Electromagnetism 1

Phys 355: E& M1

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Recap

Contents

Chapter 1

Electric Fields

1.1 Point Charges

Definition 1.1.1: Charge

An **electric charge** *is an intrinsic characteristic of fundamental particles. The unit of charge is the* **Coulomb***, C, and is defined as an ampere second*

$$i = \frac{dq}{dt} \implies dq = idt \implies 1\ C = 1\ A \cdot 1\ s \tag{1.1.1}$$

Definition 1.1.2: Coulomb's Law

Coulomb's Law *states that for two charged particles of charge q_1 and q_2, with a relative position vector $\vec{r}$ from 1 to 2 we have that the force on q_2 by q_1 is*

$$\vec{F}_{on2by1} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}\hat{r} \tag{1.1.2}$$

Definition 1.1.3: Electric Field

We define the **electric field** *at a location in space and time by*

$$\vec{E}(x,y,z,t) = \frac{\vec{F}(x,y,z,t)}{q} \tag{1.1.3}$$

where q is some non-zero test charge.

Remark 1.1.1

Note that by definition, electric field lines point in the direction positive charges feel a force. That is positive charges will accelerate along field lines while negative charges will not.

Corollary 1.1.1

By the definition of electric field, we find that the electric field generated by a point charge Q is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|^2}\hat{r} \tag{1.1.4}$$

Corollary 1.1.2: Superposition Principle

Suppose N point charges exist in some region, each creating an electric field $\vec{E}_{A_i}$ at a point A. Then the net electric field at A is

$$\vec{E}_A = \sum_{i=1}^{N} \vec{E}_{A_i} \tag{1.1.5}$$

1.2 Dipoles

Definition 1.2.1: Electric Dipole

An **electric dipole** *is a set of two equal and opposite charges separated by a small distance s.*

Definition 1.2.2: Dipole Moment

We define $\vec{p} := qs\hat{r}$ as the **dipole moment** *of the dipole, where $\hat{r}$ points from the negative charge to the positive charge.*

Proposition 1.2.1: Perpendicular Axis

It can be shown that the electric field along a line perpendicular to the charge axis and along the center of the charges is

$$\vec{E}_{net,\perp} = \frac{-k_e\vec{p}}{\left[d^2 + \left(\frac{s}{2}\right)^2\right]} \tag{1.2.1}$$

where d is the perpendicular distance to the observation point. If $d >> s$ we can approximate and write

$$\vec{E}_{net,\perp} \approx \frac{-k_e\vec{p}}{d^3} \tag{1.2.2}$$

Proposition 1.2.2: Charge Axis

For the electric field along the charge axis we, let y denote the distance (magnitude) from the center of the dipole, with $y > \frac{s}{2}$:

$$\vec{E}_{net,axis} = \frac{2k_e y}{\left(y - \frac{s}{2}\right)^2\left(y + \frac{s}{2}\right)^2}\vec{p} \tag{1.2.3}$$

If $0 \leqslant y < \frac{s}{2}$ then the above expression must be multiplied by a negative. For $y >> \frac{s}{2}$ we have *the approximation*

$$\frac{2k_e}{y^3}\vec{p} \tag{1.2.4}$$

1.3 Particles In Matter

Theorem 1.3.1: Conservation of Charge

The net charge of a system plus its surroundings cannot change. Pairwise creation and annihilation of charges is possible, but the net charge will remain the same.

1.3.1 Conductors and Insulators

Definition 1.3.1: Conductor

In a **conductor** *charges can move freely, and there are a large number of mobile charged particles. In equilibrium excess charge resides on the surface and the inner material is an equipotential.*

Definition 1.3.2: Insulator

In an **insulator** *charges cannot move freely. In equilibrium excess charge is uniformly distributed throughout the volume as polarized dipole moments of the constituent molecules. In general, the inside of an insulator is not an equipotential.*

Definition 1.3.3: Semiconductor

In a **semi-conductor** *charges can move freely but there are less of them than in conductors.*

Definition 1.3.4: Superconductors

Superconductors *are materials that are perfect conductors once a certain temperature has been reached. allowing charge to flow without hindrance.*

1.3.2 Polarization

Definition 1.3.5

A neutral (and usually symmetrical) atom is polarized by an external charge. Two opposite charges with a slight separation between them form an **electric dipole**

Definition 1.3.6: Polarizability

We quantify the amunt of polarization induced in a material by its resulting dipole moment $\vec{p}$. In general the polarized molecule has a dipole moment

$$\vec{p} = \alpha \vec{E} \tag{1.3.1}$$

where α is the **polarizability** *of the material.*

Definition 1.3.7

In conductors the sea of electrons is mobile, so the electrons will move when the conductor is polarized creating a **drift speed**

$$\overline{v} = uE_{net} \tag{1.3.2}$$

where u is the mobility of charge in the conductor.

