Phys 343 Formula Sheet

E/Ea Thompson(they/them)

August 30, 2023

0.1 Vectors

Formula 0.1 (Dot Product)**.** *For any vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, we have that the dot product between the two vectors is*

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_x w_x + v_y w_y + v_z w_z \tag{0.1}$$

Formula 0.2 (Cross Product)**.** *For any vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, we have that the cross product between the two vectors is*

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{bmatrix} v_y w_z - v_z w_y, & v_z w_x - v_x w_z, & v_x w_y - v_y w_x \end{bmatrix}^T \tag{0.2}$$

Moreover,

$$||\boldsymbol{v} \times \boldsymbol{w}|| = ||\boldsymbol{v}||||\boldsymbol{w}|| \sin(\theta) \tag{0.3}$$

and $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$. Note that the derivative when $\boldsymbol{v}$, and $\boldsymbol{w}$ are functions is

$$\frac{d}{dt}(\boldsymbol{v}(t) \times \boldsymbol{w}(t)) = \frac{d\boldsymbol{v}(t)}{dt} \times \boldsymbol{w}(t) + \boldsymbol{v}(t) \times \frac{d\boldsymbol{w}(t)}{dt} \tag{0.4}$$

0.2 Hyperbolic Trig and Trig Identities

Table 1: Functions and Derivatives

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\text{arccot}(x)$	$\frac{-1}{1+x^2}$
$\text{arcsec}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\text{arccsc}(x)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sinh(x) = \frac{e^x-e^{-x}}{2}$	$\cosh(x)$
$\cosh(x) = \frac{e^x+e^{-x}}{2}$	$\sinh(x)$
$\tanh(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$	$\text{sech}^2(x)$
$\coth(x)$	$-\text{csch}^2(x)$
$\text{sech}(x)$	$-\tanh(x)\,\text{sech}(x)$
$\text{csch}(x)$	$-\text{csch}(x)\cot(x)$
$\text{arcsinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\text{arcCosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\text{arctanh}(x)$	$\frac{1}{1-x^2}$ $(\lvert x\rvert < 1)$
$\text{arcCoth}(x)$	$\frac{1}{1-x^2}$ $(\lvert x\rvert > 1)$
$\text{arcsech}(x)$	$\frac{-1}{x\sqrt{1-x^2}}$
$\text{arcCsch}(x)$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Table 2: Trig Identities

LHS	RHS
$\sin^2(x) + \cos^2(x)$	1
$\tan^2(x) + 1$	$\sec^2(x)$
$1 + \cot^2(x)$	$\csc^2(x)$
$\sin(\alpha \pm \beta)$	$\sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$
$\cos(\alpha \pm \beta)$	$\cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\tan(\alpha \pm \beta)$	$\frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin(2x)$	$2\cos(x)\sin(x)$
$\cos(2x)$	$\cos^2(x) - \sin^2(x)$
	$2\cos^2(x) - 1$
	$1 - 2\sin^2(x)$
$\tan(2x)$	$\frac{2\tan(x)}{1 - \tan^2(x)}$
$\cosh^2(x) - \sinh^2(x)$	1
$1 - \tanh^2(x)$	$\text{sech}^2(x)$
$\coth^2(x) - 1$	$\text{csch}^2(x)$

0.3 Coordinates

Formula 0.3 (Cartesian). *Any vector* $\boldsymbol{v} \in \mathbb{R}^3$ *can be represented in cartesian coordinates,* (x, y, z)*, as*

$$\boldsymbol{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{0.5}$$

Additionally, if $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ *is a vector valued function, it has derivatives*

$$\dot{\boldsymbol{v}} = \dot{v_x}\hat{x} + \dot{v_y}\hat{y} + \dot{v_z}\hat{z} \tag{0.6}$$

and

$$\ddot{\boldsymbol{v}} = \ddot{v_x}\hat{x} + \ddot{v_y}\hat{y} + \ddot{v_z}\hat{z} \tag{0.7}$$

