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Phys 501: Relativity

– In Pursuit of Abstract Nonsence –

Wednesday 30th August, 2023

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Preface

This is a collection of notes associated with Phys 501 (Relativity) taken at the University of Calgary.

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Contents

Notation

List of common notations used in these notes.

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers

Chapter 1

Geometry of Relativity

Abstract In this chapter we begin the discussion of the relation between gravity and geometry. In particular, we remark that Gravity is simply the geometry of space time, with objects following straight paths or “geodesics" in the curved space time. For this purpose we study a few common geometries, and how we can use the notion of a Riemannian metric, or **line element**, to generalize our understanding of geometry from the Euclidean setting to spherical geometries and beyond.

1.1 Gravity and Geometry

As we shall soon be aware, gravity **is** geometry—it arises from the curvature of **spacetime**. The fact that all bodies fall with the same acceleration in a uniform gravitational field, independently of their composition, is one of the most accurately tested facts in physics. The most accurate such test is the comparison of the accelerations of the Earth and the Moon as they fall around the Sun. These match to within a fractional error of less than 1.5×10^{-13}.

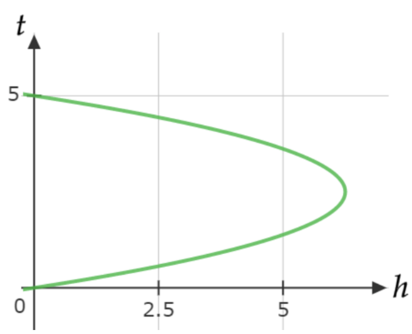

Fig. 1.1 Space-time diagram for object thrown vertically on earth with height on the horizontal.

In Einstein’s general relativity, the bodies are following a straight path in the curved spacetime produced by the Earth’s mass. Note that the exact same trajectory pictured above would occur for any other body with the same initial velocity and same initial position. This uniqueness property is not seen in all other fields. For instance, the motion of a body in a magnetic field depends on what kind of charge it has. Bodies with one sign of charge will be deflected one way, bodies with the opposite charge will be deflected the other, and bodies with no charge will not be deflected at all.

Einstein proposed that in the absence of other forces, bodies move on straight paths in this curved spacetime.

1.2 Experiments in Geometry

Geometries different from Euclid's produce different results for the sum of the interior angles of a triangle. For instance, due to the curvature of the Earth, the interior angles of a triangle on its surface would deviate from π on the order of:

$$\left|\begin{pmatrix}\text{sum of interior angles}\\ \text{of a triangle in radians}\end{pmatrix}\right| - \pi \sim \frac{(\text{area of triangle})}{R_{\oplus}^2}\left(\frac{GM_{\oplus}}{R_{\oplus}c^2}\right)$$

where $\oplus$ denotes the Earth.

1.3 Different Geometries

When we talk about straight paths in different geometries, we mean paths of shortest distance between two points, which generalized the notion of straight lines in Euclidean Geometry. In Spherical geometry these are segments of great circles (circles which cut through the origin of the sphere). Triangles on the sphere are then made of three intersecting great circles. For a spherical triangle of area A,

$$\sum_{vertices}\begin{pmatrix}\text{interior}\\ \text{angle}\end{pmatrix} = \pi + \frac{A}{a^2}$$

where a is the radius of the sphere.

Note this implies that the sum of the interior angles of a spherical triangle is always greater than π.

With a bit of geometry, the ratio of the circumference to the radius of a circle on a sphere can be calculated to be

$$\frac{C}{r} = \frac{2\pi a\sin(r/a)}{r} = 2\pi\frac{\sin(r/a)}{(r/a)}$$

In theory by surveying in three dimensions we can determine the geometry of space without needing an extra dimension. However, visualization of three dimensional geometries is in general very difficult. An example which is simpler, at least to describe, is the three-sphere. If space had such a geometry a journey in a straight line in any direction would eventuallly bring one back to the starting points. However, we can determine more detail locally. For instance, the volume inside a two-dimensional sphere of radius r in such a spatial geometry is given by

$$V = 4\pi a^3 \left\{ \frac{1}{2} \sin^{-1}(r/a) - \frac{r}{2a} \left[1 - \frac{r^2}{a^2} \right]^{1/2} \right\}$$
$$\approx \frac{4\pi r^3}{3} \left[1 + \begin{pmatrix} \text{corrections} \\ \text{of order } (r/a)^2 \end{pmatrix} \right] \quad \text{for small } r/a$$

where a is the characteristic radius of curvature of the three-sphere geometry. If the three-dimensional space had such a geometry, the characteristic radius of curvature could be determined by careful measurements of the radii and volume of two-spheres.

1.4 Specifying Geometry

One way to describe a geometry is to embed it as a surface in a higher-dimensional space whose geometry is Euclidean. However, we want an intrinsic description of geometry that makes use of just the physical dimensions that can be measured—this leads to the study of manifolds.

We could also specify a geometry by giving a small number of axioms from which the other results of geometry can be derived as theorems. However this strategy only works for some of the simplest geometries.

1.5 Coordinates and Line Element

We now investigate the a number of simple geometries with a focus on their “line elements."

1.5.1 The Euclidean Geometry of a Plane

First we must choose some coordinate system to specify the points in our geometry. For now this will be global, but in general need only be local in nature. For example we can use Cartesian coordinates, (x, y), with infinitesimals dx and dy, or polar coordinates, (r, ϕ), with infinitesimals dr and $d\phi$.

In Cartesian coordinates the distance dS between two points (x, y) and $(x + dx, y + dy)$ is specified by

$$dS = \left[(dx)^2 + (dy)^2\right]^{1/2}$$

In polar coordinates this same metric takes the form

$$dS = \left[(dr)^2 + (rd\phi)^2\right]^{1/2}$$

Note these are local representations, and hence corespond to small changes.

Now, recall the equation for a circle of origin center and radius R in Cartesian coordinates:

$$x^2 + y^2 = R^2$$

Its circumference can be obtained by integrating our metric:

$$\begin{aligned} C &= \int dS = \int \left[(dx)^2 + (dy)^2\right]^{1/2} \\ &= 2\int_{-R}^{+R} dx \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}_{x^2+y^2=R^2} \\ &= 2\int_{-R}^{+R} dx \sqrt{\frac{R^2}{R^2 - x^2}} \end{aligned}$$

Consider the change of variables $x = R\xi$, so

$$C = 2R\int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} = 2\pi R$$

Reverting to polar coordinates this task becomes even simpler:

$$C = \int dS = \int_0^{2\pi} R d\phi = 2\pi R$$

We can proceed in this fasion to obtain all the standard theorems of Euclidean plane geometry. For instance, we can define the angle of intersecting lines as the ratio of the length ΔC of the part of a circle centered on their intersection that lies between the lines to the circle's radius R:

$$\theta := \frac{\Delta C}{R}$$

In general all geometry can be reduced to relations between distances, which correspond to integrals of our metric between neighborhing points.

We also refer to these metrics as **line elements**. Conventionally we write our line element quadratically, with

$$dS^2 = dx^2 + dy^2$$

This is the **Riemannian metric** for the plane in cartesian coordinates.

1.6 The Non-Euclidean Geometry of a Sphere

Consider the surface of a two-dimensional sphere of radius a. We can use the angles (θ, ϕ) of three-dimensional polar coordinates to label points on the sphere. The distance between points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ can be seen after a little work to be

$$dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Note this is the Riemannian metric on the two-sphere induced by its embedding in three-space.

Let us consider a circle on the sphere, by which we mean the locus of points on the curface that are a constant distance along the surface from a fixed point in the surface. Note the sphere is geometrically uniform, so we may orient our polar coordinate system so that the polar axis is at the center of the circle (i.e. $\theta = 0, \pi = 0$). Then a circle is a the locus of an equation

$$\theta = \Theta$$

for Θ a constant. Along the circle $dS = a \sin \Theta d\phi$, so the circumference is

$$C = \int dS = \int_0^{2\pi} a \sin \Theta d\phi = 2\pi a \sin \Theta$$

The radius is the distance from the center to the circle along a curve for which θ varies but $d\phi = 0$. Along this curve $dS = a d\theta$, so the radius is

$$r = \int_{center}^{circle} dS = \int_0^{\Theta} a d\theta = a\Theta$$

Then the relationship between the circumference and radius of a circle in the non-Euclidean geometry of a sphere becomes

$$C = 2\pi a \sin (r/a)$$

Note when $r \ll a$, we have the approximation

$$C \approx 2\pi r$$

Note ϕ is the measure of longitude on the Earth and $\lambda = \pi/2 - \theta$ is the measure of latitude, so in this coordinate system the metric is

$$dS^2 = a^2(d\lambda^2 + \cos^2 \lambda d\phi^2)$$

1.7 The Geometry of More General Surfaces

Consider a Riemannian metric, or line element, given in local coordinates by

$$dS^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$$

for various choices of $f(\theta)$. The choice $f(\theta) = \sin \theta$ gives the geometry of the surface of a sphere. What other surfaces in three-space can be represented by this kind of metric?

