Mathematical Physics: A Complete Guide

Phys 435

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Solo Pursuit of Learning

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6.1.0 Initial Definitions and Dirichlet Conditions

Definition 6.1.1. *Sufficients conditions for which a function $f(x)$ to have its Fourier series to converge to it are known as the* **Dirichlet conditions***:*

(i) $f(x)$ *must be periodic; i.e. there exists* $p \in \mathbb{R}$ *such that* $f(x+p) = f(x)$ *for all* $x \in \mathbb{R}$.

(ii) $f(x)$ *must be continuous, except possibly at a finite number of jump (i.e. finite) discontinuities in any bounded interval.*

(iii) $f(x)$ *must be of* **bounded variation** *on any bounded interval, which is to say its total variation is finite; if f is differentiable and its derivative is Riemann-integrable on the interval, then the total variation is the absolute integral of the derivative over the interval:*

$$V_a^b(f) = \int_a^b |f'(x)| dx$$

An alternative formulation is to require that any bounded interval contains only a finite number of extrema of f.

(iv) $f(x)$ *is absolutely integrable over a period, so*

$$\int_0^p |f(x)| dx < \infty$$

If these criterions hold, the Fourier series converges to $f(x)$ at all points where the function is continuous.

Recall that a function $f(x)$ can be split into an even and odd part:

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = f_{even}(x) + f_{odd}(x)$$

Then, we can write the even component as a cosine series and the odd component as a sine series.

Proposition 6.1.1. *For any $L \in \mathbb{R}$, the set of functions*

$$\left\{1, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \cos\frac{2\pi 2x}{L}, \sin\frac{2\pi 2x}{L}, ..., \cos\frac{2\pi nx}{L}, \sin\frac{2\pi nx}{L}, ...\right\}$$

form an orthogonal set with respect to the inner product

$$\langle f, g\rangle = \int_{x_0}^{x_0+L} f(x)g(x)dx$$

for $x_0 \in \mathbb{R}$ fixed. In particular, we have

$$\int_{x_0}^{x_0+L} \sin\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}dx = 0, \quad \forall n, m \in \mathbb{N} \cup \{0\}$$

$$\int_{x_0}^{x_0+L} \cos\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}dx = \begin{cases} L & \text{if } n = m = 0, \\ \frac{L}{2} & \text{if } n = m > 0, \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{x_0}^{x_0+L} \sin\frac{2\pi nx}{L}\sin\frac{2\pi mx}{L}dx = \begin{cases} 0 & \text{if } n = m = 0, \\ \frac{L}{2} & \text{if } n = m > 0, \\ 0 & \text{if } n \neq m \end{cases}$$

Definition 6.1.2. *The classical Fourier series expansion of a function $f(x)$ is*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi xn}{L} + b_n \sin\frac{2\pi nx}{L}\right] \tag{6.1.1}$$

where a_0, a_n, b_n, for $n \geqslant 1$, are called the **Fourier coefficients**

For a periodic function $f(x)$ of period L, we use the orthogonality conditions to find the Fourier coefficients as follows:

$$a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \cdot 1dx \tag{6.1.2}$$

$$a_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \cdot \cos\frac{2\pi nx}{L}dx \tag{6.1.3}$$

$$b_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \cdot \sin\frac{2\pi nx}{L}dx \tag{6.1.4}$$

where x_0 is arbitrary, but fixed, and $n \geqslant 1$.

Symmetry Conditions

From these coefficient equations we observe that if $f(x)$ is even with respect to the origin then all sine terms, b_n, are zero. Conversely, if $f(x)$ is odd with respect to the origin then all cosine terms, a_n, are zero. We now consider a more subtle symmetry about $L/4$, where L is a period of f, so $f(x + L) = f(x)$ for all $x \in \mathbb{R}$.

Definition 6.1.3. *We say that $f(x)$ has even symmetry about $L/4$ if $f(L/4-x) = f(x-L/4)$ for all $x \in \mathbb{R}$. We say that $f(x)$ has odd symmetry about $L/4$ if $f(L/4-x) = -f(x-L/4)$.*