Formula 0.4 (Cylindrical). *Any vector* $\boldsymbol{v} \in \mathbb{R}^3$ *can be represented in cylindrical coordinates,* (ρ, ϕ, z)*, as*

$$\boldsymbol{v} = v_\rho\hat{\rho} + v_z\hat{z} \tag{0.8}$$

Additionally, if $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ *is a vector valued function, it has derivatives*

$$\dot{\boldsymbol{v}} = \dot{v_\rho}\hat{\rho} + v_\rho\dot{v_\phi}\hat{\phi} + \dot{v_z}\hat{z} \tag{0.9}$$

and

$$\ddot{\boldsymbol{v}} = (\ddot{v_\rho} - v_\rho \dot{v_\phi}^2)\hat{\rho} + (v_\rho \ddot{v_\phi} + \dot{v_\rho}\dot{v_\phi})\hat{\phi} + \ddot{v_z}\hat{z} \tag{0.10}$$

Where we define

$$v_\rho = \sqrt{v_x^2 + v_y^2} \textit{ and } v_\phi = \arctan\left(\frac{v_x}{v_y}\right) \tag{0.11}$$

Formula 0.5 (Spherical)**.** *Any vector $\boldsymbol{v} \in \mathbb{R}^3$ can be represented in spherical coordinates, (r, θ, ϕ), as*

$$\boldsymbol{v} = v_r \hat{r} \tag{0.12}$$

Additionally, if $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector valued function, it has derivatives

$$\dot{\boldsymbol{v}} = \dot{v_r}\hat{r} + v_r \dot{v_\theta}\hat{\theta} + v_r \sin(v_\theta)\dot{v_\phi}\hat{\phi} \tag{0.13}$$

and

$$\ddot{\boldsymbol{v}} = (\ddot{v_r} - v_r \dot{v_\theta}^2 - v_r \sin^2(v_\theta)\dot{v_\phi}^2)\hat{r} + (v_r \ddot{v_\theta} + \dot{v_r}\dot{v_\theta} - v_r \sin(v_\theta)\cos(v_\theta)\dot{v_\phi}^2)\hat{\theta} \tag{0.14}$$

$$+(\dot{v_r}\sin(v_\theta)\dot{v_\phi} + v_r \cos(v_\theta)\dot{v_\theta}\dot{v_\phi} + v_r \sin(v_\theta)\ddot{v_\phi})\hat{\phi} \tag{0.15}$$

Where we define

$$\boldsymbol{v} = v_r \hat{r},\ v_\theta = \arctan\left\{\frac{\sqrt{v_x^2 + v_y^2}}{v_z}\right\},\ \textit{and } v_\phi = \arctan\left(\frac{v_x}{v_y}\right) \tag{0.16}$$

Formula 0.6 (Spherical Gradient)**.**

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi} \tag{0.17}$$

Note (Inertial Frames). The center of mass of a system is an inertial reference frame.

0.4 Center of Mass

Formula 0.7 (Integral Form)**.**

$$\boldsymbol{R}_{CM} = \frac{1}{M}\int \boldsymbol{r} dm = \frac{1}{M}\int\int\int \boldsymbol{\varrho}(\boldsymbol{r})\boldsymbol{r} dV \tag{0.18}$$

Note that this integral can be done component wise for X, Y, and Z seperately (or whatever orthonormal basis you are using).

0.5 Angular Momentum

Formula 0.8 (Angular Momentum).

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} \tag{0.19}$$

and

$$\dot{\boldsymbol{l}} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\Gamma} \tag{0.20}$$

and for a rigid body rotating about a fixed axis,

$$\boldsymbol{L} = I\boldsymbol{\omega} \tag{0.21}$$

Formula 0.9 (Kepler's Law).

$$\frac{dA}{dt} = \frac{l}{2m} \tag{0.22}$$

Formula 0.10 (Angular Momentum Magnitude).

$$l = m\omega r^2 \tag{0.23}$$

0.6 Rotational Motion of Rigid Bodies

Formula 0.11 (Moment of Inertia).

$$I_A = \sum_{n=1}^{N} m_n r_n^2 \tag{0.24}$$

which gives $\boldsymbol{L}_A = I_A\boldsymbol{\omega}$, *where* $\boldsymbol{\omega}$ *is the angular velocity of the object about* A.

$$I_A = \int\int\int r^2 dV \tag{0.25}$$

where r *is the distance from a particle to the axis of rotation.*

Formula 0.12 (Kinetic Rotational Energy). *Rigid object rotating about a fixed axis*

$$T = \frac{1}{2}I\omega^2 \tag{0.26}$$

total kinetic energy of a rigid object spinning about a fixed axis:

$$T = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega_{CM}^2 \tag{0.27}$$