Definition 1.3.8: Drude Model

The **drude model** *is a classical model for the motion of electrons in conductors in the presence of an electric field, and is quantified by*

$$\frac{\Delta \vec{p}}{\Delta t} = \vec{F}_{net} \tag{1.3.3}$$

Then we can write

$$\overline{v} = \frac{eE_{net}\overline{\Delta t}}{m} \tag{1.3.4}$$

where m is the mass of the charge,= and $\overline{\Delta t}$ is the average time between collisions. From this we find the mobility of charges to be

$$u = \frac{e\overline{\Delta t}}{m} \tag{1.3.5}$$

1.4 Distributed Charges

Remark 1.4.1

To find the electric field due to a continuous charge distribution at a point A, first choose an arbitrary charge element dq on the material. Determine an expression for the direction and magnitude of the field due to dq at P using Coulomb's Law for a point charge. Break you field expression into dE_x, dE_y, and dE_z. Consider symmetry to reduce certain components to zero before calculation. Then, integrate over the material to obtain each component of the net field. In particular, for $\alpha \in \{x, y, z\}$ integrate

$$E_\alpha = \int \lambda ds, \quad E_\alpha = \int\int \eta dA, \ \textit{or } E_\alpha = \int\int\int \rho dV \tag{1.4.1}$$

defpending on the object, where $dq = \lambda ds, \eta dA, \rho dV$ depending if we have a curve, surface, *or volume.*

Proposition 1.4.1: Line of charge

The electric field about a perpendicular axis through the center of a line of charge, say the y axis, is

$$\vec{E} = \frac{2k_e\lambda}{y\sqrt{\frac{4y^2}{L^2}+1}}\hat{y} \tag{1.4.2}$$

where $\lambda = \frac{Q}{L}$, Q is the charge of the line, and L is its length. Taking the limit as L goes to infinity we have

$$\vec{E} = \frac{2k_e\lambda}{y}\hat{y} \tag{1.4.3}$$

Proposition 1.4.2: Ring of Charge

The electric field about a perpendicular axis through the center of a ring of charge, say the z axis, is

$$\vec{E} = \frac{k_eQz}{(R^2+z^2)^{3/2}}\hat{z} \tag{1.4.4}$$

where Q is the charge and R the radius of the ring.

Proposition 1.4.3: Disk of Charge

The electric field about a perpendicular axis through the center of a disk of charge, say the z axis, is

$$\vec{E} = \frac{\eta}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z} \tag{1.4.5}$$

where η is the surface charge density of the disk.

Proposition 1.4.4: Infinite Plane

Extending the last example to that of an infinite plane by taking $R \to \infty$ we have that

$$\vec{E} = \frac{\eta}{2\varepsilon_0}\hat{z} \tag{1.4.6}$$

Proposition 1.4.5: Capacitor

Using the infinite plane electric field we can approximate the electric field between a parallel plate capacitor by

$$E_{capacitor} = \frac{\eta}{\varepsilon_0} \tag{1.4.7}$$

Proposition 1.4.6: Thin Spherical Shell

Given a thin spherical shell (insulating or conducting) we have that

$$E = \begin{cases} E_{inside} = 0 & \text{if } R > r \\ E_{outside} = \frac{k_eQ}{r^2} & \text{if } R < r \end{cases} \tag{1.4.8}$$

if R is the radius of the shell.

Proposition 1.4.7: Solid Sphere

For a conducting sphere, the excess charge must remain on the sphere's surface giving the same electric field as a spherical shell. For an insulating sphere the charge is distributed uniformly through the volume giving:

$$E = \begin{cases} E_{inside} = \frac{k_e Q}{R^3} r & \text{if } R > r \\ E_{outside} = \frac{k_e Q}{r^2} & \text{if } R < r \end{cases} \tag{1.4.9}$$

Chapter 2

Electric Potentials

2.1 Potential Energy

Definition 2.1.1

Note that the Coulomb force is conservative force so we can define a potential function, called the **electric potential energy***, by*

$$\Delta U_E = -\int_A^B \vec{F}_E \cdot d\vec{r} \tag{2.1.1}$$

so that $\vec{F}_E = -\nabla U_E$*. In particular, we have that* $\Delta U_E = -W_C$ *where* W_C *is the work done by the coulomb force.*

Definition 2.1.2: Electric Potential

We define the **electric potential** *associated with an electric potential energy by*

$$\Delta V = \frac{\Delta U}{q} = -\int_A^B \vec{E}_E \cdot d\vec{r} \tag{2.1.2}$$

In particular, we define $V_\infty = 0$ *so we can express* V *at any point in space as*

$$V(B) = -\int_\infty^B \vec{E}_E \cdot d\vec{r} \tag{2.1.3}$$

The units of V *are volts,* $V = J/C$*.*

Remark 2.1.1

The electric potential decreases in the direction of the existing electric field $\vec{E}$*, while it increases in the opposite direction.*

Definition 2.1.3: Equipotentials

We define an equipotential surface to be a surface in which V = constant, so $\Delta V = 0$, $\Delta U = 0$, and $W_C = 0$.

