Complex Analysis: A Complete Guide

Complex Analysis

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Solo Pursuit of Learning

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Part 1

Chapter 1

The Complex Plain and Basic Functions

1.1.0 Complex Numbers

The complex numbers, $\mathbb{C}$, consist of all formul sums $z = x + iy$, for $x, y \in \mathbb{R}$, where $i^2 = -1$ is the root of $x^2 + 1 = 0$. Then, for multiplication we proceed by $z \cdot w = (a + ib)(x + iy) = (ax - by) + i(xb + ay)$.

Gauss concieved of $\mathbb{C}$ as $\mathbb{R}^2$ with a binary operation $*$, where $(a, b) * (x, y) = (ax - by, xb + ay)$. Then, we observe that $(1, 0) * (a, b) = (a, b)$, so $(1, 0)$ acts as 1. Moreover, $(0, 1) * (0, 1) = (-1, 0) = -(1, 0)$.

The matrix model of $\mathbb{C}$ is

$$\mathbb{C} = \left\{ \begin{bmatrix} a & -b//b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

In terms of extension fields, we can consider $\mathbb{C}$ to be $\mathbb{R}[x]/(x^2 + 1)$.

Definition 1.1.1. *If $z = x + iy$, with $x, y \in \mathbb{R}$, then we define $\mathscr{R}e(z) = x$ and $\mathscr{I}m(z) = y$.*

Definition 1.1.2. *If $z = x + iy$, we define the* **<u>conjugate</u>** *of z to be $\overline{z}x - iy$.*

Properties 1.1.3. *Let $z, w \in \mathbb{C}$.*

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\overline{\overline{z}} = z$
- $\overline{z} = 0$ *if and only if* $z = 0$
- $\mathscr{R}e(z) = \frac{z + \overline{z}}{2}$
- $\mathscr{I}m(z) = \frac{z - \overline{z}}{2i} = \frac{i(\overline{z} - z)}{2}$

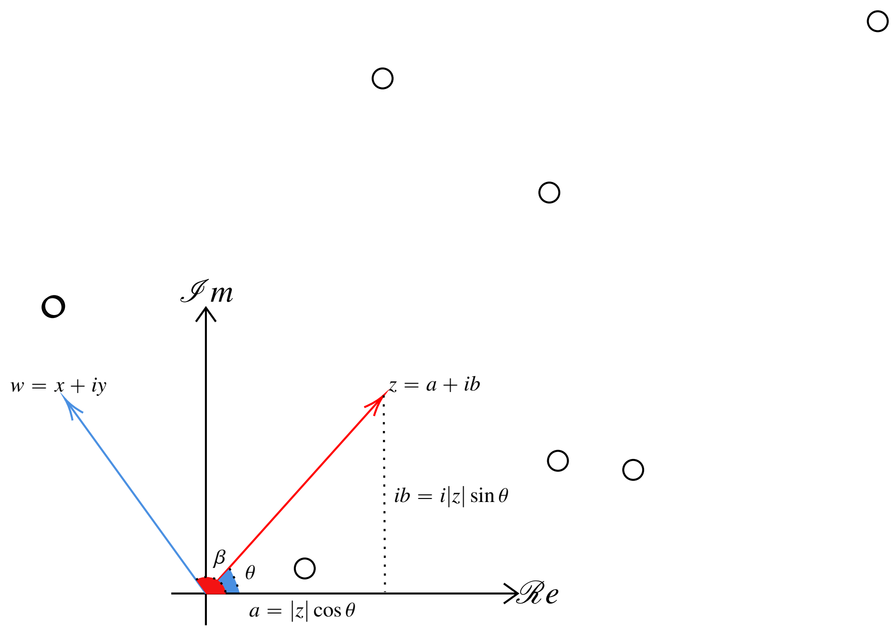
Proposition 1.1.1. *If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{z\bar{z}}$.*

Definition 1.1.4. *Let $z \in \mathbb{C}$. Then the* **modulus**, $|\cdot|$ *of $z = a + ib$ (the norm), is the length of z as a vector:*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$$

Geometry of the Complex Numbers

The complex numbers, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$, can be considered as a plane of points, or we can consider the complex numbers as vectors in the plane eminating from 0.



Noting this geometric picture, we can write $z = |z| \cos\theta + i|z| \sin\theta = |z|(\cos\theta + i \sin\theta)$. Suppose we had another complex number $w = |w|(\cos\beta + i \sin\beta)$. Then we observe that

$$\begin{aligned} zw &= |z||w|(\cos\theta + i \sin\theta)(\cos\beta + i \sin\beta) \\ &= |zw|(\cos\theta\cos\beta - \sin\theta\sin\beta + i(\sin\theta\cos\beta + \cos\theta\sin\beta)) \\ &= |zw|(\cos(\theta + \beta) + i \sin(\theta + \beta)) \end{aligned}$$

so complex multiplication aligns with angle addition in the plane.

Definition 1.1.5. *Define* **Euler's Formula**

$$e^{i\theta} = \cos\theta + i \sin\theta$$

From our above work we have that

$$zw = (|z|e^{i\theta})(|w|e^{i\beta}) = |zw|e^{i(\theta+\beta)}$$

so

$$e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}$$

The conjugate of $e^{i\theta}$ is

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$

Proposition 1.1.2. *$e^{i\theta} = e^{i\beta}$ if and only if $\theta = \beta + 2\pi k$ for some $k \in \mathbb{Z}$.*

Then we have that $e^{i\theta} = e^{-i\theta}$ if and only if $\theta = \pi k$ for some $k \in \mathbb{Z}$.

Definition 1.1.6. *Let $z \in \mathbb{C}$. The **argument** of $z = |z|e^{i\theta}$ is $\arg(z) = \{\theta + 2\pi k : k \in \mathbb{Z}\}$, and the **principal argument** of z, $Arg(z) = \theta_0$, where $z = |z|e^{i\theta_0}$ and $\theta_0 \in (-\pi, \pi]$.*

Example 1.1.1. Consider $z = -42 - 42i$. Then $|z| = 42\sqrt{2}$, and $\operatorname{Arg}(z) = -\frac{3\pi}{4}$, so $z = 42\sqrt{2}e^{-i\frac{3\pi}{4}}$, and $\arg(z) = -\frac{3\pi}{4} + 2\pi\mathbb{Z}$.

