LOGIC: A COMPLETE GUIDE

LOGIC

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Solo Pursuit of Learning

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Part I

Basic Notions

Chapter 1

§§Arguments

1.1.0 §Definitions and Examples: Arguments

Definition 1.1.1. *An* **argument** *(or* **deductive argument***) is any collections of premises, together with a conclusion.*

To be completely general, we can define an argument as a series of sentences. The sentences at the beginning are premises, and the final sentence in the series is the conclusion.

Definition 1.1.2. *A* **statement** *is a sentence which is either true or false.*

Remark 1.1.1. Questions, imperative sentences, and exclamatory sentences are all not statements. Commands are often phrased as imperative sentences.

1.2.0 §Validity

Definition 1.2.1 (Informal Validity). *An argument is said to be* **valid** *if the conclusion follows from the premises.*

Definition 1.2.2 (Informal Invalidity). *If in an argument the conclusion does not follow from the premises, then the argument is said to be* **invalid***.*

Definition 1.2.3 (Consequence). *A sentence A is a* **consequence** *of sentences $B_1, ..., B_n$ if and only if there is no case where $B_1, ..., B_n$ are all true and A is not true. (We then say that A follows from $B_1, ..., B_n$, or that $B_1, ..., B_n$* **entail** *A)*

Definition 1.2.4 (Informal Validity 2). *An argument is* **valid** *if and only if the conclusion is a consequence of the premises.*

Definition 1.2.5 (Informal Invalidity 2). *An argumnt is* **invalid** *if and only if it is not valid, i.e., it has a counter-example.*

Remark 1.2.1. An argument is ***nomologically valid*** if there are no counter-examples which obey the laws of physics. An argument is ***conceptually valid*** if there are no counter-examples that don't violate conceptual connections between words.

Remark 1.2.2. An argument is ***formally valid*** if we can describe the "form" of the argument as a logical pattern.

1.3.0 §Other Notions

Definition 1.3.1. *An argument is said to be* **sound** *if and only if it is valid and its premises are true.*

Definition 1.3.2. *An* **inductive argument** *is an argument which generalises from observations about many past cases to a conclusion about all future cases. Inductive arguments are not deductively valid.*

Definition 1.3.3 (Jointly Possible). *Sentences are* **jointly possible** *if and only if there is a case where they are all true together.*

Definition 1.3.4 (Contingent). *A sentence which is capable of being true in one case and capable of being false in another case is called* **contingent**.

Definition 1.3.5. *A sentence is a* **necessary truth** *if it is true in all cases.*

Definition 1.3.6. *A sentence is a* **necessary falsehood** *if it is false in all cases.*

Definition 1.3.7. *If two sentences have the same truth value in every case, we say that they are* **necessarily equivalent**.

Chapter 2

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Chapter 4

§§Symbolization and Ambiguity

4.1.0 §Atomic Sentences

Definition 4.1.1. *TFL* **symbolizes** *basic sentences, or sentence components, as sentence letters such as* A, P_1, P_2, P_{432} *etc. These sentence letters are* **atomic sentences** *of TFL, and when symbolizing a sentence in terms of sentence letters we provide a symbolization key, for instance:*

↳ *A: I am a cat.*

4.2.0 §Connectives

Table 4.1: Logical Connectives of TFL

Symbol	**Name**	**Rough meaning**
$\neg$	Negation	'It is not the case that...'
$\wedge$	Conjunction	'Both... and...'
$\vee$	Disjunction	'Either... or...'
$\rightarrow$	Conditional	'If ... then ...'
$\leftrightarrow$	Biconditional	'... if and only if ...'

Definition 4.2.1. *Consider formulas* $\mathscr{A}$ *and* $\mathscr{B}$*. In* $\neg\mathscr{A}$ $\mathscr{A}$ *is said to be the* **negatum**. *For* $(\mathscr{A} \wedge \mathscr{B})$, $\mathscr{A}$ *and* $\mathscr{B}$ *are called the* **conjuncts**. *For* $(\mathscr{A} \vee \mathscr{B})$, $\mathscr{A}$ *and* $\mathscr{B}$ *are called the* **disjuncts**. *For* $(\mathscr{A} \rightarrow \mathscr{B})$, $\mathscr{A}$ *is called the* **antecedent** *and* $\mathscr{B}$ *is called the* **consequent**.

4.3.0 §TFL Sentences

Definition 4.3.1. *The symbols of TFL are the atomic sentences* $(A, B, ..., Z, P_{453}, ...)$*, the connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$*, and brackets* $($, $)$.

Definition 4.3.2. *An* **expression of TFL** *is any string of symbols of TFL.*

Definition 4.3.3. *The following are the only sentences of TFL:*

1. *Every sentence letter is a sentence.*
2. *If $\mathscr{A}$ is a sentence, then $\neg\mathscr{A}$ is also a sentence.*
3. *If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then $(\mathscr{A} \wedge \mathscr{B})$ is a sentence.*
4. *If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then $(\mathscr{A} \vee \mathscr{B})$ is a sentence.*
5. *If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then $(\mathscr{A} \rightarrow \mathscr{B})$ is a sentence.*
6. *If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a sentence.*
7. *Nothing else is a sentence.*

The last sentential connective used in constructing a TFL sentence is called the **main logical operator**.

Definition 4.3.4. *The* **scope** *of a connective (in a sentence) is the subsentence for which that connective is the main logical operator.*

4.4.0 §Ambiguity

Definition 4.4.1. **Lexical ambiguity** *is when a sentence contains words which have more than one meaning.*

Definition 4.4.2. **Structural ambiguity** *occurs when a sentence can be interpreted in different ways, and depending on the interpretation, a different meaning is selected.*

4.5.0 §Object and Meta languages

Remark 4.5.1. When we want to talk about things in the world we just *use* words. When we talk about words we typically have to *mention* the words. Usually, *mentioning* is done using single quotation marks ‘’ (or double quotes if encasing single quotes).

Definition 4.5.1. *When we talk about a language the language we are talking about is called the* **object language**. *The language that we use to talk about the object language is called the* **metalanguange**.

Remark 4.5.2. In TFL sentence letters are sentences of the object language. When refering to a sentence letter in the metalanguage of English (supplemented with some symbols), we may write something along the lines of: ‘D’ is a sentence letter of TFL.

Definition 4.5.2. *We define* **metavariables** *for our augmented metalanguage English to talk about any expression of TFL:*

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, ... \tag{4.5.1}$$

In particular, '$\mathscr{A}$' is a symbo (called a **metavariable***) in the augmented English we use to talk about expressions of TFL.*

Definition 4.5.3. *Suppose we wish to symbolize the premises of an argument by $\mathscr{A}_1, ..., \mathscr{A}_n$, and the conclusion of the argument by $\mathscr{C}$. Then we will write:*

$$\mathscr{A}_1, ..., \mathscr{A}_n \therefore \mathscr{C} \tag{4.5.2}$$

The purpose of the '$\therefore$' symbol is to indicate which sentences are premises and which are conclusions. Strictly speaking '$\therefore$' is a part of our metalanguage, but we shall take the convention to not include quotation marks around the TFL sentences which flank it.

Chapter 5

§§Truth Tables

5.1.0 §Characteristic Truth Tables

Table 5.1: ***Negation***

$\mathscr{A}$	$\neg\mathscr{A}$
T	**F**
F	**T**

Table 5.2: ***Conjunction***

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \wedge \mathscr{B}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Table 5.3: ***Disjunction***

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \vee \mathscr{B}$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Table 5.4: ***Conditional***

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \rightarrow \mathscr{B}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Table 5.5: ***Biconditional***

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \leftrightarrow \mathscr{B}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

5.2.0 §Using Truth Tables

Definition 5.2.1. *A* **valuation** *is any assignment of truth values to particular sentences of TFL.*

Remark 5.2.1. Each row of a truth table represents a possible valuation.

Definition 5.2.2. *A* **complete truth table** *has a line for every possible valuation of the relevant sentence letters.*

Proposition 5.2.1. *If a complete truth table has n different sentence letters, then it must have 2^n rows.*

Remark 5.2.2. Truth tables can be used to test the validity of an argument. Simply check if there is any line in the truth table where all the premises are true and the conclusion is false - if this occurs then the argument is invalid, and if it doesn't the argument is valid.

Remark 5.2.3. If every line of a truth table is true for a sentence, then it is a tautology. Similarly, if every line is false then the sentence is a contradiction.

Remark 5.2.4. Two sentences are equivalent if their truth values on every line of a truth table are equivalent.

Remark 5.2.5. A set of sentences is jointly satisfiable if there is a line in their truth table for which all sentences are true. If no such line exists, the sentences are jointly insatisfiable.

In summary we have the following table:

Table 5.6: Truth table requirements for demonstrating logical properties

	Yes	**No**
Tautology?	complete table	one-line table
Contradiction?	complete table	one-line table
Equivalent?	complete table	one-line table
Satisfiable?	one-line table	complete table
Valid?	complete table	one-line table
Entailment?	complete table	one-line table

Chapter 6

§§Semantics

6.1.0 §Truth-functional

Definition 6.1.1. *A connective is* **truth-functional** *iff the truth value of the sentence with that connective as a main logical operator is uniquely determined by the truth value(s) of the constituent sentence(s).*

Remark 6.1.1. Truth functional connectives simply map us between truth values. When we ***symbolize*** an English sentence in TFL we ignore everything besides the contribution that the truth values of a component make to the truth value of the whole. It is important to note that TFL is unequipped to deal with meaning.