1. Since the line element is the same for all ϕ it corresponds to a surface that is axisymmetric about an axis.

2. The circumference $C(\theta)$ of a circle of constnat θ is

$$C(\theta) = \int_0^{2\pi} af(\theta)d\phi = 2\pi af(\theta)$$

3. The distance from pole to pole is

$$d = a\int_0^\pi d\theta = \pi a$$

Working these properties out we can build a picture of this surface.

Example:

Peanut Geometry. Consider the surface specified by

$$f(\theta) = \sin\theta(1 - 3\sin^2(\theta)/4)$$

Note the surface is symmetric under reflection in the equatorial plane $\theta = \pi/2$. Starting at $\theta = 0$ the circumference of the lines of constant θ first increases and then decreases with $f(\theta)$; then it increases and decreases again. At any one θ the circumference is smaller than the corresponding value on a sphere. At the equator, for instance,

$$C(\pi/2) = 2\pi a(1 - 3/4) = \frac{\pi a}{2}$$

The maximum circumference is $(8\pi/9)a$ and occurs at $\theta = \sin^{-1}(2/3) \approx 0.73$ rad. Since the distance from pole to pole is πa, this surface has an elongated “peanut" shape.

1.8 Coordinates and Invariance

Note coordinates are a systematic set of labels for a geometry, locally, but the geometry itself is invariant under any choice of smooth coordinate system. In particular, as long as the coordinate transformation about any shared neighborhood is smooth, we can use either coordinate system to produce our calculations. For instance, the coordinates (x, y) and (r, ϕ) in the plane have local coordinate transformation

$$x = r\cos\phi, \quad y = r\sin\phi$$

The point of this discussion is that the Riemannian metric dS^2, and hence the distance dS, is an invariant quantity independent of the choice of coordinates used to compute it.

Problems

1.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 2

Principles of Special Relativity

Abstract To be completed once done

2.1 Addition of Velocities and Michelson-Morley Experiment

Recall that Maxwell's equations governing electromagnetic fields do not take the same form in every intertial frame of Newtonian mechanics. Maxwell's equations, however, do imply that light travels with constant speed c in vacuum, and this is a basic parameter in these equations. But the Galilean notion of transformation between inertial frames would imply that light should travel with different speeds in different inertial frames, moving with respect to each other.

Consider v^x, v^y, v^z, components of the velocity of a particle measured in one frame, and $v^{x'}, v^{y'}, v^{z'}$, the components of the velocity measured in a frame moving with respect to the first along its x-axis with velocity v. Galilean transformations predict

$$v^{x'} = v^x - v$$

This would imply that Maxwell's equations can only be valid in one inertial frame, because they predict one velocity for light.

> Remark:
>
> In an experiment whose results were published in 1887, Albert Michelson and Edward Morley tested the Newtonian addition of velocities law for light. The theory explaining the fact Maxwell's equations can only be valid in one intertial frame was explained using a notion of the rest frame of light, called the ether. Michelson and Morley showed using the Earth's orbital velocity around the sun would imply light should be measured at different speeds at different locations in the orbt. This was not the case, so either Newtonian mechanics or Maxwell's equations had to be modified.

2.2 Einstein's Resolution

Einstein's 1905 successful modification of Newtonian mechanics is called the special theory of relativity. Einstein supposed that the velocity of light had the same value c in all inertial frames. We must now determine a new notion of velocity addition. We will also need to re-examine the Newtonian notion of absolute time.

Consider the following thought experiment. Three observers, A, B, and O, are riding a rocket of length L. O is midway between A and B. A and B each emit light signals directed along the rocket toward O. O receives the signals simultaneously. Which signal was emitted first? This must depend on the inertial frame if the velocity of light is the same in all of them.

If the rocket is at rest in the inertial frame, they must be emitted simultaneously. If the rocket is moving in the inertial frame we reason as follows: The signals are received simultaneously by O. At earlier times when the signals were emitted B was always closer to O's position at reception than A (thinking of B as on the side in which the rocket is traveling). Since both signals travel with speed c, the one from A must have been emitted earlier than the one from B because it has a longer distance to travel to reach O at the same instant as the one from B.

> Remark:
>
> Thus **two events simultaneous in one inertial frame are not simultaneous in one moving with respect to the first if the velocity of light is the same in both**.

2.3 Spacetime

Newton's first law (free particles move at constant speed on straight lines) is unchanged in special relativity. Thus we can construct inertial frames as follows: start with an origin following the straight-line trajectory of a free particle. At one moment choose three Cartesian coordinates (x, y, z) with this origin. Propagate these axes parallel to themselves as the origin moves to define (x, y, z) at later times. The result is an inertial frame.

For each inertial frame there is a notion of time t. From our previous discussion each inertial frame has a different notion of time and simultaneity. Thus inertial frames are spanned by four Cartesian coordinates, (t, x, y, z), giving us spacetime. The defining assumption of special relativity is a geometry for four-dimensional spacetime.

2.3.1 Spacetime Diagrams

A **spacetime diagram** is a plot of two of the coordinate axes of an inertial frame—two coordinate axs of spacetime. Spacetime diagrams are slices or sections of spacetime in much the same way as an xy plot is a two-dimensional slice of three-dimensional space. It is convenient to use ct rather than t as an axis, because then both have the same dimension.

> **Definition 2.3.1** An **event** is a point P in spacetime located at a particular place in space (x_P) at a particular time (t_P).

> **Definition 2.3.2** A particle describes a curve in spacetime called a **world line**. It is the curve of positions of the particle at different instants.

The slope of the world line gives the ratio c/v^i, since $d(ct)/dx_i = cdt/dx_i = c/v^i$, where zero velocity corresponds to infinite slope (no position change in time), and a velocity of c corresponds to a slope of 1. Hence light rays move along the 45^{circ} lines in a spacetime diagram.

2.3.2 The Geometry of Flat Spacetime

Consider the following thought experiment. We have two parallel mirrors separated by a distance L that are at rest in an inertial frame in wich events are described by coordinates (t, x, y, z). Take y to be the vertical direction between the mirrors and x the direction parallel to them. A light signal bounces back and forth between the mirrors. A clock measures the time interval Δt between the event A of the departure of the light ray and the event C of its return to the same point in space. These two events are separated by coordinate intervals

$$\Delta t = 2L/c, \quad \Delta x = \Delta y = \Delta z = 0$$

in the inertial frame where the mirrors are at rest.

Now consider a frame that is moving with a speed v with respect to the (t, x, y, z) inertial frame along the negative x-direction parallel to the mirrors. Write events in this frame by (t', x', y', z') wiith x' parallel to x. In this frame the mirrors are moving with speed V along the positive x'-direction. Consider the time $\Delta t'$ between the departure and return of a light ray. The light ray travels a distance $\Delta x' = v\Delta t'$ in the x'-direction between emission at A and return at C. The distance traveled in the y'-direction is L, assuming the transverse distances are the same in both inertial frames. The total distance traveled is therefore $2[L^2 + (\Delta x'/2)^2]^{1/2}$. Assuming with Einstein that the velocity of light is c in this inertial frame, the time of travel $\Delta t'$ is this distance divided by c. Thus the coordinate intervals between A and C in this frame are

$$\Delta t' = \frac{2}{c}\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}, \quad \Delta x' = v\Delta t', \quad \Delta y' = 0, \quad \Delta z' = 0$$

It follows that

$$-(c\Delta t')^2 + (\Delta x')^2 = -4[L^2 + (\Delta x'/2)^2] + (\Delta x')^2 = -4L^2 = -(c\Delta t)^2$$

This identity is key to identifying an **invariant** and to finding the line element that describes the geometry of spacetime. Since $\Delta x = 0$ and the Δy's and Δz's are zero in both frames, we can judiciously add them back into the two sides of (4.5) to find that the combination

$$\boxed{(\Delta s)^2 := -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

is the same in both frames. The quantity $(\Delta s)^2$ is **invariant** under the change in inertial frames.