Properties 1.1.7. *For $z, w \in \mathbb{C}$, $\arg(zw) = \arg(z) + \arg(w)$, but $Arg(zw) \neq Arg(z) + Arg(w)$.*

Theorem 1.1.3 (DeMoivre's Theorem)**.** *For all $n \in \mathbb{Z}$,*

$$(e^{i\theta})^n = e^{in\theta}$$

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$ then $(e^{i\theta})^1 = e^{i1\theta}$, so the base case holds. Now, suppose that the claim holds for some $n \geqslant 1$. It follows that

$$\begin{aligned}
(e^{i\theta})^{n+1} &= (e^{i\theta})^1(e^{i\theta})^n \\
&= e^{i\theta}e^{in\theta} && \text{(by I.H)} \\
&= e^{i(\theta+n\theta)} \\
&= e^{i(n+1)\theta}
\end{aligned}$$

completing the proof. ■

Definition 1.1.8. *Suppose that $n \in \mathbb{N}$, $w, z \in \mathbb{C}$ such that $z^n = w$, then z is said to be an **nth root of** w. Moreover, the set of all nth roots is dentoed $w^{1/n} \neq \sqrt[n]{w}$.*

Let $z^n = w$, where $z = \rho e^{i\theta}$. Then it follows that

$$\begin{aligned}
(\rho e^{i\theta})^n &= w \\
\rho^n e^{in\theta} &= |w|e^{i\operatorname{Arg}(w)} && \text{(by 1.1.3)}
\end{aligned}$$

This gives the two equations $\rho^n = |w|$ and $e^{in\theta} = e^{i\operatorname{Arg}(w)}$, so $\rho = \sqrt[n]{|w|}$, and

$$\begin{aligned}
n\theta &= \operatorname{Arg}(w) + 2\pi k \\
\theta &= \frac{\operatorname{Arg}(w)}{n} + \frac{2\pi k}{n}
\end{aligned}$$

This gives the following result:

Corollary 1.1.4.

$$w^{1/n} = \left\{ \sqrt[n]{|w|} e^{i\frac{\mathrm{Arg}(w)}{n}} e^{i\frac{2\pi k}{n}} : k \in \mathbb{Z} \right\}$$

Definition 1.1.9. *If $w = 1$, we have that*

$$1^{1/n} = \left\{ e^{i\frac{2\pi k}{n}} : k \in \mathbb{Z} \right\}$$

These are the **nth roots of unity**.

Then we observe that for any $w \in \mathbb{C}$,

$$w^{1/n} = \sqrt[n]{|w|} e^{i\frac{\mathrm{Arg}(w)}{n}} 1^{1/n}$$

Example 1.1.2. Consider the fourth roots of $81i$, so $(81i)^{1/4}$. Then we have that

$$\begin{aligned} (81i)^{1/4} &= \sqrt[4]{81} e^{i\frac{\pi}{8}} 1^{1/4} \\ &= 3e^{i\frac{\pi}{8}} \{1, i, -1, -i\} \end{aligned}$$

Example 1.1.3. Let $w = \exp\left(\frac{2\pi i}{6}\right)$. Then

$$1^{1/6} = \{z \in \mathbb{C} : z^6 = 1\} = \{w, w^2, w^3, w^4, w^5, w^6 = 1\}$$

Note $w = e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1+i\sqrt{3}}{2}$. Now, if we consider the polynomial $f(z) = z^6 - 1$, we now know six roots for this polynomial. Then we can factor

$$f(z) = \prod_{i=1}^{6} (z - w^i)$$

In short, to solve $z^n = \rho$, we take the nth roots of ρ, $\rho^{1/n}$.

Example 1.1.4. Consider $z^2 + bz + c = 0$. Then completing the square we obtain $z \in \left\{ \frac{-b+(b^2-4c)^{1/2}}{2} \right\}$, where

$$\left(\frac{b^2-4c}{2}\right)^{1/2} = \left\{ \begin{array}{ccl} \sqrt{\left|\frac{b^2-4c}{2}\right|} & -\sqrt{\left|\frac{b^2-4c}{2}\right|} & if\ b^2-4c \geqslant 0 \\ \sqrt{\left|\frac{b^2-4c}{2}\right|} i & -\sqrt{\left|\frac{b^2-4c}{2}\right|} i & if\ b^2-4c < 0 \end{array} \right\}$$

1.2.0 Local Inverses and Branch-cut

Definition 1.2.1. *Let $z, w \in \mathbb{C}$, then the line segment from z to w is*

$$[z, w] = \{z + t(w - z) : 0 \leqslant t \leqslant 1\}$$

where we also allow z or w to be plus or minus infinity.

Definition 1.2.2. *The* **negative slit plane**, $\mathbb{C}^-$, *is defined by* $\mathbb{C}^- = \mathbb{C}\backslash(-\infty, 0]$, *and the* **positive slit plane**, $\mathbb{C}^+$, *is defined by* $\mathbb{C}^+ = \mathbb{C}\backslash[0, \infty)$. *In general, we define*

$$C^\alpha = \mathbb{C}\backslash[0, e^{i\alpha}\infty)$$

to denote the exclusion of the ray along the α*th angle from the positive real axis.*

Question 1.2.1. What is a function?

Definition 1.2.3. *If* $f : S \to \mathbb{C}$ *is a function, and* $U \subseteq S$, *then* $f|_U : U \to \mathbb{C}$ *is defined by* $f|_U(z) = f(z)$ *for all* $z \in U$.

We may have the case that f is not injective (so it cannot be inverted). But, for a smart choice of U, we may have that $f|_U$ is one-to-one, and hence invertible. Such a restriction is known as a ***local inverse*** for f.

Rigourously, a ***branch cut*** is a curve in the complex plane such that it is possible to define a single analytic branch (sheets of a multivalued function) of a multivalued function on the plane minus that curve. That is, a branch is a way of making the multivalued function single valued, and in the context of determining inverses a branch is a choice of inverse.