Definition 6.1.2. *When we treat a TFL sentence as* **symbolizing** *an English sentence, we are stipulating that the TFL sentence is to take the same truth value as the English sentence.*

§Indicative versus Subjunctive Connectives

Definition 6.1.3. *TFL strictly uses* **indicative conditionals***, as these are truth-functional.*

Definition 6.1.4. *A* **subjunctive conditional** *is a sentence of the form 'If it were the case that P, then it would be the case that Q'.*

6.2.0 §Tautologies and Contradictions

Definition 6.2.1. *The TFL sentence $\mathscr{A}$ is a* **tautology** *(in TFL) iff it is true on every valuation.*

Remark 6.2.1. Tautology is a surrogate for necessary truth in TFL. There are necessary truths that cannot be adequately symbolized in TFL. Nonetheless, if we can adequately symbolize an English sentence in TFL and the resulting sentence is a tautology, then the English sentence expresses a necessary truth.

Definition 6.2.2. *A TFL sentence $\mathscr{A}$ is a* **contradiction** *(in TFL) iff it is false on every valuation.*

6.3.0 §Equivalence

Definition 6.3.1. *$\mathscr{A}$ and $\mathscr{B}$ are* **equivalent** *(in TFL) iff, for every valuation, their truth values agree, i.e., if there is no valuation in which they have opposite truth values.*

6.4.0 §Satisfiability

Definition 6.4.1. *$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are* **jointly satisfiable** *(in TFL) iff there is some valuation which makes them all true.*

Definition 6.4.2. *$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are* **jointly unsatisfiable** *(in TFL) iff there is no valuation which makes them all true.*

6.5.0 §Entailment and Validity

Definition 6.5.1. *The sentences $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$* **entail** *(in TFL) the sentence $\mathscr{C}$ iff no valuation of the relevant sentence letters makes all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ true and $\mathscr{C}$ false.*

Theorem 6.5.1. *If $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ entail $\mathscr{C}$, in TFL, then $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \therefore \mathscr{C}$ is valid.*

§Double Turnstile

Definition 6.5.2. *We abbreviate the sentence '$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ entail $\mathscr{C}$' by:*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vDash \mathscr{C} \tag{6.5.1}$$

The symbol '$\vDash$' is called the **double turnstile**, *and it is a symbol of our metalanguage.*

Definition 6.5.3. *When we write*

$$\vDash \mathscr{C} \tag{6.5.2}$$

we are saying there is no valuation which makes $\mathscr{C}$ false, so in particular every valuation makes it true. Thus $\mathscr{C}$ is a tautology. Equally, to say that $\mathscr{A}$ is a contradiction we may write

$$\mathscr{A} \vDash \tag{6.5.3}$$

For this says that no valuation makes $\mathscr{A}$ true.

Definition 6.5.4. *To say that $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ do not entail $\mathscr{C}$ we write*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \nvDash \mathscr{C} \tag{6.5.4}$$

Remark 6.5.1. Note that '$\rightarrow$' is a sentential connective of TFL while '$\vDash$' is a symbol of our metalanguage, augmented English. Now, observe that $\mathscr{A} \rightarrow \mathscr{C}$ is a tautology if and only if $\mathscr{A} \vDash \mathscr{C}$.

Chapter 7

§§Natural Deduction

7.1.0 §Initial Definitons and Reasoning

Definition 7.1.1. *A* **formal proof** *is a sequence of sentences, some of which are marked as being initial assumptions (or premises). The last line of the formal proof is the conclusion.*

Example 7.1.1. *Our first example will be to prove the statement*

$$\neg(A \vee B) \therefore \neg A \wedge \neg B$$

Let us begin with how we start such proofs

$$\begin{array}{r|l} 1 & \neg(A \vee B) \\ \hline \end{array}$$

The horizontal line indicates a separation between the assumptions and main body of the proof. Before we complete this proof we need to determine some laws of inference we can use.

7.2.0 §Rules of Inferences for TFL Natural Deduction

Definition 7.2.1 (Reiteration Rule). *If you have already shown something in the course of a proof, the* **reiteration rule** *allows you to repeat it on a new line. For example:*

$$\begin{array}{r|ll} n & \mathscr{A} & \\ \vdots & \vdots & \\ k & \mathscr{A} & R\, n \end{array}$$

This indicates that we have written $\mathscr{A}$ on line n. We write 'R' to indicate we are using the reiteration rule. Note that both '$\mathscr{A}$' and 'n','k' are symbols of our metalanguage

Definition 7.2.2 (Conjunction Introduction). *Our conjunction introduction rule states that for any sentences $\mathscr{A}$ and $\mathscr{B}$ of TFL:*

$$
\begin{array}{l|ll}
m & \mathscr{A} & \\
n & \mathscr{B} & \\
 & \mathscr{A} \wedge \mathscr{B} & \wedge I\ m, n
\end{array}
$$

When introducing a conjuction we reference the line number of the first conjunct first, and the second conjunct second.

Definition 7.2.3 (Conjunction Elimination). *Our conjunction elimination rule states that for any sentences $\mathscr{A}$ and $\mathscr{B}$ of TFL:*

$$
\begin{array}{l|ll}
m & \mathscr{A} \wedge \mathscr{B} & \\
 & \mathscr{A} & \wedge E\ m
\end{array}
$$

or

$$
\begin{array}{l|ll}
m & \mathscr{A} \wedge \mathscr{B} & \\
 & \mathscr{B} & \wedge E\ m
\end{array}
$$

Remark 7.2.1. There is no reason that we have to apply conjunction introduction to two separate lines. Indeed:

$$
\begin{array}{l|ll}
1 & A & \\
\hline
2 & A \wedge A & \wedge\mathrm{I}\ 1, 1
\end{array}
$$

Definition 7.2.4 (Modus Ponens). <u>**Modus Ponens**</u> *is a conditional elimination rule which states that*

$$
\begin{array}{l|ll}
m & \mathscr{A} \to \mathscr{B} & \\
n & \mathscr{A} & \\
 & \mathscr{B} & \to E\ m, n
\end{array}
$$

In citing we always cite the conditional's line number first.

Definition 7.2.5. *Conditional Introduction First, we make an additional assumption $\mathscr{A}$ (in a subproof); from the additional assumption we prove $\mathscr{B}$. From this, we know that if $\mathscr{A}$ is true, then $\mathscr{B}$ is true. This is wrapped up in the following rule:*

$$\begin{array}{r|l|l} i & \quad \mathcal{A} & \\ \hline j & \quad \mathcal{B} & \\ & \mathcal{A} \to \mathcal{B} & \to I\ i\text{–}j \end{array}$$

Remark 7.2.2. Once we have closed a subproof, we cannot use any of the assumptions or results shown in it. In particular, to cite an individual line when applying rule:

1. the line must come before the line where the rule is applied, but
2. not occur within a subproof that has been closed before the line where the rule is applied.

When we close a subproof we say we have discharged the assumptions of the subproof.

Remark 7.2.3. To cite a subproof when applyiing a rule:

1. the cited subproof must come entirely before the application of the rule where it is cited,
2. the cited subproof must not lie within some other closed subproof which is closed at the line it is cited, and
3. the last line of the cited subproof must not occur inside a nested subproof.

Definition 7.2.6 (Biconditional Introduction). *The biconditional introduction rule is similar to two conditional introductions:*

$$\begin{array}{r|l|l} i & \quad \mathcal{A} & \\ \hline j & \quad \mathcal{B} & \\ k & \quad \mathcal{B} & \\ \hline l & \quad \mathcal{A} & \\ & \mathcal{A} \leftrightarrow \mathcal{B} & \leftrightarrow I\ i\text{–}j,\ k\text{–}l \end{array}$$

Definition 7.2.7 (Biconditional Elimination). *The biconditional elimination rule works like the conditional elimination rule, but in two ways:*

$$\begin{array}{r|l|l} m & \mathcal{A} \leftrightarrow \mathcal{B} & \\ n & \mathcal{B} & \\ & \mathcal{A} & \leftrightarrow E\ m, n \end{array}$$

and

$$
\begin{array}{r|l l}
m & \mathscr{A} \leftrightarrow \mathscr{B} & \\
n & \mathscr{A} & \\
 & \mathscr{B} & \leftrightarrow E\ m, n
\end{array}
$$

Note that we always cite the line of the biconditional first when eliminating it.

Definition 7.2.8 (Disjunction Introduction). *As disjunctions are inherently weaker than initial statements of truth, we have the following disjunction rules:*

$$
\begin{array}{r|l l}
m & \mathscr{A} & \\
 & \mathscr{A} \vee \mathscr{B} & \vee I\ m
\end{array}
$$

and

$$
\begin{array}{r|l l}
m & \mathscr{A} & \\
 & \mathscr{B} \vee \mathscr{A} & \vee I\ m
\end{array}
$$

Definition 7.2.9 (Disjunction Elimination). *The disjunction elimintation rule requires subproofs, and is as follows:*

$$
\begin{array}{r|l l}
m & \mathscr{A} \vee \mathscr{B} & \\
i & \quad \mid \underline{\mathscr{A}} & \\
j & \quad \mid \mathscr{C} & \\
k & \quad \mid \underline{\mathscr{B}} & \\
l & \quad \mid \mathscr{C} & \\
 & \mathscr{C} & \vee E\ m, i\text{–}j, k\text{–}l
\end{array}
$$

Definition 7.2.10 (Negation Elimination). *With negation elimination we introduce The False, $\bot$:*

$$
\begin{array}{r|l l}
m & \neg\mathscr{A} & \\
n & \mathscr{A} & \\
 & \bot & \neg E\ m, n
\end{array}
$$

The order of the negation and statement do not matter, but we always cite the line number of the negation first.

