

$$A_n \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1_{A_n} \end{pmatrix}} A_{n-1} \oplus A_n \oplus A_n$$

Then a map $H : \mathrm{cyl}(1_{A_\bullet}) \to B_\bullet$ making the triangle commute is on each degree of the form

$$A_{n-1} \oplus A_n \oplus A_n \xrightarrow{\begin{pmatrix} (-1)^{n-1} s_{n-1} & f_n & g_n \end{pmatrix}} B_n$$

where the chain map condition reduces to

$$\partial_n^B \circ s_{n-1} + s_{n-2} \circ \partial_{n-1}^A = f_{n-1} - g_{n-1}$$

which is exactly the condition for a chain homotopy. ∎

We now prove some preliminary results on exactness and preserving chain homotopies in order to show the exactness and preservation of limits for this construction.

Lemma 1.A.15 The totalization functor $\mathrm{Tot} : \mathsf{Ch}^2(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})$ is exact.

Proof. Let

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$$

be a short exact sequence of bicomplexes in $\mathcal{A}$. This becomes a sequence of complexes

$$\mathrm{Tot}(A_1) \xrightarrow{\mathrm{Tot}(f_1)} \mathrm{Tot}(A_2) \xrightarrow{\mathrm{Tot}(f_2)} \mathrm{Tot}(A_3)$$

where at a given n,

$$\mathrm{Tot}(A_i)_n = \bigoplus_{j=0}^{n} (A_i)_{j,n-j}$$

and

$$\mathrm{Tot}(f_i)_n = \bigoplus_{j=0}^{n} (f_i)_{j,n-j}$$

Note that the sequence of bicomplexes being exact means that each component sequence in $\mathcal{A}$ is exact. Then the component sequence of the totalization at n is a finite direct sum of exact sequences, and hence exact. ∎

Lemma 1.A.16 Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor between abelian categories. Then for a category $\mathcal{A}$, $F_* : \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is exact.

Proof. Let $0 \to G_1 \xrightarrow{\eta_1} G_2 \xrightarrow{\eta_2} G_3 \to 0$ be a short exact sequence of functors from $\mathcal{A}$ to $\mathcal{B}$. Since abelian categories are finitely complete and cocomplete, finite limits and colimits in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ are computed pointwise, so it is sufficient to prove the lemma at a given $A \in \mathcal{A}$. This follows by exactness of F. ∎

In order to prove our desired exactness result we first introduce a naive notion chain complexes of functors for the subcategory of additive functors.

Lemma 1.A.17 Let $\mathrm{Fun}_{Add}(\mathcal{A}, \mathcal{C})$ be the category of additive functors between abelian categories with all natural transformations. Then we have a functor

$$\mathsf{Ch} : \mathrm{Fun}_{Add}(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}_{Add}(\mathsf{Ch}(\mathcal{A}), \mathsf{Ch}(\mathcal{C}))$$

given by sending functors to their action componentwise.

Proof. Let $\mathcal{F} \in \mathrm{Fun}_{Add}(\mathcal{A}, \mathcal{C})$. Since $\mathcal{F}$ is additive it preserves 0's and hence sends chain complexes to chain complexes. Then let $f_\bullet : A_\bullet \to A'_\bullet$ be a map of chain complexes. Then $\mathsf{Ch}(\mathcal{F})(f_\bullet)_n := \mathcal{F}(f_n)$, and since $\mathcal{F}$ is additive

$$\mathcal{F}(f_n)\mathcal{F}(\partial^A_{n+1}) - \mathcal{F}(\partial^{A'}_n)\mathcal{F}(f_{n+1}) = \mathcal{F}(f_n\partial^A_{n+1} - \partial^{A'}_n f_{n+1}) = \mathcal{F}(0) = 0$$

so $\mathsf{Ch}(\mathcal{F})(f_\bullet)$ is a chain map. Further, since $\mathsf{Ch}(\mathcal{F})$ is defined componentwise and $\mathcal{F}$ is a functor and additive, $\mathsf{Ch}(\mathcal{F})$ is a functor and additive.
Next, let $\eta : \mathcal{F} \to \mathcal{G}$ be a natural transformation between additive functors. Then define $\mathsf{Ch}(\eta)_{A_\bullet} : \mathsf{Ch}(\mathcal{F})(A_\bullet) \to \mathsf{Ch}(\mathcal{G})(A_\bullet)$ by $(\mathsf{Ch}(\eta)_{A_\bullet})_n := \eta_{A_n}$. Then $\mathsf{Ch}(\eta)_{A_\bullet}$ is a chain map by naturality of η. Further, $\mathsf{Ch}(\eta)$ is natural again by naturality of η, which makes the following diagram commute for $f_\bullet : A_\bullet \to A_\bullet$:

$$\begin{tikzcd} \mathcal{F}(A_n) \arrow[r, "\mathcal{F}(f_n)"] \arrow[d, "\eta_{A_n}"'] & \mathcal{F}(B_n) \arrow[d, "\eta_{B_n}"] \\ \mathcal{G}(A_n) \arrow[r, "\mathcal{G}(f_n)"'] & \mathcal{G}(B_n) \end{tikzcd}$$

Since $\mathsf{Ch}(\eta)$ is defined componentwise it preserves composites and identities. ■

Lemma 1.A.18 Let $F, G : \mathcal{B} \to \mathcal{C}$, $H : \mathcal{C} \to \mathcal{D}$, and $K : \mathcal{A} \to \mathcal{B}$ be additive functors. If $\eta : F \Rightarrow G$ is a natural transformation, then

$$\mathsf{Ch}(H\eta_K) = \mathsf{Ch}(H)\mathsf{Ch}(\eta)_{\mathsf{Ch}(K)}$$

Proof. From the proof of Lemma 1.A.17 we have that for a chain complex $A_\bullet$,

$$(\mathsf{Ch}(H\eta_K)_{A_\bullet})_n = H(\eta_{K(A_n)}) = H(\eta_{\mathsf{Ch}(K)(A_\bullet)_n}) = H((\mathsf{Ch}(\eta)_{\mathsf{Ch}(K)(A_\bullet)})_n) = \mathsf{Ch}(H)(\mathsf{Ch}(\eta)_{\mathsf{Ch}(K)(A_\bullet)})_n$$

■

Note that Lemma 1.A.18 shows that Ch is a strict 2-functor on the category of abelian categories with additive functors between them. We now show how additive functors acting componentwise preserve chain homotopies.

Lemma 1.A.19 Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then $\mathsf{Ch}(F) : \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{B})$ preserves chain homotopies.

Proof. Let $f, g : A_\bullet \to B_\bullet$ be chain maps with a homotopy $s_n : A_n \to B_{n+1}$ for $n \in \mathbb{Z}$ from f to g. Since F is additive we have that

$$F(\partial^B_{n+1}) \circ F(s_n) + F(s_{n-1}) \circ F(\partial^A_n) = F(\partial^B_{n+1} \circ s_n + s_{n-1} \circ \partial^A_n) = F(f_n - g_n) = F(f_n) - F(g_n)$$

Since $\mathsf{Ch}(F)(f)_n := F(f_n)$ and $\mathsf{Ch}(F)(g)_n := F(g_n)$, it follows that $F(f)$ and $F(g)$ are homotopic by $F(s_n) : F(A_n) \to F(B_{n+1})$. ■

We also have an analogous result for the chain construction from comonads.

Lemma 1.A.20 Let (C, ϵ, δ) be a comonad on $\mathcal{A}$ which is also an additive functor. Then $C^{\mathsf{Ch}} : \mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ is additive.

Proof. Let $f, g : A \to B$ in $\mathcal{A}$. Then since C is additive, so is all of its powers, so

$$C^{\mathsf{Ch}}(f+g)_n := C^n(f+g) = C^n(f) + C^n(g) = C^{\mathsf{Ch}}(f)_n + C^{\mathsf{Ch}}(g)_n$$

Thus C^{Ch} is additive. ■

By Lemma 1.3.5 and Lemma **??** it follows that $\mathsf{Ch}(C)^{\mathsf{Ch}}$ preserves homotopies. Another important result for this construction is how it is affected by isomorphisms between comonads.

Lemma 1.A.21 Let (C, ϵ, δ) be a comonad on $\mathcal{A}$ and let (C', ϵ', δ') be a comonad on $\mathcal{A}'$. Suppose $\gamma : \mathcal{A} \to \mathcal{A}'$ is an additive isomorphism of categories such that $C' = \gamma \circ C \circ \gamma^{-1}$, $\epsilon' = \gamma\epsilon_{\gamma^{-1}}$, and $\delta' = \gamma\delta'_{\gamma^{-1}}$. Then there is an equality

$$C'^{\mathsf{Ch}} = \mathsf{Ch}(\gamma) \circ C^{\mathsf{Ch}} \circ \gamma^{-1}$$

Proof. This equality is an immediate consequence of the construction of C^{Ch} and the specified equalities for the comonad natural transformations. Indeed, observe that for $A' \in \mathcal{A}'$

$$\begin{aligned}
\mathsf{Ch}(\gamma) \circ C^{\mathsf{Ch}} \circ \gamma^{-1}(A') &= \mathsf{Ch}(\gamma)(\cdots \to C^2\gamma^{-1}(A') \xrightarrow{C\epsilon_{\gamma^{-1}(A')} - \epsilon_{C\gamma^{-1}(A')}} C\gamma^{-1}(A') \xrightarrow{\epsilon_{\gamma^{-1}(A')}} \gamma^{-1}(A')) \\
&= \cdots \to \gamma C^2\gamma^{-1}(A') \xrightarrow{\gamma C\epsilon_{\gamma^{-1}(A')} - \gamma\epsilon_{C\gamma^{-1}(A')}} \gamma C\gamma^{-1}(A') \xrightarrow{\gamma\epsilon_{\gamma^{-1}(A')}} \gamma\gamma^{-1}(A') \\
&= \cdots \to C'^2(A') \xrightarrow{C'\epsilon'_{A'} - \epsilon'_{C'(A')}} C'(A') \xrightarrow{\epsilon'_{A'}} A'
\end{aligned}$$

■

Next, we have the following result on (co)limits in chain complex categories, which is a analoguous to the result for functor categories.

Lemma 1.A.22 Let $\mathcal{A}$ be an abelian category with I shaped (co)limits for a small category I. Then $\mathsf{Ch}(\mathcal{A})$ has I shaped (co)limits which are computed pointwise.

Proof. We provide the proof in the case of colimits, while the case of limits is analoguous. Let $D : I \to \mathsf{Ch}(\mathcal{A})$ be an I shaped diagram in $\mathsf{Ch}(\mathcal{A})$.
For each $n \in \mathbb{Z}$ we have a functor $(-)_n : \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$ given by projecting on the nth component. By assumption, for each $n \in \mathbb{Z}$ the colimit of $(-)_n \circ D : I \to \mathcal{A}$ exists. Denote this limit by D_n, and denote its injections by $\iota_{n,i}$, for $i \in I$. For each $i \in I$ we have an induced map $D(i)_{n+1} \to D(i)_n \to D_n$ obtained by composing with the the boundary map for $D(i)_\bullet \in \mathsf{Ch}(\mathcal{A})$ and the inclusion. By the universal property of the colimit we obtain a unique map making the square

$$\begin{tikzcd} D_{n+1} \arrow[r, dashed] & D_n \\ D(i)_{n+1} \arrow[u, "\iota_{n+1,i}"] \arrow[r, "\partial^{D(i)}_{n+1}"'] & D(i)_n \arrow[u, "\iota_{n,i}"'] \end{tikzcd}$$

commute. Let ∂_{n+1} denote this map. Then by uniqueness of this map and the fact that pasting two such squares together gives a commuting rectangle

$$\begin{tikzcd} D_{n+1} \arrow[r, dashed] & D_n \arrow[r, dashed] & D_{n-1} \\ D(i)_{n+1} \arrow[u, "\iota_{n+1,i}"] \arrow[r, "\partial^{D(i)}_{n+1}"'] & D(i)_n \arrow[u, "\iota_{n,i}"'] \arrow[r, "\partial^{D(i)}_{n}"'] & D(i)_{n-1} \arrow[u, "\iota_{n-1,i}"'] \end{tikzcd}$$

we must have that $\partial_n \circ \partial_{n+1}$ is the zero map. This implies that $(D_\bullet, \partial_\bullet)$ defines a chain complex on $\mathcal{A}$. It remains to show that it is the colimit of D. But by construction any other such chain complex with maps out of each $D(i)$ induces maps unique pointwise maps which form commuting squares in a chain map by their uniqueness. ■

This allows us to also describe how totalization behaves under the isomorphism from functors valued in chain complexes and chain complexes of functors.