Example 1.2.1. For $f(z) = z^n$, then for $U = \left\{z \in \mathbb{C} : -\frac{\pi}{n} < \operatorname{Arg}(z) < \frac{\pi}{n}\right\}$, $f|_U$ is invertible, and $f|_U^{-1}$ is called the ***principal branch***. $f|_U^{-1}$ is a choice of the nth root of $w \in \mathbb{C}^-$.

Definition 1.2.4. *The* **α-argument** *for* $\alpha \in \mathbb{R}$ *is denoted* $Arg_\alpha : \mathbb{C}^\times \to (\alpha, \alpha+2\pi)$. *In particular, for each* $z \in \mathbb{C}^\times$ *we define* $Arg_\alpha \in \arg(z)$ *such that* $z \in (\alpha, \alpha + 2\pi)$

We can give branch cuts for the nth root function which delete the ray at standard angle α. These correspond to local inverse functions $f(z) = z^n$ restricted to $\{z \in \mathbb{C}^\times : \arg(z) = (\alpha/n, (\alpha + 2\pi)/2) + 2\pi\mathbb{Z}\}$.

Square-Root Function

If we have $z^2 = w$, this is equivalent to $(|z|e^{i\theta})^2 = |w|e^{i\beta}$, so $|z|^2 = |w|$ and $e^{i2\theta} = e^{i\beta}$. Then $|z| = \sqrt{|w|}$, and $\theta = \frac{\beta}{2} + \pi k$ for $k \in \mathbb{Z}$. Then our solutions are

$$z = \sqrt{|w|}e^{i(\beta/2+\pi k)} = \sqrt{|w|}e^{i\beta/2}e^{i\pi k} = \sqrt{|w|}e^{i\beta/2}\cos(\pi k)$$

Thus, in general

$$z = \sqrt{|w|}e^{i\operatorname{Arg}(w)/2}(-1)^k = \pm\sqrt{|w|}e^{i\operatorname{Arg}(w)/2}$$

and

$$w^{1/2} = \{\sqrt{|w|}e^{i\operatorname{Arg}(w)/2}, -\sqrt{|w|}e^{i\operatorname{Arg}(w)/2}\}$$

In general we have

$$w^{1/n} = \{\sqrt[n]{w}, \zeta\sqrt[n]{w}, ..., \zeta^{n-1}\sqrt[n]{w}\}$$

where $\sqrt[n]{w} = \sqrt[n]{|w|}\exp\left(\frac{i\text{Arg}(w)}{n}\right)$ is the principal root, and $\zeta = e^{\frac{2\pi i}{n}}$ is an nth root of unity. The principal root is the local inverse for the principal branch $U = \{z : -\pi/n < \text{Arg}(z) < \pi/n\}$.

1.3.0 Complex Exponential

Definition 1.3.1. *We define the complex exponential for $z \in \mathbb{C}$ to be*

$$e^z = e^{\mathscr{R}e(z)}e^{i\mathscr{I}m(z)} = e^{\mathscr{R}e(z)}(\cos(\mathscr{I}m(z)) + i\sin(\mathscr{I}m(z)))$$

Properties 1.3.2. *Let $z = x + iy, w = a + ib \in \mathbb{C}$.*

- $e^z e^w = e^{z+w}$
- $|e^{x+iy}| = |e^x||e^{iy}| = e^x$*, which is never zero so the complex exponential is never zero. that is,*
- $e^z \neq 0$ *for all* $z \in \mathbb{C}$.
- $\arg(e^z) = \arg(e^x e^{iy}) = y + 2\pi\mathbb{Z}$.

Failure to Inject

If $e^{z_1} = e^{z_2}$, then $e^{x_1}e^{iy_1} = e^{x_2}e^{iy_2}$, so $x_1 = x_2$ and $y_1 \in y_2 + 2\pi\mathbb{Z}$. Thus, e^z has a $2\pi i$-periodicity; $e^z = e^{z+2\pi ik}$ for $k \in \mathbb{Z}$. To make the complex exponential, we must restrict the domain to some horizontal strip of height at most 2π (with endpoints not included). In particular, if we take $U = \{x + iy : -\pi < y < \pi\}$ we obtain the branch $\mathbb{C}^-$, and branch cut $(-\infty, 0]$. Then, suppose we write $e^z = w = |w|e^{i\text{Arg}(w)}$. Then a solution is $e^x = |w|$, and $y = \text{Arg}(w)$. We can then define

$$\text{Log}(w) = \ln|w| + i\text{Arg}(w) = z = x + iy$$

for $w \in \mathbb{C}^-$, which is the branch cut to the multivalued log

$$\log(z) = \ln|z| + i\arg(z)$$

taking the restriction U in the range.

1.4.0 Sine, Cosine, Cosh, Sinh

Recall $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Then we have that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

and

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

Thus, we can obtain formulas for sin and cos, $\theta \in \mathbb{C}$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Then we define:

Definition 1.4.1. *We define the complex sine and cosine, $z \in \mathbb{C}$, by*

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \tag{1.4.1}$$

and

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}} \tag{1.4.2}$$

Observe that

$$e^x = \underbrace{\frac{1}{2}(e^x + e^{-x})}_{\cosh(x)} + \underbrace{\frac{1}{2}(e^x - e^{-x})}_{\sinh(x)}$$

Definition 1.4.2. *We define the complex hyperbolic sine and hyperbolic cosine, $z \in \mathbb{C}$, by*

$$\boxed{\cosh z = \frac{e^{z} + e^{-z}}{2}} \tag{1.4.3}$$

and

$$\boxed{\sinh z = \frac{e^{z} - e^{-z}}{2}} \tag{1.4.4}$$

Then we have the identities

$$\cosh z = \cos(iz), \sinh z = -i\sin(iz)$$

and

$$\cos(z) = \cosh(iz), \sin z = -i\sinh(iz)$$

Complex Cosine is Not Bounded

Observe

$$\cos(z) = \cos(x+iy) = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}$$

Now, using angle formulas we have

$$\cos(z) = \cos(x+iy)$$

$$
\begin{aligned}
&= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y)
\end{aligned}
$$

so

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y$$

so as cosh and sinh are unbounded, so is complex cos.