Definition 7.2.11 (Negation Introduction). *In introducing a negation we start a subproof with an assumption, and show that it leads to a contradiction (The False):*

$$
\begin{array}{l|l}
i & \quad \mathscr{A} \\
j & \quad \bot \\
 & \neg\mathscr{A} \qquad \neg I\ i\text{–}j
\end{array}
$$

Definition 7.2.12 (Indirect Proof). *This rule stems from the argument that if the assumption that $\mathscr{A}$ is false leads to a contradiction, then $\mathscr{A}$ cannot be false, so it must be true:*

$$
\begin{array}{l|l}
i & \quad \neg\mathscr{A} \\
j & \quad \bot \\
 & \mathscr{A} \qquad IP\ i\text{–}j
\end{array}
$$

Definition 7.2.13 (Explosion Rule). *The explosion rule allows you to derive anything from a contradiction (The False):*

$$
\begin{array}{l|l}
m & \bot \\
 & \mathscr{A} \qquad X\ m
\end{array}
$$

where $\mathscr{A}$ is any sentence whatsoever. This rule is also known as **ex contradictione quod libet**, *"from contradiction, anything." We can rephrase this as a contradiction entails everything:*

$$\bot \vDash \mathscr{A}$$

§Additional and Derived Rules

Definition 7.2.14. *The following is a natural argument form called a* **disjunctive syllogism***:*

$$
\begin{array}{l|l}
m & \mathscr{A} \vee \mathscr{B} \\
n & \neg\mathscr{A} \\
 & \mathscr{B} \qquad DS\ m, n
\end{array}
$$

and

$$
\begin{array}{l|l}
m & \mathscr{A} \vee \mathscr{B} \\
n & \neg\mathscr{B} \\
 & \mathscr{A} \qquad DS\ m, n
\end{array}
$$

When citing we always place the disjunction first.

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be formulas of TFL, and I shall show that $\mathscr{A} \vee \mathscr{B}, \neg\mathscr{A} \vDash \mathscr{B}$:

1	$\mathscr{A} \vee \mathscr{B}$	
2	$\neg\mathscr{A}$	
3	$\quad \mathscr{B}$	
4	$\quad \mathscr{B}$	R 3
5	$\quad \mathscr{A}$	
6	$\quad \bot$	$\neg$E 2, 5
7	$\quad \mathscr{B}$	X 6
8	$\mathscr{B}$	$\vee$E 1, 3–4, 5–7

■

Definition 7.2.15. *The following is a natural argument form called* **modus tollens***:*

m	$\mathscr{A} \to \mathscr{B}$	
n	$\neg\mathscr{B}$	
	$\neg\mathscr{A}$	*MT m, n*

When citing we always place the conditional first.

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be formulas of TFL, and I shall show that $\mathscr{A} \to \mathscr{B}, \neg\mathscr{B} \vDash \neg\mathscr{A}$:

1	$\mathscr{A} \to \mathscr{B}$	
2	$\neg\mathscr{B}$	
3	$\quad \mathscr{A}$	
4	$\quad \mathscr{B}$	$\to$E 1, 3
5	$\quad \bot$	$\neg$E 2, 4
6	$\neg\mathscr{A}$	$\neg$I 3–5

■

Definition 7.2.16. *The* **double-negation elimination** *rule states that:*

$$\begin{array}{l|ll}
m & \neg\neg\mathscr{A} & \\
 & \mathscr{A} & \textit{DNE } m
\end{array}$$

Proof. Let $\mathscr{A}$ be a formula of TFL, and I shall show that $\neg\neg\mathscr{A} \vDash \mathscr{A}$:

$$\begin{array}{l|l|ll}
1 & \neg\neg\mathscr{A} & & \\
\hline
2 & & \neg\mathscr{A} & \\
\cline{3-3}
3 & & \bot & \neg\text{E } 1, 2 \\
4 & \mathscr{A} & & \text{IP } 2\text{–}3
\end{array}$$

∎

Definition 7.2.17. *The* **law of excluded middle** *states that:*

$$\begin{array}{l|l|ll}
i & & \mathscr{A} & \\
\cline{3-3}
j & & \mathscr{B} & \\
m & & \neg\mathscr{A} & \\
\cline{3-3}
n & & \mathscr{B} & \\
 & \mathscr{B} & & \textit{LEM } i\text{–}j,\ m\text{–}n
\end{array}$$

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be formulas of TFL, and I shall show that $\mathscr{A} \to \mathscr{B}, \neg\mathscr{A} \to \mathscr{B} \vDash \mathscr{B}$:

$$\begin{array}{l|l|ll}
1 & \mathscr{A} \to \mathscr{B} & & \\
2 & \neg\mathscr{A} \to \mathscr{B} & & \\
\hline
3 & & \neg\mathscr{B} & \\
\cline{3-3}
4 & & \neg\mathscr{A} & \text{MT } 1, 3 \\
5 & & \neg\neg\mathscr{A} & \text{MT } 2, 3 \\
6 & & \bot & \neg\text{E } 5, 4 \\
7 & \mathscr{B} & & \text{IP } 3\text{–}6
\end{array}$$

∎

Definition 7.2.18. **De Morgan's Laws** *state that:*

$$\begin{array}{r|ll} m & \neg(\mathscr{A} \vee \mathscr{B}) & \\ & \neg\mathscr{A} \wedge \neg\mathscr{B} & \textit{DeM } m \end{array}$$

$$\begin{array}{r|ll} m & \neg(\mathscr{A} \wedge \mathscr{B}) & \\ & \neg\mathscr{A} \vee \neg\mathscr{B} & \textit{DeM } m \end{array}$$

$$\begin{array}{r|ll} m & \neg\mathscr{A} \vee \neg\mathscr{B} & \\ & \neg(\mathscr{A} \wedge \mathscr{B}) & \textit{DeM } m \end{array}$$

and

$$\begin{array}{r|ll} m & \neg\mathscr{A} \wedge \neg\mathscr{B} & \\ & \neg(\mathscr{A} \vee \mathscr{B}) & \textit{DeM } m \end{array}$$

Remark 7.2.4. All of the 'proofs' in this section are not proofs of TFL, but rather proof schemes as they use the metavariables $\mathscr{A}$ and $\mathscr{B}$. Nonetheless, they show how any of this rules can be derived from our basic rules for specific formulas of TFL.

Chapter 8

§§Soundness and Completeness

8.1.0 §Basic Proof-Theoretic Concepts

Definition 8.1.1. *The following expression:*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vdash \mathscr{C}$$

means that there **exists** *some proof which ends with $\mathscr{C}$ whose undischarged assumptions are among $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$. When we want to say that no such proof exists, we write*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \nvdash \mathscr{C}$$

The symbol '$\vdash$' is called the **single turnstile**.

Remark 8.1.1. Note that '$\vdash$' and '$\vDash$' are *very* different things: '$\vdash$' concerns with the existence of proofs, while '$\vDash$' concerns the existence of valuations.

Definition 8.1.2. *$\mathscr{A}$ is a* **theorem** *if and only if $\vdash \mathscr{A}$; that is, there is a proof of $\mathscr{A}$ with no undischarged assumptions.*

Definition 8.1.3. *Two sentences $\mathscr{A}$ and $\mathscr{B}$ are* **provably equivalent** *if and only if each can be proved from the other; i.e., both $\mathscr{A} \vdash \mathscr{B}$ and $\mathscr{B} \vdash \mathscr{A}$.*

Definition 8.1.4. *The sentences $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are* **provably inconsistent** *if and only if a contradiction can be proved from the, i.e., $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vdash \bot$. If they are not* **inconsistent**, *we call them* **provably consistent**.

In summary we have the following table:

Table 8.1: Proof requirements for demonstrating proof theoretic properties

	Yes	**No**
Theorem?	one proof	all possible proofs
Inconsistent?	one proof	all possible proofs
Equivalent?	two proofs	all possible proofs
consistent?	all possible proofs	one proof

§Other Syntactic Concepts

Definition 8.1.5. *A sentence $\mathscr{A}$ is a* **syntactic contradiction** *in TFL if $\neg\mathscr{A}$ is a* **theorem** *(or syntactic tautology).*

Definition 8.1.6. *A sentence $\mathscr{A}$ is* **syntactically contingent** *in TFL if it is not a theorem or a contradiction.*

Definition 8.1.7. *An argument $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n, \therefore \mathscr{C}$ is* **provably valid** *in TFL if and only if there is a derivation of its conclusion from its premises.*

§Semantic versus Syntactic Definitions

Concept	Truth table (semantic) definition	Proof-theoretic (syntactic) definition
Tautology	A sentence whose truth table only has Ts under the main connective	A sentence that can be derived without any premises.
Contradiction	A sentence whose truth table only has Fs under the main connective	A sentence whose negation can be derived without any premises
Contingent sentence	A sentence whose truth table contains both Ts and Fs under the main connective	A sentence that is not a theorem or contradiction
Equivalent sentences	The columns under the main connectives are identical.	The sentences can be derived from each other
Unsatisfiable/ inconsistent sentences	Sentences which do not have a single line in their truth table where they are all true.	Sentences from which one can derive a contradiction
Satisfiable/ Consistent sentences	Sentences which have at least one line in their truth table where they are all true.	Sentences from which one cannot derive a contradiction
Valid argument	An argument whose truth table has no lines where there are all Ts under main connectives for the premises and an F under the main connective for the conclusion.	An argument where one can derive the conclusion from the premises

Table 8.2: Two ways to define logical concepts.

§Proving Logical Properties

Logical Property	To prove it present	To prove it absent
Being a Theorem	Derive the sentence	Find a false line in the truth table for the sentence
Being a Contradiction	Derive the negation of the sentence	Find a true line in the truth table for the sentence
Contingency	Find a false line and a true line in the truth table for the sentence	Prove the sentence or its negation
Equivalence	Derive each sentence from the other	Find a line in the truth tables for the sentences where they have different values
Consistency	Find a line in the truth table for the sentences where they are all true	Derive a contradiction from the sentences
Validity	Derive the conclusion from the premises	Find a line in the truth table where the premises are true and the conclusion false

Table 8.3: When to provide a truth table and when to provide a proof.