Lemma 1.A.23 For abelian categories $\mathcal{A}, \mathcal{B}$, we have a natural isomorphism

$$(\mathrm{Tot}_{\mathcal{A}})_* \circ \mathrm{Fun}^{\mathsf{Ch}} \circ \mathrm{Fun}^{\mathsf{Ch}} \cong \mathrm{Fun}^{\mathsf{Ch}} \circ \mathrm{Tot}_{\mathrm{Fun}(\mathcal{B},\mathcal{A})}$$

Proof. Let $F_{\bullet,\bullet} \in \mathsf{Ch}^2(\mathrm{Fun}(\mathcal{B}, \mathcal{A}))$. Let $F : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}^2(\mathcal{A}))$ be the associated functor under two applications of $\mathrm{Fun}^{\mathsf{Ch}}$. For $B \in \mathcal{B}$, the left functor gives the complex

$$\mathrm{Tot}_{\mathcal{A}}(F(B))_n := \bigoplus_{p+q=n} F_{p,q}(B)$$

at B, while the right functor gives the complex

$$\mathrm{Fun}^{\mathsf{Ch}}(\mathrm{Tot}_{\mathrm{Fun}(\mathcal{B},\mathcal{A})}(F_{\bullet,\bullet}))(B)_n := \left(\bigoplus_{p+q=n} F_{p,q}\right)(B) \cong \bigoplus_{p+q=n} F_{p,q}(B)$$

Under the adjunction in Lemma 1.A.12, these give functors $\mathrm{Fun}_{\mathsf{Ch}}(\mathcal{B}, \mathsf{Ch}(\mathcal{A})) \cong \mathrm{Fun}_{\mathsf{Ch}}(\mathcal{B} \times \mathbb{Z}, \mathcal{A})$. Then we can use the results in Section 1.A.4 to conclude that

$$\mathrm{Fun}^{\mathsf{Ch}}(\mathrm{Tot}_{\mathrm{Fun}(\mathcal{B},\mathcal{A})}(F_{\bullet,\bullet})) \cong \mathrm{Tot}_{\mathcal{A}}(F)$$

for all $F_{\bullet,\bullet}$, and further these isomorphisms are natural in $F_{\bullet,\bullet}$ **STILL NEED SOME CLARIFICATION**. ■

1.A.2 Properties of Dold-Kan and Simplicial Homotopies

In this section we collect properties related to the Dold-Kan equivalence, and unify notation for use in the remainder of the notes. We first recall the statement of the general Dold-Kan equivalence from [dJ05, Thm 14.24.3].

Theorem 1.A.24 For $\mathcal{A}$ an abelian category, there is an equivalence of categories $N : \mathcal{A}^{\Delta^{op}} \simeq \mathsf{Ch}(\mathcal{A}) : \Gamma$ [dJ05]. Let $\eta : 1_{\mathsf{Ch}(\mathcal{A})} \Rightarrow N \circ \Gamma$ and $\varepsilon : \Gamma \circ N \Rightarrow 1_{\mathcal{A}^{\Delta^{op}}}$ be the unit and counit for the equivalence, which can be chosen to satisfy the triangle identities. N is given explicitly on a simplicial object X by

$$N(X)_n := \begin{cases} \bigcap_{i=0}^{n-1} \ker(d_n^i) & n \geq 1 \\ X_0 & n = 0 \\ 0 & n < 0 \end{cases}$$

with differential given by $(-1)^n d_n^n : N(X)_n \to N(X)_{n-1}$. N is given on arrows by restriction. On the other hand, for a chain complex $A_\bullet$, with boundary maps $d_{A,n}$, $\Gamma(A_\bullet)$ is given on objects by

$$\Gamma(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$$

where $I_n = \{\alpha : [n] \to \mathbb{N} | \operatorname{Im}(\alpha) = [k(\alpha)]\}$ where $k(\alpha)$ is the maximum element in the image. For a monotonic map $\varphi : [m] \to [n]$, we define Γ using the universal property of the biproduct by

$$\pi_\beta \circ \Gamma(A_\bullet)(\varphi) \circ \iota_\alpha := \begin{cases} 0 & \alpha \circ \varphi \notin I_m \\ 0 & \alpha \circ \varphi \in I_m, k(\alpha \circ \varphi) \neq k(\alpha), k(\alpha) - 1 \\ 0 & k(\alpha \circ \varphi) \neq \beta \\ 1_{A_{k(\alpha)}} & \alpha \circ \varphi \in I_m, k(\alpha \circ \varphi) = k(\alpha) \\ (-1)^{k(\alpha)} d_{A,k(\alpha)} & \alpha \circ \varphi \in I_m, k(\alpha \circ \varphi) = k(\alpha) - 1 \end{cases}$$

Occasionally we will denote the Dold-Kan equivalence functors for a category $\mathcal{A}$ by $N_{\mathcal{A}}$ and $\Gamma_{\mathcal{A}}$ if multiple categories are involved, or if the category isn't clear from the context. We begin by reciting certain properties that the functors in the Dold-Kan equivalence satisfy [dJ05, Section 14.24].

Lemma 1.A.25 The functor N reflects isomorphisms, injections, and surjections.

Recall since N and Γ are equivalences, in particular this means that they are full, faithful, and essentially surjective. They also satisfy a number of other properties.

Lemma 1.A.26 For any abelian category $\mathcal{A}$, the Dold-Kan functors satisfy the following properties:

- N is exact
- N sends simplicial homotopies to chain homotopies, and hence sends simplicially homotopic maps to chain homotopic maps
- N reflects chain homotopies
- Γ sends chain homotopies to simplicial homotopies

The majority of our results will involve the interaction of N and Γ with standard functors and natural transformations, which we collect the definitions of here for simplicity:

Definition 1.A.27 Let $(-)^{\Delta^{op}} : \mathsf{2Ab} \to \mathsf{2Ab}$ denote the pseudomonad constructed in Section 1.A.1 which sends an abelian category to its category of simplicial objects, a functor to its post composition, and natural transformation to its post composition through whiskering with simplicial objects.
From Section 1.A.1 we also have a natural transformation, $\Delta_{(-)} : (-)^{\Delta^{op}} \circ (-)^{\Delta^{op}} \Rightarrow (-)^{\Delta^{op}}$, which gives the multiplication of the pseudomonad, and on objects sends a bisimplicial complex to its diagonal.
Another important pseudonatural transformation is given by $\iota_{(-)} : 1_{\mathsf{2Ab}} \to (-)^{\Delta^{op}}$, which on an abelian category $\mathcal{A}$ has component $\iota_{\mathcal{A}}$ which sends objects to the constant simplicial object for them.

The first result we give is used in Section 1.2.1 during the proof that $\triangleleft$ gives a well-defined composition on $\mathsf{AbCat}_{\mathsf{Ch}}$.

Lemma 1.A.28 For any abelian category $\mathcal{A}$, $\Gamma_{\mathcal{A}} \circ \deg_0^{\mathcal{A}} = \iota_{\mathcal{A}}$.

Proof. Let $A \in \mathcal{A}$. Then $\Gamma_{\mathcal{A}}(\deg_0^{\mathcal{A}}(A))([n]) = A$ for all n, since $\deg_0^{\mathcal{A}}(A)$ contains A concentrated in degree 0 and their is a unique $\alpha : [n] \to [0]$ for each n. Next, for each $\alpha : [m] \to [n]$, $\Gamma(\deg_0^{\mathcal{A}}(A))(\alpha) = 1_A$ from the piecewise definition of Γ on arrows.
Next, let $f : A \to B$ be a map in $\mathcal{A}$. Then $\deg_0^{\mathcal{A}}(f)$ is the map concentrated in degree 0. Then $\Gamma(\deg_0^{\mathcal{A}}(f))([n]) = f$ for each n. It follows that $\Gamma_{\mathcal{A}} \circ \deg_0^{\mathcal{A}} = \iota_{\mathcal{A}}$, as claimed. ■

In addition to the Dold-Kan correspondence, we have two other natural functors from simplicial categories to categories of chain complexes.

Proposition 1.A.29 Let $\mathcal{A}$ be an abelian category. Then there is a functor

$$\mathfrak{C} : \mathcal{A}^{\Delta^{op}} \to \mathsf{Ch}(\mathcal{A})$$

Proof. Let $A \in \mathcal{A}_0$. We define $\mathfrak{C}(A)_n := A_n$ for $n \in \mathbb{N}$, with boundary map

$$\partial_n^A : A_n \xrightarrow{\sum_{i=0}^{n}(-1)^i A(d_i^n)} A_{n-1}$$

To show that this indeed gives a boundary map observe that

$$\begin{aligned}
\left[\sum_{i=0}^{n}(-1)^i A(d_i^n)\right] \circ \left[\sum_{i=0}^{n+1}(-1)^i A(d_i^{n+1})\right] &= \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1}) \\
&= \sum_{i=0}^{n}\sum_{j>i}^{n+1}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1}) \\
&+ \sum_{i=0}^{n}\sum_{j\le i}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1}) \\
&= \sum_{i=0}^{n}\sum_{j>i}^{n+1}(-1)^{i+j} A(d_{j-1}^n) \circ A(d_i^{n+1}) \\
&+ \sum_{i=0}^{n}\sum_{j\le i}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1}) \\
&= \sum_{i=0}^{n}\sum_{j\ge i}^{n}(-1)^{i+j+1} A(d_j^n) \circ A(d_i^{n+1}) \\
&+ \sum_{i=0}^{n}\sum_{j\le i}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1}) \\
&= -\sum_{j=0}^{n}\sum_{i\le j}(-1)^{i+j} A(d_j^n) \circ A(d_i^{n+1}) \\
&+ \sum_{i=0}^{n}\sum_{j\le i}(-1)^{i+j} A(d_i^n) \circ A(d_j^{n+1})
\end{aligned}$$

$$= 0$$

For a map $f : A \to B$ of simplicial objects, $\mathfrak{C}(f)_n := f_n$, which commutes with the differentials since the composition is bilinear in an abelian category and f commutes with the face operators being a map of simplicial sets. Since the components of $\mathfrak{C}(f)$ are given by the components of f, $\mathfrak{C}$ is functorial, completing the construction. ∎

Finally, we have a functor $D : \mathcal{A}^{\Delta^{op}} \to \mathsf{Ch}(\mathcal{A})$ given by $D(A)_n$ being the colimit over the images of the degeneracies s_i^{n-1}. From [Wei94, Lem 8.3.7] we have that $\mathfrak{C} \cong D \oplus N$. Further, for all simplicial objects $A \in \mathcal{A}_0$, $D(A)$ is acyclic by [Wei94, Thm 8.3.8].