Remark:

The distance between points defining spacetime geometry must be the same in all systems of coordinates used to label the points. The **principle of relativity** requires that the line element that defines the distance should have the same form in all inertial frames. Thus, we posit the **line element of flat spacetime**:

$$\boxed{dS^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2} \tag{2.3.1}$$

Note the geometry specified by (4.8) is non-Euclidean because of the minus sign (i.e. it is a Psuedo-Riemannian metric rather than a Riemannian metric). Sometimes this is referred to as **Minkowski space**.

Lengths in spacetime are giving by the square root of the absolute value of dS^2.

Example:

The analog of a circle of radius R centered on the origin is the locus of points a constant spacetime distance from the origin. THis consists of the hyperbolas $x^2 - (ct)^2 = R^2$. The ratios of arcs along a hyperbola to R define hyperbolic angles, with the relation

$$ct = R\sinh\theta, \quad x = R\cosh\theta$$

2.3.3 Light Cones

Note, two points can be separated by distances whose square is positive, negative, or zero. When dS^2 is positive the points are said to be **spacelike separated**. When dS^2 is negative the points are said to be **timelike separated**. This can occur when $\Delta x_i = 0$ for all i, but $\Delta t \neq 0$. When $dS^2 = 0$, the two points are said to be **null separated**. Null separated points can be connected by light rays that move with speed c, so **lightlike separate** is used as a synonym.

Definition 2.3.3 The locus of points that are null separated from a point P in spacetime is its **light cone**. The light cone of P is a three-dimensional surface in four dimensional spacetime specified by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2(t - t_0)^2$$

where $P = (t_0, x_0, y_0, z_0)$.

The future light cone of P is generated by light rays that move outward from P, while the past light cone of P is generated by light rays that converge on P.

The points that are timelike separated from P lie inside the light cone ($(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < c^2(t - t_0)^2$), and the points that are spacelike separated from P lie outside the light cone ($(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 > c^2(t - t_0)^2$). The paths of light rays are straight lines in spacetime with constant slope corresponding to the speed of light, that is, along null world lines. The distance between two points along a light ray is zero!

Particles with nonzero rest mass move along **timelike world lines** that are always inside the light cone of any point along their trajectory. That way their velocity is always less than the speed of light at that point.

Entities with spacelike world lines would move always with speeds greater than that of light (we call these **tachyons**). None have ever been observed to exist, and any would conflict with other principles of physics such as causality and positive energy. Hence we ignore these moving forward.

Light cones therefore define the causal relationships between points in spacetime. An event at P can signal or influence points inside or on its future light cone, but not outside it. Information can be received at P only from events inside or on its past light cone, but not from events outside it.

Remark:

An event can be later than another spacelike separated event in one inertial frame and earlier in another. But, for two timelike separated events the notion of earlier is well-defined. This is because events to the future of P are inside its future light cone, and the inside and outside of a light cone are properties of the geometry of spacetime—the same in all frames.

Two nearby points on a timelike world line are timelike separated, $dS^2 < 0$. To measure the distance along a particle's world line, it is convenient to introduce

$$d\tau^2 := -dS^2/c^2$$

Then $d\tau$ is real with units of time. Thus a clock moving along a timelike curve measures the distance τ along it. An alternative name for this distance is the **proper time**.

2.4 Time Dilation and the Twin Paradox

2.4.1 Time Dilation

The proper time, τ_{AB}, between any two points A and B on a timelike world line can be computed from the line element as

$$\begin{aligned}\tau_{AB} &= \int_A^B d\tau = \int_A^B \left[dt^2 - (dx^2 + dy^2 + dz^2)/c^2\right]^2 \\ &= \int_{t_A}^{t_B} dt \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\}^{1/2}\end{aligned}$$

More compactly,

$$\boxed{\tau_{AB} = \int_{t_A}^{t_B} dt' \left[1 - ||\mathbf{V}(t')||^2 / c^2\right]^{1/2}} \tag{2.4.1}$$

The proper time τ_{AB} is **shorter** than the interval $t_B - t_A$ because $\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2} < 1$. This is our mathematical expression for **time dilation**, which informally says “moving clocks run slow." In differential form

$$d\tau = dt\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2}$$

Note these expressions are valid even when the velocity is dependent on time (i.e. the clock is accelerating).

2.4.2 The Twin Paradox

The time dilation equation shows that the time registered by a clocking moving between two points in space depends on the route traveled even if it returns to the same point it started from.

Consider two twins, Alic and Bob, starting from rest at one point in space at time t_1 in an inertial frame. Alice moves away from the starting point but later returns to rest at the same point at time t_2. Bob remains at rest at the starting point. The time elapsed on Bob’s clock is $t_2 - t_1$. The time elapsed on Alice’s clock is always less than this because $\sqrt{1 - ||\mathbf{V}(t')||^2 / c^2} < 1$. The moving twin ages less than the stationary twin.

Remark:

The straight line path is the longest distance between any two timelike separated points in flat four-dimensional spacetime. (a line of constant velocity)

2.5 Lorentz Boosts

2.5.1 The Connection Between Inertial Frames

Recall the principle of relativity implies that the line element must take the same form in the rectangular coordinates of any inertial frame. Thus, the transformation laws that connect different inertial coordinate frames must be among those that preserve our psuedo-riemannian metric. These are called **Lorentz transformations**.

Recall the line element of Euclidean space is left unchanged by translations and isometries (i.e. rotations and reflections). Hence spatial translations and isometries will preserve the line element of special relativistic spacetime. But what new transformations that preserve the four-dimensional flat spacetime do we obtain?

The most important examples of new transformations are the analogs of rotations between time and space. These are called **Lorentz boosts** and correspond to the uniform motion of one frame with respect to another.

Consider the analog of rotations in the (ct, x) plane. Transformations of this character that leave the metric unchanged are the analogs of rotations in Euclidean space, but now due to the minus in front of dt^2, we replace the trigonometric functions with hyperbolic functions. Specifically

$$ct' = (\cosh\theta)(ct) - (\sinh\theta)x, \quad x' = (-\sinh\theta)(ct) + (\cosh\theta)x$$

where θ can vary from $-\infty$ to $+\infty$.

Superposing the axes of (ct', x') and (ct, x), we can find that a particle at rest at the origin $x' = 0$ in (ct', x') coordinates has the ct' axis as its world line. In (ct, x) coordinates, that particle is moving with a constant speed along the x-axis. The speed v can by found by putting $x' = 0$ in our transformation above, so

$$v = c\tanh\theta$$

A particle at rest at any other value of x' in the (ct', x') coordinates moves in the x-direction with the same speed in the (ct, x) coordinates. The transformation is therefore from one inertial frame to another moving uniformly with respect to it along the x-axis with speed v.

Replacing θ by v using the above expression in our transformation, we find

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt)$$

where we have introduced

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The inverse transformation is then obtained just by changing v into $-v$. When $v/c \ll 1$, this reduces to the Galilean transformations.

2.5.2 The Relativity of Simultaneity

Recall events A and B can be simultaneous in one inertial frame and be separated by a time $\Delta t = t_B - t_A$ in another frame. This difference can be computed from the Lorentz boost connecting the two frames. If $\Delta x' = x'_B - x'_A$ is the distance between the simultaneous events in the (ct', x') frame, then

$$\Delta t = \gamma(v/c^2)\Delta x'$$

where the (ct', x') frame is moving with velocity $v\hat{x}$ with respect to the (ct, x) frame.

2.5.3 Lorentz Contraction

Consider a rod of length L_* when measured in its own rest frame. What is its length when measured in an inertial frame in which it is moving with speed V? Note the length of a rod is the distance between two simultaneous events at its ends in spacetime. But simultaneity is different in different inertial frames, so the measured length of the rod is, therefore, also different. The length L in the frame where the rod is moving is the spacetime distance between the ends of the rod at $t' = 0$.Then

$$L^2 = L_*^2 - (c\Delta t)^2$$

From our Lorentz boost equation $t' = 0$ is the line $t = (V/c^2)x$, so $\Delta t = (V/c^2)L_*$. Thus

$$\boxed{L = L_* \sqrt{1 - V^2/c^2}} \tag{2.5.1}$$

This is **Lorentz contraction**.

