Aug 6, Top continued
compactly embeddable
compactifications

continued

Thm: $X$ completely reg $\Longleftrightarrow$ $X$ is embeddable in a compact $T_2$ space.

($\Leftarrow$) compact $T_2$ implies normal, so (Urysohn) it is compl. reg, and subspace of compl. reg is completely reg.

($\Rightarrow$) Consider a collection $\mathcal{C}$ of cont functions $X \to [0,1]$ $\ni$ $\forall p \in X$, $C \subseteq X$ closed, $p \in X \setminus C$ $\exists f \in \mathcal{C}$ $\ni$ $f(p) = 1$ and $f|_C = 0$.

Claim: $\mathcal{C}$ separates pts, and $X$ has the weak top induced by $\mathcal{C}$. Then apply last thm.

pf: need to show $\{f^{-1}(V) \mid V \subseteq [0,1] \text{ open}, f \in \mathcal{C}\}$ is a base for the top on $X$.
Let $x \in \mathcal{U}$ open, $\mathcal{U} \subseteq X$. To show $\exists f \in \mathcal{C}$, $V \subseteq [0,1]$ open with $x \in f^{-1}(V) \subseteq \mathcal{U}$.
Since $x \notin \mathcal{U}^c$ there is $f \in \mathcal{C}$ $\ni$ $f(x) = 1$ and $f|_{\mathcal{U}^c} = 0$
Then $f(x) \notin \overline{f(\mathcal{U}^c)} = \{0\}$, $\mathcal{U}^c \subseteq f^{-1}(\overline{f(\mathcal{U}^c)})$, so
$\mathcal{U} \supseteq f^{-1}(\overline{f(\mathcal{U}^c)}^c)$. Let $V = \overline{f(\mathcal{U}^c)}^c = (0,1]$. ∎

Note: for ($\Rightarrow$) we only used that 1) $\mathcal{C}$ sep. pts
2) whenever $C$ is closed in $X$ and $x \notin C$ then $\exists f \in \mathcal{C}$ with $f(x) \notin \overline{f(C)}$.

If this occurs then we've shown that $X$ has the weak top induced by $\mathcal{C}$, and $e : X \to \prod_{f \in \mathcal{C}} X_f$ is an embedding

$$x \to (f(x))_{f \in \mathcal{C}}$$

For $X$ compl. reg we could have used $\mathcal{C} = C_b(X, \mathbb{R})$
(or $\mathcal{C} = C_b(X, \mathbb{C})$, also, etc) (Will use for Stone-Cech comp!)

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break: Locally compact spaces.

Def: $X$ is loc. comp $\Longleftrightarrow$ every pt $x$ has a nbhd $O_x$ with $\overline{O_x}$ compact.

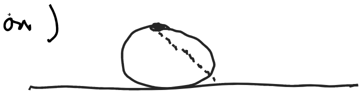
ex: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, ex: $X$ compact $\Rightarrow X$ l.c.

<u>Obs</u>: $X$ lc $\Rightarrow X$ has a compactification, so is completely reg.

A compactification of $X$ is the <u>1-point compactification</u> $X_\infty$ (Alexandrov compactification)

ex: $\mathbb{R} \hookrightarrow \mathbb{T}$

$\mathbb{C} \hookrightarrow S^2$ (2-sphere in $\mathbb{R}^3$)




<u>Def</u>: $X_\infty$ as a set $X \sqcup \{p\}$ "$p = \infty$"

open sets $:= \{ O \subseteq X \mid O \text{ open} \} \cup \{ \{p\} \cup X \setminus K \mid K \subseteq X \text{ compact} \}$

this is a top. (check)

Check: • $X_\infty$ is compact
- $X$ is dense in $X_\infty$
- $X$ is open in $X_\infty$ (so $X$ has the subspace top) so $X \overset{i}{\hookrightarrow} X_\infty$ is an embedding.
- $X$ is l.c. $T_2 \Leftrightarrow X_\infty$ is $T_2$. ($\Rightarrow$) check

($\Leftarrow$) In general if $U$ open in a l.c. space then $U$ is lc in the subspace top.

