# DIFFERENTIAL GEOMETRY: A COMPLETE GUIDE

SUBJECT

APRIL 30, 2021

ELIJAH THOMPSON,
PHYSICS AND MATH HONORS

*Solo Pursuit of Learning*

# Contents

# Part I

# Euclidean Geometry

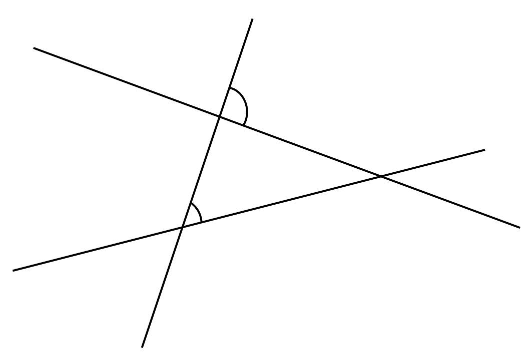
# Chapter 1

# Postulates

## 1.1 Euclid's Postulates

### Theorem: Euclid's Postulates

1. *To draw a straight line, from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a* **circle** *with any center and distance.*
4. *That all right angles are* **equal** *to another.*
5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles [in sum], then the two straight lines if produced indefinitely meet in that side on which the angles less than right angles.*



## Theorem 1.1: Playfair's Postulate

*This postulate is equivalent to Euclid's fifth postulate: Given straight line m and point P not on m, there exists a unique line n that contains P and is parallel to m.*

### 1.1.1 Hyperbolic Geometry Introduction

#### Definition 1.1.1: Hyperbolic Plane

*We define the hyperbolic plane to be the set*

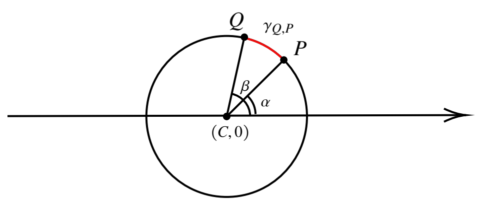
$$\mathcal{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\} \tag{1.1.1}$$

*The hyperbolic metric is defined as*

$$d_{\mathcal{H}}(\gamma) = \int_{\gamma} \frac{\sqrt{dx^2 + dy^2}}{y} \tag{1.1.2}$$

*where $\gamma$ is a curve.*

#### Proposition 1.1.1




*For $\alpha < \beta$ we have that*

$$d_{\mathcal{H}}(\gamma_{Q,P}) = \ln\left[\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right] \tag{1.1.3}$$

*Moreover, if we had a line segment from $(a, y_1)$ to $(a, y_2)$, the hyperbolic length would be*

$$d_{\mathcal{H}}(l) = \int_l \frac{1}{y}\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt = \int_{y_1}^{y_2} \frac{1}{t}dt = \ln\left(\frac{y_2}{y_1}\right) \tag{1.1.4}$$

#### Proposition 1.1.2

*Euclidean angles are the same as hyperbolic angles.*

*What is a line?*

↳ A line is a ***geodesic***, which is the shortest path with respect to the metric of the geometry (the distance function).

## Theorem 1.1.3

*In the hyperbolic plane the geodesics are either vertical lines (rays from the Euclidean respective), or semi-circles, terminating asymptotically to the x-axis.*

**Remark 1.1.1**

*Given a circle with center $(h,k)$ above the x-axis in the Euclidean plane, the image in the hyperbolic plane is a circle centered at $(H,K)$ with $H = \sqrt{k^2 - r^2}$, $R = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right)$, and $K = k$.*

**Remark 1.1.2**

*The hyperbolic half-plane satisfies Euclid's first four axioms, but the fifth (e.i Playfair's Postulate) is not satisfied in the hyperbolic half-plane.*

# Chapter 2

# Tangent and Normal Spaces

## 2.1 Notation

**Remark 2.1.1**

*In $\mathbb{R}^n$ we shall write $(p+q)^i = p^i + q^i$ (component wise addition) with i as an index and $(cp)^i = cp^i$. For $p \in \mathbb{R}^n$ we have*

$$\begin{aligned} p &= (p^1, p^2, ..., p^n) \\ &= p^1(1,0,...,0) + p^2(0,1,...,0) + ... + p^n(0,...,0,1) \\ &= \sum_{i=1}^{n} p^i e_i \end{aligned}$$

*We define the* **Kroneker Delta** *to be*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.1.1}$$

**Remark 2.1.2**

*There exists a correspondence between points and vectors based at the origin* $(0,0,...,0)$.

## 2.2 Definitions and Examples

### Definition 2.2.1

*The* **tangent space** *to* $\mathbb{R}^n$ *at a point* $p \in \mathbb{R}^n$ *is defined as*

$$T_p\mathbb{R}^n := \{p\} \times \mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\} \tag{2.2.1}$$

*Then, the* **tangent bundle** *is defined as*

$$T\mathbb{R}^n := \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n \tag{2.2.2}$$

**Remark 2.2.1**

*We give* $T_p\mathbb{R}^n$ *a vector space structure by defining an addition*

$$(p, v) + (p, w) := (p, v + w) \tag{2.2.3}$$

*for all* $v, w \in \mathbb{R}^n$*, and we define scalar multiplication by*

$$c(p, v) = (p, cv) \tag{2.2.4}$$

*for all* $c \in \mathbb{R}$*. We can also define a standard inner product on* $T_p\mathbb{R}^n$ *by*

$$(p, v) \cdot (p, w) = v \cdot w = v^T w \tag{2.2.5}$$

*as well as a norm*

$$||(p, v)|| = ||v|| \tag{2.2.6}$$

*We say that* $(p, v), (p, w) \in T_P\mathbb{R}^n$ *are orthogonal if*

$$(p, v) \cdot (p, w) = 0 = v \cdot w \tag{2.2.7}$$

*Given a subspace* $S \subset T_p\mathbb{R}^n$*, we have the orthogonal complement of S*

$$S^\perp := \{(p, w) \in T_p\mathbb{R}^n : (p, w) \cdot (p, v) = 0, \forall (p, v) \in S\} \tag{2.2.8}$$

### Definition 2.2.2

*For a curve C, we can define the tangent space at a point p on C by*

$$T_pC := \text{span}\{(p, v)\} \subset T_p\mathbb{R}^n \tag{2.2.9}$$

*where* $(p, v)$ *is tangent to C at p. Then, we have that*

$$(T_pC)^\perp = \textit{normal space to C at p} \tag{2.2.10}$$

*For* $n = 2$ *we get the normal line, for* $n = 3$ *we get the normal plane, etc.*

**Example 1**

*$\alpha(t) = (t, t^2)$, $p = \alpha(1) = (1, 1)$. Then $\alpha'(1) = \langle 1, 2 \rangle$, so $v = (p, \langle 1, 2 \rangle)$. This parametrizes a parabola in $\mathbb{R}^2$.*

# Part II

# Manifold Theory

# Chapter 3

# Manifold Definitions and Types

# Chapter 4

# Smooth Maps

# Chapter 5

# Immersions, Submersions, and Submanifolds

# Chapter 6

# Tangent Bundles

# Chapter 7

# Differential Forms

# Chapter 8

# Integration on Manifolds

# Chapter 9

# Stoke’s Theorem