2.5.4 Addition of Velocities

Consider a particle whose motion is described by $x(t), y(t), z(t)$ in one frame and $x'(t'), y'(t'), z'(t')$ in a second frame moving along the x-axis of the first with velocity v. From our Lorentz boost transformations we can compute the relation between $\mathbf{V} = d\mathbf{x}/dt$ in one frame and the velocity $\mathbf{V}' = d\mathbf{x}'/dt'$ in the other, namely

$$V^{x'} = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - v/c^2 dx)} = \frac{V^x - v}{1 - vV^x/c^2}$$

Similarly,

$$V^{y'} = \frac{V^y/\gamma}{1 - vV^x/c^2}$$
$$V^{z'} = \frac{V^z/\gamma}{1 - vV^x/c^2}$$

2.6 Units

Today the velocity of light is not measured, it is defined to be exactly the conversion factor

$$c = 299792458 \text{ m/s}$$

Measuring time in units of length means changing from the mass-length-time system of units traditional in mechanics to a mass-length system. Measuring both space and time in length units has the effect of putting $c = 1$ everywhere in our formulas. Further, in these units velocities are dimensionless.

Problems

2.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 3

Special Relativistic Mechanics

Abstract To be completed once done

3.1 Four-Vectors

Four vectors are four dimensional vectors invisioned in flat spacetime. Timelike, spacelike, and null four-vectors distinguish the direction of the vectors relative to the event they are attached to. We define the **length** of a four-vector as the absolute value of the spacetime distance between its tail and its tip. The vector operations are independent, or **invariant**, or the inertial frame.

3.1.1 Basis Four-Vecotrs

In a particular inertial frame, the standard basis consists of four unit vectors e_t, e_x, e_y, e_z pointing in the corresponding directions. We use Einstein notation and write a vector in these coordinates as

$$\mathbf{a} = a^{\alpha} e_{\alpha}$$

> Note:
> The components of a four-vector are different in different inertial frames because the coordinate basis four-vectors are different.

For example, for two inertial frames related by a uniform motino v along the x-axis, the components of a four vector $\mathbf{a}$ transform as

$$a^{t'} = \gamma(a^t - va^x), a^{x'} = \gamma(a^x - va^t), a^{y'} = a^y, a^{z'} = a^z$$

where we're taking $c = 1$.

3.1.2 Scalar Product

We define a scalar product on four vectors using the psuedo-riemannian metric on our flat spacetime. In particular, if $\mathbf{a} = a^\alpha e_\alpha, \mathbf{b} = b^\beta e_\beta$, then

$$\mathbf{a} \cdot \mathbf{b} = (e_\alpha \cdot e_\beta) a^\alpha b^\beta$$

where we write

$$\eta_{\alpha\beta} := e_\alpha \cdot e_\beta$$

Note these are the coefficients of our psuedo-riemannian metric, so

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

where $dx^\alpha dx^\beta = \frac{dx^\alpha \otimes dx^\beta + dx^\beta \otimes dx^\alpha}{2}$. Thus we have that $\eta_{tt} = -1, \eta_{xx} = \eta_{yy} = \eta_{zz} = 1$, and all other entries are zero. In particular we have a matrix representation

$$(\eta_{\alpha\beta}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As this makes no reference to a particular frame, the scalar product is an invariant.

Example:

Lorentz boosts preserve the orthogonality of coordinate axes. In particular, they are isometries. In a frame (t, x), consider a displacement **a** along t' and a displacement **b** along x' (both unit vectors). The (t', x', y', z') components of these vectors are $a^{\alpha'} = (1, 0, 0, 0)$ and $b^{\alpha'} = (0, 1, 0, 0)$. These are therefore orthogonal in the (t', x') frame. This means they are orthogonal in any other inertial frame as Lorentz boosts preserve the spacetime metric. To see this explicitly observe

$$a^\alpha = (\gamma, v\gamma, 0, 0), \;\; b^\alpha = (v\gamma, \gamma, 0, 0)$$

and so

$$ds^2(\mathbf{a}, \mathbf{b}) = -v\gamma^2 + v\gamma^2 = 0$$

as desired.

3.2 Special Relativistic Kinematics

When parameterizing a world line in spacetime we often use the proper time that gives the spacetime distance τ along the world line measured both positively and negatively from some arbitrary starting point. Thus we describe a world line by the equations

$$x^\alpha = x^\alpha(\tau)$$

Example:

A particle moves on the x-axis along a world line described parametrically by

$$t(\sigma) = a^{-1}\sinh\sigma, \quad x(\sigma) = a^{-1}\cosh\sigma$$

where a is a constant with the dimension of inverse length. The parameter $\sigma \in (-\infty, \infty)$. For each value of σ, the equation determines a point (t, x) in spacetime. As σ varies, the world line is swept out. In a spacetime diagram this is a hyperbola $x^2 - t^2 = a^{-2}$.

The world line is accelerated because it is not straight. Proper time τ alone the world line is related to σ by

$$d\tau^2 = dt^2 - dx^2 = (a^{-1}\cosh\sigma d\sigma)^2 - (a^{-1}\sinh\sigma d\sigma)^2 = (a^{-1}d\sigma)^2$$

Fixing τ to be zero when σ is zero, $\tau = a^{-1}\sigma$, and the world line can be expressed with proper time as the parameter in the form

$$t(\tau) = a^{-1}\sinh(a\tau), \quad x(\tau) = a^{-1}\cosh(a\tau)$$

The **four-velocity** of a curve x^α is the derivative of the position along the world line with respect to the proper time parameter, τ:

$$\boxed{u^\alpha = \frac{dx^\alpha}{d\tau}}$$

Using the chain rule we can express the components of the four-velocity in terms of the three-velocity $v = d\mathbf{x}/dt$ in a particular inertial frame by using the relation between t and proper time τ as follows:

$$u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - |v|^2}}$$

and, for example,

$$u^x = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{v^x}{\sqrt{1 - |v|^2}}$$

Then, with $\gamma 1/\sqrt{1 - |v|^2}$, we have

$$u^\alpha = (\gamma, \gamma v)$$

It follows immediately that the inner product of the tangent vector u is

$$u \cdot u = -1$$

so that the four-velocity is always a unit timelike four-vector. Indeed, this follows directly from

$$u \cdot u = \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -1$$

where the last equality follows from the connection $ds^2 = -d\tau^2$.

Example:

The four-velocity u of the world line discussed in the previous example has components

$$u^t \equiv dt/d\tau = \cosh(a\tau), \quad u^x \equiv dx/d\tau = \sinh(a\tau)$$

The particle's three-velocity is

$$v^x = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \tanh(a\tau)$$

This never exceeds the speed of light, $|v^x| = 1$, but approaches it at $\tau = \pm\infty$.

3.3 Special Relativistic Dynamics

3.3.1 Equation of Motion

In the absence of forces,

$$\frac{d\mathbf{u}}{d\tau} = 0$$

where **u** is the 4-velocity of the object. This is **Newton's First Law**. We next aim to introduce an analogue of Newton's second law, $\mathbf{F} = m\mathbf{a}$. This analogue must satisfy the principle of relativity, reduce to the first law when force is zero, and must reduce to the original from in any inertial frame where the speed of the particle is much less than the speed of light. THe choice

$$\boxed{m\frac{d\mathbf{u}}{d\tau} = \mathbf{F}}$$

naturally suggests itself. The constant m, which characterizes the particle's inertial properties, is called the **rest mass**, and **F** is called the **four-force**. We still need a proper choice of **F** so that this reduces to the classical case for non-relativistic situations. By introducing the **four-acceleration a** $= \frac{d\mathbf{u}}{d\tau}$ we can express this as $\mathbf{F} = m\mathbf{a}$. The normalization of the four-velocity means

$$m\frac{d(\mathbf{u}\cdot\mathbf{u})}{d\tau} = 0$$

which implies $\mathbf{u}\cdot\mathbf{a} = 0$, or

$$\mathbf{F}\cdot\mathbf{u} = 0$$

Example:

The four-acceleration of the world line described in our previous examples has components

$$a^t = du^t/d\tau = a\sinh(a\tau), \quad a^x = du^x/d\tau = a\cosh(a\tau)$$

The magnitude of this acceleration is $(\mathbf{a}\cdot\mathbf{a})^{1/2} = a$.