Claim 1.4.1.

$$
\begin{aligned}
\cos(z+w) &= \cos(z)\cos(w) - \sin(z)\sin(w) \\
\sin(z+w) &= \sin(z)\cos(w) + \sin(w)\cos(z)
\end{aligned}
$$

and

$$
\begin{aligned}
\cosh(z+w) &= \sinh z \sinh w + \cosh z \cosh w \\
\sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w
\end{aligned}
$$

Claim 1.4.2. $\cos^2 z + \sin^2 z = 1$

Proof. First, observe

$$\cos^2 z = \left[\frac{1}{2}(e^{iz} + e^{-iz})\right]^2 = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz})$$

and

$$\sin^2 z = \left[\frac{1}{2i}(e^{iz} - e^{-iz})\right]^2 = \frac{-1}{4}(e^{2iz} - 2 + e^{-2iz})$$

Hence, indeed, $\cos^2 z + \sin^2 z = 1$. ∎

1.5.0 Power Functions

Definition 1.5.1. *Let $\alpha \in \mathbb{C}$ be arbitrary. For $z \in \mathbb{C}^\times$ we define the power function z^α to be the multivalued function*

$$z^\alpha = e^{\alpha \log z}$$

Thus, the values of z^α are given by

$$
\begin{aligned}
z^\alpha &= e^{\alpha(\log|z| + i\arg(z))} \\
&= e^{\alpha \mathrm{Log}(z)} e^{2\pi i \alpha m}, m = 0, \pm 1, \pm 2, ...
\end{aligned}
$$

Consequently, the various values of z^α are obtained by multiplying the principal value $e^{\alpha \mathrm{Log}|z|}$ by the integer power of $e^{2\pi i \alpha}$. Consequently, if α is itself an integer $e^{2\pi i \alpha} = 1$, and the power function is single valued and equal to the principal value, $e^{\alpha \mathrm{Log}|z|}$. If $\alpha = 1/n$, for $n \in \mathbb{N}$, then the factor is precisely the nth roots of unity, and $z^{1/n}$ are the nth roots of unity of z.

It is important to note that the usual algebraic rules do not apply to power functions when they are multivalued.

To haze the power function move continuously with z we make the branch cut $[0, \infty)$. Then we define a continuous branch on $\mathbb{C}^+$ to be

$$w = r^\alpha e^{i\alpha\theta}, \text{ for } z = re^{i\theta}, 0 < \theta < 2\pi$$

At the top edge of the slit, $\theta = 0$, we have the usual power function $r^\alpha = e^{\alpha \mathrm{Log} r}$. At the bottom of the slit, $\theta = 2\pi$, we have the function $r^\alpha e^{2\pi i\alpha}$. For a fixed r, as θ ranges the values of $w = r^\alpha e^{i\theta\alpha}$ move continuously. Thus, the values of this branch of z^α on the bottom edge are $e^{2\pi i\alpha}$ times the values at the top edge. This multiple, $e^{2\pi i\alpha}$, is called the ***phase factor*** of z^α at $z = 0$.

For any other choice of branch, $w = r^\alpha e^{i\alpha(\theta+2\pi m)}$, the same phase factor is observed.

Lemma 1.5.1 (Phase Change Lemma). *Let $g(z)$ be a single-valued function that is defined and continuous near z_0. For any continuously varying branch of $(z - z_0)^\alpha$ the function $f(z) = (z - z_0)^\alpha g(z)$ is multiplied by the phase factor $e^{2\pi i\alpha}$ when z traverses a complete circle about z_0 in the positive direction.*

Chapter 2

Analytic Functions

2.1.0 Basic Analysis and Topology

Definition 2.1.1. *A* **sequence** *of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$, where $f(i) = a_i$ for each $i \in \mathbb{N}$. We often write this sequence as $\{a_n\}_{n=1}^{\infty}$.*

Definition 2.1.2. *We say a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit $L \in \mathbb{C}$, $a_n \to L$, as n approaches infinity, $n \to \infty$, if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, $|a_n - L| < \varepsilon$. We also write*

$$\lim_{n\to\infty} a_n = L$$

That is, if we pick a particular limiting value for the limit value, we can find a point past which the tail of the sequence is between limiting value and our limit.

Definition 2.1.3. *A sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ is said to be* **bounded** *if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|a_n| < M$ (where $|\cdot|$ is the modulus).*

That is, all points in the sequence exist in an M radius disk of the origin in the complex plane.

Example 2.1.1. Consider a sequence $a_n = (e^{ir})^n$, where $r \notin \mathbb{Q}$. Then $|a_n| = 1$, for all n. Then $\lim\limits_{n\to\infty} |a_n| = 1$, but the sequence itself does not cnverge in $\mathbb{C}$.

Theorem 2.1.1. *Suppose $s_n \to s$ and $t_n \to t$ for sequences in $\mathbb{C}$. Then*

- $s_n + t_n \to s + l$
- $cs_n \to cs$, *for all* $c \in \mathbb{C}$
- $s_n t_n \to st$
- $s_n/t_n \to s/t$, *provided* $t \neq 0$.

We remark that the squeeze theorem only applies for real-sequences, as it relies on the ordering of the reals. Moreover, the theorem that "A bounded monotone sequence of real numbers converges" also does not hold, again due to ordering.

Definition 2.1.4. *Let (a_n) be a sequence. Let $n_1 < n_2 < n_3 < ...$ be an increasing sequence of natural numbers. Then $\{a_{n_k}\}_{k=1}^{\infty}$ is a* **subsequence** *of the sequence (a_n).*

Example 2.1.2. Consider the sequence $(-1)^n$. Then we have that

$$\limsup_{n\to\infty}(-1)^n = 1, \quad \text{and} \quad \liminf_{n\to\infty}(-1)^n = -1$$

Theorem 2.1.2. *Suppose $z_n = x_n + iy_n$ for real sequences x_n and y_n. Let $z = x + iy \in \mathbb{C}$. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.*

Proof. $\implies$. Assume that $z_n \to z$. Hence, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$ then $|z_n - z| < \varepsilon$. Consider $|x_n - x|$. Then observe that $|x_n - x| = \sqrt{(x_n - x)^2} \leqslant \sqrt{(x_n - x)^2 + (y_n - y)^2} = |z_n - z$. Then for any $n \geqslant N$ we have that $|x_n - x| \leqslant |z_n - z| < \varepsilon$, and similarly, $|y_n - y| \leqslant |z_n - z| < \varepsilon$. Hence, $x_n \to x$ and $y_n \to y$, as desired.