Proposition 2.1.1: Infinite Wire

The potential difference between points at perpendicular distances r_A and r_B from an infinite line of charge is

$$\Delta V(A \to B) = 2k_e\lambda \ln\left|\frac{r_A}{r_B}\right| \tag{2.1.4}$$

Remark 2.1.2

The electric potential at any point in space (relative to infinity) due to N point charges is

$$V_{net} = \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} \frac{k_e q_i}{r_i} \tag{2.1.5}$$

Proposition 2.1.2: Ring of Charge

The potential difference between two points A and B along the line perpendicular to and through the center of a ring of charge is

$$\Delta V(A \to B) = k_e q\left[\frac{1}{\sqrt{R^2 + z_B^2}} - \frac{1}{\sqrt{R^2 + z_A^2}}\right] \tag{2.1.6}$$

Taking the limit as A goes to infinity we have

$$V_B = \frac{k_e q}{\sqrt{R^2 + z_B^2}} \tag{2.1.7}$$

Remark 2.1.3

One can determine the electric potential at a point in space using its definition as a line integral, or if the surface generating is the electric field is nice and known, then

$$V = \int dV = \int \frac{k_e}{r} dq \tag{2.1.8}$$

Proposition 2.1.3: Disk of Charge

Above a charged disk (along a line perpendicular to the disk and through its center) we have the potential

$$\Delta V(A \to B) = -\frac{\eta}{2\varepsilon_0}\left[(z_B - z_A) - (\sqrt{z_B^2 + R^2} - \sqrt{z_A^2 + R^2})\right] \tag{2.1.9}$$

and relative to infinity

$$V = \frac{\eta}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - |z|\right] \tag{2.1.10}$$

Proposition 2.1.4: Conducting Sphere

For a charged conducting sphere we have that

$$\Delta V(A \to B) = \frac{k_e Q}{r_B} - \frac{k_e Q}{r_A} \tag{2.1.11}$$

if $r_B, r_A \geqslant R$*, and*

$$V_{outside} = \frac{k_e Q}{r} \tag{2.1.12}$$

Inside the sphere we have an equipotential with

$$V_{inside} = \frac{k_e Q}{R} \tag{2.1.13}$$

Proposition 2.1.5: Insulating Sphere

Outside a charged insulating sphere we have

$$\Delta V(A \to B) = \frac{k_e Q}{r_B} - \frac{k_e Q}{r_A} \tag{2.1.14}$$

if $r_B, r_A \geqslant R$*, and*

$$V_{outside} = \frac{k_e Q}{r} \tag{2.1.15}$$

Inside the insulating sphere we do not have an equipotential and

$$V_A = \frac{k_e Q}{2R}\left[3 - \frac{r_A^2}{R^2}\right] \tag{2.1.16}$$

for $r_A < R$*.*

2.2 Energy Density and Insulators

Definition 2.2.1

An applied field polarizes an insulator. Due to this the net electric field inside the insulator is reduced due the opposing electric fields caused by the superposition of the induced dipoles, causing

$$\vec{E}_{ins} = \frac{\vec{E}_{app}}{\kappa} \tag{2.2.1}$$

where κ is the dielectric constant of the material, and

$$\Delta V_{ins} = \frac{\Delta V_{app}}{\kappa} \tag{2.2.2}$$

Definition 2.2.2: Energy Density

The **energy density** *in a region of space is given by*

$$\frac{\Delta U}{\Delta Vol} = \frac{1}{2}\varepsilon_0 E^2 \tag{2.2.3}$$

Chapter 3

Magnetic Fields

3.1 Detecting Magnetic Fields and Electron Currents

Definition 3.1.1: Electron Current

*The **electron current** i is the number of electrons per second that enter a section of a conductor. As electrons drift through the wire their collisions with atomic cores heat up the wire, preventing them from speeding up continuously.*

Definition 3.1.2: Detecting Magnetic Fields

We can use the needle of a compass to detect and measure magnetic fields, and in particular the Earth's magnetic field. If we measure the deflection θ from the direction of the Earth's magnetic field due to a perpendicular known, generated magnetic field B_{wire} we have that

$$\tan(\theta) = \frac{B_{wire}}{B_{Earth}} \tag{3.1.1}$$

3.2 Biot Savart Law

Definition 3.2.1: Biot-Savart Law

A moving charge generates a magnetic field. In particular Biot-Savart's Law states that

$$\vec{B}_{pt,charge} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} \tag{3.2.1}$$

Biot-Savart's Law is only correct for $v << c$.

3.3 Conduction and Currents

Definition 3.3.1: Model of Conduction

For conduction of electrons through a wire we model the electron current by

$$i_e = n_e A v_d = n_e A \frac{e\tau}{m} E \tag{3.3.1}$$

where n_e is the electron density and τ is the average time between collisions.

Definition 3.3.2: Current

For general **charge carriers** *in a conductor, of charge q, we have that the current in the conductor is*

$$I = |q| n A v_d \tag{3.3.2}$$

where n is the charge carrier density, A is the cross-sectional area, and v_d is the drift speed superimposed on the random motion of the charge carriers.