Formula 0.13 (Perpendicular Axis Theorem)**.** *For perpendicular axis x, y, and z, if a planar lamina lies in the xy-plane, then the perpendicular axis theorem states that*

$$I_z = I_x + I_y \tag{0.28}$$

Formula 0.14 (Angular Momentum)**.** *The total angular momentum of a system is define as*

$$\vec{L} := \sum_i \vec{l}_i = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i = \vec{R} \times M\dot{\vec{R}} + \sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i' \tag{0.29}$$

where $\vec{r}_i = \vec{R} + \vec{r}_i'$. Moreover, we have that the time derivative is

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i' \times \vec{F}_{ext,i} + \vec{R} \times \vec{F}_{ext} = \vec{\Gamma}_{ext,cm} + \vec{\Gamma}_{ext} \tag{0.30}$$

Formula 0.15 (Kinetic Energy of a Rigid Body)**.** *For a rigid body we have that the kinetic energy is*

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \underbrace{\frac{1}{2} M \dot{\vec{R}} \cdot \dot{\vec{R}}}_{\textit{Motion of CM}} + \underbrace{\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i' \cdot \dot{\vec{r}}_i'}_{\textit{Rotation about CM}} \tag{0.31}$$

Remark 0.16. Suppose we made the above derivation with $\vec{R}$ not the CM, but a point p that is momentarily at rest, so $\dot{\vec{R}} = \vec{0}$. Then

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i' \cdot \dot{\vec{r}}_i' \tag{0.32}$$

Formula 0.17 (Products and Moments of Inertia)**.** *Given a rigid object, we define the moment of inertia's of the object as*

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2), \quad I_{yy} = \sum_i m_i (x_i^2 + z_i^2), \quad I_{zz} = \sum_i m_i (x_i^2 + y_i^2) \tag{0.33}$$

and the products of inertia as

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i, \quad I_{xz} = I_{zx} = -\sum_i m_i x_i z_i, \quad I_{yz} = I_{zy} = -\sum_i m_i y_i z_i \tag{0.34}$$

For continuous mass distributions we can extend these discrete summations to continuous definite integrals.

Remark 0.18 (Mirror Symmetry). Suppose a rigid body has mirror symmetry about a plane $\alpha = 0$, for $\alpha \in \{x, y, z\}$. Then for all $\beta \in \{x, y, z\} \backslash \{\alpha\}$, the perpendicular products of inertia $I_{\alpha\beta}$ are zero. For example, if we have mirror symmetry about the $z = 0$ plane, then I_{zy} and I_{zx} are zero.

Remark 0.19 (Rotational Symmetry). Any object with rotational symmetry about an axis can be thought of as being built up with thin circular hoops. Due to this, each product of inertia is 0.

Formula 0.20 (Inertia Tensor and Angular Momentum). *For a rigid body and a given choice of orthonormal coordinates, we define the inertia tensor as*

$$\vec{\vec{I}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{0.35}$$

Then, if the body rotates with angular velocity $\vec{\omega} = [\omega_x \ \omega_y \ \omega_z]^T$, the angular momentum about the coordinate system's origin, for which the axis of rotation intersects, is

$$\vec{L} = \vec{\vec{I}}\vec{\omega} \tag{0.36}$$

Formula 0.21 (Parallel Axis Theorem). *Suppose a rigid body is rotating about a fixed axis α. Then, the moment of inertia about α is*

$$I_{\alpha\alpha} = M||proj_\alpha(\vec{R}) - \vec{R}||^2 + I_{cm,para} \tag{0.37}$$

where $I_{cm,para}$ is the moment of inertia about a parallel axis through the object's center of mass. In particular, if α is the z axis, d is the distance from the object's center of mass and the z axis, and x'_i and y'_i are coordinates in the center of mass frame about a parallel z' axis to z, then

$$I_{zz} = Md^2 + \sum_i m_i(x_i'^2 + y_i'^2) \tag{0.38}$$

Formula 0.22 (Perpendicular Axis Theorem). *Let D be a Lamina, and choose its plane to be the xy-plane. Then we have that*

$$I_{zz} = I_{xx} + I_{yy} \tag{0.39}$$

Formula 0.23 (Kinetic energy of a Rotating Rigid Body). *Given a rotating rigid object with angular velocity $\vec{\omega}$ and inertia tensor $\vec{\vec{I}}$, we have that*

$$T_{rot} = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2}\vec{\omega} \cdot (\vec{\vec{I}}\vec{\omega}) \tag{0.40}$$