$\impliedby$. Now, suppose that $x_n \to x$ and $y_n \to y$. Fix $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $n_1 \geqslant N_1$ and $n_2 \geqslant N_2$ imply $|x_{n_1} - x| < \varepsilon/2$ and $|y_{n_2} - y| < \varepsilon$. Let $N = \max(N_1, N_2)$. Then for all $n \geqslant N$,

$$|z_n - z| \leqslant |x_n - x| + |i(y_n - y)| = |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence, we conclude that $z_n \to z$, as desired. ■

Definition 2.1.5. *A sequence $\{z_n\}_{n=1}^{\infty}$ is said to be* **Cauchy** *if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$,*

$$|z_n - z_m| < \varepsilon$$

Thus, the tail of a Cauchy sequence gets arbitrarily close as we go arbitrarily far.

$\mathbb{C}$ is a complete metric space, so the Cauchy sequences are precisely the convergent sequences.

Definition 2.1.6. *An open disk of radius $\varepsilon > 0$ centered at $z_0 \in \mathbb{C}$ is defined to be*

$$D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

Definition 2.1.7. *Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. Then z_0 is an* **interior point** *if there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq S$.*

We say that S is an **open set** *in $\mathbb{C}$ if each point in S is an interior point.*

Definition 2.1.8. *We say that $S \subseteq \mathbb{C}$ is a* **closed set** *if and only if $\mathbb{C}\backslash S$ is open.*

Example 2.1.3. Consider the half-plane $S = \{z \in \mathbb{C} : \mathscr{I}m(z) \geqslant 1\}$. This is not a closed set as all points on the boundary ($\mathscr{I}m(z) = 1$) are not interior.

Definition 2.1.9. *Let $S \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then z_0 is said to be a* **limit/accumulation point** *of S if for every $\varepsilon > 0$, $D^*_\varepsilon(z_0) \cap S \neq \varnothing$.*

That is, neighborhoods of limit points always intersect the set in a point different from the limit point.

Definition 2.1.10. *Let $f : S \to \mathbb{C}$ be a complex function, $S \subseteq \mathbb{C}$. Suppose $z_0 \in \mathbb{C}$ is a limit point of S. Then we say that $f(z) \to L$, for $L \in \mathbb{C}$, if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$.*

Equivalently, the functional limit converges to L if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(D^*_\delta(z_0) \cap S) \subseteq D_\varepsilon(L)$$

Theorem 2.1.3. *Let $f : S \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be complex functions such that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$ for some $L, M \in \mathbb{C}$. Let $c \in \mathbb{C}$. Then*

- $\lim_{z\to z_0}(f(z) + g(z)) = L + M$
- $\lim_{z\to z_0} cf(z) = cL$
- $\lim_{z\to z_0} f(z)g(z) = LM$
- $\lim_{z\to z_0} f(z)/g(z) = L/M$, *provided $M \neq 0$.*

Definition 2.1.11. *We say that $f : S \to \mathbb{C}$ is continuous at $z_0 \in S$ if and only if $\lim_{z\to z_0} f(z) = f(z_0)$.*

Theorem 2.1.4. *Let $f : S \to \mathbb{C}$. Then $\lim_{z\to z_0} f(z) = L$ if and only if for all sequences z_n such that $z_n \to z_0$, $\lim_{n\to\infty} f(z_n) = L$.*

Definition 2.1.12. *A subset $S \subseteq \mathbb{C}$ is connected if and only if there exists no $A, B \subseteq \mathbb{C}$ non-empty such that $S = A \cup B$ and $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.*

Noting that connected and path-connected subsets are equivalent in $\mathbb{C}$, we formulate (path) connectedness in another way as follows:

Definition 2.1.13. *A* **polygonal chain** *P is a curve composed of a finite number of connected line segments. That is, there exist $z_0, z_1, ..., z_n \in \mathbb{C}$ such that*

$$P = [z_0, z_1] \cup [z_1, z_2] \cup ... \cup [z_{n-1}, z_n]$$

Definition 2.1.14. *A subset $U \subseteq \mathbb{C}$ is (path)* **connected** *if and only if for each $p, q \in U$, there exists a polygonal chain γ which begins at p and terminates at q, and $\gamma \subseteq U$.*

Definition 2.1.15. *A subset $D \subseteq \mathbb{C}$ is called a* **domain** *if D is both open and connected.*

Definition 2.1.16. *A* **region** *is a domain paired with some or all of its topological boundary.*

Definition 2.1.17. *Let $U \subseteq \mathbb{C}$. Then U^o denotes the* **interior** *of U, and is the set of all interior points in U.*

Definition 2.1.18. *A set $U \subseteq \mathbb{C}$ is* **compact** *if and only if (Heine-Borel) U is closed and bounded.*

Definition 2.1.19. *A set $U \subseteq \mathbb{C}$ is said to be* **star-shaped** *at a point $z_0 \in \mathbb{C}$ if $z_0 \in U$, and for any $z \in U$ we have $[z_0, z] \subseteq U$.*

Star-shaped implies simply connected: every closed curve in the region can be continuously deformed to a point.

