Complex Analysis: A Complete Guide

Complex Analysis

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Solo Pursuit of Learning

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Part I

Part 1

Chapter 1

The Complex Plain and Basic Functions

1.1.0 Complex Numbers

The complex numbers, $\mathbb{C}$, consist of all formul sums $z = x + iy$, for $x, y \in \mathbb{R}$, where $i^2 = -1$ is the root of $x^2 + 1 = 0$. Then, for multiplication we proceed by $z \cdot w = (a + ib)(x + iy) = (ax - by) + i(xb + ay)$.

Gauss concieved of $\mathbb{C}$ as $\mathbb{R}^2$ with a binary operation $*$, where $(a, b) * (x, y) = (ax - by, xb + ay)$. Then, we observe that $(1, 0) * (a, b) = (a, b)$, so $(1, 0)$ acts as 1. Moreover, $(0, 1) * (0, 1) = (-1, 0) = -(1, 0)$.

The matrix model of $\mathbb{C}$ is

$$\mathbb{C} = \left\{ \begin{bmatrix} a & -b//b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

In terms of extension fields, we can consider $\mathbb{C}$ to be $\mathbb{R}[x]/(x^2 + 1)$.

Definition 1.1.1. *If $z = x + iy$, with $x, y \in \mathbb{R}$, then we define $\mathscr{R}e(z) = x$ and $\mathscr{I}m(z) = y$.*

Definition 1.1.2. *If $z = x + iy$, we define the* **conjugate** *of z to be $\overline{z}x - iy$.*

Properties 1.1.3. *Let $z, w \in \mathbb{C}$.*

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\overline{\overline{z}} = z$
- $\overline{z} = 0$ *if and only if* $z = 0$
- $\mathscr{R}e(z) = \frac{z+\overline{z}}{2}$
- $\mathscr{I}m(z) = \frac{z-\overline{z}}{2i} = \frac{i(\overline{z}-z)}{2}$

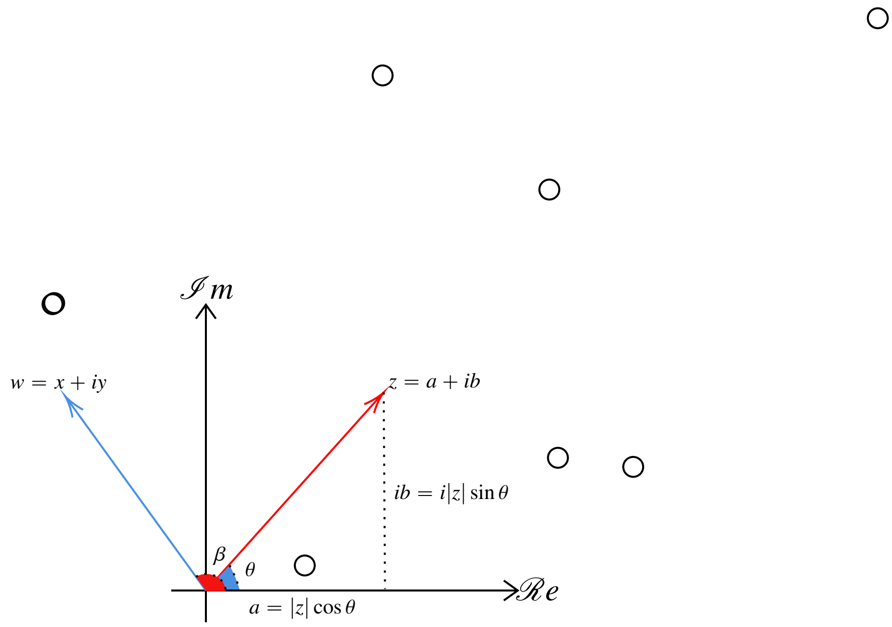
Proposition 1.1.1. *If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{z\bar{z}}$.*

Definition 1.1.4. *Let $z \in \mathbb{C}$. Then the* **modulus**, $|\cdot|$ *of $z = a + ib$ (the norm), is the length of z as a vector:*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$$

Geometry of the Complex Numbers

The complex numbers, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$, can be considered as a plane of points, or we can consider the complex numbers as vectors in the plane eminating from 0.