Proposition 3.3.1: Superposition Principle

Magnetic fields obey the superposition principle, so given N magnetic fields at a point P we have that the net magnetic field at P is

$$\vec{B}_{net} = \sum_{i=1}^{N} \vec{B}_i \tag{3.3.3}$$

Definition 3.3.3: Magnetic Field due to Currents

Given a current carrying wire with wire segment $d\vec{s}$, pointing in the direction of conventional current, we have that the magnetic field at a position $\vec{r}$ relative to the segment due to the current in that segment is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \hat{r}}{r^2} \tag{3.3.4}$$

We then integrate over the wire to get the net magnetic field.

Proposition 3.3.2: Straight Long Wire

Along a line perpendicular to a wire with current running through it, the magnitude of the magnetic field is given by

$$B_{wire} = \frac{\mu_0 I L}{2\pi d \sqrt{L^2 + d^2}} \tag{3.3.5}$$

for length L and distance d. For $L >> d$ we have that

$$B_{wire} \approx \frac{\mu_0 I}{2\pi d} \tag{3.3.6}$$

Proposition 3.3.3: Loop

The magnitude of the magnetic field along a line perpendicular to a circular loop carrying current is

$$B_{loop} = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}} \tag{3.3.7}$$

If the location is at the center of the loop ($z = 0$) then we have

$$B_{loop} = \frac{\mu_0 I}{2R} \tag{3.3.8}$$

Conversely, if $z >> R$ then we have

$$B_{loop} = \frac{\mu_0 I R^2}{2z^3} \tag{3.3.9}$$

Definition 3.3.4: Magnetic Moment

We define the magnetic moment of a closed loop carrying current as

$$\vec{\mu} = I\vec{A} \tag{3.3.10}$$

where $\vec{A}$ points in the direction of the generated magnetic field, so

$$\vec{B}_{axis} = \frac{\mu_0}{4\pi}\frac{2\vec{\mu}}{z^3} \tag{3.3.11}$$

in the case of a circular loop.

3.3.1 Atomic Structures

Definition 3.3.5: Electron Orbits

We define the magnetic dipole moment of an electron orbiting a nucleus to be

$$\mu_{e^-} = \frac{1}{2} eRv \tag{3.3.12}$$

Then the angular momentum is $L = Rmv$ since it is in circular motion, so $Rv = L/m$ and we have

$$\mu_{e^-} = \frac{1}{2} e\frac{L}{m} \tag{3.3.13}$$

Note that the angular momentum is quantized such that $L = N\hbar$ for $N \in \mathbb{N}$.

Chapter 4

Electric Fields and Circuitry

4.1 Circuits

Definition 4.1.1

A circuit is said to be in a **steady state** *if the charges are moving ($\overline{v} \neq 0$), but their drift velocities at any location do not vary in time, and there is no change in the deposits of excess charge in the circuit.*

Theorem 4.1.1: Kirchhoff Node Rule

The **Law of Conservation of Current** *states that the current is the same at all points in a current carrying wire with no junctions. Moreover,* **Kirchhoff's Node Rule** *states that at a junction*

$$\sum_i I_{in,i} = \sum_j I_{out,j} \tag{4.1.1}$$

Recall 4.1.1

Recall that electron current is equal to $i_e = nA\overline{v}$, and by the Law of Conservation of Current, even if one of the variables change, i_e must remain constant.

Remark 4.1.2

The electric field which points along the wire and causes current is due to a build of charge along the wire caused by the presence of a battery or some other potential difference. Moreover, this charge build-up changes as a gradient through the wire. Additionally, there is no current in an open circuit, but once closed it will take a few nanoseconds for charges to redistribute at the point of contact generating the desired electric field through the wire, causing current to flow.

Definition 4.1.2: Feedback

Feedback ensures equalization of current, and can be summed up as follows: surface charge will rearrange itself in such a way as to ensure that the electric field points along the wire and has appropriate magnitude to drive the appropriate amount of steady state current. If the electric field is not along the wire charges will be pushed into the wires walls, redistributing the charge and adjusting the electric field so that it doesn't point towards any walls, and instead along the wire.

Remark 4.1.3

In general, we can express current as $I = nuEA$, which will be constant along a wire without any junctions in a steady-state. Hence, if the parameters change at some type of boundary we must have that $n_1 u_1 E_1 A_1 = n_2 u_2 E_2 A_2$.

Theorem 4.1.2: Kirchhoff's Loop Rule

Kirchhoff's Loop Rule *states that the sum of the potential differences across all elements around any closed circuit loop must be zero:*

$$\sum_i \Delta V_i = 0 \tag{4.1.2}$$

This follows from the conservation of energy principle, so $\Delta U(A \to A) = 0$. Recall that potential increases when moving against the electric field and decreases when moving with the electric field.