If the object rotates about a fixed axis, say z, then $\vec{\omega} = [0 \ 0 \ \omega]^T$ and

$$T_{rot} = \frac{1}{2} I_{zz}\omega^2 \tag{0.41}$$

0.7 Physical Pendulum

Formula 0.24 (General)**.** *The equation of motion for a physical pendulum swinging about a fixed axis (say the z-axis) with no friction is*

$$\ddot{\theta} + \frac{mgd}{I_{zz}} \sin(\theta) = 0 \tag{0.42}$$

where d is the distance between the pivot point and the center of mass. We can use energy conservation to obtain

$$E = mgd(1 - \cos(\theta_0)) = \frac{1}{2} I_z z \dot{\theta}^2 + mgd(1 - \cos(\theta)) \tag{0.43}$$

Solving by separation of variables we have that

$$t = \sqrt{\frac{I_{zz}}{2mgd}} \int_0^{\theta'} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}} \tag{0.44}$$

Define a new variabel ϕ *such that* $\sin(\phi) = \frac{1}{k} \sin\left(\frac{1}{2}\theta\right)$ *with* $k = \sin\left(\frac{1}{2}\theta_0\right)$*. Note that* $\cos(\theta) = 1 - 2\sin^2(\frac{1}{2}\theta)$*, so* $\cos(\theta) - \cos(\theta_0) = 2(\sin^2(\frac{1}{2}\theta_0) - \sin^2(\frac{1}{2}\theta))$ *and we can then write* $\cos(\theta) - \cos(\theta_0) = 2k^2(1 - \sin^2(\phi)$*. Then we have that* $\cos(\phi)d\phi = \frac{1}{k}\frac{1}{2}\cos\left(\frac{1}{2}\theta\right)d\theta$*. Note*

$$d\theta = \frac{2k\cos(\phi)d\phi}{\cos\left(\frac{1}{2}\theta\right)} d\phi = \frac{2k\sqrt{1 - \sin^2(\phi)}}{\sqrt{1 - k^2\sin^2(\phi)}} d\phi$$

Then we can rewrite

$$\begin{aligned} t &= \sqrt{\frac{I_{zz}}{2mgd}} \int_0^{\phi} \frac{1}{\sqrt{2k^2(1 - \sin^2(\phi))}} \frac{2k\sqrt{1 - \sin^2(\phi)}}{\sqrt{1 - k^2\sin^2(\phi)}} d\phi \\ &= \sqrt{\frac{I_{zz}}{2mgd}} \int_0^{\phi} \frac{\sqrt{2}d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} \\ &= \sqrt{\frac{I_{zz}}{mgd}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} \end{aligned}$$

This integral is called the incomplete elliptical integral of the first kind, denoted by $F(k, \phi)$*. When* $\theta = \theta_0$*,* $\sin(\phi) = 1$*, so* $\phi = \frac{\pi}{2}$*. This is a quarter of our oscillation, so we have that*

$$T = 4\sqrt{\frac{I_{zz}}{mgd}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} \tag{0.45}$$

where the integral is now called the complete elliptical integral of the first kind, and denoted by $F(k, \pi/2)$*. As* θ_0 *becomes small,* $F(k, \pi/2) \approx \pi/2$ *and* $T \approx T_0 = 2\pi\sqrt{\frac{I_{zz}}{mgd}}$*. In particular we can write*

$$T = 4\sqrt{\frac{I_{zz}}{mgd}} F(k, \pi/2) = \frac{2}{\pi} T_0 F(k, \pi/2) \tag{0.46}$$

Formula 0.25 (Small Theta)**.** *In the case of small* θ, $\sin(\theta) \sim \theta$ *(in radians), so our equation becomes that of a simple harmonic oscillator*

$$\ddot{\theta} = -\frac{mgd}{I_{zz}}\theta \tag{0.47}$$

which has angular velocity $\omega = \sqrt{\frac{mgd}{I_{zz}}}$*. Then, for small angles the period of the physical pendulum is*

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_{zz}}{mgd}} \tag{0.48}$$

0.8 Euler's Equations

Formula 0.26 (General Case)**.** *We consider an object moving in free fall and define an inertial frame* $[x, y, z]$ *and a principal axis frame in the object,* $[\hat{e}_1, \hat{e}_2, \hat{e}_3]$*. Then since we are in a principal axis frame in the body, the inertia tensor with respect to this frame is diagonal*

$$\vec{\vec{I}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{0.49}$$

0.9 Central Forces

Formula 0.27 (Gravity)**.** *The force of gravity on a mass 2 by a mass 1, with* $\hat{r}$ *pointing from 1 to 2 is:*

$$\boldsymbol{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r} \tag{0.50}$$