Example 2.1.4. The slit complex plane $\mathbb{C}^- = \mathbb{C} \backslash (-\infty, 0]$ is star shaped, with any point along $(0, \infty)$ serving as a star center. In particular, consider the star center 1. Then, let $z \in \mathbb{C}^-$ and consider $[1, z] = \{1 + t(z-1) : 0 \leqslant t \leqslant 1\}$. Then, for each $a + ib \in [1, z]$ we have $a + ib = 1 + t(x + iy - 1) = 1 + (x-1)t + iyt$ for some $t \in [0, 1]$. For $z \in (0, \infty)$ we have that $y = 0$, and $x - 1 > -1$, so $(x-1)t > -1t > -1$, so $a + ib = a > 0$. On the other hand, if $z \in \mathbb{C}^-$ with $y \neq 0$, then $a + ib \notin (-\infty, 0]$ for all $t > 0$, and by construction $a + ib = 1$ for $t = 0$, which is again not in $(-\infty, 0]$. Thus, $[1, z] \cap (-\infty, 0] = \varnothing$ for all $z \in \mathbb{C}^-$, as claimed.

Theorem 2.1.5. *If $h : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable on D and $\nabla h = \vec{0}$ on D, then h is constant on D.*

Proof. Let $\gamma : [t_0, t_1] \to D$ be a polygonal chain in D, with $\gamma(t_0) = p$ and $\gamma(t_1) = q$. Then

$$\frac{d}{dt}(h(\gamma(t)) = \nabla h(\gamma(t)) \cdot \frac{d\gamma}{dt} = 0$$

since the gradient is zero in D. Then by properties of single variable differentiable functions, $h(\gamma(t))$ is constant so $h(p) = h(\gamma(t_0)) = h(\gamma(t_1)) = h(q)$. But this is true for all $p, q \in D$, so h is constant. ■

2.2.0 Analytic Functions

Definition 2.2.1. *Let $f : S \to \mathbb{C}$. If the limit*

$$\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

exists, we say f is **complex differentiable** *at z_0, and we denote*

$$f'(z_0) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

Futhermore, the mapping $z \mapsto f'(z)$ is the complex derivative of f.

Example 2.2.1. Consider $f(z) = 2z^2 - 1$. Then

$$\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right) = \lim_{z\to z_0}\left(\frac{2z^2-1-(2z_0^2-1)}{z-z_0}\right) = 2\lim_{z\to z_0}\left(\frac{(z-z_0)(z+z_0)}{z-z_0}\right) = 2{\cdot}2z_0 = 4z_0$$

Theorem 1 (Caratheodory Theorem).

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function $a \in D$, a limit point. Then f is **complex differentiable** *at a if and only if there exists a function $\phi : D \to \mathbb{C}$ such that*

1. *ϕ is continuous at a*
2. *$f(z) - f(a) = \phi(z)(z-a)$ for all $z \in D$.*

Proof. $\implies$. Suppose $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is complex differentiable at a. Construct $\phi(z) = \frac{f(z)-f(a)}{z-a}$ for $z \neq 0$ anf $\phi(a) = f'(z)$. Then, observe that

$$\lim_{z\to a}\phi(z) = \lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\right) = f'(a)$$

so ϕ is continuous at a. Next, for $z \neq a$ we have by definition that $f(z) - f(a) = \phi(z)(z-a)$. At $z = 0$ we just have $0 = 0$, so this also holds.

$\impliedby$. By 2. we have that $\phi(z) = \frac{f(z)-f(a)}{z-a}$ for $z \neq a$, so by continuity $\phi(a) = f'(a)$, so f' is complex differentiable at a. ■

Theorem 2.2.1. *If $f : S \to \mathbb{C}$ is complex differentiable at $a \in \mathbb{C}$, then that implies that f is continuous at $z = a$.*

Proof. By Theorem 1, there exists $\phi : S \to \mathbb{C}$, where $a \in S$, and $f(z) = f(a) + \phi(z)(z-a)$. Then,

$$\lim_{z\to a} f(z) = \lim_{z\to a}(f(a) + \phi(z)(z-a)) = f(a) + \phi(a)(a-a) = f(a)$$

which implies that f is continuous at a, as desired. ■

Assume that f, g are complex differentiable at $z = a$. Then there exist ϕ_f, ϕ_g such that $f(z) = f(a) + (z-a)\phi_f(z)$ and $g(z) = g(a) + (z-a)\phi_g(z)$ for some domain of a. Then, observe that

$$f(z)g(z) = f(a)g(a) + (z-a)[\phi_f(z)g(a) + \phi_g(z)f(a) + (z-a)\phi_f(z)\phi_g(z)]$$

We claim $\phi_{fg}(z) = \phi_f(z)g(a) + \phi_g(z)f(a) + (z-a)\phi_f(z)\phi_g(z)$. By continuity of the component functions $\phi_{fg}(z)$ is continuous at a. Hence, $f(z)g(z)$ is differentiable and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

Theorem 2.2.2. *Suppose that g is complex differentiable at a, and that f is complex differentiable at $g(a)$.*

Proof. Suppose $g'(a)$ and $f'(g(a))$ exist. Also, let $g(z) = g(a) + (z-a)\phi_g(z)$, and $f(w) = f(g(a)) + (w - g(a))\psi_f(w)$. Simply compose, set $w = g(z)$. Then

$$\begin{aligned} f(g(z)) &= f(g(a)) + (g(z) - g(a))\psi_f(g(z)) \\ &= f(g(a)) + (z-a)\phi_g(z)\psi_f(g(z)) \end{aligned}$$

Then by continuity of ϕ_g and ψ_f we have that $(f \circ g)'(a) = f'(g(a))g'(a)$, completing the proof. Moreover, $\phi_g(z)\psi_f(g(z)) = \phi_{f\circ g}(z)$. ■

Theorem 2.2.3. *Suppose that f and g are complex differentiable at a, and $c \in \mathbb{C}$. Then*

- $(f+g)'(a) = f'(a) + g'(a)$
- $(cf)'(a) = cf'(a)$
- $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- $(f/g)'(a) = (f'(a)g(a) - f(a)g'(a))/(g(a))^2$ *provided* $g(a) \neq 0$.

2.3.0 The Cauchy Riemmann Equations

Theorem 2.3.1. *If $f(z) = u(z) + iv(z)$ is a complex valued function of a complex variable, u and v real valued functions of a complex variable, and*

1. *u and v are continuously differentiable on an open set containing z_0*
2. $u_x(z_0) = v_y(z_0)$ *and* $u_y(z_0) = -v_x(z_0)$

Then $f'(z_0) = u_x(z_0) + iv_x(z_0)$ is complex differentiable.