Noting this geometric picture, we can write $z = |z|\cos\theta + i|z|\sin\theta = |z|(\cos\theta + i\sin\theta)$. Suppose we had another complex number $w = |w|(\cos\beta + i\sin\beta)$. Then we observe that

$$\begin{aligned} zw &= |z||w|(\cos\theta + i\sin\theta)(\cos\beta + i\sin\beta) \\ &= |zw|(\cos\theta\cos\beta - \sin\theta\sin\beta + i(\sin\theta\cos\beta + \cos\theta\sin\beta)) \\ &= |zw|(\cos(\theta + \beta) + i\sin(\theta + \beta)) \end{aligned}$$

so complex multiplication aligns with angle addition in the plane.

Definition 1.1.5. *Define* **Euler's Formula**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From our above work we have that

$$zw = (|z|e^{i\theta})(|w|e^{i\beta}) = |zw|e^{i(\theta+\beta)}$$

so

$$e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}$$

The conjugate of $e^{i\theta}$ is

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$

Proposition 1.1.2. *$e^{i\theta} = e^{i\beta}$ if and only if $\theta = \beta + 2\pi k$ for some $k \in \mathbb{Z}$.*

Then we have that $e^{i\theta} = e^{-i\theta}$ if and only if $\theta = \pi k$ for some $k \in \mathbb{Z}$.

Definition 1.1.6. *Let $z \in \mathbb{C}$. The* **argument** *of $z = |z|e^{i\theta}$ is $\arg(z) = \{\theta + 2\pi k : k \in \mathbb{Z}\}$, and the* **principal argument** *of z, $Arg(z) = \theta_0$, where $z = |z|e^{i\theta_0}$ and $\theta_0 \in (-\pi, \pi]$.*

Example 1.1.1. Consider $z = -42 - 42i$. Then $|z| = 42\sqrt{2}$, and $\mathrm{Arg}(z) = -\frac{3\pi}{4}$, so $z = 42\sqrt{2}e^{-i\frac{3\pi}{4}}$, and $\arg(z) = -\frac{3\pi}{4} + 2\pi\mathbb{Z}$.

Properties 1.1.7. *For $z, w \in \mathbb{C}$, $\arg(zw) = \arg(z) + \arg(w)$, but $Arg(zw) \neq Arg(z) + Arg(w)$.*

Theorem 1.1.3 (DeMoivre's Theorem)**.** *For all $n \in \mathbb{Z}$,*

$$(e^{i\theta})^n = e^{in\theta}$$

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$ then $(e^{i\theta})^1 = e^{i1\theta}$, so the base case holds. Now, suppose that the claim holds for some $n \geqslant 1$. It follows that

$$\begin{aligned}
(e^{i\theta})^{n+1} &= (e^{i\theta})^1(e^{i\theta})^n \\
&= e^{i\theta}e^{in\theta} && \text{(by I.H)} \\
&= e^{i(\theta+n\theta)} \\
&= e^{i(n+1)\theta}
\end{aligned}$$

completing the proof. ∎

Definition 1.1.8. *Suppose that $n \in \mathbb{N}$, $w, z \in \mathbb{C}$ such that $z^n = w$, then z is said to be an* **nth root of** *w. Moreover, the set of all nth roots is dentoed $w^{1/n} \neq \sqrt[n]{w}$.*

Let $z^n = w$, where $z = \rho e^{i\theta}$. Then it follows that

$$\begin{aligned}
(\rho e^{i\theta})^n &= w \\
\rho^n e^{in\theta} &= |w|e^{i\mathrm{Arg}(w)} && \text{(by 1.1.3)}
\end{aligned}$$

This gives the two equations $\rho^n = |w|$ and $e^{in\theta} = e^{i\mathrm{Arg}(w)}$, so $\rho = \sqrt[n]{|w|}$, and

$$\begin{aligned}
n\theta &= \mathrm{Arg}(w) + 2\pi k \\
\theta &= \frac{\mathrm{Arg}(w)}{n} + \frac{2\pi k}{n}
\end{aligned}$$

This gives the following result:

Corollary 1.1.4.
$$w^{1/n} = \left\{ \sqrt[n]{|w|} e^{i\frac{\mathrm{Arg}(w)}{n}} e^{i\frac{2\pi k}{n}} : k \in \mathbb{Z} \right\}$$

Definition 1.1.9. *If $w = 1$, we have that*

$$1^{1/n} = \left\{ e^{i\frac{2\pi k}{n}} : k \in \mathbb{Z} \right\}$$

These are the <u>n**th roots of unity**</u>.