Formula 0.28 (Coloumb)**.** *The Coloumb force on a charge 2 by a charge 1, with* $\hat{r}$ *pointing from 1 to 2, is:*

$$\boldsymbol{F}_C = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\hat{r} \tag{0.51}$$

0.10 Energy

Formula 0.29 (General Energy)**.**

$$\Delta E = \Delta(T + U) = W_{nc} \tag{0.52}$$

Formula 0.30 (Kinetic Energy).

$$T = \frac{1}{2}mv^2 \textit{ and } \frac{dT}{dt} = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{v} \tag{0.53}$$

Formula 0.31 (Work Energy Thm). *The change in kinetic energy between points A and B is given by:*

$$\Delta T = T_B - T_A = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = W(A \to B) \tag{0.54}$$

where $\boldsymbol{F}$ *is the net force, so* $W(A \to B)$ *can be written as:*

$$W(A \to B) = \sum_{n=1}^{N} W_n(A \to B) \tag{0.55}$$

Formula 0.32 (Potential). *Given a conservative force* $\boldsymbol{F}$*, we can define a potential function with zero potential at* $\boldsymbol{r}_0$ *by*

$$U(\boldsymbol{r}) = U(\boldsymbol{r}) - U(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F(r')} \cdot d\boldsymbol{r'} \tag{0.56}$$

Formula 0.33 (Solution for 1D Conservative Systems). *For a one dimensional conservative system, the mechanical energy is given by* $E = \frac{1}{2}m\dot{x}^2 + U$*, which can then be solved by separation of variables as*

$$t_2 - t_1 = \pm \int_{x_1}^{x_2} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}} \tag{0.57}$$

0.11 Non-inertial Frames

Formula 0.34 (Equation of Motion in a Linearly Accelerated Frame). *The equation of motion for an object with position* $\vec{r}$ *in a linearly accelerated frame, with acceleration* $\vec{A}$ *relative to an inertial frame, is*

$$m\ddot{\vec{r}} = \vec{F} + \vec{F}_{inertial} \tag{0.58}$$

where $\vec{F}$ *is the net force measured in the inertial frame, and*

$$\vec{F}_{inertial} = -m\vec{A} \tag{0.59}$$

Definition 0.35 (Angular Velocity)**.** If an object is rotating about an a line with unit vector $\hat{u}$, where the direction is given by the right hand rule, at a rate ω, then its angular velocity is given by

$$\vec{\omega} = \omega\hat{u} \tag{0.60}$$

Formula 0.36 (Rotating vector)**.** *Suppose that $\vec{r} \in \mathbb{R}^3$ is rotating along with a rigid body with angular velocity $\vec{\omega}$. Then the derivative of $\vec{r}$ is given by*

$$\dot{\vec{r}} = \omega \times \vec{r} \tag{0.61}$$

Formula 0.37 (Derivative in a rotating frame)**.** *Suppose that we have right handed orthonormal coordinate systems $S_0 = (\hat{E}_1, \hat{E}_2, \hat{E}_3)$ and $S = (\hat{e_1}, \hat{e_2}, \hat{e_3})$, where S rotates with angular velocity $\vec{\omega}$ with respect to S_0, and S and S_0 share the same origin. Then for a function $\vec{r} : \mathbb{R}^3 \to \mathbb{R}^3$, we can write $\vec{r} = A_1\hat{E}_1 + A_2\hat{E}_2 + A_3\hat{E}_3 = a_1\hat{e_1} + a_2\hat{e_2} + a_3\hat{e_3}$. Then, it follows that*

$$\frac{d\vec{r}}{dt} = \sum_{i=1}^{3} \frac{dA_i}{dt}\hat{E}_i = \sum_{i=1}^{3} \frac{da_i}{dt}\hat{e_i} + a_i(\omega \times \hat{e_i}) \tag{0.62}$$

Formula 0.38 (Equation of Motion in a Rotating Frame)**.** *If a frame S is rotating with angular velocity $\vec{\Omega}$ with respect to an inertial frame S_0, then for a position $\vec{r}$ measured in S we have the equation of motion*

$$m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf} \tag{0.63}$$

where $\vec{F}$ is the net force on the object measured in S_0,

$$\vec{F}_{cor} = -2m\vec{\Omega} \times \dot{\vec{r}} = 2m\dot{\vec{r}} \times \vec{\Omega} \tag{0.64}$$

and

$$\vec{F}_{cf} = -m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \tag{0.65}$$