Definition 2.3.1. *$g : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable on $U \subseteq \mathbb{R}^2$ if g_x and g_y exist and are continuous on U.*

Example 2.3.1. Suppose $f(z) = z^2$. Then $f(x+iy) = x^2 - y^2 + i2xy$, and let $u(z) = x^2 - y^2$ and $v(z) = 2xy$. Then $u_x(z) = 2x, u_y(z) = -2y, v_x = 2y, v_y = 2x$, so the partials are continuously differentiable. Moreover, $u_x = v_y$ and $u_y = -v_x$, so the Cauchy Riemman equations hold and $f'(z) = 2x + i2y = 2z$.

Example 2.3.2. Suppose $f(z) = e^z$. Then $f(x+iy) = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$. Let $u(z) = e^x \cos y$ and $v(z) = e^x \sin y$. Then $u_x(z) = e^x \cos y, u_y(z) = -e^x \sin y, v_x(z) = e^x \sin y, v_y(z) = e^x \cos y$. Thus, the partial derivatives are continuous on $\mathbb{C}$, and $u_x = v_y$ and $u_y = -v_y$, so $f'(z) = u_x(z) + iv_x(z) = u(z) + iv(z) = f(z)$.

Example 2.3.3. Consider $f(z) = \bar{z} = x - iy$. Then $u = x$ and $v = -y$. Now, observe $u_x = 1 \neq -1 = v_y$, so $f(z)$ is nowhere complex differentiable.

Example 2.3.4. Consider $f(z) = \cos(z) = \cos(x + iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$. Then $u = \cos(x)\cosh(y)$ and $v = -\sin(x)\sinh(y)$. Now, $u_x = -\sin(x)\cosh(y), u_y = \cos(x)\sinh(y), v_x = -\cos(x)\sinh(y), v_y = -\sin(x)\cosh(y)$. Thus, the partials are continuous and $u_x = v_y$ and $u_y = -v_x$, so the Cauchy-Reimman equations hold and $f'(z) = -\sin(x)\cosh(y) - i\cos(x)\sinh(y) = -\sin(x)\cos(iy) - \cos(x)\sin(iy) = -\sin(x + iy)$.

Definition 2.3.2. *A function $f : D \to \mathbb{C}$ is* **holomorphic** *on the domain D if and only if f is complex differentiable at each point in D.*

Definition 2.3.3. *A function $f : D \to \mathbb{C}$ is* **holomorphic** *at a point $z_0 \in D$ if there exists some open disk $D_r(z_0)$ such that $f|_{D_r(z_0)}$ is holomorphic.*

Real Differentiability

If we identify $\mathbb{C} = \mathbb{R}^2$, then $f : \mathbb{C} \to \mathbb{C}$ naturally is associated with $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Definition 2.3.4. *If $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ has a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\lim_{h \to \vec{0}} \left(\frac{F(p+h) - F(p) - L(h)}{||h||} \right) = 0$$

then F is **real-differentiable** *at $p \in \mathbb{R}^2$, and we denote $dF_p = L$. Moreover, the standard matrix for the differential, where $F = F(u, v)$, is*

$$[dF_p] = J_F(p) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \left[\begin{array}{c|c} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{array} \right]$$

Example 2.3.5. Let $F(x, y) = (x, -y)$, then $J_F(p) = \begin{bmatrix} 1 & 0 \\ & -1 \end{bmatrix}$

Definition 2.3.5. *$F(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable at $z_0 = (x_0, y_0)$ if u_x, u_y, v_x, v_y are all continuous near z_0.*

Definition 2.3.6. *If F is continuously differentiable at $z_0 \in \mathbb{R}^2$, then F is differentiable at z_0.*

Proof. Assume $F = (u, v)$ is continuously differentiable with $dF = (du, dv)$ and we can focus on u. We wish to show that

$$\lim_{h \to \vec{0}} \left(\frac{u(p+h) - u(p) - L(h)}{||h||} \right) = 0$$

Let $L(h) = \nabla u(p) \cdot h = h_1 u_x(p) + h_2 u_y(p)$. Observe that

$$\begin{aligned} u(p+h) - u(p) &= u(p + h_1 e_1 + h_2 e_2) - u(p) \\ * &= u(p + h_1 e_1 + h_2 e_2) - u(p + h_1 e_1) + u(p + h_1 e_1) - u(p) \\ &= u(p_1 + h_1, p_2 + h_2) - u(p_1 + h_1, p_2) + u(p_1 \\ &= h_2 u_y(p_1 + h_1, c_2) + h_1 u_x(c_1, p_2) \end{aligned}$$

where the last equality is by the Mean Value Theorem. Then it follows that

$$\lim_{h\to\vec{0}}\left(\frac{u(p+h)-u(p)-L(h)}{||h||}\right)=\lim_{h\to\vec{0}}\left(\frac{h_2u_y(p_1+h_1,c_2)+h_1u_x(c_1,p_2)-h_1u_x(p)-h_2u_y(p)}{||h||}\right)$$

where as $h\to\vec{0}$, c_1 and c_2 go to p_1 and p_2, respective, so by continuity of the partial derivatives this limit goes to zero. The same holds for v, so F is differentiable. ∎

In our convention we write $F(u,v)=u+iv$, so we have $e_1=1$ and $e_2=i$. Now, note that $F(x,y)=(x,-y)=x-iy=\overline{z}$, so F would be nowhere complex differentiable.