Proposition 1.A.30 Let $A_\bullet \in \mathcal{A}^{\Delta^{op}}$ be the constant simplicial object at an object $A \in \mathcal{A}_0$. Then $N(A_\bullet) \cong \deg_0^{\mathcal{A}}(A)$.

Proof. By definition we have that for any $n \geq 1$, $N(A_\bullet)_n = \bigcap_{i=0}^{n-1} \ker(1_A) \cong 0$, while $N(A_\bullet)_0 = A_0 = A$. Thus as all boundary maps are vacuously zero maps, $N(A_\bullet) \cong \deg_0^{\mathcal{A}}(A)$. ∎

To compare with the definition in [JM03] we must introduce the concept of a mapping cone for chain complexes.

Definition 1.A.31 [Wei94, Sec. 1.5.1] The mapping cone for a chain map $f_\bullet : B_\bullet \to A_\bullet$ is given by the totalization of the bicomplex

$$\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d] & B_0 \arrow[l] \arrow[d, "f_0"'] & B_1 \arrow[l, "\partial_1^B"'] \arrow[d, "f_1"'] & B_2 \arrow[l, "\partial_2^B"'] \arrow[d, "f_2"'] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d] & A_0 \arrow[l] \arrow[d] & A_1 \arrow[l, "\partial_1^A"] \arrow[d] & A_2 \arrow[l, "\partial_2^A"] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where $A_\bullet$ is located in the 0th row and $B_\bullet$ is located in the 1st row.

The boundary maps for the mapping cone can be described by a simple 2×2 matrix.

$$B_{n-1} \oplus A_n \xrightarrow{\begin{pmatrix} \partial_{n-1}^B & 0 \\ (-1)^{n-1} f_{n-1} & \partial_n^A \end{pmatrix}} B_{n-2} \oplus A_{n-1}$$

Remark:
Let $f_\bullet : B_\bullet \to \deg_0^{\mathcal{A}}(A)$ be a chain map for $A \in \mathcal{A}_0$. Then $\mathrm{cone}(f_\bullet)$ is the chain complex that in degree $n \geq 1$ is B_{n-1}, and in degree 0 is A. For $n \geq 1$ the $n+1$st boundary map is $B_n \xrightarrow{\partial_n^B} B_{n-1}$, while the 1st boundary map is $B_0 \xrightarrow{f_0} A$.

Proposition 1.A.32 Let $f_\bullet : B_\bullet \to \deg_0^{\mathcal{A}}(A)$, $g_\bullet : C_\bullet \to \deg_0^{\mathcal{A}}(A)$, and $h_\bullet : B_\bullet \to C_\bullet$ be chain maps such that $g_\bullet \circ h_\bullet = f_\bullet$. Then we have a map $\overline{h}_\bullet : \mathrm{cone}(f_\bullet) \to \mathrm{cone}(g_\bullet)$ such that $\overline{h}_n = h_{n-1} : B_{n-1} \to C_{n-1}$ for $n \geq 1$, and $\overline{h}_0 = 1_A$.

Proof. Since $h_\bullet$ is a chain map, the described map is a chain map as

$$\begin{tikzcd} B_0 \arrow[r, "f_0"] \arrow[d, "h_0"'] & A \arrow[d, equal] \\ C_0 \arrow[r, "g_0"'] & A \end{tikzcd}$$

commutes by assumption. ■

By Proposition 1.A.10 we have for a comonad (C, ϵ, δ) on a category $\mathcal{C}$ with a natural transformation $\overline{\epsilon} : C^{\bullet+1} \to 1_{\mathcal{C}}^{\bullet+1}$ to the functor which assigns the constant complex associated to each object $A \in \mathcal{C}_0$ given by successive application of the counit. Composing with the Dold-Kan equivalence $N_{\mathcal{C}}$ we obtain a functor

$$N_{\mathcal{C}} \circ C^{\bullet+1} : \mathcal{C} \to \mathsf{Ch}(\mathcal{C}), \quad N_{\mathcal{C}}(C^{\bullet+1}(A))_n = \begin{cases} \bigcap_{i=0}^{n-1} \ker(C^i \epsilon_{C^{n-i}(A)}), & n \geq 1 \\ C(A), & n = 0 \end{cases}$$

with boundary maps

$$\bigcap_{i=0}^{n} \ker(C^i \epsilon_{C^{n+1-i}(A)}) \xrightarrow{(-1)^{n+1} C^{n+1} \epsilon_A} \bigcap_{i=0}^{n-1} \ker(C^i \epsilon_{C^{n-i}(A)}), \ n \geq 1$$

and

$$\ker(\epsilon_{C(A)}) \xrightarrow{-C\epsilon_A} C(A)$$

The map $N_{\mathcal{C}}(\overline{\epsilon}) : N_{\mathcal{C}} \circ C^{\bullet+1} \to N_{\mathcal{C}} \circ 1_{\mathcal{C}}^{\bullet+1}$ is then given by $N_{\mathcal{C}}(\overline{\epsilon})_A$ which has nth component 0 for $n \geq 1$, and 0th component is given by the natural transformation ϵ_A. From the work above the chain complex $\mathrm{cone}(\overline{\epsilon})$ is given by

$$\bigcap_{i=0}^{n-1} \ker(C^i \epsilon_{C^{n-i}(A)}) \xrightarrow{(-1)^n C^n \epsilon_A} \cdots \xrightarrow{-C^3 \epsilon_A} \ker(\epsilon_{C^2(A)}) \cap \ker(C\epsilon_{C(A)}) \xrightarrow{C^2 \epsilon_A} \ker(\epsilon_{C(A)}) \xrightarrow{-C\epsilon_A} C(A) \xrightarrow{\epsilon_A} A$$

On the other hand, using the functor $\mathfrak{C}$ we also obtain

$$\mathfrak{C}_{\mathcal{C}} \circ C^{\bullet+1} : \mathcal{C} \to \mathsf{Ch}(\mathcal{C}), \quad \mathfrak{C}_{\mathcal{C}}(C^{\bullet+1}(A))_n = C^{n+1}(A)$$

with boundary maps

$$C^{n+2}(A) \xrightarrow{\sum_{i=0}^{n}(-1)^i C^i \epsilon_{C^{n-i}(A)}} C^{n+1}(A)$$

Note that for $n=0$ we have $C^2(A) \xrightarrow{\epsilon_{C(A)} - C\epsilon_A} C(A)$. We then have a natural transformation

$$\mathfrak{C}_{\mathcal{C}} \circ C^{\bullet+1} \to N_{\mathcal{C}} \circ 1_{\mathcal{C}}^{\bullet+1}$$

Given by zero maps in degrees $n \geq 1$, and ϵ_A in degree 0, which is a map of chain complexes for each A since

$$\epsilon_A \circ (\epsilon_{C(A)} - C\epsilon_A) = \epsilon_A \circ \epsilon_{C(A)} - \epsilon_A \circ C\epsilon_A = 0$$

by naturality of ϵ. Let us denote this map by $\widetilde{\epsilon} : \mathfrak{C}_{\mathcal{C}} \circ C^{\bullet+1} \to N_{\mathcal{C}} \circ 1_{\mathcal{C}}^{\bullet+1}$. Taking the mapping cone we obtain the chain complex cone$(\widetilde{\epsilon})$ which at A is given by

$$\cdots \to C^3(A) \xrightarrow{\epsilon_{C^2(A)} - C\epsilon_{C(A)} + C^2\epsilon_A} C^2(A) \xrightarrow{\epsilon_{C(A)} - C\epsilon_A} C(A) \xrightarrow{\epsilon_A} A$$

Note that this is precisely the functor C^{Ch}.

Let $\iota_{\mathcal{C},C^{\bullet+1}} : N_{\mathcal{C}} \circ C^{\bullet+1} \to \mathfrak{C}_{\mathcal{C}} \circ C^{\bullet+1}$ denote the natural inclusion of the normalized chain complex into the full chain complex, and let $\pi_{\mathcal{C},C^{\bullet+1}} : \mathfrak{C} \circ C^{\bullet+1} \to N_{\mathcal{C}} \circ C^{\bullet+1}$ be the natural projection. Then $\pi_{\mathcal{C}} \circ \iota_{\mathcal{C}} = 1_{N_{\mathcal{C}} \circ C^{\bullet+1}}$. From our above work we obtain corresponding maps of augmented chain complexes with one direction composing to the identity and the other giving the composite

$$\begin{tikzcd}
\cdots \arrow[r] & C^3(A) \arrow[r, "\epsilon_{C^2(A)} - C\epsilon_{C(A)} + \epsilon_{C^2(A)}"] \arrow[d, "\pi_{\mathcal{C},C^{\bullet+1}(A)_3}"'] & C^2(A) \arrow[r, "\epsilon_{C(A)} - C\epsilon_A"] \arrow[d, "\pi_{\mathcal{C},C^{\bullet+1}(A)_2}"'] & C(A) \arrow[r, "\epsilon_A"] \arrow[d, equal] & A \arrow[d, equal] \\
\cdots \arrow[r] & \ker(\epsilon_{C^2(A)}) \cap \ker(C\epsilon_{C(A)}) \arrow[r, "C^2\epsilon_A"] \arrow[d, "\iota_{\mathcal{C},C^{\bullet+1}(A)_3}"'] & \ker(\epsilon_{C(A)}) \arrow[r, "-C\epsilon_A"] \arrow[d, "\iota_{\mathcal{C},C^{\bullet+1}(A)_2}"'] & C(A) \arrow[r, "\epsilon_A"] \arrow[d, equal] & A \arrow[d, equal] \\
\cdots \arrow[r] & C^3(A) \arrow[r, "\epsilon_{C^2(A)} - C\epsilon_{C(A)} + \epsilon_{C^2(A)}"'] & C^2(A) \arrow[r, "\epsilon_{C(A)} - C\epsilon_A"'] & C(A) \arrow[r, "\epsilon_A"'] & A
\end{tikzcd}$$

We wish to show that this gives a natural chain homotopy equivalence. However this follows from more general results.

Proposition 1.A.33 Let $f_\bullet : B_\bullet \to \deg_0^{\mathcal{A}}(A)$, $g_\bullet : C_\bullet \to \deg_0^{\mathcal{A}}(A)$, and $h_\bullet, k_\bullet : B_\bullet \to C_\bullet$ be chain maps such that $g_\bullet \circ h_\bullet = f_\bullet = g_\bullet \circ k_\bullet$. If $h_\bullet$ and $k_\bullet$ are chain homotopy equivalent, then $\overline{h}_\bullet$ and $\overline{k}_\bullet$ are also chain homotopy equivalent.

Proof. Let $s_n : B_n \to C_{n+1}$ be chain homotopies from $h_\bullet$ to $k_\bullet$, so

$$\partial_{n+1}^C \circ s_n + s_{n-1} \circ \partial_n^B = h_n - k_n$$

Then we define $\overline{s}_n : B_{n-1} \to C_n$ by $\overline{s}_{n+1} = s_n$ for $n \geq 0$, and $\overline{s}_0 : A \to C_0$ equal to 0 so that $g_0 \circ 0 = 1_A - 1_A = 0$, and

$$\partial_1^C \circ s_0 + f_0 \circ 0 = \partial_1^C \circ s_0 = h_0 - k_0$$

Thus we obtain a homotopy from $\overline{h}_\bullet$ to $\overline{k}_\bullet$, as desired. ■

The desired result now follows by [GJ09, Thm 2.5].

