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# Algebraic Geometry: A Complete Guide

## – In Pursuit of Abstract Nonsense –

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# Preface

This text consists of a collection of Algebraic Geometry notes taken at the University of Calgary.

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month year

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# Part I

# 3264: Schubert Calculus

# Chapter 1

# Introduction

Throughout this book a **scheme** $X$ refers to a separated scheme of finite type over an algebraically closed field $k$ of characteristic 0.

**Definition 1.1** A scheme $X$ over $S$, $p : X \to S$, is said to be **separated** if the diagonal morphism $\Delta : X \to X \times_S X$, given by the universal property of the fiber product of $p$ along itself and the identity $1_X : X \to X$, is a **closed immersion**.

A **closed immersion** $f : Z \to X$ is a map of locally ringed spaces such that $f$ is a homeomorphism, $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Z$ is surjective with kernel $\mathcal{I}$, and the kernel $\mathcal{I}$ is locally generated by sections as a $\mathcal{O}_X$-module, i.e. for every $x \in X$ there exists an open neighborhood $U$ of $x$ such that $\mathcal{I}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules, i.e. there exists a set $I$, and global sections $s_i \in \Gamma(U, \mathcal{I}|_U), i \in I$ such that the map

$$\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{I}$$

is surjective.

**Definition 1.2** An $R$-scheme $X$, for $R$ a ring, is called a $R$-scheme of **finite type** if there exists a finite affine covering $(X_i)_{i \in I}$ of $X$ such that the $R$-algebras $\Gamma(X_i, \mathcal{O}_X)$ is a finite type $R$-algebra (i.e. finitely generated as an $R$-algebra)

In practice we will only consider quasi-projective schemes.

**Definition 1.3** We say a scheme $X$ over a ring $R$ is **quasi-projective** if $X$ is a quasicompact open subscheme of a projective $A$-scheme, where a projective $A$ scheme is isomorphic to Proj $S_\bullet$, where $S_\bullet$ is some finitely generated graded ring over $A$.

**Definition 1.4** A scheme $X$ is said to be **integral** if it is non-empty and $\mathcal{O}_X(U)$ is an integral domain for every non-empty open set $U$ - equivalently $X$ is irreducible and reduced, where $X$ is reduced if each $\mathcal{O}_X(U)$ is reduced. In other words, the nilradical of each $\mathcal{O}_X(U)$ is trivial (no nilpotent elements).

A **variety** will mean an integral scheme. If $X$ is a variety we write $k(X)$ for the field of rational functions on $X$ (i.e. the field of fractions of $\Gamma(X, \mathcal{O}_X)$).

**Definition 1.5** If $X$ is a scheme, then a $\mathcal{O}_X$-module $\mathcal{F}$ is **quasicoherent** if for every affine open subset Spec $A \subset X$, $\mathcal{F}|_{\text{Spec } A} \cong \widetilde{M}$ for some $A$ module $\widetilde{M}$. $\mathcal{F}$ is said to further be **coherent** if for every affine open Spec $A$, $\Gamma(\text{Spec } A, \mathcal{F})$ is a coherent $A$-module (i.e. it is finitely generated and for any map $A^{\oplus p} \to \Gamma(\text{Spec } A, \mathcal{F})$, the kernel is finitely generated).

**Definition 1.6** If $V$ is a vector space, its **projectivization** refers to the scheme $\text{Proj}(\text{Sym } V^*)$, where Sym $V$ is the symmetric algebra of $V$. The closed points of this space correspond to one-dimensional subspaces of $V$.

If $X, Y \subset \mathcal{P}^n$ are subvarieties, we define the **join** of $X$ and $Y$, denoted $\overline{X, Y}$, to be the closure of the union of lines meeting $X$ and $Y$ at distinct points. If $X \subset \mathcal{P}^n$, this is just the cone over $Y$ with vertex $X$; if $X$ and $Y$ are both linear subspaces, this is simply their span.

*Remark 1.1* There is a one-to-one correspondence between vector bundles on a scheme $X$ and locally free sheaves on $X$.

**Definition 1.7** A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be **locally free** if for every point $x \in X$, there exists a set $I$ and an open neighborhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.

# Chapter 2

# Chow Ring

## 2.1 The Chow Ring

Throughout we assume that $k$ is an algebraically closed field.

### 2.1.1 Cycles

Let $X$ be an algebraic.

**Definition 2.1** The **group of cycles** on $X$, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties (reduced irreducible subschemes) of $X$. The group $Z(X)$ is graded by dimension.

A cycle $Z = \sum n_i Y_i$, where the $Y_i$ are subvarieties, is **effective** if the coefficients $n_i$ are nonnegative. A **divisor**, or **Weil divisor**, is an $(n-1)$-cycle on a pure $n$-dimensional scheme.

To any closed subscheme $Y \subset X$ we associate an effective cycle $\langle Y \rangle$: if $Y \subset X$ is a subscheme, and $Y_1, ..., Y_s$ are the irreducible components of the reduced scheme $Y_{red}$, then, because our schemes are Noetherian, each local ring $O_{Y,Y_i}$ has a finite composition series. Writing $l_i$ for its length, we define $\langle Y \rangle = \sum l_i Y_i$.

### 2.1.2 Rational Equivalence

A **Chow group** of $X$ is the group of cycles of $X$ modulo **rational equivalence**. Informally, two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them.

**Definition 2.2** Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

where $t_0, t_1 \in \mathcal{P}^1$ and $\Phi$ is a subvariety of $\mathcal{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. We say that two cycles are **rationally equivalent** if their difference is in $\mathrm{Rat}(X)$, and we say that two subschemes are rationally equivalent if their associated cycles are rationally equivalent.

**Definition 2.3** The **Chow group** of $X$ is the quotient

$$A(X) = Z(X)/\mathrm{Rat}(X)$$

the **group of rational equivalence classes of cycles on** $X$. If $Y \in Z(X)$ is a cycle, we write $[Y] \in A(X)$.

## Chapter 3

# First Examples

## Chapter 4

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## Chapter 6

# Projective Bundles and their Chow Rings

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## Chapter 8

# Maps of Curves and Projective Spaces