Differential Geometry: A Complete Guide

Subject

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Solo Pursuit of Learning

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Part I

Euclidean Geometry

Chapter 1

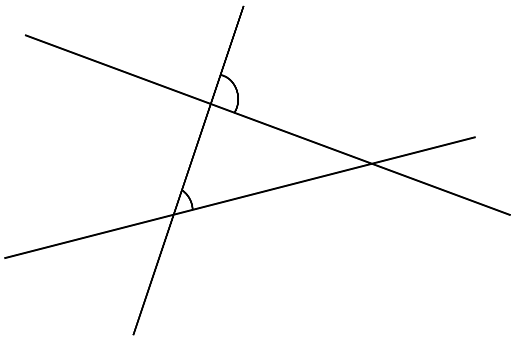
Postulates

1.1.0 Euclid's Postulates

Theorem 1.

Euclid's Postulates

1. *To draw a straight line, from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a* **circle** *with any center and distance.*
4. *That all right angles are* **equal** *to another.*
5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles [in sum], then the two straight lines if produced indefinitely meet in that side on which the angles less than right angles.*



Theorem 2 (Playfair's Postulate).

This postulate is equivalent to Euclid's fifth postulate: Given straight line m *and point* P *not on* m*, there exists a unique line* n *that contains* P *and is parallel to* m*.*

Hyperbolic Geometry Introduction

Definition 1.1.1 (Hyperbolic Plane). *We define the hyperbolic plane to be the set*

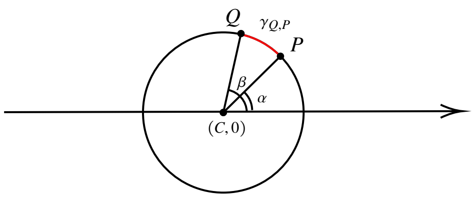
$$\mathcal{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\} \tag{1.1.1}$$

The hyperbolic metric is defined as

$$d_{\mathcal{H}}(\gamma) = \int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} \tag{1.1.2}$$

where γ is a curve.

Proposition 1.1.1.



For $\alpha < \beta$ we have that

$$d_{\mathcal{H}}(\gamma_{Q,P}) = \ln\left[\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right] \tag{1.1.3}$$

Moreover, if we had a line segment from (a, y_1) to (a, y_2), the hyperbolic length would be

$$d_{\mathcal{H}}(l) = \int_l \frac{1}{y}\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt = \int_{y_1}^{y_2} \frac{1}{t}dt = \ln\left(\frac{y_2}{y_1}\right) \tag{1.1.4}$$

Proposition 1.1.2. *Euclidean angles are the same as hyperbolic angles.*

What is a line?

↳ A line is a ***geodesic***, which is the shortest path with respect to the metric of the geometry (the distance function).

Theorem 1.1.3. *In the hyperbolic plane the geodesics are either vertical lines (rays from the Euclidean respective), or semi-circles, terminating asymptotically to the x-axis.*

Remark 1.1.1. Given a circle with center (h, k) above the x-axis in the Euclidean plane, the image in the hyperbolic plane is a circle centered at (H, K) with $H = \sqrt{k^2 - r^2}$, $R = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right)$, and $K = k$.

Remark 1.1.2. The hyperbolic half-plane satisfies Euclid's first four axioms, but the fifth (e.i Playfair's Postulate) is not satisfied in the hyperbolic half-plane.

Chapter 2

Tangent and Normal Spaces

2.1.0 Notation

Remark 2.1.1. In $\mathbb{R}^n$ we shall write $(p+q)^i = p^i + q^i$ (component wise addition) with i as an index and $(cp)^i = cp^i$. For $p \in \mathbb{R}^n$ we have

$$\begin{aligned} p &= (p^1, p^2, ..., p^n) \\ &= p^1(1,0,...,0) + p^2(0,1,...,0) + ... + p^n(0,...,0,1) \\ &= \sum_{i=1}^{n} p^i e_i \end{aligned}$$

We define the ***Kroneker Delta*** to be

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.1.1}$$

Remark 2.1.2. There exists a correspondence between points and vectors based at the origin $(0,0,...,0)$.

2.2.0 Definitions and Examples

Definition 2.2.1. *The* **tangent space** *to* $\mathbb{R}^n$ *at a point* $p \in \mathbb{R}^n$ *is defined as*

$$T_p\mathbb{R}^n := \{p\} \times \mathbb{R}^n = \{(p,v) : v \in \mathbb{R}^n\} \tag{2.2.1}$$

Then, the **tangent bundle** *is defined as*

$$T\mathbb{R}^n := \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n \tag{2.2.2}$$

Remark 2.2.1. We give $T_p\mathbb{R}^n$ a vector space structure by defining an addition

$$(p,v) + (p,w) := (p, v+w) \tag{2.2.3}$$

for all $v, w \in \mathbb{R}^n$, and we define scalar multiplication by

$$c(p, v) = (p, cv) \tag{2.2.4}$$

for all $c \in \mathbb{R}$. We can also define a standard inner product on $T_p\mathbb{R}^n$ by

$$(p, v) \cdot (p, w) = v \cdot w = v^T w \tag{2.2.5}$$

as well as a norm

$$||(p, v)|| = ||v|| \tag{2.2.6}$$

We say that $(p, v), (p, w) \in T_P\mathbb{R}^n$ are orthogonal if

$$(p, v) \cdot (p, w) = 0 = v \cdot w \tag{2.2.7}$$

Given a subspace $S \subset T_p\mathbb{R}^n$, we have the orthogonal complement of S

$$S^\perp := \{(p, w) \in T_p\mathbb{R}^n : (p, w) \cdot (p, v) = 0, \forall (p, v) \in S\} \tag{2.2.8}$$

Definition 2.2.2. *For a curve C, we can define the tangent space at a point p on C by*

$$T_pC := \operatorname{span}\{(p, v)\} \subset T_p\mathbb{R}^n \tag{2.2.9}$$

where (p, v) is tangent to C at p. Then, we have that

$$(T_pC)^\perp = \textit{normal space to } C \textit{ at } p \tag{2.2.10}$$

For $n = 2$ we get the normal line, for $n = 3$ we get the normal plane, etc.

Example 2.2.1. $\alpha(t) = (t, t^2)$, $p = \alpha(1) = (1, 1)$. Then $\alpha'(1) = \langle 1, 2\rangle$, so $v = (p, \langle 1, 2\rangle)$. This parametrizes a parabola in $\mathbb{R}^2$.

Part II

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