Lemma 1.A.34 The inclusion $\iota_{\mathcal{C}} : N_{\mathcal{C}} \to \mathfrak{C}_{\mathcal{C}}$ and projection $\pi_{\mathcal{C}} : \mathfrak{C}_{\mathcal{C}} \to N_{\mathcal{C}}$ constitute a natural chain homotopy equivalence.

We will show this explicitly in the case of general abelian categories **TBC**.

1.A.3 General Polynomial Approximations from Cotriples

In this section we develop of a general theory of polynomial approximations associated with additive adjunctions between abelian categories. To this end let

$$(\eta, \epsilon) : \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "L"'] \end{tikzcd}$$

be an adjunction between abelian categories such that L and R are additive functors. Let $C = L \circ R$ be the comonad with co-unit $\epsilon : L \circ R \to 1_{\mathcal{A}}$ and co-multiplication $\delta = L\eta_R : L \circ R \to L \circ R \circ L \circ R$. Note C is additive being the composite of additive functors. By Lemmas 1.A.18 1.A.19 1.A.20 we have another additive functor C^{Ch} and an adjunction

$$\begin{tikzcd} \mathsf{Ch}(\mathcal{A}) \arrow[r, bend right, "\mathsf{Ch}(R)"'] \arrow[r, phantom, "\perp"] & \mathsf{Ch}(\mathcal{B}) \arrow[l, bend right, "\mathsf{Ch}(L)"'] \end{tikzcd}$$

with comonad $\mathsf{Ch}(C)$ which preserves chain homotopies. As in the case of cross-effects we define the notion of degree.

Definition 1.A.35 We say a chain complex $A_\bullet \in \mathsf{Ch}(\mathcal{A})$ is of degree C if $\mathsf{Ch}(R)(A_\bullet)$ is chain contractible.

Next we can define the polynomial approximation associated with the comonad C.

Definition 1.A.36 We define the polynomial approximation for C, $P_C : \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})$, by the composite

$$P_C = \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(C^{\mathsf{Ch}})$$

By Lemmas 1.A.19 1.A.20 P_C is additive (exact if C is exact), and further preserves chain homotopies. We now prove some commutation type relations.

Lemma 1.A.37 Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then

$$\mathsf{Ch}(F) \circ \mathrm{Tot}_{\mathcal{A}} \cong \mathrm{Tot}_{\mathcal{B}} \circ \mathsf{Ch}^2(F)$$

Proof. Let $A_{\bullet,\bullet} \in \mathsf{Ch}(\mathcal{A})$. Then since F is additive it preserves finite direct sums, so

$$\mathsf{Ch}(F)(\mathrm{Tot}_{\mathcal{A}}(A_{\bullet,\bullet}))_n = F\left(\bigoplus_{p+q=n} A_{p,q}\right) \cong \bigoplus_{p+q=n} F(A_{p,q}) = \mathrm{Tot}_{\mathcal{B}} \circ \mathsf{Ch}^2(F)(A_{\bullet,\bullet})_n$$

and since $\mathsf{Ch}(F)$ and $\mathsf{Ch}^2(F)$ act term-wise on differentials, it follows that

$$\mathsf{Ch}(F)(\mathrm{Tot}_{\mathcal{A}}(A_{\bullet,\bullet})) \cong \mathrm{Tot}_{\mathcal{B}} \circ \mathsf{Ch}^2(F)(A_{\bullet,\bullet})$$

Further, since this isomorphism is induced by the unique map between biproducts it follows that we have the desired natural isomorphism. ∎

Lemma 1.A.38 Suppose $F : \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})$ such that F is additive, $F \circ \mathsf{Ch}(C) \cong \mathsf{Ch}(C) \circ F$ under this isomorphism $F\mathsf{Ch}(\epsilon)_{A_\bullet}$ becomes $\mathsf{Ch}(\epsilon)_{F(A_\bullet)}$. Then $\mathsf{Ch}(F) \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \cong \mathsf{Ch}(C^{\mathsf{Ch}}) \circ F$.

Proof. Let $A_\bullet \in \mathsf{Ch}(\mathcal{A})$. Recall by Lemma 1.3.5 we have $\mathsf{Ch}(C^{\mathsf{Ch}}) \cong \mathsf{Ch}(C)^{\mathsf{Ch}}$. Then we have that $\mathsf{Ch}(F) \circ \mathsf{Ch}(C)^{\mathsf{Ch}}(A_\bullet)$ gives the bicomplex

$$\cdots \longrightarrow F(\mathsf{Ch}(C)^2(A_\bullet)) \xrightarrow{F(\mathsf{Ch}(C)\epsilon_{A_\bullet}) - F(\epsilon_{\mathsf{Ch}(C)(A_\bullet)})} F(\mathsf{Ch}(C)(A_\bullet)) \xrightarrow{F(\epsilon_{A_\bullet})} F(A_\bullet)$$

The under the isomorphism $F \circ \mathsf{Ch}(C) \cong \mathsf{Ch}(C) \circ F$ this bicomplex becomes

$$\cdots \longrightarrow \mathsf{Ch}(C)^2(F(A_\bullet)) \xrightarrow{\mathsf{Ch}(C)\epsilon_{F(A_\bullet)} - \epsilon_{\mathsf{Ch}(C)(F(A_\bullet))}} F(\mathsf{Ch}(C)(A_\bullet)) \xrightarrow{\epsilon_{F(A_\bullet)}} F(A_\bullet)$$

which is exactly $\mathsf{Ch}(C)^{\mathsf{Ch}} \circ F \cong \mathsf{Ch}(C^{\mathsf{Ch}}) \circ F$. ∎

Lemma 1.A.39 We have a natural isomorphism $\mathrm{Tot}_{\mathcal{A}} \circ \mathrm{Tot}_{\mathsf{Ch}(\mathcal{A})} \cong \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(\mathrm{Tot}_{\mathcal{A}})$

Proof. The proof is primarily a bookeeping of indices. Let $A_{\bullet,\bullet,\bullet} \in \mathsf{Ch}^3(\mathcal{A})$ denote $((A_\bullet)_\bullet)_\bullet$. First note that

$$\mathrm{Tot}_{\mathsf{Ch}(\mathcal{A})}(A_{\bullet,\bullet,\bullet})_n = \bigoplus_{p,q=n} A_{\bullet,p,q}$$

with differentials

$$\partial^{(0,-1,-1)}_{\bullet,n} = (\partial^{(0,-1,0)}_{\bullet,p,q} + (-1)^{p-1}\partial^{(0,0,-1)}_{\bullet,p-1,q+1})_{p+q=n}$$

and

$$\partial^{(-1,0,0)}_{m,n} = \bigoplus_{p+q=n} \partial^{(-1,0,0)}_{m,p,q}$$

Next, we have that

$$\begin{aligned}
\mathrm{Tot}_{\mathcal{A}} \circ \mathrm{Tot}_{\mathsf{Ch}(\mathcal{A})}(A_{\bullet,\bullet,\bullet})_n &= \bigoplus_{p+q=n} \mathrm{Tot}_{\mathsf{Ch}(\mathcal{A})}(A_{\bullet,\bullet,\bullet})_{p,q} \\
&= \bigoplus_{p+q=n} \left(\bigoplus_{r+s=p} A_{\bullet,r,s} \right)_q \\
&= \bigoplus_{p+q=n} \bigoplus_{r+s=p} A_{q,r,s} \\
&\cong \bigoplus_{r+s+q=n} A_{q,r,s}
\end{aligned}$$

with maps

$$\begin{aligned}
\partial^{tottot}_n &= (\partial^{(-1,0,0)}_{p,q} + (-1)^{p-1}\partial^{(0,-1,-1)}_{p-1,q+1})_{p+q=n} \\
&= \left(\bigoplus_{r+s=q} \partial^{(-1,0,0)}_{p,r,s} + (-1)^{p-1}(\partial^{(0,-1,0)}_{p-1,r,s} + (-1)^{r-1}\partial^{(0,0,-1)}_{p-1,r-1,s+1})_{r+s=q+1} \right)_{p+q=n} \\
&= \left(\partial^{(-1,0,0)}_{p,r,s} + (-1)^{p-1}\partial^{(0,-1,0)}_{p-1,r+1,s} + (-1)^{p+r-1}\partial^{(0,0,-1)}_{p-1,r,s+1} \right)_{p+r+s=n}
\end{aligned}$$

after re-ordering. On the other hand,

$$\mathsf{Ch}(\mathrm{Tot}_{\mathcal{A}})(A_{\bullet,\bullet,\bullet})_{p,q} = \mathrm{Tot}_{\mathcal{A}}(A_{\bullet,\bullet,q})_p = \bigoplus_{r+s=p} A_{r,s,q}$$

with differentials

$$\mathrm{Tot}_{\mathcal{A}}(\partial^{(0,0,-1)}_{\bullet,\bullet,q})_p = \bigoplus_{r+s=p} \partial^{(0,0,-1)}_{r,s,q}$$

and

$$\partial^{(-1,-1,0)}_{p,q} = (\partial^{(-1,0,0)}_{r,s,q} + (-1)^{r-1}\partial^{(0,-1,0)}_{r-1,s+1,q})_{r+s=p}$$

It follows that

$$\begin{aligned}
\mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(\mathrm{Tot}_{\mathcal{A}})(A_{\bullet,\bullet,\bullet})_n &= \bigoplus_{p+q=n} \mathrm{Tot}_{\mathcal{A}}(A_{\bullet,\bullet,q})_p \\
&= \bigoplus_{p+q=n} \left(\bigoplus_{r+s=p} A_{r,s,q} \right)
\end{aligned}$$

with maps

$$\partial^{Chtot}_n = (\partial^{(-1,-1,0)}_{p,q} + (-1)^{p-1}\partial^{(0,0,-1)}_{p-1,q+1})_{p+q=n}$$

$$= \left((\partial^{(-1,0,0)}_{r,s,q} + (-1)^{r-1}\partial^{(0,-1,0)}_{r-1,s+1,q})_{r+s=p} + (-1)^{p-1} \bigoplus_{r+s=p-1} \partial^{(0,0,-1)}_{r,s,q+1} \right)_{p+q=n}$$
$$= \left(\partial^{(-1,0,0)}_{r,s,q} + (-1)^{r-1}\partial^{(0,-1,0)}_{r-1,s+1,q} + (-1)^{r+s-1}\partial^{(0,0,-1)}_{r-1,s,q+1} \right)_{r+s+q=n}$$

which agrees with our previous result. Thus up to a natural isomorphism corresponding to re-ordering the direct sums involved in the totalization, the two procedures yield identical results. ■

These results imply that when their hypotheses (in particular $P_C \circ \mathsf{Ch}(C) \cong \mathsf{Ch}(C) \circ P_C$) hold we have a sequence of natural isomorphisms

$$\begin{aligned} P_C \circ P_C &= \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \circ \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \\ &\cong \mathrm{Tot}_{\mathcal{A}} \circ \mathrm{Tot}_{\mathsf{Ch}(\mathcal{A})} \circ \mathsf{Ch}(\mathsf{Ch}(C^{\mathsf{Ch}})) \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \\ &\cong \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(\mathrm{Tot}_{\mathcal{A}}) \circ \mathsf{Ch}(\mathsf{Ch}(C^{\mathsf{Ch}})) \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \\ &\cong \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(P_C) \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \\ &\cong \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \circ P_C \\ &= P_C \circ P_C \end{aligned}$$

Note that we have a natural transformation $I_C : \deg_0^{\mathcal{A}} \to C^{\mathsf{Ch}}$ such that at $A \in \mathcal{A}_0$, $I_{C,A}$ is

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[d, equal] \\ \cdots \arrow[r] & C^2(A) \arrow[r, "\epsilon_{C(A)} - C\epsilon_A"'] & C(A) \arrow[r, "\epsilon_A"'] & A \end{tikzcd}$$

Then we define $p_C := \mathrm{Tot}_{\mathcal{A}}(\mathsf{Ch}(I_C)) : 1_{\mathsf{Ch}(\mathcal{A})} \to P_C$ using the isomorphism $\mathrm{Tot}_{\mathcal{A}} \circ \deg_0^{\mathsf{Ch}(\mathcal{A})} \cong 1_{\mathsf{Ch}(\mathcal{A})}$.

