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Phys 501: Relativity

– In Pursuit of Abstract Nonsence –

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Preface

This is a collection of notes associated with Phys 501 (Relativity) taken at the University of Calgary.

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Notation

List of common notations used in these notes.

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers

Chapter 1

Geometry of Relativity

Abstract In this chapter we begin the discussion of the relation between gravity and geometry. In particular, we remark that Gravity is simply the geometry of space time, with objects following straight paths or “geodesics" in the curved space time. For this purpose we study a few common geometries, and how we can use the notion of a Riemannian metric, or **line element**, to generalize our understanding of geometry from the Euclidean setting to spherical geometries and beyond.

1.1 Gravity and Geometry

As we shall soon be aware, gravity **is** geometry—it arises from the curvature of **spacetime**. The fact that all bodies fall with the same acceleration in a uniform gravitational field, independently of their composition, is one of the most accurately tested facts in physics. The most accurate such test is the comparison of the accelerations of the Earth and the Moon as they fall around the Sun. These match to within a fractional error of less than 1.5×10^{-13}.

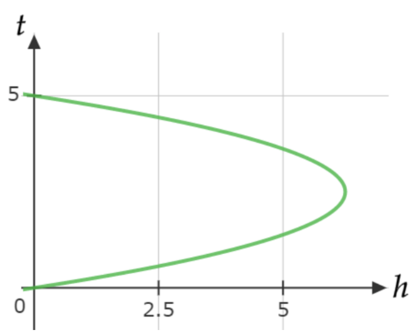

Fig. 1.1 Space-time diagram for object thrown vertically on earth with height on the horizontal.

In Einstein’s general relativity, the bodies are following a straight path in the curved spacetime produced by the Earth’s mass. Note that the exact same trajectory pictured above would occur for any other body with the same initial velocity and same initial position. This uniqueness property is not seen in all other fields. For instance, the motion of a body in a magnetic field depends on what kind of charge it has. Bodies with one sign of charge will be deflected one way, bodies with the opposite charge will be deflected the other, and bodies with no charge will not be deflected at all.

Einstein proposed that in the absence of other forces, bodies move on straight paths in this curved spacetime.

1.2 Experiments in Geometry

Geometries different from Euclid's produce different results for the sum of the interior angles of a triangle. For instance, due to the curvature of the Earth, the interior angles of a triangle on its surface would deviate from π on the order of:

$$\left|\begin{pmatrix}\text{sum of interior angles}\\ \text{of a triangle in radians}\end{pmatrix}\right| - \pi \sim \frac{(\text{area of triangle})}{R_\oplus^2}\left(\frac{GM_\oplus}{R_\oplus c^2}\right)$$

where $\oplus$ denotes the Earth.

1.3 Different Geometries

When we talk about straight paths in different geometries, we mean paths of shortest distance between two points, which generalized the notion of straight lines in Euclidean Geometry. In Spherical geometry these are segments of great circles (circles which cut through the origin of the sphere). Triangles on the sphere are then made of three intersecting great circles. For a spherical triangle of area A,

$$\sum_{vertices}\begin{pmatrix}\text{interior}\\ \text{angle}\end{pmatrix} = \pi + \frac{A}{a^2}$$

where a is the radius of the sphere.

Note this implies that the sum of the interior angles of a spherical triangle is always greater than π.

With a bit of geometry, the ratio of the circumference to the radius of a circle on a sphere can be calculated to be

$$\frac{C}{r} = \frac{2\pi a\sin(r/a)}{r} = 2\pi\frac{\sin(r/a)}{(r/a)}$$

In theory by surveying in three dimensions we can determine the geometry of space without needing an extra dimension. However, visualization of three dimensional geometries is in general very difficult. An example which is simpler, at least to describe, is the three-sphere. If space had such a geometry a journey in a straight line in any direction would eventuallly bring one back to the starting points. However, we can determine more detail locally. For instance, the volume inside a two-dimensional sphere of radius r in such a spatial geometry is given by

$$
\begin{aligned}
V &= 4\pi a^3 \left\{ \frac{1}{2} \sin^{-1}(r/a) - \frac{r}{2a} \left[1 - \frac{r^2}{a^2} \right]^{1/2} \right\} \\
&\approx \frac{4\pi r^3}{3} \left[1 + \begin{pmatrix} \text{corrections} \\ \text{of order } (r/a)^2 \end{pmatrix} \right] \quad \text{for small } r/a
\end{aligned}
$$

where a is the characteristic radius of curvature of the three-sphere geometry. If the three-dimensional space had such a geometry, the characteristic radius of curvature could be determined by careful measurements of the radii and volume of two-spheres.