8.2.0 §Soundness

Definition 8.2.1. *A proof system is* **sound** *if there are no derivations of arguments that can be shown to be invalid by truth tables. Demonstrating that a proof system is sound consists of showing that every possible proof is the proof of a valid argument. Symbolically, we wish to show valid$_{\vdash}$ implies valid$_{\vDash}$.*

§Proof Sketch: Soundness

Consider a base class of one-line proofs, one for each of our eleven rules of inference. The members of this class would look like $\mathscr{A}, \mathscr{B} \vdash \mathscr{A} \wedge \mathscr{B}$; $\mathscr{A} \wedge \mathscr{B} \vdash \mathscr{A}$; $\mathscr{A} \vee \mathscr{B}, \neg\mathscr{A} \vdash \mathscr{B}$, ... etc. Note that this proof is in our metalanguage, since TFL does not have the capability to talk about itself.

One can use truth tables to show that each of these one-line proofs in this base class are valid$_{\vDash}$.

Next, we must show that adding lines to a valid$_{\vDash}$ proof will not change it into a invalid$_{\vDash}$ one. This would need to be done for each of our eleven rules of inference. Completing this process also completes are proof that valid$_{\vdash}$ implies valid$_{\vDash}$.

8.3.0 §Completeness

Definition 8.3.1. *A proof system has the property of* **completeness** *if and only if there is a derivation of every semantically valid argument. This is in general very difficult to prove, and amounts to showing that the rules of inference we have defined for our proof system are sufficient.*

Remark 8.3.1. TFL is an example of a proof system which has both the property of soundness and the property of completeness.

Part III

First-Order Logic

Chapter 9

§§Building Blocks of FOL

9.1.0 §Names

Definition 9.1.1. *In English, a* **singular term** *is a word or phrase that refers to a* specific *person, place, or thing.*

Definition 9.1.2. *In FOL, our* **names** *are lower-case letters 'a' through to 'r', possibly with the addition of subscripts. Each name must pick out* **exactly** *on thing (like a function).*

9.2.0 §Predicates

Definition 9.2.1. *In FOL,* **predicates** *are captical letters A through Z, with or without subscripts. They can be thought of as representing things which combine with singular terms to make sentences.*

9.3.0 §Quantifiers

Definition 9.3.1. *In FOL, the symbol '$\forall$' is called the* **universal quantifier**.

Remark 9.3.1. A quantifier must always be followed by a ***variable***. In FOL, variables are italic lowercase letters 's' through 'z', with or without subscripts.

Definition 9.3.2. *In FOL, the symbol '$\exists$' is called the* **existential quantifier**.

Remark 9.3.2. In general, $\forall x \neg \mathscr{A}$ is logically equivalent to $\neg \exists x \mathscr{A}$, and $\neg \forall x \mathscr{A}$ is logically equivalent to $\exists x \neg \mathscr{A}$.

9.4.0 §Domains

Definition 9.4.1. *In FOL, the* **domain** *is the collection of things that we are talking about. The quantifiers in an argument* range over *its domain. A domain must have* at least *one member. Every name must pick out* exactly *one member of the domain, but a member of the domain may be picked out by one name, many names, or none at all.*

Definition 9.4.2. *A predicate that applies to nothing in the domain is called an* **empty predicate**.

Remark 9.4.1. When $\mathscr{F}$ is an empty predicate, any sentence $\forall x(\mathscr{F} \to ...)$ is vacuously true.

Definition 9.4.3. *A k-place predicate is a predicate $P(x_1, ..., x_k)$ which can take in k sentence letters.*

9.5.0 §Identity

Definition 9.5.1. *The symbol '$=$' is a two-place predicate of meaning:*

$$x = y : \underline{\qquad}_x \text{ is identical to } \underline{\qquad}_y$$

Chapter 10

§§Sentences of FOL and Ambiguity

10.1.0 §Expressions

Definition 10.1.1. *There are six types of symbols in FOL:*

1. ***Predicates***: $A, B, C, ..., Z$, *or with subscripts, as needed:* $A_1, B_1, Z_1, A_2, A_{25}, ...$
2. ***Names***: $a, b, c, ..., r$, *pr wotj subscripts, as needed* $a_1, b_{224}, h_7, m_{32}, ...$
3. ***Variables*** s, t, u, v, w, x, y, z, *or with subscripts, as needed* $x_1, y_1, z_1, x_2, ...$
4. ***Connectives*** $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
5. ***Brackets*** $($, $)$
6. ***Quantifiers*** $\forall, \exists$

We define an **expression of FOL** *as any string of symbols of FOL.*

10.2.0 §Terms and Formulas

Definition 10.2.1. *A* **term** *is any name or any variable.*

Definition 10.2.2. *We define the* **atomic formulas** *of FOL as follows:*

1. *Any sentence letter is an atomic formula.*
2. *If* $\mathscr{R}$ *is an* n*-place predicate and* $t_1, t_2, ..., t_n$ *are terms, then* $\mathscr{R}(t_1, t_2, ..., t_n)$ *is an atomic formula.*
3. *If* t_1 *and* t_2 *are terms, then* $t_1 = t_2$ *is an atomic formula.*
4. *Nothing else is an atomic formula.*

Definition 10.2.3. *We define formulas in FOL recursively as follows:*

1. *Every atomic formula is a formula*
2. *If $\mathscr{A}$ is a formula, then $\neg\mathscr{A}$ is a formula.*
3. *If $\mathscr{A}$ and $\mathscr{B}$ are formulas, then $(\mathscr{A} \wedge \mathscr{B})$ is a formula.*
4. *If $\mathscr{A}$ and $\mathscr{B}$ are formulas, then $(\mathscr{A} \vee \mathscr{B})$ is a formula.*
5. *If $\mathscr{A}$ and $\mathscr{B}$ are formulas, then $(\mathscr{A} \rightarrow \mathscr{B})$ is a formula.*
6. *If $\mathscr{A}$ and $\mathscr{B}$ are formulas, then $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a formula.*
7. *If $\mathscr{A}$ is a formula and x is a variable, then $\forall x\mathscr{A}$ is a formula.*
8. *If $\mathscr{A}$ is a formula and x is a variable, then $\exists x\mathscr{A}$ is a formula.*
9. *Nothing else is a formula.*

Definition 10.2.4. *The* **main logical operator** *in a formula is the operator that was introduced most recently, when that formula was constructed using the recursion rules.*

The **scope** *of a logical operator in a formula is the subformula for which that operator is the main logical operator.*

10.3.0 §Sentences and Free Variables

Definition 10.3.1. *An occurrence of a variable x is* **bound** *if and only if it falls within the scope of either $\forall x$ or $\exists x$. An occurrence of a variable which is not bound is* **free**.

Definition 10.3.2. *A* **sentence** *of FOL is any formula of FOL that contains no free variables.*

10.4.0 §Definite Descriptions

Definition 10.4.1. **Definite descriptions** *are meant to pick out a* unique *object.*

Definition 10.4.2. *Russel's Analysis treats definite descriptions in FOL as follows*

the F is G ***iff*** *there is at least one F, and*
there is at most one F, and
every F is G

Chapter 11

§§Extensionality and Interpretations

11.1.0 §Extensionality

Remark 11.1.1. FOL has no resources for dealing with nuances of meaning. When we interpret FOL, all we are considering is what the predicates are true of, regardless of whether we specify these things directly or indirectly.

Definition 11.1.1. *The things a predicate is true of are known as the* **extension** *of that predicate. We say that FOL is an* **extensional language** *because FOL does not represent differences of meaning between predicates that have the same extension.*

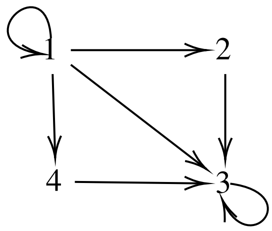
Remark 11.1.2. The ***identity of indiscernibles*** is a philosophical claim that if two objects are true of exactly the same sentences, then they are the very same object. Our logic will not subscribe to this claim-we will leave the possibility that distinct objects can be true of the same things.

11.2.0 §Interpretations

Definition 11.2.1. *In FOL, we define an* **interpretation** *as consisting of four things:*

1. *the specification of a domain*
2. *for each sentence letter we care to consider, a truth value*
3. *for each name that we care to consider, an assignment of exactly one object within the domain*
4. *for each predicate that we care to consider (apart from '$=$'), a specification of what things (in what order) the predicate is to be true of. (We don't specify an interpretation of '$=$', since it has a* fixed *interpretation.)*

Remark 11.2.1. One way to specify an interpretation is with a symbolization key. We can also present interpretations diagrammatically. Suppose we want to consider just a single two-place predicate, ‘$R(x,y)$’. THen we can represent it just by drawing an arrow between two objects, and stipulate that ‘$R(x,y)$’ is to hold of x and y just in case there is an arrow running from x to



y in our diagram. As an example we may offer:

Chapter 12

§§Truth and Models

12.1.0 §Truth

§Atomic Sentences

Definition 12.1.1. *When $\mathscr{R}$ is an n-place predicate and $a_1, a_2, ..., a_n$ are names, the sentence $\mathscr{R}(a_1, a_2, ..., a_n)$ is true in an interpretation if and only if $\mathscr{R}$ is true of the objects named by $a_1, a_2, ..., a_n$ (in that order) in that interpretation.*

Definition 12.1.2. *For any names a and b, $a = b$ is true in an interpretation if and only if a and b name the very same object in that interpretation.*

§Sentential Connectives

Definition 12.1.3. *Given any sentences $\mathscr{A}$ and $\mathscr{B}$ of FOL,*

1. *$\mathscr{A} \wedge \mathscr{B}$ is true in an interpretation if and only if both $\mathscr{A}$ is true and $\mathscr{B}$ is true in that interpretation*
2. *$\mathscr{A} \vee \mathscr{B}$ is true in an interpretation if and only if either $\mathscr{A}$ is true or $\mathscr{B}$ is true in that interpretation*
3. *$\neg\mathscr{A}$ is true in an interpretation if and only if $\mathscr{A}$ is false in that interpretation*
4. *$\mathscr{A} \rightarrow \mathscr{B}$ is true in an interpretation if and only if either $\mathscr{A}$ is false or $\mathscr{B}$ is true in that interpretation*
5. *$\mathscr{A} \leftrightarrow \mathscr{B}$ is true in an interpretation if and only if $\mathscr{A}$ has the same truth value as $\mathscr{B}$ in that interpretation*

§Quantifiers as the Main Logical Operator

Definition 12.1.4. *Suppose that $\mathscr{A}$ is a formula containing at least one occurence of the variable x, and that x is* **free** *in $\mathscr{A}$. We will write this thus:*

$$\mathscr{A}(...x...x...)$$

Suppose also that c is a name. Then we will write:

$$\mathscr{A}(...c...c...)$$

for the formula we obtain by replacing every *occurrence of x in $\mathscr{A}$ with c. The resulting formula is called a* **substitution instance** *of $\forall x\mathscr{A}$ and $\exists x\mathscr{A}$. Also, c is called the* **instantiating name**.