Definition 4.1.3

The rate of energy generation (power) due to a battery with potential difference ϵ is

$$\frac{dU}{dt} = P = i\epsilon \tag{4.1.3}$$

Similarly, the rate of energy dissipation across a resistor with potential difference V is $P = iV$.

4.2 Currents and Current Density

Definition 4.2.1: Electric Current

We define electric current by

$$i = \frac{dq}{dt} \implies q = \int dt = \int_0^t i(t)dt \tag{4.2.1}$$

Definition 4.2.2: Current Density

We define the current density vector $\vec{J}$ such that it is in the direction of the moving charges. Then the current is

$$i = \int \vec{J} \cdot d\vec{A} \tag{4.2.2}$$

If the current is uniform and parallel to A then $i = JA$, so $J = \frac{i}{A}$.

Corollary 4.2.1: Drift Speed

Recall that $i = nAev_d$. It then follows that

$$J = nev_d \tag{4.2.3}$$

Remark 4.2.1

The charge carrier density n can be expressed by

$$n = density \times \frac{N_A z}{Molar\ mass} \tag{4.2.4}$$

where N_A is Avogadro's number.

4.3 Resistance and Resistivity

Definition 4.3.1

The resistance of an object is defined as

$$R = \frac{V}{i} \tag{4.3.1}$$

Note that the resistance depends on both the material and geometry of the object. Equivalently the resistence can be written as

$$R = \rho \frac{L}{A} \tag{4.3.2}$$

where ρ is the **resistivity** *of the material, and is equal to $\rho = \frac{E}{J}$ (it is a property of the material). We can also define* **conductivity** *which is $\sigma = \frac{1}{\rho}$.*

Theorem 4.3.1: Ohm's Law

Ohm's Law *asserts that the resistance in* **ohmic materials** *is approximately constant so the current depends linearly on the potential difference across a circuit element. Conducting materials are ohmic. Note that microscopically we can write*

$$\rho = \frac{m}{e^2 n\tau} \tag{4.3.3}$$

where τ is the average time between collisions.

Remark 4.3.1

The resistivity of a material is a function of the temperature, and in particular

$$\rho = \rho_0(1 + \alpha(T - T_0)) \tag{4.3.4}$$

For metals we have $\alpha > 0$ while for semi-conductors $\alpha < 0$.

4.4 Superconductors

Definition 4.4.1: Superconductors

Superconductors *are a class of metals and compounds whose resistance decreases to zero once a certain critical temperature T_C is reached. T_C is sensitive to chemical composition, pressure, and molecular structure. A* **type 1 superconductor** *is characterized by*

1. *Sharp transition to superconducting state*
2. *Requires extremely cold temperatures to become superconducting*
3. *Perfect diamagnetism - (magnets kill the superconducting effect)*
4. *BCS theory - electrons team up in "Cooper pairs"*
5. *Soft superconductors*

A **type 2 superconductor** *is characterized by*

1. *Except for Vanadium, Technetium and Niobium, metallic compounds and alloys and perovskites*
2. *Higher T_C than type 1*
3. *Mechanism not completely understood*
4. *Meissner effect*
5. *Hard superconductors*

4.5 Batteries and Circuit Calculations

Definition 4.5.1: EMF

The **electromotive force***, emf, is the energy per unit charge that is converted reversibly from chemical, mechanical, or other forms of energy into electrical energy in a battery. Real batteries have internel resistence r, so the voltage provided to the circuit is* $\epsilon - ir$*, not simply the emf* ϵ*.*

Proposition 4.5.1: Equivalent Resistence

Series wiring means that the circuit elements are connected in such a way that there is the same current running through each device. If resistors are connected in series we have the equivalent resistence

$$R_{eq} = \sum_{i=1}^{N} R_i \tag{4.5.1}$$

Next, parallel wiring means that the devices are connected in such a way that the same voltage is applied across each device. In this case the equivalent resistence for resistors connected in parallel is

$$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} \tag{4.5.2}$$

4.6 Capacitance

Definition 4.6.1: Capacitance

The charge on a capacitor plate and the potential difference across the capacitor are proportional. The proportionality constant is the **capacitance***, C, of the capacitor, such that*

$$C = \frac{q}{\Delta V} \tag{4.6.1}$$

The unit of capacitance is farads, $F = C/V$

Remark 4.6.1

To calculate the capacitance of a capacitor we determine the charge on the surface using $Q = \eta A$*, then we calculate the potential difference between the capacitor plates before taking the ration* $Q/\Delta V$*.*

Proposition 4.6.1: Parallel Plate

The capacitance of a parallel plate capacitor is

$$C = \frac{\varepsilon_0 A}{d} \tag{4.6.2}$$

Proposition 4.6.2: Cylindrical

The capacitance of a cylindrical capacitor with inner cylinder of radius a and an outer cylinder of radius b is

$$C = \frac{2\pi L\varepsilon_0}{\ln\left|\frac{b}{a}\right|} \tag{4.6.3}$$

Proposition 4.6.3: Spherical

The capacitance of a spherical capacitor with inner radius a and an outer radius b is