3.3.2 Energy-Momentum

If the **four-momentum** is defined by

$$\mathbf{p} = m\mathbf{u}$$

then the equation of motion can be written

$$\frac{d\mathbf{p}}{d\tau} = \mathbf{F}$$

From normalization of the four-velocity

$$\mathbf{p}^2 = \mathbf{p} \cdot \mathbf{p} = -m^2$$

Then in an inertial frame where the three-velocity is $\mathbf{v}$,

$$p^t = \frac{m}{\sqrt{1 - ||\mathbf{v}||^2}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - ||\mathbf{v}||^2}}$$

For small speeds $||\mathbf{v}|| \ll 1$,

$$p^t = m + \frac{1}{2}m||\mathbf{v}||^2 + \cdots, \quad \mathbf{p} = m\mathbf{v} + \cdots$$

Note p^t reducs to the kinetic energy plus the rest mass. For this reason $\mathbf{p}$ is also called the **energy-momentum four-vector**, and its components in an inertial frame are written

$$p^\alpha = (E, \mathbf{p}) = (m\gamma, m\gamma\mathbf{v})$$

Then by our normalization

$$E = \sqrt{m^2 + ||\mathbf{p}||^2}$$

where $\mathbf{p}$ is our three-momentum. For a particle at rest this reduces to the usual $E = mc^2$ in standard units.

In a particular inertial frame define the three-force $\mathbf{F}$ as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

Evidently, $\mathbf{f} = d\mathbf{p}/d\tau = (d\mathbf{p}/dt)(dt/d\tau) = \gamma\mathbf{F}$. The four-orce can be written in terms of the three-force as

$$\boxed{\mathbf{F} = (\gamma\mathbf{F} \cdot \mathbf{v}, \gamma\mathbf{F})}$$

where $\mathbf{v}$ is the particle's three-velocity.

> Example:
>
> A particle with charge q and rest mass m moves in a uniform magnetic field $\mathbf{B}$ with total energy E. Electromagnetism is unchanged in special relativity so that the three-force on a charged particle in a magnetic field is
>
> $$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

The particle moves in a circular orbit of radius R at constant speed, obeying the familiar equation of motion. Therefore

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-||\mathbf{v}||^2}}\right) = \frac{m}{\sqrt{1-||\mathbf{v}||^2}}\frac{d\mathbf{v}}{dt}$$

The centripetal acceleration $d\mathbf{v}/dt$ is given by the usual, purely kinematic relation $||\mathbf{v}||^2/R$. Therefore,

$$\frac{m\gamma||\mathbf{v}||^2}{R} = q||\mathbf{v}||B$$

Thus

$$R = \frac{m||\mathbf{v}||\gamma}{qB} = \frac{||\mathbf{p}||}{qB} = \frac{\sqrt{E^2-m^2}}{qB}$$

which relates the radius to the total energy. The components of the four-force are $f^t = \gamma\mathbf{F}\cdot\mathbf{v} = 0$, and a radial component

$$f^r = \gamma F^r = qVB\gamma = \frac{qB}{m}\sqrt{E^2-m^2}$$

3.4 Variational Principle for Newtonian Mechanics

Consider the simple case of a particle of mass m moving in one dimension in a potential $V(x)$, whose equations of motion are summarized by the Lagrangian:

$$L(\dot{x},x) = \frac{1}{2}m\dot{x} - V(x)$$

where the dot denotes a time derivative. Newton's law $m\ddot{x} = -dV/dx$ can be expressed as Lagrange's equation

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0$$

Consider the possible paths between a point x_A at time t_A and a point x_B at time t_B. For each path construct a real number called its **action**:

$$S[x(t)] = \int_{t_A}^{t_B} dt L(\dot{x}(t), x(t))$$

Theorem 3.4.1 Variational Principle for Newtonian Mechanics A particle moves between a point in space at one time and another point in space at a later time so as to extremize the action in between.

An extremum can be characterized as the place where the first variation of the function vanishes,

$$\delta f = \sum_{a=1}^{n}\frac{\partial f}{\partial x^\alpha}\delta x^a = 0$$

The extrema of the action functional $S[x(t)]$ are defined by the vanishing of its first-order variation $\delta S[x(t)]$ for arbitrary variations $\delta x(t)$ of the path connecting (x_A, t_A) to (x_B, t_B). To compute $\delta S[x(t)]$ just substitute $x(t) + \delta x(t)$ for $x(t)$ in the definition of the action, expand to first order in $\delta x(t)$, and integrate once by parts to find:

$$\begin{aligned}\delta S[x(t)] &= \int_{t_A}^{t_B} dt \left[\frac{\partial L}{\partial \dot{x}(t)}\delta \dot{x}(t) + \frac{\partial L}{\partial x(t)}\delta x(t)\right] \\ &= \frac{\partial L}{\partial \dot{x}(t)}\delta x(t)\bigg|_{t_A}^{t_B} + \int_{t_A}^{t_B} dt \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}(t)}\right) + \frac{\partial L}{\partial x(t)}\right]\delta x(t)\end{aligned}$$

Note that variations of the path that connects x_A at t_A to x_B at t_B necessarily vanish at the endpoints, so the first term vanishes. The remaining term has to vanish for arbitrary $\delta x(t)$ that meet these conditions for $\delta S[x(t)]$ to vanish. This can only happen if

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0$$

If the Lagrangian is a function of n coordinates $x^a(t)$ and their time derivatives, its extrema satisfy the n equations

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) + \frac{\partial L}{\partial x^a} = 0$$

$a = 1, ..., n$.

3.5 Variational Principle for Free Particle Motion

Recall the straight lines along which free particles move in spacetime are paths of longest proper time between two events.

> **Theorem 3.5.1** Variational Principle for Free Particle Motion The world line of a free particle between two timelike separated points extremizes the proper time between them.

Note the proper time between two points A and B along some path is

$$\tau_{AB} = \int_A^B [dt^2 - dx^2 - dy^2 - dz^2]^{1/2}$$

Parameterizing the worldline with parameter σ such that $\sigma = 0$ at A and $\sigma = 1$ at B, we have that

$$\tau_{AB} = \int_0^1 d\sigma \left[\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{dx}{d\sigma}\right)^2 - \left(\frac{dy}{d\sigma}\right)^2 - \left(\frac{dz}{d\sigma}\right)^2\right]^{1/2}$$

We seek the world lines that extremize τ_{AB}. Lagrange's equations take the form

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right) + \frac{\partial L}{\partial x^\alpha} = 0$$

with

$$L = \left[\left(\frac{dt}{d\sigma} \right)^2 - \left(\frac{dx}{d\sigma} \right)^2 - \left(\frac{dy}{d\sigma} \right)^2 - \left(\frac{dz}{d\sigma} \right)^2 \right]^{1/2} = \left[-\eta_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right]^{1/2}$$

The Lagrange equations for a free particle imply

$$\frac{d^2 x^\alpha}{d\tau^2} = 0$$

3.6 Light Rays

3.6.1 Zero Rest Mass Particles

We now consider particles that move at the speed of light, $v = 1$, along null world lines.

Evidently the proper time can no longer be used as a parameter along the world line of a light ray—the proper time interval between any two points on it is zero. However, there are many other parameters that could be used. For example, the curve

$$x = t$$

which has $v = 1$ could be written parametrically as

$$x^\alpha = u^\alpha \lambda$$

where λ is the parameter and $u^\alpha = (1, 1, 0, 0)$. The four-vector u is a tangent-four vector $u^\alpha = dx^\alpha / d\lambda$. However, here u is a null vector. Therefore, we now have

$$u \cdot u = 0$$

With this choice of parameterization

$$\frac{du}{d\lambda} = 0$$

This is not true for every choice of parameterization. This is the same motion as for a particle—parameters which do this are called **affine parameters**.