Then we observe that for any $w \in \mathbb{C}$,

$$w^{1/n} = \sqrt[n]{|w|} e^{i\frac{\mathrm{Arg}(w)}{n}} 1^{1/n}$$

Example 1.1.2. Consider the fourth roots of $81i$, so $(81i)^{1/4}$. Then we have that

$$\begin{aligned} (81i)^{1/4} &= \sqrt[4]{81} e^{i\frac{\pi}{8}} 1^{1/4} \\ &= 3e^{i\frac{\pi}{8}} \{1, i, -1, -i\} \end{aligned}$$

Example 1.1.3. Let $w = \exp\left(\frac{2\pi i}{6}\right)$. Then

$$1^{1/6} = \{z \in \mathbb{C} : z^6 = 1\} = \{w, w^2, w^3, w^4, w^5, w^6 = 1\}$$

Note $w = e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1+i\sqrt{3}}{2}$. Now, if we consider the polynomial $f(z) = z^6 - 1$, we now know six roots for this polynomial. Then we can factor

$$f(z) = \prod_{i=1}^{6} (z - w^i)$$

In short, to solve $z^n = \rho$, we take the nth roots of ρ, $\rho^{1/n}$.

Example 1.1.4. Consider $z^2 + bz + c = 0$. Then completing the square we obtain $z \in \left\{ \frac{-b+(b^2-4c)^{1/2}}{2} \right\}$, where

$$\left(\frac{b^2-4c}{2}\right)^{1/2} = \left\{ \begin{array}{ccl} \sqrt{\left|\frac{b^2-4c}{2}\right|} & -\sqrt{\left|\frac{b^2-4c}{2}\right|} & if\ b^2-4c \geqslant 0 \\ \sqrt{\left|\frac{b^2-4c}{2}\right|}i & -\sqrt{\left|\frac{b^2-4c}{2}\right|}i & if\ b^2-4c < 0 \end{array} \right\}$$

1.2.0 Local Inverses and Branch-cut

Definition 1.2.1. *Let $z, w \in \mathbb{C}$, then the line segment from z to w is*

$$[z, w] = \{z + t(w - z) : 0 \leqslant t \leqslant 1\}$$

where we also allow z or w to be plus or minus infinity.

Definition 1.2.2. *The* **negative slit plane**, $\mathbb{C}^-$, *is defined by* $\mathbb{C}^- = \mathbb{C}\backslash(-\infty, 0]$, *and the* **positive slit plane**, $\mathbb{C}^+$, *is defined by* $\mathbb{C}^+ = \mathbb{C}\backslash[0, \infty)$. *In general, we define*

$$C^\alpha = \mathbb{C}\backslash[0, e^{i\alpha}\infty)$$

to denote the exclusion of the ray along the α*th angle from the positive real axis.*

Question 1.2.1. What is a function?

Definition 1.2.3. *If* $f : S \to \mathbb{C}$ *is a function, and* $U \subseteq S$, *then* $f|_U : U \to \mathbb{C}$ *is defined by* $f|_U(z) = f(z)$ *for all* $z \in U$.

We may have the case that f is not injective (so it cannot be inverted). But, for a smart choice of U, we may have that $f|_U$ is one-to-one, and hence invertible. Such a restriction is known as a ***local inverse*** for f.

Rigourously, a ***branch cut*** is a curve in the complex plane such that it is possible to define a single analytic branch (sheets of a multivalued function) of a multivalued function on the plane minus that curve. That is, a branch is a way of making the multivalued function single valued, and in the context of determining inverses a branch is a choice of inverse.

Example 1.2.1. For $f(z) = z^n$, then for $U = \left\{z \in \mathbb{C} : -\frac{\pi}{n} < \text{Arg}(z) < \frac{\pi}{n}\right\}$, $f|_U$ is invertible, and $f|_U^{-1}$ is called the ***principal branch***. $f|_U^{-1}$ is a choice of the nth root of $w \in \mathbb{C}^-$.

Definition 1.2.4. *The* **α-argument** *for* $\alpha \in \mathbb{R}$ *is denoted* $Arg_\alpha : \mathbb{C}^\times \to (\alpha, \alpha+2\pi)$. *In particular, for each* $z \in \mathbb{C}^\times$ *we define* $Arg_\alpha \in \arg(z)$ *such that* $z \in (\alpha, \alpha + 2\pi)$

We can give branch cuts for the nth root function which delete the ray at standard angle α. These correspond to local inverse functions $f(z) = z^n$ restricted to $\{z \in \mathbb{C}^\times : \arg(z) = (\alpha/n, (\alpha + 2\pi)/2) + 2\pi\mathbb{Z}\}$.