Definition 12.1.5. *Take any object in the domain, say, d, and a name c which is not already assigned by the interpretation. If our interpretation is $\mathbf{I}$, then we can consider the interpretation $\mathbf{I}[d/c]$ which is just like $\mathbf{I}$ except it* also *assigns the name c to the object d. Then we can say that d* **satisfies** *the formula $\mathscr{A}(...x...x...)$ in the interpreation $\mathbf{I}$ if, and only if, $\mathscr{A}(...c...c...)$ is true in $\mathbf{I}[d/c]$. (We also say that $\mathscr{A}(...x...x...)$ is* **true of** *d)*

Definition 12.1.6. *The interpretation $\mathbf{I}[d/c]$ is just like the interpretation $\mathbf{I}$ except it also assigns the name c to the object d.*

An object d **satisfies** *$\mathscr{A}(...x...x...)$ in interpretation $\mathbf{I}$ if and only if $\mathscr{A}(...c...c...)$ is true in $\mathbf{I}[d/c$.*

Definition 12.1.7. *$\forall x\mathscr{A}(...x...x...)$ is true in an interpretation if and only if every object in the domain* **satisfies** *$\mathscr{A}(...x...x...)$.*

$\exists x\mathscr{A}(...x...x...)$ is true in an interpretation if and only if at least one object in the domain satisfies $\mathscr{A}(...x...x...)$.

12.2.0 §Semantic Concepts

Definition 12.2.1. *In FOL the symbolization*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vDash \mathscr{C}$$

means that there is no interpretation in which all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are true and in which $\mathscr{C}$ is false.

Definition 12.2.2. *Derivatively to the last definition,*

$$\vDash \mathscr{A}$$

means that $\mathscr{A}$ is true in every interpretation.

Definition 12.2.3. *An FOL sentence $\mathscr{A}$ is a* **validity** *if and only if $\mathscr{A}$ is true in every interpretation; i.e., $\vDash \mathscr{A}$.*

Definition 12.2.4. *An FOL sentence $\mathscr{A}$ is a* **contradiction** *if and only if $\mathscr{A}$ is false in every interpretation; i.e., $\vDash \neg\mathscr{A}$.*

Definition 12.2.5. *$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \therefore \mathscr{C}$ is* **valid in FOL** *if and only if there is no interpretation in which all of the premises are true and the conclusion is false; e.e., $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vDash \mathscr{C}$. It is* **invalid in FOL** *otherwise.*

Definition 12.2.6. *Two FOL sentences $\mathscr{A}$ and $\mathscr{B}$ are* **equivalent** *if and only if they are true in exactly the same interpretations as each other; i.e., both $\mathscr{A} \vDash \mathscr{B}$ and $\mathscr{B} \vDash \mathscr{A}$.*

Definition 12.2.7. *The FOL sentences $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are* **jointly satisfiable** *if and only if some interpretation makes all of them true. They are* **jointly unsatisfiable** *if and only if there is no such interpretation.*

12.3.0 §Working with Interpretations

Remark 12.3.1. To show that $\mathscr{A}$ is not a validity, it suffices to find an interpretation where $\mathscr{A}$ is false.

TO show that $\mathscr{A}$ is not a contradiction, it suffices to find an interpretation where $\mathscr{A}$ is true.

Remark 12.3.2. To show that $\mathscr{A}$ and $\mathscr{B}$ are not logically equivalent, it suffices to find an interpretation where oen is true and the other is false.

Remark 12.3.3. If some interpretation makes all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ true and $\mathscr{C}$ false, then:

1. $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \therefore \mathscr{C}$ is invalid; and
2. $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \nvDash \mathscr{C}$; and
3. $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n, \neg\mathscr{C}$ are jointly consistent (satisfiable).

Definition 12.3.1. *An interpretation which refutes a claim (to logical truth, say, or to entailment) is called a* **counter-interpretation**, *or a* **counter-model**.

Remark 12.3.4. If you want to infer from the absence of an entailment in FOL to the invalidity of some English argument, then you need to argue that nothing important is lost in the way you have symbolized the English argument.

Remark 12.3.5. We must reason about all interpretations if we wish to show:

1. that a sentence is a contradiction; for this requires that it is false in *every* interpretation.
2. that two sentences are logically equivalent; for this requires that they have the same truth value in *every* interpretation.
3. that some sentences are jointly unsatisfiable; for this requires that there is no interpretation in which all of those sentences are true together; i.e. that, in *every* interpretaation, at least one of those sentences is false.

4. that an argument is valid; for this requires that the conclusion is true in *every* interpretation where the premises are true.

5. that some sentences entail another sentence.

Table 12.1: Interpretation requirements for demonstrating FOL semantic properties

	Yes	**No**
Validity?	all interpretations	one counter-interpretation
Contradiction?	all interpretations	one counter-interpretation
Equivalent?	all interpretations	one counter-interpretation
Satisfiable?	one interpretation	all interpretations
Valid?	all interpretations	one counter-interpretation
Entailment?	all interpretations	one counter-interpretation

Chapter 13

§§Natural Deduction in FOL

13.1.0 §Basic Rules of FOL

Definition 13.1.1. *The universal elimination rule ($\forall E$) is given by:*

$$
\begin{array}{r|ll}
m & \forall x \mathcal{A}(...x...x...) & \\
 & \mathcal{A}(...c...c...) & \forall E\, m
\end{array}
$$

The point of this rule of inference is that you can obtain any *substitution instance* of a universally quantified formula: replace *every* instance of the quantified variable with any name you like.

Recall, that as with any rule of inference, it can only be applied if the universal quantifier is the main logical operator.

Definition 13.1.2. *Where $\mathcal{A}$ is a sentence containing the name c, we can emphasize this by writing '$\mathcal{A}(...c...c)$'. We write '$\mathcal{A}(...x...c...)$' to indicate any formula obtained by replacing* some or all *of the instances of the name c with the variable x.*

Definition 13.1.3. *Using the previously defined notation, we write the existential introduction rule as:*

$$
\begin{array}{r|ll}
m & \mathcal{A}(...c...c...) & \\
 & \exists x \mathcal{A}(...x...c...) & \exists I\, m
\end{array}
$$

where x must not *occur in $\mathcal{A}(...c...c...)$.*

Remark 13.1.1. It is clear from these rules of inference that if we want them to always apply and be valid, we must accept that as a matter of logic alone, there exists something rather than nothing. However, this is completely reasonable as we have already stipulated that domains be non-empty.

In order to consider notions involving empty domains, we would need a more complicated proof system than FOL gives us.

Definition 13.1.4. *Note that if we know a statement holds of an* arbitrary *name, we can conclude it holds of everything. This motivates the rule of inference for universal introduction:*

$$
\begin{array}{l|ll}
m & \mathscr{A}(...c...c...) & \\
 & \forall x\mathscr{A}(...x...x...) & \forall I\ m
\end{array}
$$

where c must not occur in any undischarged assumption, and x must not occur in $\mathscr{A}(...c...c...)$. Note also that we are replacing all *instances of c in $\mathscr{A}(...c...c...)$ with x in this rule.*

The name in the above rule of inference may occur in *discharged* assumptions, even though it is not allowed to occur in any *undischarged* assumptions.

Definition 13.1.5. *Noting that if we know something satisfies F, and every F is G, we can conclude something satisfies G. This reasoning inspires our rule of inference for existential elimination:*

$$
\begin{array}{l|ll}
m & \exists x\mathscr{A}(...x...x...) & \\
i & \quad \mid \underline{\mathscr{A}(...c...c...)} & \\
j & \quad \mid \mathscr{B} & \\
 & \mathscr{B} & \exists E\ m,\ i\text{–}j
\end{array}
$$

where c must not occur in any assumption undischarged before line i, c must not occur in $\exists x\mathscr{A}(...x...x...)$, and c must not occur in $\mathscr{B}$.

If we want to squeeze information out of an existential quantifier, it is advisable choose a new name for our substitution instance (in general), to operate in the utmost generality.

Definition 13.1.6. *The identity introduction rule is given by:*

$$
\begin{array}{l|ll}
m & c = c & =I
\end{array}
$$

Notice that this rule does not require referring to any prior lines.

Definition 13.1.7. *The identity elimination rule is given by:*

$$
\begin{array}{l|ll}
m & a = b & \\
n & \mathscr{A}(...a...a...) & \\
 & \mathscr{A}(...b...a...) & =E\ m,\ n
\end{array}
$$

Note the notation implies we can replace one or more '$\mathscr{a}$''s in the sentencewith '$\mathscr{b}$''s. Symmetrically we have:

$$
\begin{array}{r|ll}
m & \mathscr{a} = \mathscr{b} & \\
n & \mathscr{A}(...\mathscr{b}...\mathscr{b}...) & \\
 & \mathscr{A}(...\mathscr{a}...\mathscr{b}...) & =E\ m, n
\end{array}
$$

which is sometimes called **Leibniz's Law**.

