Physics 375: Waves and Optics

Phys 375

September 2, 2023

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Solo Pursuit of Learning

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0.1.0 Review of Oscillations

0.1.1 Mass Spring System

Consider a mass spring system with mass M and a massless, frictionless spring of spring constant k_s. Letting $x = 0$ denote the equilibrium position of the mass spring system the force on the mass due to the spring, when located at a position x, is given by **Hooke's law**:

$$F = -k_s x \tag{0.1.1}$$

Note that this describes a restorative force, directed towards the equilibrium position of the mass spring system. We may also consider the length of the spring, s, in which case if the equilibrium length of the spring is l, Hooke's law may be expressed as

$$F = k_s(l - s) \tag{0.1.2}$$

In general, a **restoring force** is a force acting on a system which seeks to return the system to some stable equilibrium state.

The oscillation frequency, ν, for the mass spring system can be given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k_s}{M}} \quad (\text{cycles/sec} = Hz) \tag{0.1.3}$$

Applying Newton's second law we can derive this expression through the equation of motion for the mass:

$$M\frac{d^2x}{dt^2} = -k_s x \tag{0.1.4}$$

Provided $k_s > 0$, this system has an oscillatory solution. If we consider the initial condition $x(0) = x_0$ and $\dot{x}(0) = v$, then

$$x(t) = x_0 \cos(\omega t) + \frac{v}{\omega}\sin(\omega t) \tag{0.1.5}$$

where $\omega = \sqrt{\frac{k_s}{M}}$ is the angular frequency of the system, giving the previous frequency $\nu = \frac{\omega}{2\pi}$.

Recall that the spring force is conservative (i.e. corresponds to an exact form), and so it has a potential function described by $F = -\nabla U$, which implies:

$$U = \frac{1}{2}k_s x^2 \tag{0.1.6}$$

We can consider the oscillation in a mass spring system as being done by the energy, as potential energy is converted to kinetic and then back, with the total energy always being conserved since we are in an isolated system with a conservative force being the only force present.

0.1.2 Other Mechanical Oscillation Systems

Next let us consider a pendulum consisting of a massless string of length l from its support, and a mass M attached at the base of the string. The restoring force in this context is the gravitational force on the mass M. Explicitly, accounting for the constraining force for the string, the net restoring force when the mass is at an angle θ is:

$$F = -Mg\sin\theta \tag{0.1.7}$$

where we are using circular coordinates with the angle measured counter clockwise. Using the arc-length change of variables for the circular pendulum path, $x(t) = l\theta(t)$, the equation of motion can be written as

$$Ml\frac{d^2\theta}{dt^2} = -Mg\sin(\theta) \tag{0.1.8}$$

Note this will indeed give an equation of motion independent of the mass M. However, since this is a **nonlinear differential equation** its solution is not as simple as the mass spring system. In fact, the solution cannot be expressed in a closed form. However, provided $|\theta| \ll 1$ *rad* we can Taylor expand $\sin\theta$ to obtain an approximate linear form of this system:

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta \tag{0.1.9}$$

Thus, for small angles we obtain the approximate solution:

$$\theta(t) \approx \theta_0\cos(\omega t) + \frac{\dot{\theta}_0}{\omega}\sin(\omega t), \quad \omega = \sqrt{\frac{g}{l}} \tag{0.1.10}$$

Next let us consider a balance wheel, such as in a watch, which oscillates about about its center with a spiral spring to provide a restoring torque. Let I (kg·m^2) denote the moment of inertia of the wheel. We can describe the restoring torque provided by the spring using $\tau = -k_\tau\theta$ where k_τ is the torsional constant and θ is the rotational angle of the wheel balance measured from the equilibrium (zero torsion) angular position.

Recall that the equation of motion for a rotational system such as this is given by:

$$I\frac{d^2\theta}{dt^2} = \tau = -k_\tau\theta \tag{0.1.11}$$

This again describes an oscillation, now with angular frequency:

$$\omega = \sqrt{\frac{k_\tau}{I}} \tag{0.1.12}$$

0.1.3 Electromagnetic Oscillation

Oscillatory behaviour can be also found in circuits, such as an isolated LC circuit, which will oscillate with angular frequency:

$$\omega = \frac{1}{\sqrt{LC}} \tag{0.1.13}$$

The energies oscillating in these systems are electric and magnetic potential energies stored in the capacitor and inductor, respectively.

Consider the scenario of a capacitor charged to a charge q_0 connected to an inductor of inductance L with the closing of a switch. We can describe the voltage across the capacitor as:

$$V_C(t) = \frac{q(t)}{C} \tag{0.1.14}$$

while the potential across the inductor is described by

$$V_L(t) = -L\frac{dI(t)}{dt} \tag{0.1.15}$$

so by Kirchhoff's voltage law

$$L\frac{d^2q}{dt^2} = -\frac{q}{C} \tag{0.1.16}$$

where $I(t) = -\frac{dq(t)}{dt}$ when considering a discharging capacitor. This gives the angular frequency described previously.

Recall that the electric potential energy stored in a capacitor may be described as:

$$U_e = \frac{q^2(t)}{2C} \tag{0.1.17}$$

while the magnetic potential energy stored in an inductor is given by:

$$U_m = \frac{1}{2}LI^2(t) \tag{0.1.18}$$

0.1.4 Damped Oscillation

Consider a capacitor C with a charge q_0 which is connected to an inductor L through a finite resistance R. Using Kirchhoff's voltage law we can again write:

$$\frac{q(t)}{C} - RI(t) - L\frac{dI(t)}{dt} = 0 \tag{0.1.19}$$

Recalling $I(t) = -dq(t)/dt$, this becomes a second order linear differential equation with constant coefficients:

$$\frac{d^2q(t)}{dt^2} + \frac{R}{L}\frac{dq(t)}{dt} + \frac{1}{LC}q(t) = 0 \tag{0.1.20}$$

We can solve this system using the method of the Laplace transform, but for the current study this is sufficient. In the case of weak damping, $R \ll \sqrt{\frac{2L}{C}}$, this system has an approximate solution:

$$q(t) = q_0e^{-\gamma t}\cos\omega t \tag{0.1.21}$$

where $\gamma = \frac{R}{2L}$ is the damping constant.