1.4 Specifying Geometry

One way to describe a geometry is to embed it as a surface in a higher-dimensional space whose geometry is Euclidean. However, we want an intrinsic description of geometry that makes use of just the physical dimensions that can be measured—this leads to the study of manifolds.

We could also specify a geometry by giving a small number of axioms from which the other results of geometry can be derived as theorems. However this strategy only works for some of the simplest geometries.

1.5 Coordinates and Line Element

We now investigate the a number of simple geometries with a focus on their “line elements."

1.5.1 The Euclidean Geometry of a Plane

First we must choose some coordinate system to specify the points in our geometry. For now this will be global, but in general need only be local in nature. For example we can use Cartesian coordinates, (x, y), with infinitesimals dx and dy, or polar coordinates, (r, ϕ), with infinitesimals dr and $d\phi$.

In Cartesian coordinates the distance dS between two points (x, y) and $(x + dx, y + dy)$ is specified by

$$dS = \left[(dx)^2 + (dy)^2 \right]^{1/2}$$

In polar coordinates this same metric takes the form

$$dS = \left[(dr)^2 + (r d\phi)^2 \right]^{1/2}$$

Note these are local representations, and hence corespond to small changes.

Now, recall the equation for a circle of origin center and radius R in Cartesian coordinates:

$$x^2 + y^2 = R^2$$

Its circumference can be obtained by integrating our metric:

$$\begin{aligned} C &= \int dS = \int \left[(dx)^2 + (dy)^2\right]^{1/2} \\ &= 2\int_{-R}^{+R} dx \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}_{x^2+y^2=R^2} \\ &= 2\int_{-R}^{+R} dx \sqrt{\frac{R^2}{R^2 - x^2}} \end{aligned}$$

Consider the change of variables $x = R\xi$, so

$$C = 2R\int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} = 2\pi R$$

Reverting to polar coordinates this task becomes even simpler:

$$C = \int dS = \int_0^{2\pi} R d\phi = 2\pi R$$

We can proceed in this fasion to obtain all the standard theorems of Euclidean plane geometry. For instance, we can define the angle of intersecting lines as the ratio of the length ΔC of the part of a circle centered on their intersection that lies between the lines to the circle's radius R:

$$\theta := \frac{\Delta C}{R}$$

In general all geometry can be reduced to relations between distances, which correspond to integrals of our metric between neighborhing points.

We also refer to these metrics as **line elements**. Conventionally we write our line element quadratically, with

$$dS^2 = dx^2 + dy^2$$

This is the **Riemannian metric** for the plane in cartesian coordinates.

1.6 The Non-Euclidean Geometry of a Sphere

Consider the surface of a two-dimensional sphere of radius a. We can use the angles (θ, ϕ) of three-dimensional polar coordinates to label points on the sphere. The distance between points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ can be seen after a little work to be

$$dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Note this is the Riemannian metric on the two-sphere induced by its embedding in three-space.

Let us consider a circle on the sphere, by which we mean the locus of points on the curface that are a constant distance along the surface from a fixed point in the surface. Note the sphere is geometrically uniform, so we may orient our polar coordinate system so that the polar axis is at the center of the circle (i.e. $\theta = 0, \pi = 0$). Then a circle is a the locus of an equation

$$\theta = \Theta$$

for Θ a constant. Along the circle $dS = a \sin \Theta d\phi$, so the circumference is

$$C = \int dS = \int_0^{2\pi} a \sin \Theta d\phi = 2\pi a \sin \Theta$$

The radius is the distance from the center to the circle along a curve for which θ varies but $d\phi = 0$. Along this curve $dS = ad\theta$, so the radius is

$$r = \int_{center}^{circle} dS = \int_0^{\Theta} ad\theta = a\Theta$$

Then the relationship between the circumference and radius of a circle in the non-Euclidean geometry of a sphere becomes

$$C = 2\pi a \sin (r/a)$$

Note when $r \ll a$, we have the approximation

$$C \approx 2\pi r$$

Note ϕ is the measure of longitude on the Earth and $\lambda = \pi/2 - \theta$ is the measure of latitude, so in this coordinate system the metric is

$$dS^2 = a^2(d\lambda^2 + \cos^2 \lambda d\phi^2)$$

1.7 The Geometry of More General Surfaces

Consider a Riemannian metric, or line element, given in local coordinates by

$$dS^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$$

for various choices of $f(\theta)$. The choice $f(\theta) = \sin \theta$ gives the geometry of the surface of a sphere. What other surfaces in three-space can be represented by this kind of metric?