Formula 0.39 (Height of the tides)**.** *Suppose we have two celestial bodies with masses M_1 and M_2, radii R_1 and R_2, and distance d_0 between their centers. Then the max height of the tides on M_1 due to the gravitational force from M_2 is*

$$h = \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_1}{d_0}\right)^3 R_1 \tag{0.66}$$

Formula 0.40 (Free-fall Acceleration on Earth)**.** *Consider an object falling on Earth at an angle θ from the Earth's axis of rotation. Then its initial force is given by*

$$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} \tag{0.67}$$

Then write

$$\vec{g} = \frac{\vec{F}_{eff}}{m} = \vec{g}_0 + \Omega^2 R\sin(\theta)\hat{\rho} \tag{0.68}$$

Splitting into components we have

$$g_{rad} = g_0 - \Omega^2 R\sin^2(\theta) \tag{0.69}$$

and

$$g_{tan} = \Omega^2 R\sin(\theta)\cos(\theta) \tag{0.70}$$

Note that the angle of deviation from the radial acceleration due to gravity is

$$\tan(\alpha) = \frac{g_{tan}}{g_{rad}} = \frac{\Omega^2 R\sin(\theta)\cos(\theta)}{g_0 - \Omega^2 R\sin^2(\theta)} \approx \frac{\Omega^2 R\sin(\theta)\cos(\theta)}{g_0} \tag{0.71}$$

The product $\sin(\theta)\cos(\theta)$ *is maximal for* $\theta = \frac{\pi}{4}$*, so*

$$\tan(\alpha_{max}) \approx \frac{\Omega^2 R}{2g_0} \tag{0.72}$$

Tgeb α_{max} *is a small angle so* $\tan(\alpha_{max}) \approx \alpha_{max}$ *in rad, so* $\alpha_{max} \approx 0.0017$ *rad (*≈ 0.1 *degrees)*

0.12 Variational Calculus and Lagrangians

Formula 0.41 (Euler-Lagrange Equation)**.** *Suppose that f is a function of $q_1, q_2, ..., q_n$, $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$ and possibly t, where $n \in \mathbb{Z}$ and $n \geq 1$. Then, define*

$$S(q_i, \dot{q}_i, t) = \int_{t_1}^{t_2} f(q_i, \dot{q}_i, t)dt \tag{0.73}$$

Then the S is stationary when the functions q_i and $\dot{q}_i$ satisfy the Euler-Lagrange equations

$$\frac{\partial f}{\partial q_i} = \frac{d}{dt}\left[\frac{\partial f}{\dot{q}_i}\right],\ i \in \{1, 2, ..., n\} \tag{0.74}$$

Formula 0.42 (Lagrangian)**.** *The* ***Lagrangian*** *for a conservative system is defined to be*

$$\mathcal{L} = T - U \tag{0.75}$$

*Moreover, for any **holonomic** system (degrees of freedom = # of generalized coordinates), Newton's Second Law is equivalent to the n lagrange equations*

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right],\ i \in \{1, 2, ..., n\} \tag{0.76}$$

where q_i are the generalized coordinates of the system. This is equivalent to Hamilton's principle which states that system's evolve in time such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t) dt \tag{0.77}$$

is stationary.

0.13 Hamltonian Formulaism

Formula 0.43 (Hamiltonian)**.** *We define the **Hamiltonian** for a system with generalized coordinates q_i and generalized momenta $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ by*

$$\mathcal{H} := \left[\sum_i p_i \dot{q}_i\right] - \mathcal{L} \tag{0.78}$$

It is important to note that the Hamiltonian must be expressed solely in terms of the generalized momenta, p_i, and the generalized coordinates, q_i. From the Hamiltonian we obtain the pair of first order partial differential equations:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i},\ \textit{and}\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i},\ (i \in \{1, 2, .., \}) \tag{0.79}$$

When the transformation between the generalized coordinates q_i and the cartesian coordinates e_i are independent of time, then the Hamiltonian is equal to

$$\mathcal{H} = T + U \tag{0.80}$$

the total energy of the system. Moreover, if the Lagrangian does not depend explicitly on time, then the Hamiltonian is conserved.