13.2.0 §Derived Rules in FOL

Definition 13.2.1. *We can pass negations through quantified sentences as follows:*

$$
\begin{array}{r|ll}
m & \forall x \neg \mathscr{A} & \\
 & \neg \exists x \mathscr{A} & CQ\ m
\end{array}
$$

and

$$
\begin{array}{r|ll}
m & \neg \exists x \mathscr{A} & \\
 & \forall x \neg \mathscr{A} & CQ\ m
\end{array}
$$

as well as

$$
\begin{array}{r|ll}
m & \exists x \neg \mathscr{A} & \\
 & \neg \forall x \mathscr{A} & CQ\ m
\end{array}
$$

and

$$
\begin{array}{r|ll}
m & \neg \forall x \mathscr{A} & \\
 & \exists x \neg \mathscr{A} & CQ\ m
\end{array}
$$

13.3.0 §Proof Theory Semantics in FOL

Definition 13.3.1. *Given sentences $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n, \mathscr{C}$ of FOL, we write*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vdash \mathscr{C}$$

to mean that there exists *some proof which ends with* $\mathscr{C}$ *and whose only undischarged assumptions are among* $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$. *This is a* **proof-theoretic notion**.

Definition 13.3.2. *Conversely to the last definition, the notation*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vDash \mathscr{C}$$

means that no valuation (or interpretation) makes all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ *true and* $\mathscr{C}$ *false. This is a* **semantic notion** *- it concerns the assignments of truth and falsity to sentences.*

THESE ARE DIFFERENT NOTIONS.

Theorem 13.3.1. *In FOL we have the following result:*

$$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vdash \mathscr{B} \textbf{ iff } \mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vDash \mathscr{B}$$

This shows that, whilst provability and entailment are very different *notions, they are* extensionally *equivalent. As such:*

1. *An argument is* valid *iff the conclusion can be proved from the premises (i.e. it is a theorem)*
2. *Two sentences are* logically equivalent *iff they are* provably equivalent
3. *Sentences are* satisfiable *iff they are* not provably inconsistent

Question	Yes	No
Is $\mathscr{A}$ a ***validity***?	give a proof which shows $\vdash \mathscr{A}$	give an interpretation in which $\mathscr{A}$ is false
Is $\mathscr{A}$ a ***contradiction***?	give a proof which shows $\vdash \neg\mathscr{A}$	give an interpretation in which $\mathscr{A}$ is true
Are $\mathscr{A}$ and $\mathscr{B}$ ***equivalent***?	give two proofs; one for $\mathscr{A} \vdash \mathscr{B}$ and one for $\mathscr{B} \vdash \mathscr{A}$	give an interpretation in which $\mathscr{A}$ and $\mathscr{B}$ have different truth values
Are $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ ***jointly satisfiable***?	give an interpretation in which all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are true	prove a contradiction from the assumptions $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ (that is give a proof for $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \vdash \bot$)
Is $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \therefore \mathscr{C}$ ***valid***?	give a proof with assumptions $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ and concluding with $\mathscr{C}$	give an interpretation in which all of $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ are true and $\mathscr{C}$ is false

Table 13.1: Methods of showing logical concepts

Chapter 14

§§Soundness and Completeness

Chapter 15

§§Models of Theories and Interpretations

Part IV

Modal Logic

Chapter 16

§§General Concepts

16.1.0 §Introduction to Modal Logic

Modal logic (ML) is the logic of ***modalities***, ways in which a statement can be true.

Example 16.1.1. **Necessity** *and* **possibility** *are two such modalities: a statement can be true, but it can also be necessarily true (true no matter how the world might have been). A possible statement may not actually be true, but it might have been true. We use* $\Box$ *to express necessity, and* $\Diamond$ *to express possibility. So* $\Box\mathscr{A}$ *can be read as* "it is necessarily the case that $\mathscr{A}$," *and* $\Diamond\mathscr{A}$ *as* "it is possibly the case that $\mathscr{A}$."

There are also many kinds of necessity. ML is able to deal with any of these kinds of necessary truth. We start with a basic set of rules that govern $\Box$ and $\Diamond$, and then we add more rules as needed to fit the kind of modality we are interested in.

We actually do not nead to read $\Box$ and $\Diamond$ as necessity and possibility. All we need is choose the right rules for different readings of $\Box$ and $\Diamond$.

Definition 16.1.1. *A* **modal formula** *is one that includes modal operators such as* $\Box$ *and* $\Diamond$*. Depending on the interpretation we assign to* $\Box$ *and* $\Diamond$*, different modal formulas will be provable or valid.*

16.2.0 §The Language of Modal Logic

The language of ML is an extension of the language of TFL. If we build off of FOL, we would have ***Quantified Modal Logic (QML)***.

Definition 16.2.1. *ML has an infinite stock of* **atoms** *(just like in TFL). These are written as capital letters, with or without numerical subscripts:* $A, B, ..., A_1, B_1, ...$

Definition 16.2.2. *For the rules of ML we take all of the rules of TFL, and add two more for* $\Box$ *and* $\Diamond$*:*

1. *Every atom of ML is a sentence of ML*
2. *If $\mathcal{A}$ is a sentence of ML, then $\neg\mathcal{A}$ is a sentence of ML*
3. *If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \wedge \mathcal{B})$ is a sentence of modal logic*
4. *If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \vee \mathcal{B})$ is a sentence of ML*
5. *If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \rightarrow \mathcal{B})$ is a sentence of ML*
6. *If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \leftrightarrow \mathcal{B})$ is a sentence of ML*
7. *If $\mathcal{A}$ is a sentence of ML, then $\Box\mathcal{A}$ is a sentence of ML*
8. *If $\mathcal{A}$ is a sentence of ML, then $\Diamond\mathcal{A}$ is a sentence of ML*
9. *Nothing else is a sentence of ML*

Example 16.2.1.

$$A, (P \vee Q), \Box A, (C \vee \Box D),$$
$$\Box\Box(A \rightarrow R), \Box \Diamond (S \wedge (Z \leftrightarrow (\Box W \vee \Diamond Q)))$$

16.3.0 §Natural Deduction for ML

As before, we will write $\mathcal{A}_1, ..., \mathcal{A}_n \vdash \mathcal{C}$ to express that $\mathcal{C}$ can be proven from $\mathcal{A}_1, ..., \mathcal{A}_n$. However, since we will be looking at many different systems of ML, we will add a subscript to '$\vdash$' to indicate which system we are working with.

§System K

First, the system **K** includes all of the natural deduction rules of TFL, including the derived rules as well as the basic ones. We then add two additional rules for $\Box$ and a special kind of subproof:

Definition 16.3.1. *A* **strict subproof** *is one of the form:*

$$\begin{array}{l|l|l}
m & \quad \Box & \\
n & \quad \mathcal{A} & \\
 & \Box\mathcal{A} & \Box I\ m\text{–}n
\end{array}$$

No line above m may be cited by any rule within the strict subproof begun at line m, unless the rule explicitly allows it.

These allow us to reason and prove things about alternate possibilities. What we can prove inside a strict subproof holds in any alternate possibility, in particular, in alternate possibilities where the assumptions in force in our proof may not hold.

Due to this formulation, the idea is that if $\mathscr{A}$ is a theorem, then $\Box\mathscr{A}$ should be a theorem too.

Definition 16.3.2. *The $\Box$ elimination rule is given by:*

m	$\Box\mathscr{A}$	
	$\quad\Box$	
n	$\quad\mathscr{A}$	$\Box E\ m$

$\Box E$ can only be applied if the line m (containing $\Box\mathscr{A}$) lies outside *of the strict subproof in which line n* falls, *and this strict subproof is not itself part of a strict subproof not containing m (i.e. you can't apply it in nested strict subproofs in which $\Box\mathscr{A}$ is not in the one right before where we apply $\Box E$)*

Example 16.3.1. *The following is known as the distribution rule:*

1	$\Box(A \to B)$	
2	$\quad\Box A$	
3	$\quad\quad\Box$	
4	$\quad\quad A$	$\Box E\ 2$
5	$\quad\quad A \to B$	$\Box E\ 1$
6	$\quad\quad B$	$\to E\ 4, 5$
7	$\quad\Box B$	
8	$\Box A \to \Box B$	$\to I\ 2\text{–}7$

§Possibility

Definition 16.3.3. *We can* define *possibility in terms of necessity:*

$$\Diamond\mathscr{A} =_{df} \neg\Box\neg\mathscr{A}$$

In other words, to say that $\mathscr{A}$ is possibly true, *is to say that $\mathscr{A}$ is* not necessarily false.

Definition 16.3.4. *TO introduce $\Diamond$ into the system* **K**, *we introduce the following definitional rules:*

$$\begin{array}{r|ll} m & \neg\Box\neg\mathscr{A} & \\ & \Diamond\mathscr{A} & \text{Def}\Diamond\ m \end{array}$$

$$\begin{array}{r|ll} m & \Diamond\mathscr{A} & \\ & \neg\Box\neg\mathscr{A} & \text{Def}\Diamond\ m \end{array}$$

We could leave our rules for **K** here, but for convenience we shall add some *Modal Conversion* rules:

Definition 16.3.5.

$$\begin{array}{r|ll} m & \neg\Box\mathscr{A} & \\ & \Diamond\neg\mathscr{A} & MC\ m \end{array}$$

$$\begin{array}{r|ll} m & \Diamond\neg\mathscr{A} & \\ & \neg\Box\mathscr{A} & MC\ m \end{array}$$

$$\begin{array}{r|ll} m & \neg\Diamond\mathscr{A} & \\ & \Box\neg\mathscr{A} & MC\ m \end{array}$$

$$\begin{array}{r|ll} m & \Box\neg\mathscr{A} & \\ & \neg\Diamond\mathscr{A} & MC\ m \end{array}$$

It can be proven that $\Diamond A \leftrightarrow \neg\Box\neg A$. So in particular, we could have started with $\Diamond$ and defined $\Box$ as $\Box\mathscr{A} =_{df} \neg\Diamond\neg\mathscr{A}$.