1. Since the line element is the same for all ϕ it corresponds to a surface that is axisymmetric about an axis.

2. The circumference $C(\theta)$ of a circle of constnat θ is

$$C(\theta) = \int_0^{2\pi} af(\theta)d\phi = 2\pi af(\theta)$$

3. The distance from pole to pole is

$$d = a \int_0^\pi d\theta = \pi a$$

Working these properties out we can build a picture of this surface.

> Example:
> **Peanut Geometry.** Consider the surface specified by
>
> $$f(\theta) = \sin\theta(1 - 3\sin^2(\theta)/4)$$
>
> Note the surface is symmetric under reflection in the equatorial plane $\theta = \pi/2$. Starting at $\theta = 0$ the circumference of the lines of constant θ first increases and then decreases with $f(\theta)$; then it increases and decreases again. At any one θ the circumference is smaller than the corresponding value on a sphere. At the equator, for instance,
>
> $$C(\pi/2) = 2\pi a(1 - 3/4) = \frac{\pi a}{2}$$
>
> The maximum circumference is $(8\pi/9)a$ and occurs at $\theta = \sin^{-1}(2/3) \approx 0.73$ rad. Since the distance from pole to pole is πa, this surface has an elongated “peanut" shape.

1.8 Coordinates and Invariance

Note coordinates are a systematic set of labels for a geometry, locally, but the geometry itself is invariant under any choice of smooth coordinate system. In particular, as long as the coordinate transformation about any shared neighborhood is smooth, we can use either coordinate system to produce our calculations. For instance, the coordinates (x, y) and (r, ϕ) in the plane have local coordinate transformation

$$x = r\cos\phi, \quad y = r\sin\phi$$

The point of this discussion is that the Riemannian metric dS^2, and hence the distance dS, is an invariant quantity independent of the choice of coordinates used to compute it.

Problems

1.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 2

Principles of Special Relativity

Abstract To be completed once done

2.1 Addition of Velocities and Michelson-Morley Experiment

Recall that Maxwell's equations governing electromagnetic fields do not take the same form in every intertial frame of Newtonian mechanics. Maxwell's equations, however, do imply that light travels with constant speed c in vacuum, and this is a basic parameter in these equations. But the Galilean notion of transformation between inertial frames would imply that light should travel with different speeds in different inertial frames, moving with respect to each other.

Consider v^x, v^y, v^z, components of the velocity of a particle measured in one frame, and $v^{x'}, v^{y'}, v^{z'}$, the components of the velocity measured in a frame moving with respect to the first along its x-axis with velocity v. Galilean transformations predict

$$v^{x'} = v^x - v$$

This would imply that Maxwell's equations can only be valid in one inertial frame, because they predict one velocity for light.

Remark:

In an experiment whose results were published in 1887, Albert Michelson and Edward Morley tested the Newtonian addition of velocities law for light. The theory explaining the fact Maxwell's equations can only be valid in one intertial frame was explained using a notion of the rest frame of light, called the ether. Michelson and Morley showed using the Earth's orbital velocity around the sun would imply light should be measured at different speeds at different locations in the orbt. This was not the case, so either Newtonian mechanics or Maxwell's equations had to be modified.

2.2 Einstein's Resolution

Einstein's 1905 successful modification of Newtonian mechanics is called the special theory of relativity. Einstein supposed that the velocity of light had the same value c in all inertial frames. We must now determine a new notion of velocity addition. We will also need to re-examine the Newtonian notion of absolute time.

Consider the following thought experiment. Three observers, A, B, and O, are riding a rocket of length L. O is midway between A and B. A and B each emit light signals directed along the rocket toward O. O receives the signals simultaneously. Which signal was emitted first? This must depend on the inertial frame if the velocity of light is the same in all of them.

If the rocket is at rest in the inertial frame, they must be emitted simultaneously. If the rocket is moving in the inertial frame we reason as follows: The signals are received simultaneously by O. At earlier times when the signals were emitted B was always closer to O's position at reception than A (thinking of B as on the side in which the rocket is traveling). Since both signals travel with speed c, the one from A must have been emitted earlier than the one from B because it has a longer distance to travel to reach O at the same instant as the one from B.

