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Algebraic Geometry: A Complete Guide

– In Pursuit of Abstract Nonsense –

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Preface

This text consists of a collection of Algebraic Geometry notes taken at the University of Calgary.

Place(s), *Firstname Surname*
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Part I

3264: Schubert Calculus

Chapter 1

Introduction

Throughout this book a **scheme** X refers to a separated scheme of finite type over an algebraically closed field k of characteristic 0.

Definition 1.1 A scheme X over S, $p : X \to S$, is said to be **separated** if the diagonal morphism $\Delta : X \to X \times_S X$, given by the universal property of the fiber product of p along itself and the identity $1_X : X \to X$, is a **closed immersion**.

A **closed immersion** $f : Z \to X$ is a map of locally ringed spaces such that f is a homeomorphism, $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Z$ is surjective with kernel $\mathcal{I}$, and the kernel $\mathcal{I}$ is locally generated by sections as a $\mathcal{O}_X$-module, i.e. for every $x \in X$ there exists an open neighborhood U of x such that $\mathcal{I}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules, i.e. there exists a set I, and global sections $s_i \in \Gamma(U, \mathcal{I}|_U), i \in I$ such that the map

$$\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{I}$$

is surjective.

Definition 1.2 An R-scheme X, for R a ring, is called a R-scheme of **finite type** if there exists a finite affine covering $(X_i)_{i \in I}$ of X such that the R-algebras $\Gamma(X_i, \mathcal{O}_X)$ is a finite type R-algebra (i.e. finitely generated as an R-algebra)

In practice we will only consider quasi-projective schemes.

Definition 1.3 We say a scheme X over a ring R is **quasi-projective** if X is a quasicompact open subscheme of a projective A-scheme, where a projective A scheme is isomorphic to Proj $S_\bullet$, where $S_\bullet$ is some finitely generated graded ring over A.

Definition 1.4 A scheme X is said to be **integral** if it is non-empty and $\mathcal{O}_X(U)$ is an integral domain for every non-empty open set U - equivalently X is irreducible and reduced, where X is reduced if each $\mathcal{O}_X(U)$ is reduced. In other words, the nilradical of each $\mathcal{O}_X(U)$ is trivial (no nilpotent elements).

A **variety** will mean an integral scheme. If X is a variety we write $k(X)$ for the field of rational functions on X (i.e. the field of fractions of $\Gamma(X, \mathcal{O}_X)$).

Definition 1.5 If X is a scheme, then a $\mathcal{O}_X$-module $\mathcal{F}$ is **quasicoherent** if for every affine open subset Spec $A \subset X$, $\mathcal{F}|_{\text{Spec } A} \cong \widetilde{M}$ for some A module $\widetilde{M}$. $\mathcal{F}$ is said to further be **coherent** if for every affine open Spec A, $\Gamma(\text{Spec } A, \mathcal{F})$ is a coherent A-module (i.e. it is finitely generated and for any map $A^{\oplus p} \to \Gamma(\text{Spec } A, \mathcal{F})$, the kernel is finitely generated).

Definition 1.6 If V is a vector space, its **projectivization** refers to the scheme $\text{Proj}(\text{Sym } V^*)$, where Sym V is the symmetric algebra of V. The closed points of this space correspond to one-dimensional subspaces of V.

If $X, Y \subset \mathcal{P}^n$ are subvarieties, we define the **join** of X and Y, denoted $\overline{X, Y}$, to be the closure of the union of lines meeting X and Y at distinct points. If $X \subset \mathcal{P}^n$, this is just the cone over Y with vertex X; if X and Y are both linear subspaces, this is simply their span.

Remark 1.1 There is a one-to-one correspondence between vector bundles on a scheme X and locally free sheaves on X.

Definition 1.7 A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be **locally free** if for every point $x \in X$, there exists a set I and an open neighborhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.

Chapter 2

Chow Ring

2.1 The Chow Ring

Throughout we assume that k is an algebraically closed field.

2.1.1 Cycles

Let X be an algebraic.

Definition 2.1 The **group of cycles** on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties (reduced irreducible subschemes) of X. The group $Z(X)$ is graded by dimension.

A cycle $Z = \sum n_i Y_i$, where the Y_i are subvarieties, is **effective** if the coefficients n_i are nonnegative. A **divisor**, or **Weil divisor**, is an $(n-1)$-cycle on a pure n-dimensional scheme.

To any closed subscheme $Y \subset X$ we associate an effective cycle $\langle Y \rangle$: if $Y \subset X$ is a subscheme, and $Y_1, \ldots, Y_s$ are the irreducible components of the reduced scheme Y_{red}, then, because our schemes are Noetherian, each local ring O_{Y,Y_i} has a finite composition series. Writing l_i for its length, we define $\langle Y \rangle = \sum l_i Y_i$.

2.1.2 Rational Equivalence

A **Chow group** of X is the group of cycles of X modulo **rational equivalence**. Informally, two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them.

Definition 2.2 Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

where $t_0, t_1 \in \mathcal{P}^1$ and Φ is a subvariety of $\mathcal{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. We say that two cycles are **rationally equivalent** if their difference is in $\mathrm{Rat}(X)$, and we say that two subschemes are rationally equivalent if their associated cycles are rationally equivalent.

Definition 2.3 The **Chow group** of X is the quotient

$$A(X) = Z(X)/\mathrm{Rat}(X)$$

the **group of rational equivalence classes of cycles on** X. If $Y \in Z(X)$ is a cycle, we write $[Y] \in A(X)$.

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