Observe that

$$P_C(p_C) : P_C \to P_C^2, \quad P_C(p_C) = \mathrm{Tot}_{\mathcal{A}}\mathsf{Ch}(C^{\mathsf{Ch}})\mathrm{Tot}_{\mathcal{A}}\mathsf{Ch}(I_C)$$

I claim that applying our sequence of isomorphisms which give an automorphism of P_C^2 we transform $P_C(p_C)$ into p_{C,P_C}.

First applying $\mathsf{Ch}(I_C)$ to $A_\bullet$ we obtain the map of bicomplexes

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & A_2 \arrow[dl, equal] \arrow[dd] \\
C^2(A_2) \arrow[rr, "\epsilon_{C(A_2)} - C\epsilon_{A_2}"] \arrow[dd] & & C(A_2) \arrow[rr, "\epsilon_{A_2}"] \arrow[dd] & & A_2 \arrow[dd] & \\
 & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & A_1 \arrow[dl, equal] \arrow[dd] \\
C^2(A_1) \arrow[rr, "\epsilon_{C(A_1)} - C\epsilon_{A_1}"] \arrow[dd] & & C(A_1) \arrow[rr, "\epsilon_{A_1}"] \arrow[dd] & & A_1 \arrow[dd] & \\
 & 0 \arrow[rr] \arrow[dl] & & 0 \arrow[rr] \arrow[dl] & & A_0 \arrow[dl, equal] \\
C^2(A_0) \arrow[rr, "\epsilon_{C(A_0)} - C\epsilon_{A_0}"'] & & C(A_0) \arrow[rr, "\epsilon_{A_0}"'] & & A_0 &
\end{tikzcd}$$

where the vertical maps are the $A_\bullet$ boundary maps possibly after applying C^n. Taking the totalization we obtain the map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, "i_3"] & A_1 \arrow[r, "\partial_1^A"] \arrow[d, "i_2"] & A_0 \arrow[d, equal] \\
\cdots \arrow[r] & C^2(A_0) \oplus C(A_1) \oplus A_2 \arrow[r, "\zeta_2"'] & C(A_0) \oplus A_1 \arrow[r, "\zeta_1"'] & A_0
\end{tikzcd}$$

where ζ_n is the $n \times n+1$ matrix

$$\begin{pmatrix}
\sum_{i=0}^{n-1}(-1)^i C^i \epsilon_{C^{n-1-i}(A_0)} & (-1)^{n-1} C^{n-1}(\partial_1^A) & 0 & \cdots & 0 \\
0 & \sum_{i=0}^{n-2}(-1)^i C^i \epsilon_{C^{n-2-i}(A_1)} & (-1)^{n-2} C^{n-2}(\partial_2^A) & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_{A_{n-1}} & \partial_n^A
\end{pmatrix}$$

Applying $\mathsf{Ch}(C^{\mathsf{Ch}})$ we obtain the map of bicomplexes, where we use the fact that C is additive so preserves direct sums, and by Lemma **TBD** the co-unit turns into a direct sum of co-units.

$$\begin{tikzcd}
& C^2(A_2) \arrow[rr, "\epsilon_{C(A_2)} - C\epsilon_{A_2}"] \arrow[dl, "i_3"'] \arrow[dd] & & C(A_2) \arrow[rr, "\epsilon_{A_2}"] \arrow[dl, "i_3"'] \arrow[dd] & & A_2 \arrow[dl, "i_3"] \arrow[dd, "\partial_2^A"] \\
C^4(A_0) \oplus C^3(A_1) \oplus C^2(A_2) \arrow[rr, "\bigoplus_{j=0}^2 (\epsilon_{C^{3-j}(A_j)} - C\epsilon_{C^{2-j}(A_j)})"] \arrow[dd, "C^2(\zeta_2)"'] & & C^3(A_0) \oplus C^2(A_1) \oplus C(A_2) \arrow[rr, "\bigoplus_{j=0}^2 \epsilon_{C^{2-j}(A_j)}"] \arrow[dd, "C(\zeta_2)"] & & C^2(A_0) \oplus C(A_1) \oplus A_2 \arrow[dd, "\zeta_2"] & \\
& C^2(A_1) \arrow[rr, "\epsilon_{C(A_1)} - C\epsilon_{A_1}"] \arrow[dl, "i_2"'] \arrow[dd] & & C(A_1) \arrow[rr, "\epsilon_{A_1}"] \arrow[dl, "i_2"'] \arrow[dd] & & A_1 \arrow[dl, "i_2"] \arrow[dd, "\partial_1^A"] \\
C^3(A_0) \oplus C^2(A_1) \arrow[rr, "(\epsilon_{C^2(A_0)} - C\epsilon_{C(A_0)}) \oplus (\epsilon_{C(A_1)} - C\epsilon_{A_1})"'] \arrow[dd, "C^2(\zeta_1)"'] & & C^2(A_0) \oplus C(A_1) \arrow[rr, "\epsilon_{C(A_0)} \oplus \epsilon_{A_1}"] \arrow[dd, "C(\zeta_1)"] & & C(A_0) \oplus A_1 \arrow[dd, "\zeta_1"] & \\
& C^2(A_0) \arrow[rr, "\epsilon_{C(A_0)} - C\epsilon_{A_0}"'] \arrow[dl, equal] & & C(A_0) \arrow[rr, "\epsilon_{A_0}"'] \arrow[dl, equal] & & A_0 \arrow[dl, equal] \\
C^2(A_0) \arrow[rr, "\epsilon_{C(A_0)} - C\epsilon_{A_0}"'] & & C(A_0) \arrow[rr, "\epsilon_{A_0}"'] & & A_0 &
\end{tikzcd}$$

Finally, taking the totalization of this map of bicomplexes we obtain the map of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & C^2(A_0) \oplus C(A_1) \oplus A_2 \arrow[r, "\zeta_2"] \arrow[d, "i_1 \oplus i_1 \oplus 1_{A_2}"'] & C(A_0) \oplus A_1 \arrow[r, "\zeta_1"] \arrow[d, "i_1 \oplus 1_{A_1}"] & A_0 \arrow[d, equal] \\
\cdots \arrow[r] & C^2(A_0) \oplus (C^2(A_0) \oplus C(A_1)) \oplus (C^2(A_0) \oplus C(A_1) \oplus A_2) \arrow[r, "\omega_2"'] & C(A_0) \oplus (C(A_0) \oplus A_1) \arrow[r, "\omega_1"'] & A_0
\end{tikzcd}$$

where ω_n is given by the formal $n \times n+1$ matrix

$$\begin{pmatrix}
\sum_{i=0}^{n-1} (-1)^i C^i \epsilon_{C^{n-1-i}(A_0)} & (-1)^{n-1} C^{n-1}(\zeta_1) & 0 & \cdots & 0 \\
0 & \bigoplus_{j=0}^{1} \sum_{i=0}^{n-2} (-1)^i C^i \epsilon_{C^{n-1-i-j}(A_j)} & (-1)^{n-2} C^{n-2}(\zeta_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \bigoplus_{j=0}^{n-1} \epsilon_{C^{n-1-j}(A_j)} & \zeta_n
\end{pmatrix}$$

and $C^{n-i}(\zeta_i)$ is the $i \times i+1$ matrix

$$\begin{pmatrix}
C^{n-i} \sum_{k=0}^{i-1} (-1)^k C^k \epsilon_{C^{i-1-k}(A_0)} & C^{n-i}(-1)^{i-1} C^{i-1}(\partial_1^A) & 0 & \cdots & 0 \\
0 & C^{n-i} \sum_{k=0}^{i-2} (-1)^k C^k \epsilon_{C^{i-2-k}(A_1)} & C^{n-i}(-1)^{i-2} C^{i-2}(\partial_2^A) & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & C^{n-i}(\epsilon_{A_{i-1}}) & C^{n-i}(\partial_i^A)
\end{pmatrix}$$

Note that the complex

$$\cdots \longrightarrow C^2(A_0) \oplus C(A_1) \oplus A_2 \xrightarrow[\zeta_2]{} C(A_0) \oplus A_1 \xrightarrow[\zeta_1]{} A_0$$

is precisely $\mathrm{Tot}_{\mathcal{A}}\mathsf{Ch}(C^{\mathsf{Ch}})(A_\bullet) = P_C(A_\bullet)$. In the case that $A_\bullet = \deg_0^{\mathcal{A}}(A)$, $P_C(\deg_0^{\mathcal{A}}(A))$ simplifies to $C^{\mathsf{Ch}}(A)$

$$\cdots \xrightarrow{\epsilon_{C^2(A)} - C\epsilon_{C(A)} + C^2\epsilon_A} C^2(A) \xrightarrow{\epsilon_{C(A)} - C\epsilon_A} C(A) \xrightarrow{\epsilon_A} A$$

On the other hand, $\mathsf{Ch}(I_C)$ at this complex gives the following map of bicomplexes, using the same properties as in the previous case:

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & C^2(A_0)\oplus C(A_1)\oplus A_2 \arrow[dl, equal] \arrow[dd, "\zeta_2"] \\
C^4(A_0)\oplus C^3(A_1)\oplus C^2(A_2) \arrow[rr, "\bigoplus_{i=0}^3(\epsilon_{C^{4-i}(A_i)} - C\epsilon_{C^{3-i}(A_i)})"] \arrow[dd, "C^2(\zeta_2)"] & & C^3(A_0)\oplus C^2(A_1)\oplus C(A_2) \arrow[rr, "\epsilon_{C^2(A_0)}\oplus\epsilon_{C(A_1)}\oplus\epsilon_{A_2}"'] \arrow[dd, "C(\zeta_2)"] & & C^2(A_0)\oplus C(A_1)\oplus A_2 \arrow[dd, "\zeta_2"] & \\
 & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[rr] \arrow[dl] \arrow[dd] & & C(A_0)\oplus A_1 \arrow[dl, equal] \arrow[dd, "\zeta_1"] \\
C^3(A_0)\oplus C^2(A_1) \arrow[rr, "(\epsilon_{C^2(A_0)} - C\epsilon_{C(A_0)})\oplus(\epsilon_{C(A_1)} - C\epsilon_{A_1})"'] \arrow[dd, "C^2(\zeta_1)"] & & C^2(A_0)\oplus C(A_1) \arrow[rr, "\epsilon_{C(A_0)}\oplus\epsilon_{A_1}"'] \arrow[dd, "C(\zeta_1)"] & & C(A_0)\oplus A_1 \arrow[dd, "\zeta_1"] & \\
 & 0 \arrow[rr] \arrow[dl] & & 0 \arrow[rr] \arrow[dl] & & A_0 \arrow[dl, equal] \\
C^2(A_0) \arrow[rr, "\epsilon_{C(A_0)} - C\epsilon_{A_0}"'] & & C(A_0) \arrow[rr, "\epsilon_{A_0}"'] & & A_0 &
\end{tikzcd}$$

Finally, taking totalization this map becomes the following map of complexes:

$$\begin{tikzcd}
C^2(A_0)\oplus C(A_1)\oplus A_2 \arrow[r, "\zeta_2"] \arrow[d, "i_3\oplus i_2\oplus 1_{A_2}"] & C(A_0)\oplus A_1 \arrow[r, "\zeta_1"] \arrow[d, "i_2\oplus 1_{A_1}"] & A_0 \arrow[d, equal] \\
C^2(A_0)\oplus(C^2(A_0)\oplus C(A_1))\oplus(C^2(A_0)\oplus C(A_1)\oplus A_2) \arrow[r, "\omega_2"'] & C(A_0)\oplus(C(A_0)\oplus A_1) \arrow[r, "\omega_1"'] & A_0
\end{tikzcd}$$

Now we try to understand our automorphism of P_C^2 and how it helps with this picture

We now prove a general proposition in analogy with [BJO+18, Prop 4.5].