> Remark:
>
> Thus **two events simultaneous in one inertial frame are not simultaneous in one moving with respect to the first if the velocity of light is the same in both**.

2.3 Spacetime

Newton's first law (free particles move at constant speed on straight lines) is unchanged in special relativity. Thus we can construct inertial frames as follows: start with an origin following the straight-line trajectory of a free particle. At one moment choose three Cartesian coordinates (x, y, z) with this origin. Propagate these axes parallel to themselves as the origin moves to define (x, y, z) at later times. The result is an inertial frame.

For each inertial frame there is a notion of time t. From our previous discussion each inertial frame has a different notion of time and simultaneity. Thus inertial frames are spanned by four Cartesian coordinates, (t, x, y, z), giving us spacetime. The defining assumption of special relativity is a geometry for four-dimensional spacetime.

2.3.1 Spacetime Diagrams

A **spacetime diagram** is a plot of two of the coordinate axes of an inertial frame—two coordinate axs of spacetime. Spacetime diagrams are slices or sections of spacetime in much the same way as an xy plot is a two-dimensional slice of three-dimensional space. It is convenient to use ct rather than t as an axis, because then both have the same dimension.

> **Definition 2.3.1** An **event** is a point P in spacetime located at a particular place in space (x_P) at a particular time (t_P).

> **Definition 2.3.2** A particle describes a curve in spacetime called a **world line**. It is the curve of positions of the particle at different instants.

The slope of the world line gives the ratio c/v^i, since $d(ct)/dx_i = cdt/dx_i = c/v^i$, where zero velocity corresponds to infinite slope (no position change in time), and a velocity of c corresponds to a slope of 1. Hence light rays move along the 45^{circ} lines in a spacetime diagram.

2.3.2 The Geometry of Flat Spacetime

Consider the following thought experiment. We have two parallel mirrors separated by a distance L that are at rest in an inertial frame in wich events are described by coordinates (t, x, y, z). Take y to be the vertical direction between the mirrors and x the direction parallel to them. A light signal bounces back and forth between the mirrors. A clock measures the time interval Δt between the event A of the departure of the light ray and the event C of its return to the same point in space. These two events are separated by coordinate intervals

$$\Delta t = 2L/c, \quad \Delta x = \Delta y = \Delta z = 0$$

in the inertial frame where the mirrors are at rest.

Now consider a frame that is moving with a speed v with respect to the (t, x, y, z) inertial frame along the negative x-direction parallel to the mirrors. Write events in this frame by (t', x', y', z') wiith x' parallel to x. In this frame the mirrors are moving with speed V along the positive x'-direction. Consider the time $\Delta t'$ between the departure and return of a light ray. The light ray travels a distance $\Delta x' = v\Delta t'$ in the x'-direction between emission at A and return at C. The distance traveled in the y'-direction is L, assuming the transverse distances are the same in both inertial frames. The total distance traveled is therefore $2[L^2 + (\Delta x'/2)^2]^{1/2}$. Assuming with Einstein that the velocity of light is c in this inertial frame, the time of travel $\Delta t'$ is this distance divided by c. Thus the coordinate intervals between A and C in this frame are

$$\Delta t' = \frac{2}{c}\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}, \quad \Delta x' = v\Delta t', \quad \Delta y' = 0, \quad \Delta z' = 0$$

It follows that

$$-(c\Delta t')^2 + (\Delta x')^2 = -4[L^2 + (\Delta x'/2)^2] + (\Delta x')^2 = -4L^2 = -(c\Delta t)^2$$

This identity is key to identifying an **invariant** and to finding the line element that describes the geometry of spacetime. Since $\Delta x = 0$ and the Δy's and Δz's are zero in both frames, we can judiciously add them back into the two sides of (4.5) to find that the combination

$$\boxed{(\Delta s)^2 := -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

is the same in both frames. The quantity $(\Delta s)^2$ is **invariant** under the change in inertial frames.

Remark:

The distance between points defining spacetime geometry must be the same in all systems of coordinates used to label the points. The **principle of relativity** requires that the line element that defines the distance should have the same form in all inertial frames. Thus, we posit the **line element of flat spacetime**:

$$\boxed{dS^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2} \tag{2.3.1}$$

Note the geometry specified by (4.8) is non-Euclidean because of the minus sign (i.e. it is a Psuedo-Riemannian metric rather than a Riemannian metric). Sometimes this is referred to as **Minkowski space**.