Square-Root Function

If we have $z^2 = w$, this is equivalent to $(|z|e^{i\theta})^2 = |w|e^{i\beta}$, so $|z|^2 = |w|$ and $e^{i2\theta} = e^{i\beta}$. Then $|z| = \sqrt{|w|}$, and $\theta = \frac{\beta}{2} + \pi k$ for $k \in \mathbb{Z}$. Then our solutions are

$$z = \sqrt{|w|}e^{i(\beta/2+\pi k)} = \sqrt{|w|}e^{i\beta/2}e^{i\pi k} = \sqrt{|w|}e^{i\beta/2}\cos(\pi k)$$

Thus, in general

$$z = \sqrt{|w|}e^{i\text{Arg}(w)/2}(-1)^k = \pm\sqrt{|w|}e^{i\text{Arg}(w)/2}$$

and

$$w^{1/2} = \{\sqrt{|w|}e^{i\text{Arg}(w)/2}, -\sqrt{|w|}e^{i\text{Arg}(w)/2}\}$$

In general we have

$$w^{1/n} = \{\sqrt[n]{w}, \zeta\sqrt[n]{w}, ..., \zeta^{n-1}\sqrt[n]{w}\}$$

where $\sqrt[n]{w} = \sqrt[n]{|w|} \exp\left(\frac{i\mathrm{Arg}(w)}{n}\right)$ is the principal root, and $\zeta = e^{\frac{2\pi i}{n}}$ is an nth root of unity. The principal root is the local inverse for the principal branch $U = \{z : -\pi/n < \mathrm{Arg}(z) < \pi/n\}$.

1.3.0 Complex Exponential

Definition 1.3.1. *We define the complex exponential for $z \in \mathbb{C}$ to be*

$$e^z = e^{\mathscr{R}e(z)} e^{i\mathscr{I}m(z)} = e^{\mathscr{R}e(z)}(\cos(\mathscr{I}m(z)) + i\sin(\mathscr{I}m(z)))$$

Properties 1.3.2. *Let $z = x + iy, w = a + ib \in \mathbb{C}$.*

- $e^z e^w = e^{z+w}$
- $|e^{x+iy}| = |e^x||e^{iy}| = e^x$*, which is never zero so the complex exponential is never zero. that is,*
- $e^z \neq 0$ *for all* $z \in \mathbb{C}$*.*
- $\arg(e^z) = \arg(e^x e^{iy}) = y + 2\pi\mathbb{Z}$*.*

Failure to Inject

If $e^{z_1} = e^{z_2}$, then $e^{x_1}e^{iy_1} = e^{x_2}e^{iy_2}$, so $x_1 = x_2$ and $y_1 \in y_2 + 2\pi\mathbb{Z}$. Thus, e^z has a $2\pi i$-periodicity; $e^z = e^{z+2\pi ik}$ for $k \in \mathbb{Z}$. To make the complex exponential, we must restrict the domain to some horizontal strip of height at most 2π (with endpoints not included). In particular, if we take $U = \{x + iy : -\pi < y < \pi\}$ we obtain the branch $\mathbb{C}^-$, and branch cut $(-\infty, 0]$. Then, suppose we write $e^z = w = |w|e^{i\mathrm{Arg}(w)}$. Then a solution is $e^x = |w|$, and $y = \mathrm{Arg}(w)$. We can then define

$$\mathrm{Log}(w) = \ln|w| + i\mathrm{Arg}(w) = z = x + iy$$

for $w \in \mathbb{C}^-$, which is the branch cut to the multivalued log

$$\log(z) = \ln|z| + i\arg(z)$$

taking the restriction U in the range.