Proposition 1.A.40 For $A_\bullet \in \mathsf{Ch}(\mathcal{A})$,

(i) The chain complex $P_C(A_\bullet)$ is degree C.

(ii) If $A_\bullet$ is degree C then the map $p_{C,A_\bullet} : A_\bullet \to P_C(A_\bullet)$ is a chain homotopy equivalence.

(iii) The pair $(P_C(A_\bullet), p_{C,A_\bullet} : A_\bullet \to P_C(A_\bullet))$ is universal up to chain homotopy equivalence with respect to degree C chain complexes with maps from $A_\bullet$.

Proof. We proceed with the proof in parts.

(i) By Lemma 1.A.37 we have the natural isomorphism

$$\mathsf{Ch}(R) \circ \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}(C^{\mathsf{Ch}}) \cong \mathrm{Tot}_{\mathcal{A}} \circ \mathsf{Ch}^2(R) \circ \mathsf{Ch}(C^{\mathsf{Ch}})$$

Since Tot preserves chain homotopies it is sufficient to show that for any $A_\bullet \in \mathsf{Ch}(\mathcal{A})$,

$$\mathsf{Ch}^2(R) \circ \mathsf{Ch}(C^{\mathsf{Ch}})(A_\bullet)$$

is contractible. By Lemma 1.1.17 $\mathsf{Ch}(R) \circ C^{\mathsf{Ch}}$ has a natural chain homotopy $s_k : RC^k \to RC^{k+1}$ given by η_{RC^k}. Note that if $A_\bullet \in \mathsf{Ch}(\mathcal{A})$ is a chain complex, then by naturality the square

$$\begin{tikzcd} R(C^m(A_n)) \arrow[r, "R(C^m(\partial_n^A))"] \arrow[d, "s_{m,A_n}"'] & R(C^m(A_{n-1})) \arrow[d, "s_{m,A_{n-1}}"] \\ R(C^{m+1}(A_n)) \arrow[r, "R(C^{m+1}(\partial_n^A))"'] & R(C^{m+1}(A_{n-1})) \end{tikzcd}$$

commutes. In particular, this implies that s_k defines a natural contraction of bicomplexes. Since Tot preserves natural chain homotopies we obtain the desired result.

(ii) Let $A_\bullet \in \mathsf{Ch}(\mathcal{A})$ be degree C, so that $\mathsf{Ch}(C)(A_\bullet)$ is chain contractible. Since C is additive, $\mathsf{Ch}(C)$ preserves chain homotopies so $\mathsf{Ch}(C)^n(A_\bullet)$ is chain contractible for all $n \geq 1$. Then $\mathsf{Ch}(C^{\mathsf{Ch}})(A_\bullet)$ is a first-quadrant bicomplex such that every row except the zeroth row is contractible. Therefore, by Corollary 1.6.6 we have that the natural inclusion $p_{C,A_\bullet} = \mathrm{Tot}_{\mathcal{A}}(\mathsf{Ch}(I_C)) : A_\bullet \to P_C(A_\bullet)$ is a chain homotopy equivalence, as desired.

(iii) To prove the final claim fix $A_\bullet, B_\bullet \in \mathsf{Ch}(\mathcal{A})$ such that $B_\bullet$ is degree C. Further, let $\tau : A_\bullet \to B_\bullet$ be a chain map. Then by naturality of p_C we have the commuting square

$$\begin{tikzcd} A_\bullet \arrow[r, "\tau"] \arrow[d, "p_{C,A_\bullet}"'] & B_\bullet \arrow[d, "p_{C,B_\bullet}"] \\ P_C(A_\bullet) \arrow[r, "P_C(\tau)"'] & P_C(B_\bullet) \end{tikzcd}$$

Let $s_C : P_C \to 1_{\mathsf{Ch}(\mathcal{A})}$ be the natural homotopy inverse of p_C for degree C complexes. Setting $\tau^\# = s_{C,B_\bullet} \circ P_C(\tau)$ we obtain

$$\tau^\# \circ p_{C,A_\bullet} = s_{C,B_\bullet} \circ P_C(\tau) \circ p_{C,A_\bullet} = s_{C,B_\bullet} \circ p_{C,B_\bullet} \circ \tau \simeq_{\mathsf{Ch}} \tau$$

so τ factors through $p_{C,A_\bullet}$ up to chain homotopy.

To show uniqueness for the universal property let $\sigma : P_C(A_\bullet) \to B_\bullet$ such that $\sigma \circ p_{C,A_\bullet} \simeq_{\mathsf{Ch}} \tau$. Observe that we have a commuting rectangle

$$\begin{tikzcd} A_\bullet \arrow[r, "p_{C,A_\bullet}"] \arrow[d, "p_{C,A_\bullet}"'] & P_C(A_\bullet) \arrow[r, "\sigma"] \arrow[d, "p_{C,P_C(A_\bullet)}"] & B_\bullet \arrow[d, "p_{C,B_\bullet}"] \\ P_C(A_\bullet) \arrow[r, "P_C(p_{C,A_\bullet})"'] & P_C(P_C(A_\bullet)) \arrow[r, "P_C(\sigma)"'] & P_C(B_\bullet) \end{tikzcd}$$

where from part (ii) the maps $p_{C,P_C(A_\bullet)}$ and $p_{C,B_\bullet}$ are chain homotopy equivalences.

■

To Do: Develop Theory, Check quasi-isomorphisms

1.A.4 (co)Limit Constructions on Functors

In this section we prove some general results on naturality of (co)limit constructions.

Lemma 1.A.41 Let $\mathcal{C}$ be a category with J shaped limits. Then given a choice of (co)limit for each diagram, there exists a functor $\varprojlim : \mathcal{C}^J \to \mathcal{C}$ which is unique up to unique natural isomorphism (resp. colim).

Proof. Let $\varprojlim$ be defined on objects based on a choice of limit for each diagram. Then, let $\alpha : F \Rightarrow G : J \to \mathcal{C}$ be a map of diagrams (i.e. a natural transformation) and let $\pi_F : \Delta_{\varprojlim(F)} \Rightarrow F$ and $\pi_G : \Delta_{\varprojlim(G)} \Rightarrow G$ be the limit cones. Then $\alpha \circ \pi_F : \Delta_{\varprojlim(F)} \Rightarrow G$ witnesses $\varprojlim(F)$ as a cone over G, so by the universal property there exists a unique map $\varprojlim(\alpha) : \varprojlim(F) \to \varprojlim(G)$ which commutes with the projections. By uniqueness this assignment is functorial, as desired. The colimit case follows by duality. The second claim follows from a more general result which we show next. ■

Before proving the next result note that for each $D \in \mathcal{C}^J$ we have the cone map $\omega_D : \Delta_{\varprojlim(D)} \to D$. Further, if $\alpha : D \to E$ is a map of diagrams, $\alpha \circ \omega_D = \omega_E \circ \Delta_{\varprojlim(\alpha)}$, so ω defines a functor $\mathcal{C}^J \to \mathcal{C}^J$.

Lemma 1.A.42 Let $\mathcal{C}$ be a category with J-shaped limits. Let $\gamma : \mathcal{C}^J \to \mathcal{C}$ be a functor such that each $\gamma(D)$ is a cone with cone map $\Gamma_D : \Delta_{\gamma(D)} \to D$, and for $\alpha : D \to E$ a map of diagrams, $\gamma(\alpha) : \gamma(D) \to \gamma(E)$ is a map of cones, that is to say $\Gamma_E \circ \Delta_{\gamma(\alpha)} = \alpha \circ \Gamma_D$. Then there exists a unique natural transformation $\tau : \gamma \to \varprojlim$ such that $\omega \circ \Delta_\tau = \Gamma$.

Proof. Let $\gamma : \mathcal{C}^J \to \mathcal{C}$ be as in the statement of the Lemma. Let $\omega : \mathcal{C}^J \to \mathcal{C}^J$ be as above. Then for each $D \in \mathcal{C}^J$ we have a unique map $\tau_D : \gamma(D) \to \varprojlim(D)$ such that $\omega_D \circ \Delta_{\tau_D} = \Gamma_D$. All that remains is to show that this is natural. Let $\alpha : D \to E$ be a map of diagrams. Then observe that $\gamma(D)$ becomes a cone over E via $\alpha \circ \Gamma_D = \Gamma_E \circ \Delta_{\gamma(\alpha)}$. Then we have a unique map $\gamma(D) \to \varprojlim(E)$ commuting with these projections. Since $\alpha \circ \Gamma_D = \Gamma_E \circ \Delta_{\gamma(\alpha)} = \omega_E \circ \Delta_{\tau_E \circ \gamma(\alpha)}$, it follows that the unique map is $\tau_E \circ \gamma(\alpha)$. On the other hand

$$\alpha \circ \Gamma_D = \alpha \circ \omega_D \circ \Delta_{\tau_D} = \omega_E \circ \Delta_{\varprojlim(\alpha)} \circ \Delta_{\tau_D}$$

so by uniqueness we obtain the commuting square

$$\begin{tikzcd} \gamma(D) \arrow[r, "\gamma(\alpha)"] \arrow[d, "\tau_D"'] & \gamma(E) \arrow[d, "\tau_E"] \\ \varprojlim(D) \arrow[r, "\varprojlim(\alpha)"'] & \varprojlim(E) \end{tikzcd}$$

which completes the proof that τ is natural. ■

We can use this result to now describe limits in Functor categories.

Lemma 1.A.43 Let $\mathcal{C}$ be a category with J-shaped (co)limits. Then $\mathrm{Fun}(\mathcal{A},\mathcal{C})$ has J-shaped (co)limits and the functor $\overset{Fun}{\varprojlim} : \mathrm{Fun}(\mathcal{A},\mathcal{B})^J \to \mathrm{Fun}(\mathcal{A},\mathcal{B})$ (resp. colim) then for any $X \in \mathcal{A}_0$

$$\mathrm{ev}_X \circ \overset{Fun}{\varprojlim} \cong \varprojlim \circ (\mathrm{ev}_X)_*$$

Proof. Let $D : J \to \mathrm{Fun}(\mathcal{A},\mathcal{C})$ be a J-shaped diagram. Then for any $X \in \mathcal{A}_0$ we have that $\mathrm{ev}_X \circ D$ is a J-shaped diagram in $\mathcal{C}$. Define $F : \mathcal{A} \to \mathcal{C}$ by $F(X) = \lim_{\leftarrow}(\mathrm{ev}_X \circ D)$ on objects. If $f : X \to Y$, then we have a natural transformation $\mathrm{ev}_f : \mathrm{ev}_X \to \mathrm{ev}_Y$ such that for any natural $\alpha : G \to H$ we have the commutative diagram

$$\begin{tikzcd} G(X) \arrow[r, "(\mathrm{ev}_f)_G"] \arrow[d, "\alpha_X"'] & G(Y) \arrow[d, "\alpha_Y"] \\ H(X) \arrow[r, "(\mathrm{ev}_f)_H"'] & H(Y) \end{tikzcd}$$

by the naturaly of α. Explicitly, ev is a functor $\mathcal{A} \to \mathrm{Fun}(\mathrm{Fun}(\mathcal{A},\mathcal{C}),\mathcal{C})$. This gives a natural transformation $\varprojlim(\mathrm{ev}_f)_* : \varprojlim \circ (\mathrm{ev}_X)_* \to \varprojlim \circ (\mathrm{ev}_Y)_*$ which at D gives a map $\lim_{\leftarrow}(\mathrm{ev}_f)_*(D) : \lim_{\leftarrow}(\mathrm{ev}_X \circ D) \to \lim_{\leftarrow}(\mathrm{ev}_Y \circ D)$ which we define to be $F(f)$ and is functorial in f since ev is. This defines a functor $F : \mathcal{A} \to \mathcal{C}$.