Lengths in spacetime are giving by the square root of the absolute value of dS^2.

Example:

The analog of a circle of radius R centered on the origin is the locus of points a constant spacetime distance from the origin. THis consists of the hyperbolas $x^2 - (ct)^2 = R^2$. The ratios of arcs along a hyperbola to R define hyperbolic angles, with the relation

$$ct = R\sinh\theta, \quad x = R\cosh\theta$$

2.3.3 Light Cones

Note, two points can be separated by distances whose square is positive, negative, or zero. When dS^2 is positive the points are said to be **spacelike separated**. When dS^2 is negative the points are said to be **timelike separated**. This can occur when $\Delta x_i = 0$ for all i, but $\Delta t \neq 0$. When $dS^2 = 0$, the two points are said to be **null separated**. Null separated points can be connected by light rays that move with speed c, so **lightlike separate** is used as a synonym.

Definition 2.3.3 The locus of points that are null separated from a point P in spacetime is its **light cone**. The light cone of P is a three-dimensional surface in four dimensional spacetime specified by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2(t - t_0)^2$$

where $P = (t_0, x_0, y_0, z_0)$.

The future light cone of P is generated by light rays that move outward from P, while the past light cone of P is generated by light rays that converge on P.

The points that are timelike separated from P lie inside the light cone ($(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < c^2(t - t_0)^2$), and the points that are spacelike separated from P lie outside the light cone ($(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 > c^2(t - t_0)^2$). The paths of light rays are straight lines in spacetime with constant slope corresponding to the speed of light, that is, along null world lines. The distance between two points along a light ray is zero!

Particles with nonzero rest mass move along **timelike world lines** that are always inside the light cone of any point along their trajectory. That way their velocity is always less than the speed of light at that point.

Entities with spacelike world lines would move always with speeds greater than that of light (we call these **tachyons**). None have ever been observed to exist, and any would conflict with other principles of physics such as causality and positive energy. Hence we ignore these moving forward.

Light cones therefore define the causal relationships between points in spacetime. An event at P can signal or influence points inside or on its future light cone, but not outside it. Information can be received at P only from events inside or on its past light cone, but not from events outside it.

Remark:

An event can be later than another spacelike separated event in one inertial frame and earlier in another. But, for two timelike separated events the notion of earlier is well-defined. This is because events to the future of P are inside its future light cone, and the inside and outside of a light cone are properties of the geometry of spacetime—the same in all frames.

Two nearby points on a timelike world line are timelike separated, $dS^2 < 0$. To measure the distance along a particle's world line, it is convenient to introduce

$$d\tau^2 := -dS^2/c^2$$

Then $d\tau$ is real with units of time. Thus a clock moving along a timelike curve measures the distance τ along it. An alternative name for this distance is the **proper time**.

2.4 Time Dilation and the Twin Paradox

2.4.1 Time Dilation

The proper time, τ_{AB}, between any two points A and B on a timelike world line can be computed from the line element as

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left[dt^2 - (dx^2 + dy^2 + dz^2)/c^2\right]^2$$
$$= \int_{t_A}^{t_B} dt \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\}^{1/2}$$

More compactly,

$$\boxed{\tau_{AB} = \int_{t_A}^{t_B} dt' \left[1 - ||\mathbf{V}(t')||^2 / c^2\right]^{1/2}} \tag{2.4.1}$$

The proper time τ_{AB} is **shorter** than the interval $t_B - t_A$ because $\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2} < 1$. This is our mathematical expression for **time dilation**, which informally says “moving clocks run slow." In differential form

$$d\tau = dt\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2}$$

Note these expressions are valid even when the velocity is dependent on time (i.e. the clock is accelerating).

2.4.2 The Twin Paradox

The time dilation equation shows that the time registered by a clocking moving between two points in space depends on the route traveled even if it returns to the same point it started from.

Consider two twins, Alic and Bob, starting from rest at one point in space at time t_1 in an inertial frame. Alice moves away from the starting point but later returns to rest at the same point at time t_2. Bob remains at rest at the starting point. The time elapsed on Bob's clock is $t_2 - t_1$. The time elapsed on Alice's clock is always less than this because $\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2} < 1$. The moving twin ages less than the stationary twin.

> Remark:
>
> The straight line path is the longest distance between any two timelike separated points in flat four-dimensional spacetime. (a line of constant velocity)

2.5 Lorentz Boosts

2.5.1 The Connection Between Inertial Frames

Recall the principle of relativity implies that the line element must take the same form in the rectangular coordinates of any inertial frame. Thus, the transformation laws that connect different inertial coordinate frames must be among those that preserve our psuedo-riemannian metric. These are called **Lorentz transformations**.