1.4.0 Sine, Cosine, Cosh, Sinh

Recall $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Then we have that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

and

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

Thus, we can obtain formulas for sin and cos, $\theta \in \mathbb{C}$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Then we define:

Definition 1.4.1. *We define the complex sine and cosine, $z \in \mathbb{C}$, by*

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \tag{1.4.1}$$

and

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}} \tag{1.4.2}$$

Observe that

$$e^x = \underbrace{\frac{1}{2}(e^x + e^{-x})}_{\cosh(x)} + \underbrace{\frac{1}{2}(e^x - e^{-x})}_{\sinh(x)}$$

Definition 1.4.2. *We define the complex hyperbolic sine and hyperbolic cosine, $z \in \mathbb{C}$, by*

$$\boxed{\cosh z = \frac{e^{z} + e^{-z}}{2}} \tag{1.4.3}$$

and

$$\boxed{\sinh z = \frac{e^{z} - e^{-z}}{2}} \tag{1.4.4}$$

Then we have the identities

$$\cosh z = \cos(iz), \sinh z = -i\sin(iz)$$

and

$$\cos(z) = \cosh(iz), \sin z = -i\sinh(iz)$$

Complex Cosine is Not Bounded

Observe

$$\cos(z) = \cos(x + iy) = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}$$

Now, using angle formulas we have

$$\cos(z) = \cos(x + iy)$$

$$
\begin{aligned}
&= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y)
\end{aligned}
$$

so

$$
|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y
$$

so as cosh and sinh are unbounded, so is complex cos.

Claim 1.4.1.

$$
\begin{aligned}
\cos(z+w) &= \cos(z)\cos(w) - \sin(z)\sin(w) \\
\sin(z+w) &= \sin(z)\cos(w) + \sin(w)\cos(z)
\end{aligned}
$$

and

$$
\begin{aligned}
\cosh(z+w) &= \sinh z \sinh w + \cosh z \cosh w \\
\sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w
\end{aligned}
$$

Claim 1.4.2. $\cos^2 z + \sin^2 z = 1$

Proof. First, observe

$$
\cos^2 z = \left[\frac{1}{2}(e^{iz} + e^{-iz})\right]^2 = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz})
$$

and

$$
\sin^2 z = \left[\frac{1}{2i}(e^{iz} - e^{-iz})\right]^2 = \frac{-1}{4}(e^{2iz} - 2 + e^{-2iz})
$$

Hence, indeed, $\cos^2 z + \sin^2 z = 1$. ■

1.5.0 Power Functions

Definition 1.5.1. *Let $\alpha \in \mathbb{C}$ be arbitrary. For $z \in \mathbb{C}^\times$ we define the power function z^α to be the multivalued function*

$$
z^\alpha = e^{\alpha \log z}
$$

Thus, the values of z^α are given by

$$
\begin{aligned}
z^\alpha &= e^{\alpha(\log|z| + i\arg(z))} \\
&= e^{\alpha \mathrm{Log}(z)} e^{2\pi i \alpha m}, m = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

Consequently, the various values of z^α are obtained by multiplying the principal value $e^{\alpha \mathrm{Log}|z|}$ by the integer power of $e^{2\pi i\alpha}$. Consequently, if α is itself an integer $e^{2\pi i\alpha} = 1$, and the power function is single valued and equal to the principal value, $e^{\alpha \mathrm{Log}|z|}$. If $\alpha = 1/n$, for $n \in \mathbb{N}$, then the factor is precisely the nth roots of unity, and $z^{1/n}$ are the nth roots of unity of z.

It is important to note that the usual algebraic rules do not apply to power functions when they are multivalued.

To haze the power function move continuously with z we make the branch cut $[0,\infty)$. Then we define a continuous branch on $\mathbb{C}^+$ to be

$$w = r^{\alpha}e^{i\alpha\theta}, \text{ for } z = re^{i\theta}, 0 < \theta < 2\pi$$

At the top edge of the slit, $\theta = 0$, we have the usual power function $r^{\alpha} = e^{\alpha \mathrm{Log} r}$. At the bottom of the slit, $\theta = 2\pi$, we have the function $r^{\alpha}e^{2\pi i\alpha}$. For a fixed r, as θ ranges the values of $w = r^{\alpha}e^{i\theta\alpha}$ move continuously. Thus, the values of this branch of z^{α} on the bottom edge are $e^{2\pi i\alpha}$ times the values at the top edge. This multiple, $e^{2\pi i\alpha}$, is called the ***phase factor*** of z^{α} at $z = 0$.

For any other choice of branch, $w = r^{\alpha}e^{i\alpha(\theta+2\pi m)}$, the same phase factor is observed.

Lemma 1.5.1 (Phase Change Lemma). *Let $g(z)$ be a single-valued function that is defined and continuous near z_0. For any continuously varying branch of $(z-z_0)^{\alpha}$ the function $f(z) = (z-z_0)^{\alpha}g(z)$ is multiplied by the phase factor $e^{2\pi i\alpha}$ when z traverses acomplete circle about z_0 in the positive direction.*

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