Next, note that for each $j \in J$ and each $X \in \mathcal{A}_0$ we have a unique map $\pi_{j,X} : F(X) \to D_j(X)$ by the universal property. If $f : X \to Y$ in $\mathcal{A}$, then we have the commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\pi_{j,X}"'] & F(Y) \arrow[d, "\pi_{j,Y}"] \\ D_j(X) \arrow[r, "D_j(f)"'] & D_j(Y) \end{tikzcd}$$

by definition of f in terms of $\lim_{\leftarrow}$. This implies that we have a natural transformation $\pi_j : F \to D_j$. Further, for each X the $\pi_{j,X}$ witness $F(X)$ as a limit cone, so π_J commutes with all triangles for D.

If $G : \mathcal{A} \to \mathcal{C}$ with $p_j : G \to D_j$ is another cone over D, for each $X \in \mathcal{A}_0$ we have a unique map $\alpha_X : G(X) \to F(X)$ by the universal property of the limit. Further, by the uniqueness of the map into a limit which commutes with the associated triangles we have a commuting square

$$\begin{tikzcd} G(X) \arrow[r, "G(f)"] \arrow[d, "\alpha_X"'] & G(Y) \arrow[d, "\alpha_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

so $\alpha : G \to F$ is a natural transformation. Further, α is unique since each it is unique at each component.

Finally, by our choice we have that

$$\mathrm{ev}_X \circ \overset{Fun}{\lim_{\leftarrow}}(D) = \lim_{\leftarrow}(\mathrm{ev}_X \circ D)$$

so we have equality. By Lemma 1.A.41 we have uniqueness up to unique natural isomorphism for the limit functor, so in general we have the isomorphism described in the question. ■

Corollary 1.A.44 Let $I : \mathcal{A} \to \mathsf{Cat}//\mathcal{B}$ be a functor into the lax-slice 2-category viewed as a 1-category. Suppose $\mathcal{B}$ has all colimits of shape $I(A)_0$ for each $A \in \mathcal{A}_0$. Then we have a functor $H_I : \mathcal{A} \to \mathcal{B}$ given by $H_I(A) = \lim\limits_{\rightarrow}(I(A))$ and for $f : A \to A'$, $I(f)_0 : I(A)_0 \to I(A')_0$ and $I(f)_1 : I(A)_1 \to I(A')_1 \circ I(f)_0$, $H_I(f)$ is the unique map making the diagram

$$\begin{tikzcd} H_I(A) \arrow[r, dashed] & H_I(A') \\ I(A)_j \arrow[u] \arrow[r, "(I(f)_1)_j"'] & I(A')_{I(f)_0(j)} \arrow[u] \end{tikzcd}$$

commute. Then if $F : \mathcal{B} \to \mathcal{C}$ preserves all colimits of shape $I(A)_0$, $F \circ H_I \cong H_{F_* \circ I}$ where $F_* : \mathsf{Cat}//\mathcal{B} \to \mathsf{Cat}//\mathcal{C}$ is pushforward of lax-slice 2-categories.

Proof. H_I as defined is a functor. Then for all $A \in \mathcal{A}_0$ we have that $F \circ H_I(A) = F(\lim\limits_{\rightarrow}(I(A))) \cong \lim\limits_{\rightarrow}(F \circ I(A))$. Further, for $f : A \to A'$ in $\mathcal{A}$, and all $j \in \mathrm{Ob}(I(A)_0)$, we have that the outer rectangle

$$\begin{tikzcd} F(H_I(A)) \arrow[r, "F(H_I(f))"] & F(H_I(A')) \\ H_{F_*\circ I}(A) \arrow[u, "\cong"'] \arrow[r, "H_{F_*\circ I}(f)"'] & H_{F_*\circ I}(A') \arrow[u, "\cong"] \\ F(I(A)_j) \arrow[u] \arrow[r, "F((I(f)_1)_j)"'] \arrow[uu, bend left] & F(I(A')_{I(f)_0(j)}) \arrow[u] \arrow[uu, bend right] \end{tikzcd}$$

commutes by functoriality of F. Then by the uniqueness of the map out of a colimit we have that the upper square commutes, so we obtain the desired natural isomorphism. ■

1.A.5 Translating Constructions in Goodwillie Calculus III

In this section we translate some of the methods in [Goo03] for use i our proof of the universality of P_n.

Chapter 2

Side Work

2.1.0 Pseudo-monad Attempts

2.1.1 Pseudomonad (attempt)

Let $2\mathsf{Ab}$ denote the (large) 2-category of abelian categories, arbitrary functors between them, and natural transformations. We can consider $2\mathsf{Ab}$ as an object in the Gray-category Bicat of bicategories, pseudofunctors, pseudonatural transformations, and modifications. Then $\mathsf{Ch}(-)$ is a pseudomonad on this 2-category. Explicitly, $\mathsf{Ch}(-)$ is a pseudofunctor in $\mathsf{Bicat}(2\mathsf{Ab}, 2\mathsf{Ab})$ defined as follows:

1. On 0-cells, $\mathsf{Ch}(-)$ sends an abelian category $\mathcal{A}$ to the category $\mathsf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$, concentrated in non-negative degree.

2. Given abelian categories $\mathcal{A}, \mathcal{B}$, we have a functor $\mathsf{Ch}_{\mathcal{A},\mathcal{B}} : \mathcal{B}^{\mathcal{A}} \to \mathsf{Ch}(\mathcal{B})^{\mathsf{Ch}(\mathcal{A})}$ given as follows:

   (a) On 0-cells (1-cells of the underlying bicategory) $\mathsf{Ch}_{\mathcal{A},\mathcal{B}}$ sends a functor $F : \mathcal{A} \to \mathcal{B}$ to its prolongation $\mathsf{Ch}(F) : \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{B})$. The prolongation is defined in terms of the Dold-Kan equivalence 1.A.24 as follows

   $$\mathsf{Ch}(F) : \mathsf{Ch}(\mathcal{A}) \xrightarrow{\Gamma} \mathcal{A}^{\Delta^{op}} \xrightarrow{F_*} \mathcal{B}^{\Delta^{op}} \xrightarrow{N} \mathsf{Ch}(\mathcal{A})$$

   (b) On 1-cells (2-cells of the underlying bicategory), $\mathsf{Ch}_{\mathcal{A},\mathcal{B}}$ sends a natural transformation $\gamma : F \Rightarrow G$ to a natural transformation $\mathsf{Ch}(\gamma)$ such that for $A_\bullet \in \mathsf{Ch}(\mathcal{A})_0$,

   $$\mathsf{Ch}(\gamma)_{A_\bullet} : \mathsf{Ch}(F)(A_\bullet) \to \mathsf{Ch}(G)(A_\bullet) := N(\gamma_{\Gamma(A_\bullet)})$$

3. We define $m(F,G) := NG_*\eta_{F_*\Gamma} : \mathsf{Ch}(G) \circ \mathsf{Ch}(F) \Rightarrow \mathsf{Ch}(G \circ F)$

$$\begin{tikzcd}
 & & & \mathsf{Ch}(\mathcal{B}) \arrow[dr, "\Gamma"] \arrow[d, phantom, "\Downarrow \eta_{\mathcal{B}}"] & & & \\
\mathsf{Ch}(\mathcal{A}) \arrow[r, "\Gamma"] & \mathcal{A}^{\Delta^{op}} \arrow[r, "F_*"] & \mathcal{B}^{\Delta^{op}} \arrow[ur, "N"] \arrow[rr, equal, bend right] & {} & \mathcal{B}^{\Delta^{op}} \arrow[r, "G_*"] & \mathcal{C}^{\Delta^{op}} \arrow[r, "N"] & \mathsf{Ch}(\mathcal{C})
\end{tikzcd} \tag{2.1}$$

4. For each $\mathcal{A} \in \mathsf{2Ab}$ an invertible 2-cell $i := \varepsilon : 1_{\mathsf{Ch}(\mathcal{A})} \Rightarrow \mathsf{Ch}(1_{\mathcal{A}})$

$$\begin{tikzcd}
\mathsf{Ch}(\mathcal{A}) \arrow[rr, "1_{\mathsf{Ch}(\mathcal{A})}"] \arrow[dr, "\Gamma"'] & {} \arrow[d, phantom, "\Downarrow \varepsilon"] & \mathsf{Ch}(\mathcal{A}) \\
 & \mathcal{A}^{\Delta^{op}} \arrow[ur, "N"'] &
\end{tikzcd} \tag{2.2}$$

together with the following monad data

1. A 2-cell (i.e. pseudonatural transformation) $\eta : 1_{\mathsf{2Ab}} \Rightarrow \mathsf{Ch}$ given by the following data:
   (a) For each abelian cat $\mathcal{A}$, a functor $\eta_{\mathcal{A}} := \deg_0^{\mathcal{A}} : \mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ sending an object to the chain complex
   $$\deg_0^{\mathcal{A}}(A)_n := \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$
   and a map to its action on degree zero.
   (b) For each functor $F : \mathcal{A} \to \mathcal{B}$ a natural transformation $\eta_F = \deg_0^F : \deg_{\mathcal{B}} \circ F \Rightarrow \mathsf{Ch}(F) \circ \deg_{\mathcal{A}}$
   $$\begin{tikzcd}
   \mathcal{A} \arrow[r, "F"] \arrow[d, "\deg_0^{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\deg_0^{\mathcal{B}}"] \arrow[dl, phantom, "\Leftarrow \deg_0^F"] \\
   \mathsf{Ch}(\mathcal{A}) \arrow[r, "\mathsf{Ch}(F)"'] & \mathsf{Ch}(\mathcal{B})
   \end{tikzcd}$$
   with components given by identities since $\mathsf{Ch}(F)(\deg_0^{\mathcal{A}}(A)) = \deg_0^{\mathcal{A}} F(A)$.

