Separation

Def: A top space X is T_0 iff whenever $p, q \in X$ $\exists$ an open set of at least one of these that does not contain the other

Obs: every subspace, and products of T_0 spaces are T_0.
(check)

A space is not T_0 then there are $p, q \in X$ $\ni$
a) $\overline{p} = \overline{q}$
$\Leftrightarrow$ b) $q \in \overline{p}$ and $p \in \overline{q}$.

Obs: T_0 and $T_3 \Rightarrow T_2$
pf: Let $p, q \in X$ so $\exists$ U_p nbhd of p, $q \notin U_p$.
Then $q \in U_p^c$, a closed set. Hence p, U_p^c are contained in disjoint open sets ∎

Obs: X is $T_3 \Leftrightarrow$ whenever $x \in U$ open then $\exists V$ open $\ni x \in V \subseteq \overline{V} \subseteq U$.
(try)

Def: X is <u>completely regular</u> if it's T_0 and whenever $p \in X$, closed $C \subseteq X$ then $f \in \mathcal{C}_{\mathbb{R}}(X)$, $f(X) \subseteq [0,1]$, $f(p) = 1$, $f|_C = 0$.

Note: X comp reg $\Rightarrow T_3$, so X is T_2.
NB: some sources do not include T_0 (i.e., T_2) for complet. reg.

ex: X a set, $|X| > 1$, X trivial top, then X is not T_0
$\{\emptyset, X\} = \tau$.
XXX
but, it is completely reg!

Properties: subspaces of completely reg space is comp reg (do)
⊗ products —— " ——————— (will go ⊗ through)

Thm X is completely reg $\Leftrightarrow$ X is embeddable in a compact T_2 space.

Def: X <u>embeds</u> in a top space Y if X is homeomorphic to a subspace of Y (with subspace top
i.e., $\exists f : X \to Y$, continuous, 1-1 and f is open map to $f(X)$

Def: X is loc. compact if given $p \in X$ then there is an open nbhd V of p with $\overline{V}$ compact.

Facts: X loc. comp $T_2 \Rightarrow X$ is normal and X is compl. reg.

The one pt compactification of X (for X l.c. T_2), denoted by X_∞, is $X \sqcup \{p_\infty\}$ as a set, with the topology τ given by

1) O open in X then O open in X_∞

2) the open nbhds of p_∞ are $X_\infty \setminus K$ where K compact in X.

There is a maximal compactification of X, X lc T_2, called the Stone-Cech compactification, βX.

$\beta \mathbb{N}$: worth becoming familiar with.

Obs: X is lc T_2, $C_0(X) = \{f : X_\infty \to \mathbb{K} \mid f \text{ cont}, f(p_\infty) = 0\}$

$$\overset{\text{def}}{\Longleftrightarrow} (=) \{f : X \to \mathbb{K} \mid \forall \epsilon > 0 \ \exists K \subseteq X \text{ compact with } |f(x)| < \epsilon \ (x \in X \setminus K)\}.$$

$$C_b(X) = \{f : X \to \mathbb{K} \mid f \text{ continuous}, \|f\|_\infty < \infty\} \overset{\text{Thm}}{=} C(\beta X).$$

Recall: $\prod X_\alpha$ has the weak top using $\pi_\beta : \prod X_\alpha \to X_\beta$

- If X has the weak top induced by $g_\alpha : X \to Y_\alpha$, then a map
$$\begin{tikzcd} Y \arrow[d, "h"'] & \\ X \arrow[r, "g_\alpha"'] & Y_\alpha \end{tikzcd}$$
is cont $\Longleftrightarrow g_\alpha \circ h$ are cont.

- $A \subseteq X$ has the subspace top equiv to A has the weak top from $\iota : A \to X$. $(\iota^{-1}(O) = O \cap A)$.

Therefore a map $f : X \to Y$ is cont

$$\Longleftrightarrow \begin{tikzcd} X \arrow[d] & \\ f(X) \arrow[r, hook, "\iota"'] & Y \end{tikzcd} \qquad f : X \to f(X) \text{ is cont (where } f(X) \text{ has subspace top)}$$

Thm Let $\{f_\alpha : X \to X_\alpha\}$ a collection of functions.
The map $e : X \to \prod X_\alpha$ given by $x \mapsto (f_\alpha(x))$ is an embedding $\iff$ X has the weak top induced by $\{f_\alpha\}$ and the collection f_α separates pts.

Def: i.e.; if $x \neq y \in X$ then $\exists f_\alpha \ni f_\alpha(x) \neq f_\alpha(y)$

- e is 1-1 $\iff$ f_α sep. pts (fine)

Aside: $\tilde{e} : X \to e(X)$
$e : X \to \prod X_\alpha$
$\iota \circ \tilde{e} = e$

$(\Rightarrow)$ - X has the same topology as $e(X)$
- Note: X has the weak top induced by
$\pi_\alpha \circ \iota \circ \tilde{e} = \pi_\alpha \circ e = f_\alpha$

since $e(X)$ has weak top induced by $\pi_\alpha \circ \iota$

$(\Leftarrow)$ If X has the $\{f_\alpha\}$ weak top, then to show $\tilde{e}$ is cont and open.

- $\tilde{e}$ is cont $\iff$ $\iota \circ \tilde{e} = e$ cont (as $e(X) \overset{\iota}{\hookrightarrow} \prod X_\alpha$ has weak top using ι)

$\iff$ $\underbrace{\pi_\alpha \circ e}_{= f_\alpha}$ cont $\forall \alpha$

- $\tilde{e}$ is open: sts (since $\tilde{e}$ is 1-1, onto) that a subbasic open set $f_\alpha^{-1}(O_\alpha)$ is mapped to an open set.
O_α open in X_α

$\tilde{e}\left(f_\alpha^{-1}(O_\alpha)\right) = \tilde{e}\left(\pi_\alpha \circ \iota \circ \tilde{e}\right)^{-1}(O_\alpha) = \tilde{e}\left(\tilde{e}^{-1}\left(\iota^{-1}\left(\pi_\alpha^{-1}(O_\alpha)\right)\right)\right)$

$= \iota^{-1}\left(\pi_\alpha^{-1}(O_\alpha)\right)$, which is open! ∎