3.6.2 Energy, Momentum, Frequency, and Wave Vector

Photons and neutrinos carry energy and three-momentum. In any inertial frame, the energy of a photon E is connected to its frequency ω by the relation

$$E = \hbar\omega$$

Note that the three-velocity is given by $v = p/E$. Since $|v| = 1$, this implies that $|p| = E$ for a photon, so the three-momentum can be written

$$p = \hbar k$$

where k points in the direction of propagation, has magnitude $|k| = \omega$, and is called the **wave three-vector**. In any inertial frame the components of the four-momentum of a photoncan therefore be written

$$\boxed{p^\alpha = (E, p) = (\hbar\omega, \hbar k) = \hbar k^\alpha}$$

where $\mathbf{k}$ is called the **wave four-vector**. Evidently,

$$\mathbf{p} \cdot \mathbf{p} = \mathbf{k} \cdot \mathbf{k} = 0$$

This implies that photons have zero rest mass, like all particles moving at the speed of light. The tangent vector $\mathbf{u}$ could be chosen so as to coincide with either $\mathbf{p}$ or $\mathbf{k}$ by adjusting the normalization of the affine parameter λ. Then the equation of motion can be written as

$$\frac{d\mathbf{p}}{d\lambda} = 0, \quad \text{or} \quad \frac{d\mathbf{k}}{d\lambda} = 0$$

where λ is an affine parameter.

3.6.3 Doppler Shift and Relativistic Beaming

Consider a source that emits photons of frequency ω in all directions in the source's rest frame. Suppose that in another frame the source is moving with speed v along the x'-axis. What frequency will be observed for a photon that makes an angle α' with the direction of motion? Let $k^\alpha = (\omega, k)$ be the components of the wave four-vector $\mathbf{k}$ of the photon in the frame of the source and $k'^\alpha = (\omega', k')$ the components in the frame of the observer. Then

$$\omega = \gamma(\omega' - v k^{x'})$$

But $k'^x = \omega' \cos\alpha'$, where α' is the angle between the x'-axis and the direction of the photon in the observer's frame. Thus,

$$\boxed{\omega = \omega \frac{\sqrt{1 - v^2}}{1 - v\cos\alpha'}}$$

is formula for the relativstic Dopller shift. For small v this is approximately

$$\omega' \approx \omega(1 + v\cos\alpha')$$

When $\alpha' = 0$, the photon is emitted in the same direction that the source is moving and there is a blue shift of $\Delta\omega w' = +V\omega$ in the frequency of the photon, and when $\alpha' = \pi$, the photon is moving opposite to the source and there is a red shift of $\Delta\omega' = -V\omega$.

When the photons are transverse to the direction of motion of the source, $\alpha' = \pi/2$, they are still redshifted. This is called the **transverse Doppler shift**, and the equation shows this is just time-dilation.

Suppose a photon makes an angle α with the x-axis in the source frame, where $\cos\alpha = k^x/\omega$. In the observer's frame, the angle it makes with the x'-axis is $\cos\alpha' = k'^x/\omega'$. The Lorentz transformation between these two frames shows

$$\cos\alpha' = \frac{\cos\alpha + v}{1 + v\cos\alpha}$$

Thus the half of the photons emitted in the forward hemisphere in the source frame ($|\alpha| < \pi/2$) are seen by the observer to be emitted in a smaller cone $|\alpha'| < \alpha'_{1/2}$, where $\cos\alpha'_{1/2} = v$. For v close to 1 this opening angle will be small. Photons are thus beamed along the direction of the source by its motion. The Doppler shift implies that the energy of the photons in the forward direction is greater than that in the backward direction, meaning that the **intensity** of the radiation is even more concentrated along the direction of motion. A uniformly radiating body moving toward you is brighter than if it is moving away. This is the phenomenon of **relativistic beaming**.

3.7 Observers and Observations

Note that the energy of a particle measured by an observer at rest in an inertial frame is the component of the particle's four-momentum along the time axis of that frame. But, how can we compute the predictions of accelerated observers?

This is especially important for general relativity since there are no global inertial frames, but rather only local inertial frames in the neighborhood of each point and the neighborhood of the world lines of freely falling observers. Recall the path of an observer through spacetime is a timelike world line. We think of the observer carrying a laboratory along the world line which is arbitrarily small. Inside the laboratory the observer makes measurements by means of clocks and rulers.

An observer carries along four orthogonal unit four-vectors $e_{\hat{0}}, e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}}$, which define a time direction and three spatial directions, respectively, to which the observer will refer all measurements. The time-like unit four-vector $e_{\hat{0}}$ will be tangent to the observer's world line since that is the direction a clock at rest in the laboratory is moving in spacetime. Since the observer's four-velocity is a unit tangent vector

$$e_{\hat{0}} = u_{obs}$$

Only if the laborator is at rest in an inertial frame will the $e_{\hat{\alpha}}$ point along the axes of an inertial frame.

Example:

Consider the observer moving along the accelerated world line described in our previous examples. What are the components of a set of orthonormal basis four-vectors for this observer in the inertial frame? These four-vectors will vary with the observer's proper time. First,

$$(e_{\hat{0}}(\tau))^{\alpha} = u^{\alpha}_{obs}(\tau) = (\cosh(a\tau), \sinh(a\tau), 0, 0)$$

The only conditions on the other three four-vectors $e_{\hat{i}}(\tau)$ are that they be orthogonal to $e_{\hat{0}}(\tau)$, orthogonal to each other, and of unit length. An easy choice is taking $e_{\hat{2}}(\tau)$ and $e_{\hat{3}}(\tau)$ to point in the y- and z-directions, and take $e_{\hat{1}}(\tau) = (f(\tau), g(\tau), 0, 0)$, where

$$-\cosh(a\tau)f(\tau) + \sinh(a\tau)g(\tau) = 0$$

and

$$-f(\tau)^2 + g(\tau)^2 = 1$$

This can be accomplished with $f(\tau) = \sinh(a\tau)$ and $g(\tau) = \cosh(a\tau)$.

Note that, for instance, the energy of a particle measured by an accelerating observer is the component of the particle's four-momentum $\mathbf{p}$ along the basis four-vector $e_{\hat{0}}$. In particular

$$\mathbf{p} = p^{\hat{\alpha}} e_{\hat{\alpha}}$$

We can compute these using the scalar products with the orthonormal basis four-vectors of the observer,

$$\eta_{\alpha\beta} p^{\hat{\alpha}} = \mathbf{p} \cdot e_{\hat{\beta}}$$

for $\alpha \neq 0$, and we add a minus sign for $\alpha = 0$. In particular, the energy of the particle measured by an observer with four-velocity u_{obs} is the first of these, or

$$E = -\mathbf{p} \cdot u_{obs}$$

Example:

Consider a particle at rest in some inertial frame. An observer is moving with velocity v in this frame so that the observer's world line intersects the partile's. From the observer's point of view the particle moves through the observer's laboratory. What energy of the particle would be measured? The particle will move through the laboratory with speed v and so the mesaured energy will be

$$E = m\gamma$$

where m is the particle's rest mass.
To see how this comes by scalar products, observe that in the particle's rest frame

$$\mathbf{p} = (m, 0, 0, 0)$$

In the same frame, the four-velocity of the observer is

$$e_{\hat{0}} = u_{obs} = (\gamma, V\gamma, 0, 0)$$

The energy of the measured observer is again $E = -\mathbf{p} \cdot u_{obs} = m\gamma$.