0.14 Two-body Problem

Formula 0.44. *Consider a system of two masses with a conservative central force between the masses which only depends on their distance. Hence, we can write the potential as*

$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$. Let $\vec{R}$ be the position of the center of mass and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Then we can write

$$\begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \end{cases} \tag{0.81}$$

*We define the **reduced mass of the system** as*

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{0.82}$$

Then we can write $T = \frac{1}{2}M\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}}$. Then, we have that

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}} - U(r) = \mathcal{L}_{cm} + \mathcal{L}_{rel} \tag{0.83}$$

The Lagrange equation for the center of mass is $M\ddot{\vec{R}} = \vec{0}$, so $M\dot{\vec{R}}$ is constant. Choosing the center of mass inertial coordinate system,

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}} - U(r) \tag{0.84}$$

Then, we have that $\mu\ddot{\vec{r}} = -\nabla U(r)$. Moreover, the angular momentum of the system (which is constant since there is no external force and the internal forces are central) can be written as

$$L = \vec{r} \times \mu\dot{\vec{r}}$$

Then we have

$$T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$$

in the CM frame. Solving the Lagrange equations we find

$$\mu r^2\dot{\phi} = \ell$$

is constant, where l is the magnitude of the angular momentum, and substituting this into the radial equation

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{du}{dr}$$

gives

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{dU}{dr}$$

where the first term is the centrifugal force. The centrifugal force can be associated with the potential

$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

wher $F_{cf} = -\frac{d}{dr}U_{cf}$. Then we have that

$$\mu\ddot{r} = -\frac{d}{dr}(U_{cf} + U(r)) = -\frac{d}{dr}U_{eff} \tag{0.85}$$

We assume that the central force is of the form $F(r) = -\frac{\gamma}{r^2}$ for γ a constant, then we find that

$$r(\phi) = \frac{c}{1 + \varepsilon \cos(\phi)} \tag{0.86}$$

where $\varepsilon = \frac{A\ell^2}{\gamma\mu}$, for A an integration constant, is the eccentricity of our orbit, and

$$c = \frac{\ell^2}{\gamma\mu}$$

is the latus rectum. For $\varepsilon < 1$ we have an ellipse, for $\varepsilon = 1$ a parabola, and for $\varepsilon > 1$ a hyperbola. For $\varepsilon < 1$ we also have

$$r_{min} = \frac{c}{1+\varepsilon}, \text{ and } r_{max} = \frac{c}{1-\varepsilon} \tag{0.87}$$

Formula 0.45 (Kepler's Laws)**.** *Kepler's first law states that the orbit of a planet around the sun is an ellipse with the sun at one of the focal points (i.e. $\varepsilon < 1$). For a the semi-major axis of the ellipse, the distance from the center C to the focal point F is $a\varepsilon$, and then $r_{min} = a(1-\varepsilon)$ and $r_{max} = a(1+\varepsilon)$.*

The line between two masses orbiting each other trace out equal areas in equal amounts of time, such that

$$\frac{dA}{dt} = \frac{\ell}{2\mu} \tag{0.88}$$

For a semi-minor axis of b, we have $b = a\sqrt{1-\varepsilon^2}$. Then, the period of the elliptical orbit is

$$\tau^2 = 4\pi^2 \frac{a^3 c\mu^2}{\ell^2} = 4\pi^2 \frac{a^3\mu}{\gamma} \tag{0.89}$$

where $c = a(1-\varepsilon^2)$. In the case of gravity, so $\gamma = Gm_1m_2$, we have

$$\tau^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3 \tag{0.90}$$

Formula 0.46 (Energy)**.** *Given a general inverse square force law $F(r) = \frac{-\gamma}{r^2}$, we have that the total energy of the orbit is*

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1) \tag{0.91}$$

eccentricity	energy	orbit
$\varepsilon = 0$	$E < 0$	circular
$0 < \varepsilon < 1$	$E < 0$	elliptical
$\varepsilon = 1$	$E = 0$	parabollic
$\varepsilon > 1$	$E > 0$	hyperbolic

Formula 0.47 (Change of Orbits)**.** *To change from one orbit to another we require the continuity condition*

$$\frac{c_1}{1+\varepsilon_1\cos(\theta_0-\delta_1)} = \frac{c_2}{1+\varepsilon_2\cos(\theta_0-\delta_2)} \tag{0.92}$$