In other words, necessity and possibility are exactly as fundamental as each other.

16.4.0 §System T

The system **T** is obtained by adding the following rule to **K**:

Definition 16.4.1.

$$\begin{array}{r|ll} m & \Box\mathscr{A} & \\ n & \mathscr{A} & R\boldsymbol{T}\ m \end{array}$$

The lien n on which the rule RT is applied must **not** *lie in a strict subproof that begins after line m.*

Adding this rule allows us to prove things like $\Box A \to A$ in **T**, which was previously impossible to prove in **K**.

16.5.0 System S4

Chapter 17

§§Models

Chapter 18

§§Frames

Chapter 19

§§Completeness

Chapter 20

§§Algebras

Chapter 21

§§First-Order Modal Logic

Part V

Computability

Chapter 22

§§Turing Computability

Chapter 23

§§Uncomputability

Chapter 24

§§Recursion and Recursive Functions

Chapter 25

§§Undecidability

Chapter 26

§§First-Order Logic Revisited

Chapter 27

§§Second-Order Logic

Chapter 28

§§Definability

Part VI

Metatheory

Chapter 29

§§Normal Forms

29.1.0 §Disjunctive Normal Form

Definition 29.1.1. *We will say that a sentence is in* **disjunctive normal form** *if and only if it meets all of the following requirements:*

(DNF1) No connectives occur in the sentence other than negations, conjunctions, and disjunctions;

(DNF2) Every occurence of negation has minimal scope (i.e. any '$\neg$' is immediately followed by an atomic sentence);

(DNF3) No disjunction occurs within the scope of any conjunction.

Example 29.1.1.

1. A
2. $(A \wedge (\neg B \wedge C))$
3. $(A \wedge B) \vee (A \wedge \neg B)$
4. $(A \wedge B) \vee (A \wedge (B \wedge (C \wedge (\neg D \wedge \neg E))))$
5. $A \vee (C \wedge (\neg P_{234} \wedge (P_{233} \wedge Q))) \vee \neg B$

Notation 29.1.1. We write '$\pm\mathscr{A}$' to indicate that $\mathscr{A}$ is an atomic sentence which may or may not be prefaced with an occurence of negation.

Remark 29.1.2. From the previous notation, a sentence in disjunctive normal form has the following shape:

$$(\pm\mathscr{A}_1 \wedge \ldots \wedge \pm\mathscr{A}_i) \vee (\pm\mathscr{A}_{i+1} \wedge \ldots \wedge \pm\mathscr{A}_j) \vee \ldots \vee (\pm\mathscr{A}_{m+1} \wedge \ldots \wedge \pm\mathscr{A}_n)$$

Theorem 1 (Disjunctive Normal Form Theorem).
For any sentence, there is a logically equivalent sentence in disjunctive normal form.

Truth Tables Proof. Pick any arbitrary sentence, $\mathscr{S}$, and let $\mathscr{A}_1, ..., \mathscr{A}_n$ be the atomic sentences that occur in $\mathscr{S}$. To obtain a sentence in DNF that is logically equivalent to $\mathscr{S}$, we consider $\mathscr{S}$'s truth table. There are two cases to consider:

1. $\mathscr{S}$ is false on every line of its truth table. Then, $\mathscr{S}$ is a contradiction. In that case, the contradiction $(\mathscr{A}_1 \wedge \neg\mathscr{A}_1)$ is in DNF and logically equivalent to $\mathscr{S}$.
2. $\mathscr{S}$ is true on at least one line of its truth table. For each line i of the truth table, let $\mathscr{B}_i$ be a conjunction of the form
$$(\pm\mathscr{A}_1 \wedge ... \wedge \pm\mathscr{A}_n)$$
where the following rules determine whether or not to include a negation in front of each atomic sentence:
   (a) $\mathscr{A}_m$ is a conjunct of $\mathscr{B}_i$ if and only if $\mathscr{A}_m$ is true on line i.
   (b) $\neg\mathscr{A}_m$ is a conjunct of $\mathscr{B}_i$ if and only if $\mathscr{A}_m$ is false on line i.

   Given these rules, $\mathscr{B}_i$ is true on and only on line i of the truth table which considers all possible valuations of $\mathscr{A}_1, ..., \mathscr{A}_n$ (i.e. $\mathscr{S}'$s truth table).

   Next, let $i_1, ..., i_m$ be the numbers of the lines of the truth table where $\mathscr{S}$ is true. Now let $\mathscr{D}$ be the sentence:
$$\mathscr{B}_{i_1} \vee \mathscr{B}_{i_2} \vee ... \vee \mathscr{B}_{i_m}$$
   Since $\mathscr{S}$ is true on at least one line of its truth table, $\mathscr{D}$ is indeed well-defined.

   By construction, $\mathscr{D}$ is in DNF. Moreover, by construction, for each line i of the truth table: $\mathscr{S}$ is true on line i of the truth table if and only if one of $\mathscr{D}$'s disjuncts (namely, $\mathscr{B}_i$) is true on, and only on, line i. Hence $\mathscr{S}$ and $\mathscr{D}$ have the same truth table, and so are logically equivalent.

∎

29.2.0 §Conjunctive Normal Form

Definition 29.2.1. *A sentence is in* **conjunctive normal form** *if and only if it meets all of the following requirements:*

(CNF1) No connectives occur in the sentence other than negations, conjunctions and disjunctions;

(CNF2) Every occurence of negation has minimal scope;

(CNF3) No conjunction occurs within the scope of any disjunction.

Remark 29.2.1. Generally, a sentence in CNF is of the shape

$$(\pm\mathscr{A}_1 \vee ... \vee \pm\mathscr{A}_i) \wedge (\pm\mathscr{A}_{i+1} \vee ... \vee \pm\mathscr{A}_j) \wedge ... \wedge (\pm\mathscr{A}_{m+1} \vee ... \vee \pm\mathscr{A}_n)$$

where each $\mathscr{A}_k$ is an atomic sentence.

Theorem 2 (Conjunctive Normal Form Theorem).

For any sentence, there is a logically equivalent sentence in conjunctive normal form.

Truth Table proof. Given a TFL sentence, $\mathcal{S}$, we first write down the complete truth table for $\mathcal{S}$. If $\mathcal{S}$ is true on every line of the truth table, then $\mathcal{S}$ and $(\mathcal{A}_1 \vee \mathcal{A}_1)$ are logically equivalent.

If $\mathcal{S}$ is false on at least one line of the truth table then, for every line on the truth table where $\mathcal{S}$ is false, write down a disjunction $(\pm\mathcal{A}_1 \vee ... \vee \pm\mathcal{A}_n)$ which is also false on (and only on) that line. Let $\mathcal{C}$ be the conjunction of all of these disjuncts; by construction, $\mathcal{C}$ is in CNF and $\mathcal{S}$ and $\mathcal{C}$ are logically equivalent. ■

Chapter 30

§§Functional Completeness

30.1.0 §Definitions and Main Theorem

Definition 30.1.1. *We say that some set of connectives are* **jointly functionally complete** *if and only if, for any possible truth table, there is a sentence containing only those connectives with that truth table.*

Theorem 3 (Functional Completeness Theorem).
The connectives of TFL are jointly functionally complete. Indeed, the following pairs of connective are jointly functionally complete:

1. *'$\neg$' and '$\vee$'*
2. *'$\neg$' and '$\wedge$'*
3. *'$\neg$' and '$\rightarrow$'*

Proof. Subsidiary Result 1: functional completeness of '$\neg$' and '$\vee$'. Observe that the scheme that we generate, using the truth table method of proving the DNF Theorem, will only contain the connectives '$\neg$', '$\wedge$', and '$\vee$'. So it suffices to show that there is an equivalent scheme which contains only '$\neg$' and '$\vee$'. To show this we simply consider that

$$(\mathscr{A} \wedge \mathscr{B}) \quad \text{and} \quad \neg(\neg\mathscr{A} \vee \neg\mathscr{B})$$

are logically equivalent.

Subsidiary Result 2: functional completeness of '$\neg$' and '$\wedge$'. Exactly as in Subsidiary Result 1, making use of the fact that

$$(\mathscr{A} \vee \mathscr{B}) \quad \text{and} \quad \neg(\neg\mathscr{A} \wedge \neg\mathscr{B})$$

are logically equivalent.

Subsidiary Result 3: functional completeness of '$\neg$' and '$\rightarrow$'. Exactly as in Subsidiary Result 1, making use of the equivalences:

$$(\mathscr{A} \vee \mathscr{B}) \quad \text{and} \quad (\neg\mathscr{A} \rightarrow \mathscr{B})$$

$$(\mathscr{A} \wedge \mathscr{B}) \quad \text{and} \quad \neg(\mathscr{A} \rightarrow \neg\mathscr{B})$$

■

30.2.0 §Individually Functionally Complete Connectives

Definition 30.2.1. *A connective is* **individually functionally complete** *if any truth table can be constructed using only it and valuations of atomic sentences.*

Definition 30.2.2. *The connective '↑' is truth functionally complete, with characteristic truth table:*

Table 30.1: '↑'

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \uparrow \mathscr{B}$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

This is often called 'the Sheffer stroke'. It is also commonly called 'nand' as its characteristic truth table is the negation of the truth table for '∧'.

Proof. It is sufficicent to show that '¬' and '∨' can be represented by '↑', which indeed they can as seen in the the following equivalences:

$$\neg\mathscr{A} \quad \text{and} \quad (\mathscr{A} \uparrow \mathscr{A})$$
$$(\mathscr{A} \vee \mathscr{B}) \quad \text{and} \quad ((\mathscr{A} \uparrow \mathscr{A}) \uparrow (\mathscr{B} \uparrow \mathscr{B}))$$

■

Definition 30.2.3. *The connective '↓' is truth functionally complete, with characteristic truth table:*

Table 30.2: '↓'

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \downarrow \mathscr{B}$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

This is often called 'Peirce arrow'. It is also commonly called 'nor' as its characteristic truth table is the negation of the truth table for '∨'.