2. A 2-cell $m : \mathsf{Ch} \circ \mathsf{Ch} \Rightarrow \mathsf{Ch}$ given by the following data:
   (a) For every abelian category $\mathcal{A}$ a functor $m_{\mathcal{A}} := \mathrm{Tot}_{\mathcal{A}} : \mathsf{ChCh}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})$ given by the totalization. Explicitly, for $A_{\bullet,\bullet} \in \mathsf{ChCh}(\mathcal{A})$, we set
   $$m_{\mathcal{A}}(A_{\bullet,\bullet})_n := \bigoplus_{i+j=n} A_{i,j}$$
   with differential given by the components:
   $$(d_n)_{r,s} := (-1)^s d^h_{r+1,s} + (-1)^{s+1} d^v_{r,s+1}$$

For a map $F : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$, we set

$$m_{\mathcal{A}}(F) : \mathrm{Tot}_{\mathcal{A}}(A_{\bullet,\bullet}) \to \mathrm{Tot}_{\mathcal{A}}(B_{\bullet,\bullet})$$

with nth component given by $\bigoplus_{i+j=n} F_{i,j}$

(b) For each functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, a natural transformation $m_F : m_{\mathcal{B}} \circ \mathsf{Ch}^2(F) \Rightarrow \mathsf{Ch}(F) \circ m_{\mathcal{A}}$

$$\begin{tikzcd}
\mathsf{Ch}^2(\mathcal{A}) \arrow[r, "\mathsf{Ch}^2(F)"] \arrow[d, "m_{\mathcal{A}}"'] & \mathsf{Ch}^2(\mathcal{B}) \arrow[d, "m_{\mathcal{B}}"] \\
\mathsf{Ch}(\mathcal{A}) \arrow[r, "\mathsf{Ch}(F)"'] & \mathsf{Ch}(\mathcal{B})
\end{tikzcd}$$

given by **TBD**

3. An invertible 3-cell (i.e. modification) $\mu : m \circ \mathsf{Ch}m \Rrightarrow m \circ m_{\mathsf{Ch}}$ given by the following data:

   (a) For each abelian category $\mathcal{A}$, a natural transformation $\mu_{\mathcal{A}} : m_{\mathcal{A}} \circ \mathsf{Ch}m_{\mathcal{A}} \Rightarrow m_{\mathcal{A}} \circ m_{\mathsf{Ch}(\mathcal{A})}$

4. An invertible 3-cell $\lambda : m \circ \deg_0^{\mathsf{Ch}} \Rrightarrow 1_{\mathsf{Ch}}$ given by:

   (a) For each abelian category $\mathcal{A}$, a natural transformation $\lambda_{\mathcal{A}} : m_{\mathcal{A}} \circ \deg_0^{\mathsf{Ch}(\mathcal{A})} \Rightarrow 1_{\mathsf{Ch}(\mathcal{A})}$ with components given by identities.

5. And an invertible 3-cell $\rho : 1_{\mathsf{Ch}} \Rrightarrow m \circ \mathsf{Ch}\deg_0$ given by:

   (a) For each abelian category $\mathcal{A}$, a natural transformation $\rho_{\mathcal{A}} : 1_{\mathsf{Ch}(\mathcal{A})} \Rightarrow m_{\mathcal{A}} \circ \mathsf{Ch}(\deg_0^{\mathcal{A}})$. First, we observe that the following diagram commutes up to a unique natural isomorphism specified by the universal property of the kernel

$$\begin{tikzcd}
\mathcal{A}^{\Delta^{op}} \arrow[r, "(\deg_0^{\mathcal{A}})^{\Delta^{op}}"] \arrow[d, "N_{\mathcal{A}}"'] & \mathsf{Ch}(\mathcal{A})^{\Delta^{op}} \arrow[d, "N_{\mathsf{Ch}(\mathcal{A})}"] \arrow[dl, phantom, "\overset{\simeq}{\Leftarrow}"] \\
\mathsf{Ch}(\mathcal{A}) \arrow[r, "\deg_0^{\mathsf{Ch}(\mathcal{A})}"'] & \mathsf{Ch}^2(\mathcal{A})
\end{tikzcd}$$

   Additionally, from the previous 3-cell we have that $m_{\mathcal{A}} \circ \deg_0^{\mathcal{A}}(N\Gamma(A_\bullet)) = N\Gamma(A_\bullet)$, so the components of ρ are given by the $\eta : 1_{\mathsf{Ch}(\mathcal{A})} \Rightarrow N_{\mathcal{A}}\Gamma_{\mathcal{A}}$ from the Dold-Kan equivalence composed with the above natural isomorphism. **Need to make more explicit and show coherence diagram**

It remains to show that this data satisfies the necessary coherence diagrams. We shall show these in a sequences of lemmas.

2.1.2 Quotient Monad

Although we have yet to show that $\mathsf{Ch}(-)$ defines a pseudomonad on $2\mathsf{Ab}$, we claim that it does define a monad on the 1-category AbCat consisting of abelian categories and natural isomorphism classes of functors. Since horizontal composition of natural transformations is functorial, the partition given by natural isomorphism classes of functors is associated with a congruence relation, and hence AbCat is a well-defined 1-category. We will denote the isomorphism class of a functor F by $[F]$ throughout.

We show in a sequence of lemmas that $\mathsf{Ch}(-)$ is a well-defined monad on AbCat, define on objects as before and defined on natural isomorphism classes of functors by $\mathsf{Ch}([F]) := [\mathsf{Ch}(F)]$. In order to show that this is well-defined we first demonstrate Ch is strictly functorial on isomorphism classes:

Lemma 2.1.1 Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Then $[\mathsf{Ch}(G) \circ \mathsf{Ch}(F)] = [\mathsf{Ch}(G \circ F)]$.

Proof. Using $NG_*\eta_{F_*\Gamma}$, as in Equation (2.1), we have that $\mathsf{Ch}(G) \circ \mathsf{Ch}(F)$ and $\mathsf{Ch}(G \circ F)$ are naturally isomorphic. ■

With this functoriality result we can show that Ch is a well-defined functor on the quotient category.

Lemma 2.1.2 Ch defines a functor on AbCat.

Proof. It remains to show that Ch is well-defined on arrows, and sends identities to identities, since Lemma 2.1.1 provides functoriality. Let $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ be a natural isomorphism. Then $\mathsf{Ch}(\alpha) = N\alpha_\Gamma : \mathsf{Ch}(F) \Rightarrow \mathsf{Ch}(G)$ is a natural transformation, and further, for any $A_\bullet \in \mathsf{Ch}(\mathcal{A})_0$,

$$N(\alpha_{\Gamma(A_\bullet)}) \circ N(\alpha^{-1}_{\Gamma(A_\bullet)}) = N(\alpha_{\Gamma(A_\bullet)}) \circ \alpha^{-1}_{\Gamma(A_\bullet)} = N(1_{G(\Gamma(A_\bullet))}) = 1_{\mathsf{Ch}(G)(A_\bullet)}$$

using functoriality of N. The other composition is identical, so $\mathsf{Ch}(\alpha)$ is a natural isomorphism, implying $[\mathsf{Ch}(F)] = [\mathsf{Ch}(G)]$.
Finally, consider an identity functor $1_\mathcal{A}$. Then using the invertible 2-cell ε in Equation (2.2) we witness that $[1_{\mathsf{Ch}(\mathcal{A})}] = [\mathsf{Ch}(1_\mathcal{A})]$, so $\mathsf{Ch}(-)$ is a well-defined functor on AbCat. ■

It remains to show that the functor $\mathsf{Ch}(-)$ has the structure of a monad. We take the unit and multiplication 2-cells to be defined as in our description of the possible 2-monad $\mathsf{Ch}(-)$ on $2\mathsf{Ab}$. As $\deg_0$ is already natural, we need only show that m is natural in AbCat, and that the appropriate monad laws hold.

Lemma 2.1.3 Viewed as a map in AbCat, $m : \mathsf{Ch}^2 \Rightarrow \mathsf{Ch}$ is a natural transformation.

Proof. Explicitly, for each abelian category $\mathcal{A}$, $m_{\mathcal{A}} = [\mathrm{Tot}_{\mathcal{A}}]$ is an isomorphism class of functors. Showing naturality is then equivalent to showing the following equation for any $F : \mathcal{A} \to \mathcal{B}$

$$[\mathrm{Tot}_{\mathcal{B}} \circ \mathsf{Ch}^2(F)] = [\mathsf{Ch}(F) \circ \mathrm{Tot}_{\mathcal{A}}]$$

Recall $[\mathsf{Ch}(F)] = [N_{\mathcal{B}} \circ F_* \circ \Gamma_{\mathcal{A}}]$ and $[\mathsf{Ch}^2(F)] = [N_{\mathsf{Ch}(\mathcal{B})} \circ (N_{\mathcal{B}})_* \circ (F_*)_* \circ (\Gamma_{\mathcal{A}})_* \circ \Gamma_{\mathsf{Ch}(\mathcal{A})}]$.
TBD ■

Chapter 3

Project Notes and Future Directions

3.1.0 Current Questions

- Look up mapping cone and its relation to homotopy limits in an abelian category
- Look up homotopy limits

3.2.0 Meeting Notes

3.2.1 September 27 Notes

- Began looking through BJORT[BJO⁺18] section 2, and in particular the notion of a cross-effect
- Went over preliminary definitions, such as that of an abelian category
- Analyzed the inductive definition of the cross-effect functor, and determined how it is explicitly constructed.

3.2.2 October 4 Notes

- Went through the proofs of Lemma 2.4 and Proposition 2.5 ourselves and argued for the naturality of the counit.

3.2.3 October 18 Notes

- In the paper we begin with a functor $F : \mathcal{B} \to \mathcal{A}$, and produce a functor $D_1F : \mathcal{B} \to \mathsf{Ch}\mathcal{A}$, but this results in issues of composition and functoriality if we have another functor $G : \mathcal{C} \to \mathcal{B}$. Although we can consider $D_1(F \circ G)$, $D_1(F) \circ D_1(G)$ is not well typed as $D_1(F) : \mathcal{B} \to \mathsf{Ch}\mathcal{A}$ and $D_1(G) : \mathcal{C} \to \mathsf{Ch}\mathcal{B}$.
- Question:

  > To what degree is $\mathsf{Ch} : \mathsf{AbCat} \to \mathsf{AbCat}$ a monad on AbCat? (In fact it is a psuedo-monad, and we must be careful on how maps on 1-cells and 2-cells is defined)

- We are not going to work with AbCat directly, but rather a quotient of AbCat.
- In this context we can ask if chain homotopy equivalences are pointwise, or can be promoted to being natural?
-

3.3.0 To-do List

This section lists tasks which are yet to be completed from previous meetings

Completed Sec 2:

1. The cross effects operation in Definition 2.1 is explicitly defined to be functorial in the X variables. Is it also functorial in F? That is, is cr_n a functor from the category $Fun(B, A)$ to $Fun(B^n, A)$ (these categories have functors as objects and natural transformations as morphisms). See remarks before Lemma 2.4. What else needs to be verified?
2. Verify that the counit in Remark 2.8 of [BJO+18] is natural in X. Is it also a natural transformation $\mathrm{cr}_n \Rightarrow \mathrm{id}$?
3. Is the contracting chain homotopy in Lemma 2.9 of [BJO+18] natural in A?

Completed* Sec 3:

1. In Observation 3.1, we claim that Ch is a pseudomonad [BJO+18]. Is it? This should be viewed with skepticism.
2. In Observation 3.1 [BJO+18], we claim that there is a quotient monad Ch acting on the category of abelian categories and isomorphism classes of functors. Show that there is such a monad.

3. While you are at it, please make sure you understand the phrase "here we are not interested in the 2-dimensional aspects." What are these two dimensional aspects, and what is the consequence of ignoring them?
4. In definition 3.2 [BJO+18], we use natural isomorphism classes. What happens if you use pointwise defined isomorphism classes?
5. At the top of page 388 (following the proof of Lemma 3.4) [BJO+18], we establish an equivalence relation on $\mathsf{AbCat}_{\mathsf{Ch}}$. Do both pointwise defined chain homotopy equivalences and natural chain homotopy equivalences result in an equivalence relation on this category?
6. Is it possible to alter Definition 3.5 [BJO+18] to use natural chain homotopy equivalence classes instead of pointwise chain homotopy equivalence classes?

Not completed Sec 4:

- Go through section 4 and try to go through with the example of the identity in mind.

Separate To-Do:

- Make a list/section of lemmas for Γ and N in the Dold-Kan equivalence

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