For a tangential thrust we can take, without loss of generality, $\phi_0 = \delta_1 = \delta_2 = 0$, when at perigee, so the continuity condition simplifies to

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2} \tag{0.93}$$

Define the ratio of the speeds before and after the thrust is applied to be

$$\lambda := \frac{v_2}{v_1} \tag{0.94}$$

If $\lambda > 1$ the thrust is forward and the satellite gains speed. If $0 < \lambda < 1$ then the thrust was backward and the satellite lost speed. At perigee (or apogee) we have $\ell = \mu r v$. The value of r will not changing during the impulse (since we assume it to be instantaneous), and we may assume that the change in μ is negligible. Then we have that

$$\ell_2 = \lambda\ell_1 \tag{0.95}$$

Moreover, since c is proportional to ℓ^2, we have that

$$c_2 = \lambda^2 c_1 \tag{0.96}$$

It follows that

$$\frac{1}{1+\varepsilon_1} = \frac{\lambda^2}{1+\varepsilon_2} \tag{0.97}$$

so

$$\varepsilon_2 = \lambda^2\varepsilon_1 + (\lambda^2 - 1) \tag{0.98}$$

0.15 Euler's Equations

Formula 0.48. *Euler's equation states that for an object either pivoting about a fixed point or without any fixed point (fre falling rotating object) we have that if the object rotates with angular velocity $\vec{\omega}$ in a frame with its principal axes, then*

$$\vec{L} = \begin{pmatrix}\lambda_1\omega_1\\ \lambda_2\omega_2\\ \lambda_3\omega_3\end{pmatrix} \tag{0.99}$$

since $\vec{\vec{I}}$ is diagonal. Then we have that

$$\dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma} \tag{0.100}$$

where $\vec{\Gamma}$ is the torque as measured in an inertial frame. Then we have the component Euler Equatiosn

$$\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \Gamma_1 \tag{0.101}$$
$$\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 = \Gamma_2 \tag{0.102}$$
$$\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 = \Gamma_3 \tag{0.103}$$

For the special case of $\vec{\Gamma} = \vec{0}$ we have

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 \tag{0.104}$$
$$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1 \tag{0.105}$$
$$\lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2 \tag{0.106}$$

Formula 0.49. *If $\vec{\omega}$ is parallel to a principal axis the time derivatives of its components are zero in the case of zero torque. Next, if $\vec{\omega} = [\omega_1\ \omega_2\ \omega_3]^T$ such that $|\omega_1|, |\omega_2| << |\omega_3|$, we can approximate $\omega_1\omega_2$ as 0 so $\dot{\omega}_3 \approx 0$. THen we have the equations*

$$\lambda_1\dot{\omega}_1 = [(\lambda_2 - \lambda_3)\omega_2]\omega_3 \tag{0.107}$$
$$\lambda_2\dot{\omega}_2 = [(\lambda_3 - \lambda_1)\omega_3]\omega_1 \tag{0.108}$$

So we obtain

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_3)(\lambda_3 - \lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1 \tag{0.109}$$

where this equation gives a solution for a harmonic oscillator if and only if either $\lambda_3 > \lambda_2, \lambda_1$ or $\lambda_3 < \lambda_2, \lambda_1$. That is, we have stable rotation with ω_1 remaining small if the object is rotating about its principal axis with the smallest or largest moment of inertia. We have unstable rotation otherwise.

Formula 0.50. *Consider $\lambda_1 = \lambda_2 = \lambda$, so $\dot{\omega}_3 = 0$. Define*

$$\Omega_b = \frac{(\lambda - \lambda_3)\omega_3}{\lambda} \tag{0.110}$$

Then we have that

$$\dot{\omega_1} = \Omega_b\omega_2 \tag{0.111}$$

and

$$\dot{\omega_2} = -\Omega_b\omega_1 \tag{0.112}$$

Upon solving we obtain

$$\vec{\omega} = \begin{pmatrix} \omega_0\cos(\Omega_b t) \\ -\omega_0\sin(\Omega_b t) \\ \omega_3 \end{pmatrix} \tag{0.113}$$

and

$$\vec{L} = \begin{pmatrix} \lambda\omega_0\cos(\Omega_b t) \\ -\lambda\omega_0\sin(\Omega_b t) \\ \lambda_3\omega_3 \end{pmatrix} \tag{0.114}$$

It follows that $\vec{L}, \vec{\omega}$ and $\hat{e}_3$ lie in the same plane.