$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}} \tag{4.6.4}$$

Theorem 4.6.4

The equivalent capacitance of capacitors connected in series is

$$\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i} \tag{4.6.5}$$

The equivalent capacitance of capacitors connected in parallel is

$$C_{eq} = \sum_{i=1}^{N} C_i \tag{4.6.6}$$

Definition 4.6.2: Energy Stored in an Electric Field

The potential energy stored in the capacitor upon charging (due to external forces) is

$$U = \frac{1}{2}CV^2 \tag{4.6.7}$$

The energy density is defined to be

$$u = \frac{U}{Ad} = \frac{CV^2}{2Ad} \tag{4.6.8}$$

For a parallel plate capacitor we have

$$u = \frac{1}{2}\varepsilon_0 E^2 \tag{4.6.9}$$

Definition 4.6.3: Dielectrics

If a dielectric κ is inserted between the plates of a capacitor while a battery is connected the voltage across the capacitor will initially decrease before increasing back to its original value, giving

$$C' = \kappa \frac{q}{\Delta V} = \kappa C \tag{4.6.10}$$

Due to this, the potential energy stored in the capacitor becomes

$$U' = \frac{1}{2} C' \Delta V^2 = \kappa \frac{1}{2} C \Delta V^2 = \kappa U \tag{4.6.11}$$

If no battery is connected the potential difference will not increase back to its original energy and the potential energy stored in the capacitor becomes

$$U' = \frac{1}{2} Q \Delta V' = \frac{1}{\kappa} \frac{1}{2} Q \Delta V = \frac{U}{\kappa} \tag{4.6.12}$$

Remark 4.6.2

Capacitors with different dielectrics side by side act like capacitors in parallel each with a different dielectric (and a different area). Capacitors with dielectrics stacked on top of each other act like capacitors in series each with a different dielectric in it (and a different width).

4.7 RC Circuits

Definition 4.7.1

During the charging phase of the capacitor we have that

$$q(t) = \epsilon C \left(1 - e^{-t/RC}\right) \tag{4.7.1}$$

and

$$V_C(t) = \epsilon \left(1 - e^{-t/RC}\right) \tag{4.7.2}$$

During discharge we have

$$q(t) = \epsilon C e^{-t/RC} \tag{4.7.3}$$

and

$$V_C(t) = \epsilon e^{-t/RC} \tag{4.7.4}$$

Chapter 5

Magnetic Force

5.1 Force on a particle

Definition 5.1.1: Magnetic Force

The magnetic force on a point charge q in the presence of a magnetic field $\vec{B}$ is

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{5.1.1}$$

Remark 5.1.1

In the presence of a uniform magnetic field a moving charge will often exhibit helical or circular motion. If $\vec{v} \perp \vec{B}$ then $qvB = \frac{mv^2}{r}$ so we have a radius of orbit of

$$r = \frac{mv}{qB} \tag{5.1.2}$$

and period

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB} \tag{5.1.3}$$

In general if angle$(\vec{v}, \vec{B}) = \phi$ then

$$r = \frac{mv\sin(\phi)}{qB} \tag{5.1.4}$$

but still

$$T = \frac{2\pi m}{qB} \tag{5.1.5}$$

Definition 5.1.2: Relativistic

Relativistic momentum is given by $p = \gamma m v$. Classically we have

$$\left|\frac{d\vec{p}}{dt}\right| = p\omega \tag{5.1.6}$$

with $\omega = \frac{v}{R}$. Relativistically

$$\left|\frac{d\vec{p}}{dt}\right| = \omega\gamma m v = |q|vB\sin(\phi) \tag{5.1.7}$$

so we get $\omega = \frac{|q|B}{\gamma m}$ so $T = \frac{2\pi}{\omega} = \frac{2\pi\omega m}{qB}$

Definition 5.1.3: Lorentz Force

The **Lorentz Force** *on a point charge in teh presenc of an electric and magnetic field is*

$$\vec{F} = \vec{F}_E + \vec{F}_B = q\vec{E} + q\vec{v} \times \vec{B} \tag{5.1.8}$$

Remark 5.1.2

In a velocity selector we have that the Lorentz Force is zero.

Remark 5.1.3: Cyclotrons

In a cyclotron a charged particle is accelerated while going in cicular-like paths at a fixed frequency $f = f_{osc}$. Recall $f = \frac{\omega}{2\pi} = \frac{|q|B}{2\pi m}$, and $E_{K,final} = q\Delta v \cdot n$ where n is the number of passes through the accelerating gap.

5.2 Magnetic Force on Wires

Definition 5.2.1

The magnetic force on a segment of current carrying wire is

$$d\vec{F} = id\vec{L} \times \vec{B} \tag{5.2.1}$$

where $d\vec{L}$ points in the direction of conventional current. We then integrate along the wire to get the total force. For a straight current carrying wire in a uniform magnetic field

$$\vec{F} = I\vec{L} \times \vec{B} \tag{5.2.2}$$

5.3 Hall Effect

Claim 5.3.1: Hall Effect

The **Hall Effect** *asserts that if a current carrying wire is subject to a uniform perpendicular magnetic field, the charge carriers will experience a magnetic force causing a potential difference to be created across the width of the wire. With width w, this* **hall voltage** *is*

$$V_H = wv_dB = \frac{IB}{dne} \tag{5.3.1}$$

where $v_d = \frac{I}{Ane} = \frac{I}{wdne}$.