$\boxed{\text{check}} \longrightarrow$

ex: $[0,1)_\infty = [0,1]$ i.e., $[0,1) \hookrightarrow [0,1]$

<u>Obs</u>: If $X$ is l.c. $C_0(X,\mathbb{R})$, $C_0(X,\mathbb{C}) = \{ f: X \to \mathbb{F} \mid f \text{ vanishes at } \infty \}$

$= \{ f \in C(X_\infty, \mathbb{F}) \mid f(\infty) = 0 \}$ an ideal in $C(X_\infty, \mathbb{F})$ (closed 2-sided * ideal in $\| \|_\infty$-norm)

<u>Note</u>: If $X$ is [compact] top space then $M_x = \{ f: X \to \mathbb{F} \mid f \text{ cont}, f(x) = 0 \} \trianglelefteq C(X, \mathbb{F})$.
$x \in X$

Also $\{ f: X \to \mathbb{F} \mid f \text{ bnd, cont}, f(x) = 0 \} \trianglelefteq C_b(X, \mathbb{F})$

$\to C_0(X) \ni f \longrightarrow \tilde{f} \in C(X_\infty)$

with $\tilde{f}: X_\infty \to \mathbb{F}$ defined by $\begin{cases} \tilde{f}|_X = f \\ \tilde{f}(\infty) = 0 \end{cases}$

$\boxed{\text{check}}$: $\tilde{f}$ <u>is cont</u>

Def: A <u>compactification</u> of $X$ is a pair $(K, h)$

$K$ compact, $h: X \to K$ embedding (cont, 1-1, $h^{-1}: \underset{\text{subspace}}{h(X)} \to X$ cont)

and $h(X)$ dense in $K$

<u>Obs</u>: Last Thm showed $\Leftrightarrow$ $X$ embedds in a compact $T_2$

ex: If $X$ is compact and $\{x_n\}$ a countably infinite dense set $h: \mathbb{N} \to \{x_n\} \subseteq X$ is a compactification of $\mathbb{N}$

$\Leftrightarrow$ $h$ embeding

$\Leftrightarrow$ $\{x_n\}$ is discrete in subspace top.

So $[0,1]$ is not a compactification of $\mathbb{N}$, but $[0,1]$ is a compactification of $\mathbb{Q} \cap [0,1]$

ex $X$ compl. reg: then $e: X \to \prod_{f \in C_b(X)} I_f$, $I_f$ = closed bnd interval $\supseteq f(X)$

$$x \to (f(x))_{f \in C_b(X, \mathbb{R})}$$

Def: $\overline{e(X)} = \beta X$, the Stone-Čech compactification.

<u>Note</u>: If $X$ is compact then $\beta X = \overline{e(X)} \underset{\text{use } T_2}{=} e(X) = X$

contrast with $X_\infty = X \sqcup \underset{\text{open}}{\{p\}}$

Thm: If $X$ is comp. reg, $K$ compact then
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & K \\ \beta(X) \arrow[ur, dotted, "(\exists!)\, F"'] \end{tikzcd}$$
$f$ ← cont

(like free object) so huge

$F$ extends $f$ ; $F|_X = f$ ; $F$ is unique ← $\boxed{\text{check}}$

Obs: $C_b(X, \mathbb{C}) \xrightarrow{\cong} C(\beta X, \mathbb{C})$
$f \longrightarrow F$

is norm preserving ($\| \ \|_\infty$), $\mathbb{C}$-linear, $*$-homo, 1-1, onto

$\boxed{\text{check}}$ → ( use $X$ dense in $\beta X$ ; so, for example
$fg \to FG$ since $(FG)|_X = F|_X \cdot G|_X = f \cdot g$ )

Def of max ideal, prime ideal in ring with unit
(2-sided)