Example:

Consider an observer following the world line of our past examples. Suppose they observe the light from a star that remains stationary at the origin of the inertial frame, emitting light steadily. Assume for simplicity that the light is emitted at a single optical frequency ω_* in the rest frame of the star. What frequency $\omega(\tau)$ will the observer measure?
In the inertial frame in which the star is stationary the wave four-vector $\mathbf{k}$ of a photon reaching the observer has components $k^{\alpha} = (\omega_*, \omega_*, 0, 0)$. The observed frequency $\omega(\tau)$ could be worked out by transforming these components into the instantaneous rest frame of the observer at proper time τ. But it is easier to note that $E = \hbar\omega$ for photons, and use

$$\omega(\tau) = -\mathbf{k} \cdot u_{obs}$$

Explicitly this gives

$$\omega(\tau) = \omega_* [\cosh(a\tau) - \sinh(a\tau)] = \omega_* \exp(-\alpha\tau)$$

Appendix

3.7.1 Magnetism as a Relativistic Phenomenon

Consider a string of positive charges moving moving along to the right at speed v. Assume the charges are closed enough together so that we may regard them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$, proceeding to the left at the same speed v. We then have a net current to the right of magnitude

$$I = 2\lambda v$$

Consider also a charge a distance s away of charge q traveling to the right at speed $u < v$. Because the two line charges cancel there is no electrical force on q. However, if we switch the the frame of the charqe, q is at rest, and by the velocity addition rule the velocities of the positive and negative lines are

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

Because v_- is greater than v_+, the Lorentz contraction of the spacing between the negative charges is more severe than that between the positive charges; in this frame, therefore, the wire carries a net negative charge! In fact,

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0$$

where

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}$$

and λ_0 is the charge density of the positive line in its own rest system. Thus

$$\lambda = \gamma\lambda_0$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

After some algebra

$$\gamma_{\pm} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}$$

Evidently, then, the net line charge in the rest frame of the charge is

$$\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-)$$

Remark:

As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

The line charge sets up an electric field

$$E = \frac{\lambda_{tot}}{2\pi\epsilon_0 s}$$

so there is an electrical force on q in its rest frame

$$F = qE = -\frac{\lambda v}{\pi \epsilon_0 c^2 s^2} \frac{qu}{\sqrt{1 - u^2/c^2}}$$

But if there is a force on q in its rest frame, there must be a force in the original frame. As F is perpendicular to u, the force in the original frame is

$$F' = \sqrt{1 - u^2/c^2} F = -\frac{\lambda v}{\pi \epsilon_0 c^2} \frac{qu}{s}$$

Taken together, then, electrostatics and relativity imply the existence of another force. This is of course the magnetic force. Using $c^2 = 1/(\epsilon_0 \mu_0)$, and expressing λv in terms of the current, this gives the familiar form

$$F' = -qu \left(\frac{\mu_0 I}{2\pi s} \right)$$

3.7.2 How the Fields Transform

Consider two inertial frames S and S' moving relative to one another. Consider also the uniform electric field in a region between the plates of a large parallel-plate capacitor. Say the capacitor is at rest in S_0 and carries surface charges $\pm\sigma_0$. Then

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0} \hat{y}$$

Suppose system S is moving to the right at speed v_0. In this system the plates are moving to the left, but the field still takes the form

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{y}$$

the only difference is the value of the surface charge σ.

Now, the total charge on each plate is invariant, and the width w is unchanged, but the length l is Lorentz-contracted by a factor

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}$$

so the charge per unit area is increased by a factor γ_0:

$$\sigma = \gamma_0 \sigma_0$$

Accordingly,

$$\mathbf{E}_\perp = \gamma_0 \mathbf{E}_{0,\perp}$$

where the $\perp$ indicates we are looking at components of $\mathbf{E}$ perpendicular to the direction of motion of S. To get the rule for parallel components consider the capacitor lined up with the yz plane. This time it is the plate separation d that is Lorentz-contracted, whereas l and w and hence also σ are the same in both frames. Since the field does not depend on d it follows that

$$\mathbf{E}_{||} = \mathbf{E}_{0,||}$$

Now suppose we started in frame S, so in addition to the electric field

$$E_y = \sigma/\epsilon_0$$

there is a **magnetic field** due to the surface currents

$$\mathbf{K}_{\pm} = \mp\sigma v_0 \hat{x}$$

By the right-hand rule, this field points in the negative z direction; its magnitude is given by Ampere's law:

$$B_z = -\mu_0 \sigma v_0$$

In a third system, S', traveling with speed v relative to S, the fields would be

$$E'_y = \sigma'/\epsilon_0, \quad B'_z = -\mu_0 \sigma' v'$$

where v' is the velocity of S relative to S_0:

$$v' = \frac{v + v_0}{1 + v v_0/c^2}, \quad \gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}}$$

and

$$\sigma' = \gamma' \sigma_0$$

In view of these we have

$$E'_y = \gamma(E_y - vB_z), \quad B' = \gamma(B_z - (v/c^2)E_y)$$

To do other components like E_z and B_y we simply align the same capacitor parallel to the xy plane instead of the xz plane. The fields in S are then

$$E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0 \sigma v_0$$

After some more work, we obtain the transformation rules

$$E'_x = E_x, \quad E'_y = \gamma(E_y - vB_z), \quad E'_z = \gamma(E_z + vB_y)$$
$$B'_x = B_x, \quad B'_y = \gamma(B_y + (v/c^2)E_z), \quad B'_z = \gamma(B_z - (v/c^2)E_y)$$

If $B = 0$ in S, then

$$B' = -\frac{1}{c^2}(\mathbf{v} \times \mathbf{E}')$$

If $E = 0$ in S, then

$$E' = \mathbf{v} \times \mathbf{B}$$

3.7.3 The Field Tensor

Note that **E** and **B** do not transform like the spatial parts of two 4-vecotrs. What sort of an object is this, which has six components and transforms in this way? It is an antisymmetric second-rank tensor.

Remember that a 4-vector transforms by the rule

$$\overline{a}^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}$$

where Λ is the Lorents transformation matrix from S to $\overline{S}$. If $\overline{S}$ is moving in the x direction at speed v, Λ has the form

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where Λ^{μ}_{ν} is the entry in row μ, column ν. For a $(0,2)$-tensor $g_{\alpha\beta}$ we have that

$$\overline{g}_{\mu\nu} = \Lambda^{\alpha}_{\mu} g_{\alpha\beta} \Lambda^{\beta}_{\nu}$$

A symmetric $(0,2)$-tensor is symmetric if $g_{\alpha\beta} = g_{\beta\alpha}$, and is antisymmetric if $g_{\alpha\beta} = -g_{\beta\alpha}$.

Definition 3.7.1 The **Field tensor** $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

We also have the **dual tensor**

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Problems

3.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 4

Gravity as Geometry

Abstract To be completed once done

4.1 The Equivalence Principle

The equivalence principle is regarded as a heuristic idea whose central content is incorporated automatically and precisely in general relativity where appropriate. The equality of gravitational and inertial mass is essential for this argument. Einstein's equivalence principle is the idea that there is no experiment that can distinguish a uniform acceleration from a uniform gravitational field. This implies that light falls in a gravitational field.

4.2 Clocks in a Gravitational Field

When the gravitational field is nonuniform the equivalence principle holds only for experiments in laboratories that are small enough and that take place over a short enough period of time that no nonuniformities in Φ can be detected.

> Remark:
>
> The **Equivalence Principle** states that experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiments in an intertial frame in empty space.

4.3 The Global Positioning System

4.4 Spacetime is Curved

> Question?
>
> What is the explanation of the difference between the rates at which signals are emitted and received at two different gravitational potentials?

One explanation is that gravity affects the rates at which clocks run. In the absence of any gravitational field, two clocks at rest in an inertial frame of flat spacetime both keep track of the time of that frame. In the presence of a gravitational field, the spacetime remains flat, but clocks run at a rate that is a factor $(1 + \Phi/c^2)$ different from their rates in empty spacetime, where Φ is the gravitational potential at the location of the clock. Clocks run faster where Φ is positive and slower where Φ is negative.

However, it is simpler, more economical, and ultimately more powerful to recognize that clocks correctly measure timelike distances in spacetime and that its geometry is curved.

4.5 Newtonian Gravity in Spacetime Terms

We consider a simple model that will introduce a clight curvature that will explain geometrically the behaviour of clocks we have been discussing. We specify the **Static Weak Field Metric** by

$$\boxed{ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)(dx^2 + dy^2 + dz^2)} \tag{4.5.1}$$

where the gravitational potential $\Phi(x^i)$ is a function of position satisfying the Newtonian field equation and assumed to vanish at infinity. For example, outside Earth $\Phi(r) = -GM_\oplus/r$.