5.3.1 Motional EMF

Proposition 5.3.2

For a conductor of length L moving velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$, Hall's Effect predicts a potential difference $\Delta V = EL$ to be created along the length of the conductor where $E = vB$. It follows that the induced motional emf is

$$\epsilon = \Delta V = vBL \tag{5.3.2}$$

Remark 5.3.1

If the conductor in the previous proposition is connected to a conducting wire, creating a circuit that changes size, current is induced. Moreover, this current causes a magnetic force $F_B = ILB$ on the bar in the opposite direction of its motion. Note that the current induced is

$$I = \frac{\Delta V}{R} = \frac{vBL}{R} \tag{5.3.3}$$

Then, the acceleration of the bar due to the magnetic force is

$$a = \frac{ILB}{m} = \frac{L^2B^2v}{mR} = \frac{dv}{dt} \tag{5.3.4}$$

Note that the rate of change of the flux through the circuit and the accompying emf across the bar are proportional.

Remark 5.3.2

If a force $F = ILB$ is applied to the bar to keep it moving at constant velocity, the power delivered to the circuit is

$$P = Fv = \frac{L^2B^2v}{R}v = \frac{(LBv)^2}{R} = \frac{\Delta V^2}{R} = I\Delta V \tag{5.3.5}$$

(Note that the power dissipated by the circuit is $P = IV$, so this checks out)

5.4 Flux

Definition 5.4.1

The **magnetic flux** *through a surface is defined by the surface integral*

$$\Phi_B = \int\int \vec{B} \cdot d\vec{A} \tag{5.4.1}$$

5.4.1 Induction

Theorem 5.4.1: Faraday's Law of Induction

Faraday's Law of Induction *state that the emf induced in a circuit is given by*

$$\epsilon = -N\frac{d\Phi_B}{dt} \tag{5.4.2}$$

where N is the number of loops in the circuit.

Theorem 5.4.2: Lenz' Law

Lenz' Law *states that current induced in a circuit by a changing flux will be such that it opposes the change.*

Theorem 5.4.3

The electric field induced around a loop of radius r in a changing magnetic field is

$$E = -\frac{dB}{dt}\frac{r}{2} \tag{5.4.3}$$

5.5 Magnetic Torque

Definition 5.5.1

The **magnetic torque** *on a current carrying loop is defined to be*

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{5.5.1}$$

where as before $\vec{\mu} = I\vec{A}$, which points in the direction of the generated magnetic field.

Definition 5.5.2

The work needed to rotate a current carrying loop from an angle θ_i to θ_f is

$$W = \Delta U_m = \int_{\theta_i}^{\theta_f} \tau d\theta = -\mu B \cos(\theta)\Big|_{\theta_i}^{\theta_f} \tag{5.5.2}$$

Hence, in general we define the potentional energy for a magnetic dipole subject to a torque by

$$U_m = -\vec{\mu} \cdot \vec{B} \tag{5.5.3}$$

Chapter 6

Field Patterns in Space and Maxwell's Equations

6.1 Gauss' Law

Definition 6.1.1

The **electric flux** *through a surface is defined by the surface integral*

$$\Phi_E = \int\int \vec{E} \cdot d\vec{A} \tag{6.1.1}$$

Theorem 6.1.1: Gauss' Law

Gauss' Law *states that the electric flux through a closed surface is equal to the charge enclosed divided by the permittivity of free space:*

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \tag{6.1.2}$$

Remark 6.1.1

If $\vec{E}$ and $\vec{A}$ are parallel, perpendicular, or in general constant over some closed surface, we can use Gauss' Law to determine $\vec{E}$.

Theorem 6.1.2: Gauss' Law for Magnetism

For magnetism Gauss' Law is a statement of the fact that there are no magnetic monopoles

$$\oiint \vec{B} \cdot d\vec{A} = 0 \tag{6.1.3}$$

6.2 Ampere's Law

Theorem 6.2.1: Ampere's Law

Ampere's Law *states that given any closed loop C, we have*

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \tag{6.2.1}$$

6.3 Divergence Forms

Definition 6.3.1: Gauss' Law Divergence

Gauss' Law can be expressed in terms of the divergence of $\vec{E}$ using the divergence theorem for surface integrals, giving

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{6.3.1}$$

and sicne $\vec{E} = -\nabla V$ we have

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{6.3.2}$$

where ρ is the charge density in the region.

Definition 6.3.2: Ampere's Law Curl

Ampere's Law can be expressed in terms of the curl of $\vec{B}$ using Stoke's Theorem for line integrals, which gives

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{6.3.3}$$

where $\vec{J}$ is the charge density.