Proof. It is sufficicent to show that '$\neg$' and '$\wedge$' can be represented by '$\downarrow$', which indeed they can as seen in the the following equivalences:

$$\begin{array}{rcl} \neg\mathscr{A} & \text{and} & (\mathscr{A} \downarrow \mathscr{A}) \\ (\mathscr{A} \wedge \mathscr{B}) & \text{and} & ((\mathscr{A} \downarrow \mathscr{A}) \downarrow (\mathscr{B} \downarrow \mathscr{B})) \end{array}$$

■

Theorem 30.2.1. *'$\vee$', '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' are not functionally complete by themselves. The* **only** *two-place connectives which are individually functionally complete are '$\uparrow$' and '$\downarrow$'.*

Chapter 31

§§Soundness

The aim of this chapter is to prove that the TFL proof system is sound

31.1.0 §Definitions and Proof

Definition 31.1.1. *Let Γ be a list of sentences. A formal proof system is* **sound** *(relative to a given semantics) if and only if, whenever there is a formal proof of $\mathscr{C}$ from assumptions among Γ, then Γ genuinely entails $\mathscr{C}$ (given that semantics).*

Theorem 4 (Soundness Theorem).

For any sentences Γ and $\mathscr{C}$: if $\Gamma \vdash \mathscr{C}$, then $\Gamma \vDash \mathscr{C}$.

Definition 31.1.2. *Say that a line of a proof is* **shiny** *if and only if the assumptions on which that line depends tautologically entail the sentence on that line. (this terminology is not necessarily standard)*

Lemma 31.1.1 (Shininess Lemma)**.** *Every line of every TFL-proof is shiny.*

Proof of Soundness Theorem. Suppose $\Gamma \vdash \mathscr{C}$. Then there is a TFL-proof, which $\mathscr{C}$ appearing on its last line, whose only undischarged assumptions are among Γ. The Shininess Lemma tells us that every line on every TFL-proof is shiny. So this last line is shiny, i.e. $\Gamma \vDash \mathscr{C}$. ■

Definition 31.1.3. *We say a rule of inference is* **rule-sound** *if and only if for all TFL-proofs, if we obtain a line on a TFL-proof by applying that rule, and every earlier line in the TFL-proof is shiny, then our new line is also shiny.*

Shininess Lemma. Fix any line, say line n, on any TFL-proof. The sentence written on line n must be obtained using a formal inference rule which is rule-sound. This is to say that, if every earlier line is shiny, then line n itself is shiny. Hence, by strong induction on the length of TFL-proofs every line of every TFL-proof is shiny. ■

Remark 31.1.1. We shall use 'δ_i' to abbreviate the assumptions on which line i depends in a TFL-proof.

Proof of Rule-Soundness.

Claim 31.1.2 (1). *Introducing an assumption is rule-sound.*

If $\mathscr{A}$ is introduced as an assumption on line n, then $\mathscr{A}$ is among Δ_n, and so $\delta_n \vDash \mathscr{A}$.

Claim 31.1.3 (2). $\wedge I$ *is rule-sound.*

Consider any application of $\wedge I$ in any TFL-proof. To show that $\wedge I$ is rule-sound, we assume that every line before line n is shiny; and we aim to show that line N is shiny, i.e. that $\Delta_n \vDash \mathscr{A} \wedge \mathscr{B}$.

So, let v be any valuation that makes all of Δ_n true. We first show that v makes $\mathscr{A}$ true. Note that all Δ_i are among Δ_n for $i < n$. By hypothesis, line i is shiny. So any valuation that makes all of Δ_i true makes $\mathscr{A}$ true. Since v makes all of Δ_i true, it makes $\mathscr{A}$ true too. Similarly, we see that v makes $\mathscr{B}$ true. Consequently, v makes $\mathscr{A} \wedge \mathscr{B}$ true. So any valuation that makes all of the sentences among Δ_n true also makes $\mathscr{A} \wedge \mathscr{B}$ true. That is: line n is shiny.

Claim 31.1.4 (3). $\wedge E$ *is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that $\wedge E$ is used on line n. Let v be any valuation that makes all of Δ_n true. Note that all of Δ_i are among Δ_n for $i < n$. By hypothesis, line i is shiny. So any valuation that makes all of Δ_i true makes $\mathscr{A} \wedge \mathscr{B}$ true. So v makes $\mathscr{A} \wedge \mathscr{B}$ true, and hence makes $\mathscr{A}$ and $\mathscr{B}$ true. So $\Delta_n \vDash \mathscr{A}$ (or $\mathscr{B}$)

Claim 31.1.5 (4). $\vee I$ *is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that $\vee I$ is used on line n. Let v be any valuation that makes all of Δ_n true. Since Δ_i are among Δ_n for $i < n$, line i is shiny by hypothesis. So any valuation that makes all Δ_i true makes either $\mathscr{A}$ or $\mathscr{B}$ true. So v makes either $\mathscr{A}$ or $\mathscr{B}$ true. So $\Delta_n \vDash \mathscr{A} \vee \mathscr{B}$.

Claim 31.1.6 (5). $\vee E$ *is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that $\vee E$ is used on line n.

m	$\mathscr{A} \vee \mathscr{B}$	
i	$\quad\mathscr{A}$	
j	$\quad\mathscr{C}$	
k	$\quad\mathscr{B}$	
l	$\quad\mathscr{C}$	
n	$\mathscr{C}$	$\vee$E m, i–j, k–l

i Let v be any valuation that makes all of Δ_n true. Note that all Δ_m are among Δ_n for $m < n$. By hypothesis, line m is shiny. So any valuation that makes Δ_n true makes $\mathscr{A} \vee \mathscr{B}$ true. So in particular, v makes $\mathscr{A} \vee \mathscr{B}$ true, and hence either v makes $\mathscr{A}$ true, or v makes $\mathscr{B}$ true.

Case 1: v makes $\mathscr{A}$ true. All of the Δ_i are among Δ_n, with the possible exception of $\mathscr{A}$. Since v makes all of Δ_n true, and also makes $\mathscr{A}$ true, v makes all of Δ_i true. Now, by assumption, line $j < n$ is shiny (in the subproof); so $\Delta_j \vDash \mathscr{C}$. But the sentences Δ_i are just the sentences Δ_j, so $\Delta_i \vDash \mathscr{C}$. So, any valuation that makes all of Δ_i true makes $\mathscr{C}$ true. But v is just such a valuation. So v makes $\mathscr{C}$ true.

Case 2: v makes $\mathscr{B}$ true. Reasoning in exactly the same way, considering lines k and l (in the subproof), v makes $\mathscr{C}$ true.

Either way, v makes $\mathscr{C}$ true. So $\Delta_n \vDash \mathscr{C}$.

Claim 31.1.7 (6). *$\neg E$ is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that $\neg E$ is used on line n.

i	$\mathscr{A}$	
j	$\neg\mathscr{A}$	
n	$\bot$	$\neg$E i, j

Note that all of Δ_i and Δ_j are among Δ_n. By hypothesis, lines i and j are shiny. So any valuation which makes all of Δ_n true would have to make both $\mathscr{A}$ and $\neg\mathscr{A}$ true. But no valuation can do that. So no valutaion makes all of Δ_n true. So $\Delta_n \vDash \bot$, vacuously.

Claim 31.1.8 (7). *X is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that X is used on line n.

i	$\bot$	
n	$\mathscr{A}$	X i

Note that Δ_i is among Δ_n. By hypothesis, line i is shiny. So any valuation which makes all of Δ_n true would have to make $\bot$ true. But no valuation can do that. So no valutaion makes all of Δ_n true. So $\Delta_n \vDash \mathscr{A}$, vacuously.

Claim 31.1.9 (8). *$\neg I$ is rule-sound.*

Assume that every line before line n on some TFL-proof is shiny, and that $\neg I$ is used on line n.

i	$\quad\mathscr{A}$	
j	$\quad\bot$	
n	$\neg\mathscr{A}$	$\neg$I i–j

Let v be any valutation that makes all of Δ_n true. Note that all of Δ_i are among Δ_n, with the possible exception of $\mathscr{A}$ itself. By hypothesis, line j is shiny. But no valutation can make '$\bot$' true, so no valutation can make all of Δ_j true. Since the sentences Δ_i are just the sentences Δ_j, no valuation can make all of Δ_i true. Since v makes all of Δ_n true, it must therefore make $\mathscr{A}$ false, and so make $\neg\mathscr{A}$ true. So $\Delta_n \vDash \neg\mathscr{A}$.

Claim 31.1.10 (9)**.** *IP is rule-sound.*

(To be completed)

Claim 31.1.11 (10)**.** $\rightarrow$ *I is rule-sound.*

(To be completed)

Claim 31.1.12 (11)**.** $\rightarrow$ *E is rule-sound.*

(To be completed)

Claim 31.1.13 (12)**.** $\leftrightarrow$ *I is rule-sound.*

(To be completed)

Claim 31.1.14 (13)**.** $\iff$ *E is rule-sound.*

(To be completed)

Claim 31.1.15 (14)**.** *All of the derived rules of our proof system are rule-sound.*

Suppose that we used a derived rule to obtain some sentence, $\mathscr{A}$, on line n of some TFL-proof, and that every earlier line is shiny. Every use of a derived rule can be replaced with multiple uses of basic rules. That is to say, we could have used basic rule to write $\mathscr{A}$ on some line $n+k$, without introducing any further assumptions. So, applying our individual results that all basic rules are rule-sound sever times ($k + 1$ times, in fact), we can see that line $n + k$ is shiny. Hence, the derived rule is rule-sound. ■

Appendices