4.5.1 Rates of Emission and Reception

Consider signals propagating along the x-axis emitted at one location, x_A, and received at another, x_B. The world line of a light signal won't be a 45° straight line, as in flat spacetime. But the world lines of both signals will have the same shape because the geometry is independent of t. The signals are therefore received at B with the same coordinate separation Δt as they were emitted with at A. But a coordinate separation Δt corresponds to two different proper time intervals at the two locations. The coordinate separations between the two emissions at location x_A are Δt and $\Delta x = \Delta y = \Delta z = 0$. The proper time separation $\Delta\tau_A$ between these events is $d\tau^2 = -ds^2/c^2$ so

$$\Delta\tau_A = \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t$$

accurate to order $1/c^2$, where $\Phi_A = \Phi(x_A, 0, 0)$. Similarly, on reception

$$\Delta\tau_B = \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t$$

Eliminating Δt between these two relations gives

$$\Delta\tau_B = \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A$$

4.5.2 Newtonian Motion in Spacetime Terms

The principle that a free particle follows a path of extremal proper time between ny two points also gives the motion of a particle in a gravitational potential Φ in the spacetime geometry described previously. The proper time between two points A and B in spacetime depends on the world line between them and is given by

$$\tau_{AB} = \int_A^B \left[\left(1 + \frac{2\Phi}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)\right]^{1/2}$$

integrated along the world line connecting A and B. All of our considerations have been accurate only to first order in $1/c^2$, and to that order this becomes under a t parameterization

$$\tau_{AB} \approx \int_A^B dt\left[1 - \frac{1}{c^2}\left(\frac{1}{2}\,||\mathbf{v}||^2 - \Phi\right)\right]$$

The world line that extremizes the proper time between A and B will extremize the combination

$$\int_A^B dt\left(\frac{1}{2}\,||\mathbf{v}||^2 - \Phi\right)$$

since the first term in the original integral doesn't depend on which world line is traveled. The conditions for an extremum are Lagrange's equations, following from the Lagrangian

$$L\left(\frac{d\mathbf{x}}{dt}, \mathbf{x}\right) = \frac{1}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 - \Phi(\mathbf{x}, t)$$

If multiplied by the mass, this is just the Lagrangian for a nonrelativistic particle moving in the gravitational potential Φ. Lagrange's equations imply

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi$$

which, when both sides are multiplied by m is just $\mathbf{F} = m\mathbf{a}$.

Table 4.1 Newtonian and Geometric Formulations of Gravity Compared

	Newtonian	Geometric Newtonian	General Relativity
What a mass does	Produces a field Φ causing a force on other masses $-m\nabla\Phi$	Curves spacetime $ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)(cdt)^2 + \left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)$	Curves spacetime
Motion of a particle	$m\mathbf{a} = \mathbf{F}$	Curve of extremal proper time (first order in $1/c^2$)	Curve of extremal proper time
Field equation	$\nabla^2\Phi = +4\pi G\mu$	$\nabla^2\Phi = +4\pi G\mu$	Einstein's equation

The Newtonian gravitational law is inconsistent with the principles of special relativity because it specifies an instantaneous interaction between bodies. The asymmetry between space and time in the metric shows this in anothe rway. Even in a geometric formulation Newtonian gravity is inconsistent with special relativity. A fully relativisticm geometric theory of gravity would treat space and time on a symmetric footing.

Appendix

Problems

4.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 5

Description of Curved Spacetime

Abstract To be completed once done

5.1 Coordinates

A line element specifies a geometry, but many different line elements describe the same spacetime geometry because different coordinate systems can be used. A good coordinate system provides unique labels for each point in spacetime. However, most coordinate systems only do this locally. Even polar coordinates fails to uniquely label points on the $\theta = 0$ axis. The singularities in most coordinate systems mean that different overlapping coordinate patches must be used to cover spacetime so that every point is labeled by a nonsingular set of coordinates.

5.2 Metric

To describe a general geometry we use a system of four coordinates, x^α, to label the points and specify the line element giving the distance, ds^2, between nearby points separated by coordinate intervals dx^α. That line element will have the form

$$\boxed{ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta}$$

where $g_{\alpha\beta}(x)$ is a symmetric, position-dependent matrix called the **metric**.

5.3 The Summation Convention

1. The location of the indices must be respected: superscripts for coordinates and vector components and subscripts for the metric.

2. Repeated indices always occur in superscript-subscript pairs and imply summation.

3. Indices that are not summed are called free indices.

5.4 Light Cones and World Lines

Points separated from P by infinitesimal coordinate intervals dx^α can be timelike separated, spacelike separated, or null separated as the square of their distance away defined by the metric satisfies

$$ds^2 < 0 \qquad \text{(timelike separation)}$$
$$ds^2 = 0 \qquad \text{(null separation)}$$
$$ds^2 > 0 \qquad \text{(spacelike separation)}$$

Light rays move along null curves in spacetime along which $ds^2 = 0$.

Particles move on timelike world lines which can be specified parametrically by four functions $x^\alpha(\tau)$ of the distance τ along them.

Definition 5.4.1 The distance between a point A and a point B along a timelike worl line is given by

$$\tau_{AB} = \int_A^B [-g_{\alpha\beta}(x) dx^\alpha dx^\beta]^{1/2}$$

where the integral is along the world line.

The global arrangement of light cones is called the spacetime's **causal structure**.

5.5 Length, Area, Volume, and Four-Volume for Diagonal Metrics

In this section suppose $ds^2 = g_{\alpha\alpha} dx^\alpha dx^\alpha$ is a diagonal metric. The proper lengths of a segment will be of the form $d\ell^1 = \sqrt{g_{11}} dx^1$. Since the coordinates are orthogonal, the area element created by the region spanned by two line segments is

$$dA = d\ell^2 d\ell^3 = \sqrt{g_{11} g_{22}} dx^1 dx^2$$

For three-volume

$$dV = \sqrt{g_{11} g_{22} g_{33}} dx^1 dx^2 dx^3$$

For a metric of signature $(1, 3)$, the four-volume is

$$dv = \sqrt{-\det(g_{\alpha\beta})} d^4x$$

Appendix

Problems

5.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

Chapter 6

Geodesics

Abstract To be completed once done

6.1 The Geodesic Equation

Both experimentally and theoretically, the curved spacetimes of general relativity are explored by studying how test particles and light rays move through them.

> **Theorem 6.1.1 (Variational Principle for Free Test Particle Motion)** The world line of a free test particle between two timelike separated points extremizes the proper time between them.

Extremal proper time world lines are called **geodesics**, and the equations of motion that determine them comprise the **geodesic equation**.

For a general metric $g_{\alpha\beta}$, we consider the variational principle

$$\delta \int (-g_{\alpha\beta} dx^\alpha dx^\beta)^{1/2} = 0$$

which has equations of motion

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the metric. We can describe a world line parametrically by giving the four coordinates x^α as a function of a parameter σ that varies between $\sigma = 0$ at a point A to $\sigma = 1$ at a point B. We can rewrite the proper time between A and B as

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2}$$

Applying Euler Lagrange equations we have that

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right] = \frac{\partial \mathcal{L}}{\partial x^\alpha}$$

The Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ are constructed from the metric and its first derivatives. In terms of the four-velocity $u^{\alpha} = dx^{\alpha}/d\tau$ we have

$$\frac{du^{\alpha}}{d\tau} = -\Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma}$$

The Christoffel symbols may be taken to be symmetric in the lower two indices. We can express the Christoffel symbols by

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

6.2 Solving the Geodesic Equation–Symmetries and Conservation Laws

The normalization of the four-velocity gives the equation

$$-1 = u \cdot u = g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}$$

Energy is conserved when there is a symmetry under displacements in time, linear momentum is conserved when there is a symmetry under displacements in space, and angular momentum is conserved when there is a symmetry under rotations.

A **Killing vector** is a general way of characterizing symmetry in any coordinate system. A symmetry implies a conserved quantity along a geodesic. In an arbitrary coordinate system, a conserved quantity along a geodesic is a Killing vector ξ such that

$$\xi \cdot u = \text{constant}$$

6.3 Null Geodesics

Light rays move along null world lines for which $ds^2 = 0$. If $x^{\alpha}(\lambda)$ is the path of a light ray parameterized by some λ and $u^{\alpha} := dx^{\alpha}/d\lambda$ is the tangent vector, then

$$u \cdot u = g_{\alpha\beta}(x)\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

The equation for **null geodesics** is

$$\frac{d^2x^{\alpha}}{d\lambda^2} = -\Gamma^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda}$$

Note the affine parameter λ is not a spacetime distance. Rather it is a parameter chosen so that the equation above takes the form of the geodesic equation.

Problems

6.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.