Definition 6.3.3: Ampere-Maxwell Law

For non-continuous conduction currents (i.e. through capacitors), Ampere's Law can be extended to the **Ampere-Maxwell Law**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc} + \mu_c \varepsilon_0 \frac{d\Phi_E}{dt} \tag{6.3.4}$$

where the second term is known as the **displacement current**.

Definition 6.3.4: Faraday's Law of Induction Integral

Faraday's Law of Induction can be stated as the closed loop integral

$$\oint \vec{E} \cdot d\vec{s} = emf = -N\frac{d\Phi_B}{dt} \tag{6.3.5}$$

where for an electric field in space without any conducting loops present

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{6.3.6}$$

6.4 Inductors

Definition 6.4.1

An inductor (solenoid) is used to produce a desired uniform magnetic field $\vec{B}$. The **inductance** *associated with an inductor is defined to be*

$$L := \frac{N\Phi_B}{i} \tag{6.4.1}$$

where i is the current in the solenoid and N is the number of loops. In particular if we consider the length l near the middle of a solenoid, and write $n = N/l$ for the loop density, we have $B = \mu_0 in$ so

$$L = \mu_0 n^2 lA \tag{6.4.2}$$

Theorem 6.4.1: Self-Inductance

If two coils (inductors) are near each other, a current i in one produces a magnetic flux through the second. If i changes the induced emf will appear in the second coil, which causes a current, and hence a changing flux in the first, which produces another emf, this time in the first coil. This process is called **self-induction**. *For any inductor $Li = N\Phi_B$, so by Faraday's Law*

$$\epsilon_L = -N\frac{d\Phi_B}{dt} = -L\frac{di}{dt} \tag{6.4.3}$$

Remark 6.4.1

Inductors oppose the change of current in a circuit. That is, if current increases due to a battery, the inductor will oppose the battery, while if current is decreasing the inductor will preserve the current.

Remark 6.4.2

The rate at which energy is stored in the magnetic field of the inductor is

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \tag{6.4.4}$$

Then, the potential energy stored is

$$U_B = \frac{1}{2}Li^2 \tag{6.4.5}$$

Moreover, the energy density is

$$u_B = \frac{U_B}{Al} = \frac{Li^2}{2lA} \tag{6.4.6}$$

where $L/l = \mu_0 n^2 A$, *so* $u_B = \frac{1}{2}\mu_0 n^2 i^2$, *which implies that for a solenoid with* $B = \mu_0 in$, $u_B = \frac{B^2}{2\mu_0}$.

Remark 6.4.3: LC Circuit

Consider a circuit with a capacitor and inductor, starting with the capacitor fully charged. By Kirchhoff's Loop rule we have that

$$\Delta V_C + \Delta V_L = \frac{q}{C} - L\frac{di}{dt} = 0 \tag{6.4.7}$$

Substituting $i = -\frac{dq}{dt}$ *(since the capacitor is discharging) we get*

$$\frac{q}{C} = -L\frac{d^2q}{dt^2} \tag{6.4.8}$$

so

$$\frac{d^2q}{dt^2} = \frac{-1}{LC}q \tag{6.4.9}$$

which is the form of a harmonic oscillator. With initial condition $q(0) = Q_0$ *we have that*

$$q(t) = Q_0 \cos(\omega t + \phi_0) \tag{6.4.10}$$

where evidentally $\phi_0 = 0$ *by the initial condition. Differentiating and substituting* $i = -\frac{dq}{dt}$ *we obtain*

$$i(t) = i_{max} \sin(\omega t + \phi_0) \tag{6.4.11}$$

where $i_{max} = Q_0\omega$.

Remark 6.4.4: RLC Circuit

An RLC circuit acts like a damping harmonic oscillator. The current is the same through each circuit component

$$i(t) = i_0 \sin(\omega t + \phi_0) \tag{6.4.12}$$

where $\omega^2 = \frac{1}{LC}$. *Moreover, the voltage across the resistor*

$$V_R(t) = Ri(t) \tag{6.4.13}$$

is in phase with the current, while the voltage across the inductor

$$V_L = -L\frac{di}{dt} = -Li_0\omega\cos(\omega t + \phi_0) \tag{6.4.14}$$

is shifted, along with the voltage across the capacitor

$$V_C = \frac{q}{C} = \frac{i_0}{\omega}\cos(\omega t + \phi_0) \tag{6.4.15}$$

Chapter 7

Electromagnetic Radiation

7.1 Accelerated Charges

Definition 7.1.1

If a charge q is accelerated briefly by $\vec{a}$, then at an angle θ around the particle relative to the acceleration vector, we have electric radiation of

$$\vec{E}_{rad} = k_e \frac{-q\vec{a}_\perp}{c^2 r} \tag{7.1.1}$$

where $a_\perp = a\sin(\theta)$ and points in the direction $\vec{a} - proj_{\vec{r}}(\vec{a})$ (making a right triangle with one angle equal to θ).