Recall the line element of Euclidean space is left unchanged by translations and isometries (i.e. rotations and reflections). Hence spatial translations and isometries will preserve the line element of special relativistic spacetime. But what new transformations that preserve the four-dimensional flat spacetime do we obtain?

The most important examples of new transformations are the analogs of rotations between time and space. These are called **Lorentz boosts** and correspond to the uniform motion of one frame with respect to another.

Consider the analog of rotations in the (ct, x) plane. Transformations of this character that leave the metric unchanged are the analogs of rotations in Euclidean space, but now due to the minus in front of dt^2, we replace the trigonometric functions with hyperbolic functions. Specifically

$$ct' = (\cosh\theta)(ct) - (\sinh\theta)x, \quad x' = (-\sinh\theta)(ct) + (\cosh\theta)x$$

where θ can vary from $-\infty$ to $+\infty$.

Superposing the axes of (ct', x') and (ct, x), we can find that a particle at rest at the origin $x' = 0$ in (ct', x') coordinates has the ct' axis as its world line. In (ct, x) coordinates, that particle is moving with a constant speed along the x-axis. The speed v can by found by putting $x' = 0$ in our transformation above, so

$$v = c\tanh\theta$$

A particle at rest at any other value of x' in the (ct', x') coordinates moves in the x-direction with the same speed in the (ct, x) coordinates. The transformation is therefore from one inertial frame to another moving uniformly with respect to it along the x-axis with speed v.

Replacing θ by v using the above expression in our transformation, we find

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt)$$

where we have introduced

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The inverse transformation is then obtained just by changing v into $-v$. When $v/c \ll 1$, this reduces to the Galilean transformations.

2.5.2 The Relativity of Simultaneity

Recall events A and B can be simultaneous in one inertial frame and be separated by a time $\Delta t = t_B - t_A$ in another frame. This difference can be computed from the Lorentz boost connecting the two frames. If $\Delta x' = x'_B - x'_A$ is the distance between the simultaneous events in the (ct', x') frame, then

$$\Delta t = \gamma(v/c^2)\Delta x'$$

where the (ct', x') frame is moving with velocity $v\hat{x}$ with respect to the (ct, x) frame.

2.5.3 Lorentz Contraction

Consider a rod of length L_* when measured in its own rest frame. What is its length when measured in an inertial frame in which it is moving with speed V? Note the length of a rod is the distance between two simultaneous events at its ends in spacetime. But simultaneity is different in different inertial frames, so the measured length of the rod is, therefore, also different. The length L in the frame where the rod is moving is the spacetime distance between the ends of the rod at $t' = 0$.Then

$$L^2 = L_*^2 - (c\Delta t)^2$$

From our Lorentz boost equation $t' = 0$ is the line $t = (V/c^2)x$, so $\Delta t = (V/c^2)L_*$. Thus

$$\boxed{L = L_* \sqrt{1 - V^2/c^2}} \tag{2.5.1}$$

This is **Lorentz contraction**.

2.5.4 Addition of Velocities

Consider a particle whose motion is described by $x(t), y(t), z(t)$ in one frame and $x'(t'), y'(t'), z'(t')$ in a second frame moving along the x-axis of the first with velocity v. From our Lorentz boost transformations we can compute the relation between $\mathbf{V} = d\mathbf{x}/dt$ in one frame and the velocity $\mathbf{V}' = d\mathbf{x}'/dt'$ in the other, namely

$$V^{x'} = \frac{dx'}{dt'} = \frac{\gamma(dx - v dt)}{\gamma(dt - v/c^2 dx)} = \frac{V^x - v}{1 - vV^x/c^2}$$

Similarly,

$$V^{y'} = \frac{V^y/\gamma}{1 - vV^x/c^2}$$
$$V^{z'} = \frac{V^z/\gamma}{1 - vV^x/c^2}$$

2.6 Units

Today the velocity of light is not measured, it is defined to be exactly the conversion factor

$$c = 299792458 \text{ m/s}$$

Measuring time in units of length means changing from the mass-length-time system of units traditional in mechanics to a mass-length system. Measuring both space and time in length units has the effect of putting $c = 1$ everywhere in our formulas. Further, in these units velocities are dimensionless.

Problems

2.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.