We consider the sine terms of $g(x) = f(x-L/4)$, and substitute $s = x - L/4$:

$$\begin{aligned} b_n &= \frac{2}{L}\int_{x_0}^{x_0+L} f(x-L/4)\sin\frac{2\pi nx}{L}dx \\ &= \frac{2}{L}\int_{x_0-L/4}^{x_0-L/4+L} f(s)\sin\left[\frac{2\pi ns}{L}+\frac{\pi n}{2}\right]ds \\ &= \frac{2}{L}\int_{x_0}^{x_0+L} f(s)\sin\left[\frac{2\pi ns}{L}+\frac{\pi n}{2}\right]ds \end{aligned}$$

where the limits of integration can be changed since f is periodic. We observe that

$$\sin\left[\frac{2\pi ns}{L}+\frac{\pi n}{2}\right] = \sin\frac{2\pi ns}{L}\cos\frac{\pi n}{2}+\cos\frac{2\pi ns}{L}\sin\frac{\pi n}{2}$$

so the trigonometric portion of the integrand is odd if n is even and even if n is odd. Then if $f(s)$ is even and n is even the integral is zero, and similarly if $f(s)$ is odd and n is odd the integral is zero. For the cosine coefficients we have

$$\cos\left[\frac{2\pi ns}{L}+\frac{\pi n}{2}\right] = \cos\frac{2\pi ns}{L}\cos\frac{\pi n}{2}-\sin\frac{2\pi ns}{L}\sin\frac{\pi n}{2}$$

which is even if n is even and odd if n is odd. Then if $f(s)$ is even and n is odd, the terms a_n are zero, and if $f(s)$ is odd and n is even, the terms a_n are zero. In summary:

- If $f(x)$ is even about $L/4$, then $a_{2n-1} = 0$ and $b_{2n} = 0$ for all $n \geqslant 1$,
- If $f(x)$ is odd about $L/4$, then $a_{2n} = 0$ and $b_{2n+1} = 0$ for all $n \geqslant 0$.

6.2.0 Discontinuous and Non-Periodic Functions

Discontinuities

Definition 6.2.1. *The error term for the Fourier series representation of f when expressed as a partial sum with highest term N is*

$$E_N(x) = \left|f(x) - \left[\frac{a_0}{2}+\sum_{n=1}^{N}\left(a_n\cos\frac{2\pi nx}{L}+b_n\sin\frac{2\pi nx}{L}\right)\right]\right|$$

If $f(x)$ is discontinuous at a point a in the domain of interest, then the Fourier series for $f(x)$ does not produce a discontinuity at a but rather converges to the value

$$\frac{1}{2}[f(a+)+f(a-)]$$

where $f(a+) = \lim_{x \to a^+} f(x)$ is the one-sided limit from above, and $f(a-) = \lim_{x \to a^-} f(x)$ is the one-sided limit from below. Then, there exists sequences u_n and v_n such that $u_n, v_n \to a$, with $u_n < a$ for all n and $v_n > a$ for all n and

$$E_N(u_N) \approx 0.9|f(a-) - f(a+)| \quad E_N(v_N) \approx 0.9|f(a-) - f(a+)|$$

so the maximum value of the error $E_N(x)$ near a does not approach zero as $N \to \infty$, but rather occurs closer and closer to a, and is essentially independent of N. This is known as the ***Gibbs' phenomenon***.

Non-Periodic Functions

We often wish to analysize non-periodic functions using Fourier analysis, and this can be done by using appropriate periodic extensions:

Theorem 6.2.1. *Suppose h is differentiable on $[0, L]$; that is, $h'(x)$ exists for $0 < x < L$, and the one-sided derivatives*

$$h'_+ = \lim_{x \to 0^+} \frac{h(x) - h(0)}{x} \quad \text{and} \quad h'_-(L) = \lim_{x \to L^-} \frac{h(x) - h(L)}{x - L}$$

both exist.

- *Let O denote the odd periodic extension of h to $(-\infty, \infty)$ defined by*

$$O(x) = \begin{cases} h(x), & 0 \leqslant x \leqslant L \\ -h(-x), & -L < x < 0, \end{cases} \quad \text{and } O(x + 2L) = O(x), \quad \forall x \in \mathbb{R}$$

  Then O is differentiable on $(-\infty, \infty)$ if and only if

$$h(0) = h(L) = 0$$

- *Let E denote the even periodic extension of h to $(-\infty, \infty)$, defined by*

$$E(x) = \begin{cases} h(x), & 0 \leqslant x \leqslant L \\ h(-x), & -L < x < 0, \end{cases} \quad \text{and } E(x + 2L) = E(x), \quad \forall x \in \mathbb{R}$$

  Then E is differentiable on $(-\infty, \infty)$ if and only if

$$h'_+(0) = h'_-(L) = 0$$

6.3.0 Integration and Differentiation

Theorem 6.3.1. *If $f(x)$ satisfies the Dirichlet conditions, then integrating the Fourier series of $f(x)$ term by term produces a Fourier series which converges to the integral of $f(x)$, modulo an arbitrary constant.*

Theorem 6.3.2. *If $f(x)$ satisfies the Dirichlet conditions, is differentiable, and $f'(x)$ satisfies the Dirichlet conditions, then the Fourier series obtained by differentiating f's Fourier series term by term converges to $f'(x)$.*

For general functional series we have the following important result:

Theorem 6.3.3. *A convergent infinite series*

$$W(z) = \sum_{n=1}^{\infty} w_n(z)$$

can be differentiated term by term on a closed interval $[z_1, z_2]$ to obtain

$$W'(z) = \sum_{n=1}^{\infty} w_n'(z)$$

provided that w_n' is continuous on $[z_1, z_2]$ and there exists a sequence M_n of constants such that $\sum_{n=1}^{\infty} M_n$ converges and

$$|w_n'(z)| \leqslant M_n, \quad z_1 \leqslant z \leqslant z_2, \quad n = 1, 2, 3, ...$$

6.4.0 Complex Fourier Series

Recall that by DeMoivre's Formula we have the following for the complex exponential:

$$\exp\{ix\} = \cos x + i \sin x$$

Definition 6.4.1. *For a function $f(x)$ of period L, its complex Fourier series expansion is given by*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp\left\{\frac{2\pi inx}{L}\right\}$$

We remark that the set of functions

$$\left\{1, \exp\left\{\frac{2\pi x}{L}\right\}, \exp\left\{\frac{2\pi 2x}{L}\right\}, ... \exp\left\{\frac{2\pi nx}{L}\right\}, ...\right\}$$

forms an orthogonal set under the complex inner product

$$\langle f, g\rangle = \int_{x_0}^{x_0+L} f(x)\overline{g(x)}dx$$

for x_0 fixed with the relation

$$\int_{x_0}^{x_0+L} \exp\left\{\frac{2\pi inx}{L}\right\} \exp\left\{-\frac{2\pi imx}{L}\right\} dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Then the Fourier coefficients are given by

$$c_n = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\exp\left\{-\frac{2\pi i n x}{L}\right\}dx \tag{6.4.1}$$

We expand the complex exponential in the Fourier series as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n \exp\left\{\frac{2\pi n x}{L}\right\} &= \sum_{n=1}^{\infty} c_{-n}\exp\left\{\frac{-2\pi n x}{L}\right\} + c_0 + \sum_{n=1}^{\infty} c_n \exp\left\{\frac{2\pi n x}{L}\right\} \\
&= c_0 + \sum_{n=1}^{\infty}\left[c_{-n}\left(\cos\frac{2\pi n x}{L} - i\sin\frac{2\pi n x}{L}\right) + c_n\left(\cos\frac{2\pi n x}{L} - i\sin\frac{2\pi n x}{L}\right)\right] \\
&= c_0 + \sum_{n=1}^{\infty}\left[(c_{-n}+c_n)\cos\frac{2\pi n x}{L} + (ic_n - ic_{-n})\sin\frac{2\pi n x}{L}\right]
\end{aligned}$$

From this expansion we find that

$$\begin{aligned}
c_0 &= \frac{a_0}{2} \\
c_{-n} + c_n &= a_n \\
ic_n - ic_{-n} &= b_n
\end{aligned}$$

for $n \geqslant 1$. Then we have that

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{and} \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

It follows that if $f(x)$ is real, so a_n and b_n are real, $c_{-n} = \overline{c_n}$.

Parseval's Theorem

Theorem 1 (Parseval's Theorem).

Suppose that $A(x)$ and $B(x)$ are two complex valued functions on $\mathbb{R}$ of period $2L$ that are square integrable with respect to the Lebesgue measure over intervals of period length with complex Fourier series

$$A(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left\{\frac{i\pi n x}{L}\right\}, \quad \textit{and} \quad B(x) = \sum_{n=-\infty}^{\infty} b_n \exp\left\{\frac{i\pi n x}{L}\right\}$$

Then

$$\sum_{n=-\infty}^{\infty} a_n\overline{b_n} = \frac{1}{2L}\int_{-L}^{L} A(x)\overline{B(x)}dx \tag{6.4.2}$$

Proof. Suppose $A(x)$ and $B(x)$ are as above, with corresponding Fourier series, and observe that

$$\frac{1}{2L}\int_{-L}^{L} A(x)\overline{B(x)}dx = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2L}\int_{-L}^{L}\overline{B(x)}\exp\left\{\frac{2\pi i n x}{2L}\right\}dx$$

$$= \sum_{n=-\infty}^{\infty} a_n \overline{\left[\frac{1}{2L} \int_{-L}^{L} B(x) \exp\left\{ \frac{-2\pi inx}{2L} \right\} dx \right]}$$
$$= \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

as desired. ■

As a corollary, we have that for any function $f(x)$ of period L,

$$\frac{1}{L} \int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

This says that the sum of the moduli squared of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ over one period.

Chapter 7

Integral Transforms

7.1.0 Fourier Transform

7.2.0 Laplace Transform

Chapter 8

Separation of Variables and Other Methods

8.1.0 Separation of Variables

8.2.0 Applying Integral Transforms

8.3.0 Inhomogeneous Problems

Appendices