Cauchy Reimman Equations Sketch

Suppose f is complex differentiable at z_0, so by Theorem 1 we have $f(z_0+h)-f(z_0)=\phi(z_0+h)h$. Then, I claim for $f:\mathbb{R}^2\to\mathbb{R}^2$, $df_{z_0}(h)=f'(z_0)h$, shifting between $\mathbb{C}$ and $\mathbb{R}^2$. For $h\neq 0$ we have

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)-f'(z_0)h}{|h|}&=\frac{\phi(z_0+h)h-f'(z_0)h}{|h|}\\&=\frac{h}{|h|}(\phi(z_0+h)-f'(z_0))\\&\leqslant\frac{|h|}{|h|}(\phi(z_0+h)-f'(z_0))=0\end{aligned}$$

so f is real differentiable at z_0 with $df_{z_0}(h)=f'(z_0)h$. Let $f'(z_0)=a+ib$. Then

$$f'(z_0)h=ah_1-bh_2+i(ah_2+bh_1)=\begin{bmatrix}a&-b\\b&a\end{bmatrix}\begin{bmatrix}h_1&h_2\end{bmatrix}$$

So in particular

$$df_{z_0}=\begin{bmatrix}u_x&u_y\\v_x&v_y\end{bmatrix}=\begin{bmatrix}a&-b\\b&a\end{bmatrix}$$

so we attain the Cauchy Reimman equations $u_x=v_y$ and $u_y=-v_x$. Reversing this argument we attain that if a function is continuously real-differentiable and satisfies the Cauchy Reimman equations, then the function is complex differentiable.

Theorem 2.3.2. *If $f=(u,v)=u+iv$ is real differentiable on a domain D and $u_x=v_y, u_y=-v_x$ at $z_0\in D$, then $f'(z_0)=u_x(z_0)+iv_x(z_0)$.*

Proof. Let $u_x(z_0)=a$ and $v_x(z_0)=b$. Then

$$df_{z_0}=\begin{bmatrix}u_x(z_0)&u_y(z_0)\\v_x(z_0)&v_y(z_0)\end{bmatrix}=\begin{bmatrix}a&-b\\b&a\end{bmatrix}$$

We have

$$df_{z_0}h=\begin{bmatrix}a&-b\\b&a\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix}$$

$$= \begin{bmatrix} ah_1 - bh_2 \\ bh_1 + ah_2 \end{bmatrix}$$
$$= (ah_1 - bh_2) + i(bh_1 + ah_2) = (a + ib)(h_1 + ih_2) = (a + ib)h$$

We claim that $f'(z_0) = a + ib$. Observe

$$\lim_{h\to 0} \left(\frac{f(z_0 + h) - f(z_0) - (a + ib)h}{h} \right) = 0$$

Moreover, since $\lim\limits_{h\to 0} \frac{(a+ib)h}{h} = a + ib$, so by algebraic properties of limits we have that

$$\lim_{h\to 0} \left(\frac{f(z_0 + h) - f(z_0)}{h} \right) = a + ib = u_x(z_0) + iv_x(z_0)$$

■

Then, we can write

$$\boxed{\frac{df}{dz} = f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

for a holomorphic function on some domain.

2.4.0 Analytic Functions (cont.)

Definition 2.4.1. *If $f : \mathbb{C} \to \mathbb{C}$ and f is holomorphic on $\mathbb{C}$, then f is* **entire**. *We say $f \in \mathcal{O}(\mathbb{C})$. If f is holomorphic on D, we write $f \in \mathcal{O}(D)$.*

Theorem 2.4.1. *If $f \in \mathcal{O}(D)$ and $f = \mathscr{R}e(f)$, then f is constant.*

Proof. First, this implies $f(z) = u(z)$. Then $\nabla u = \langle u_x, u_y\rangle$ and $\nabla v = \langle v_x, v_y\rangle$. But $v = 0$, so $\nabla v = \vec{0}$, and $u_x = v_y, u_y = -v_x$ by the Cauchy Riemman equations so $\nabla u = \vec{0}$. Thus, we have that u and v are constant, so f is constant by Theorem 2.1.5. ■

If $f = (u, v)$ is continuously differentiable at (a, b), then for (h_1, h_2) near (a, b) we have that

$$f(a + h_1, b + h_2) \approx f(a, b) + J_f(a, b) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

Assume the cauchy riemman equations hold. Then

$$J_f(a, b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2 + b^2} \begin{bmatrix} a/\sqrt{a^2 + b^2} & -b/\sqrt{a^2 + b^2} \\ b/\sqrt{a^2 + b^2} & a/\sqrt{a^2 + b^2} \end{bmatrix}$$

where the matrix on the right, R, is orthogonal ($RR^T = I$), and the term in front is a dilation. Hence, at an infinitesimal level changes in holomorphic functions correspond to rotations and dilations. Also, observe that $\det(J_f(z_0)) = a^2 + b^2 = |f'(z_0)|^2$.

Theorem 2.4.2. *$f'(z_0) \neq 0$ if and only if $\det(J_f(z_0)) \neq 0$, for f complex differentiable at z_0.*

Theorem 2.4.3. *Of $f(z)$ is analytic on a domain D, $z_0 \in D$, and $f'(z_0) \neq 0$. Then there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq D$ such that $f|_{D_\varepsilon(z_0)}$ is injective, the image $V = f(D_\varepsilon(z_0))$ is open and the inverse function $f^{-1} : V \to U$ is analytic and satisfies*

$$(f^{-1})'(f(z)) = 1/f'(z) \quad \text{for } z \in D_\varepsilon(z_0)$$

Example 2.4.1. Let $\mathrm{Log}(z) = w$, so $e^w = z$. Then $e^w \frac{dw}{dz} = 1$, so $\frac{d}{dz}(\mathrm{Log}(z)) = 1/z$ in $\mathbb{C}^-$. In general, this can be extended to arbitrary branches and slit complex plains such that the function is single valued and continuous.

Example 2.4.2. Consider the power function on a particular branch:

$$\begin{aligned}\frac{d}{dz}(z^c) = \frac{d}{dz}(e^{c\mathrm{Log}(z)}) &= e^{c\mathrm{Log}(z)}\frac{d}{dz}(c\mathrm{Log}(z))\\ &= e^{c\mathrm{Log}(z)}\frac{c}{z} = ce^{c\mathrm{Log}(z)}e^{-\mathrm{Log}(z)}\\ &= ce^{(c-1)\mathrm{Log}(z)} = cz^{c-1}\end{aligned}$$

on the domain of $\mathbb{C}$ for which the particular branch of the log is holomorphic.

Chapter 3

Line Integrals and Harmonic Functions

Chapter 4

Complex Integration and Analyticity

Chapter 5

Power Series

Chapter 6

Laurent Series and Isolated Singularities

Chapter 7

The Residue Calculus

Part II

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