

E/Ea Thompson

# Category Theory: A Viewpoint on Math

– In Pursuit of Abstract Nonsense –

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# Preface

This text consists of a collection of category theory notes recorded based on Emily Riehl's lecture notes and *Categories for the Working Mathematician*.

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## Chapter 1

# Categories and Fundamental Examples

### 1.1 Category Definitions

**Definition 1.1** A category  $\mathbf{C}$  is given by the following data:

1. a class  $\mathbf{Ob}(\mathbf{C})$  of objects of  $\mathbf{C}$
2. a family  $\mathbf{Hom}_{\mathbf{C}}$  associating with each pair  $A, B \in \mathbf{C}$  a class  $\mathbf{Hom}_{\mathbf{C}}(A, B)$  of morphisms from  $A$  to  $B$

so that:

1. For a morphism  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$ , we say that  $A$  is the domain object of  $f$  and codomain object of  $f$ , and we write  $f : A \rightarrow B$ .
2. For each object  $A$  of  $\mathbf{C}$  there is a designated identity morphism  $\text{Id}_A \in \mathbf{Hom}_{\mathbf{C}}(A, A)$  (i.e.  $\text{Id}_A : A \rightarrow A$ ).
3. For all  $A, B, C \in \mathbf{Ob}(\mathbf{C})$  a mapping

$$\circ : \mathbf{Hom}_{\mathbf{C}}(B, C) \times \mathbf{Hom}_{\mathbf{C}}(A, B) \rightarrow \mathbf{Hom}_{\mathbf{C}}(A, C)$$

called composition exists, and is defined such that for all  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$  and  $g \in \mathbf{Hom}_{\mathbf{C}}(B, C)$ , the following diagram commutes:

$$\begin{array}{ccc} & B & \\ f \nearrow & & \searrow g \\ A & \xrightarrow{g \circ f} & C \end{array}$$

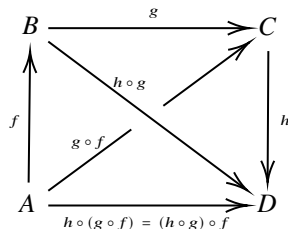
such that  $g \circ f \in \mathbf{Hom}_{\mathbf{C}}(A, C)$  is called the composite morphism.

## CHAPTER 1. CATEGORIES AND FUNDAMENTAL EXAMPLES

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This data is subject to the following axioms:

1. For any objects  $A$  and  $B$  and morphism  $f : A \rightarrow B$ , the composites  $\text{Id}_B \circ f$  and  $f \circ \text{Id}_A$  are equal to  $f$ .
2. For any composable triple of morphisms  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ , and  $h : C \rightarrow D$ , the composites  $h \circ (g \circ f)$  and  $(h \circ g) \circ f$  are equal. In particular, the following diagram commutes:



That is, the composition law is associative and unital with the identity morphisms serving as two-sided identities.

*Remark 1.1* The objects of a category are in bijective correspondence with the identity morphisms, which are uniquely determined by the property that they serve as two-sided identities with composition. Thus, we can define a category as a collection of morphisms with a partially defined composition operation that has certain special morphisms which are used to recognize composable pairs and which serve as two-sided identities.

*Example 1.1*

1. **Set** is the category with sets as its objects and set-theoretic functions, with specified domain and codomain, as its morphisms
2. **Top** is the category with topological spaces as its objects and continuous functions as its morphisms.
3. **Set**<sub>\*</sub> (**Top**<sub>\*</sub>) are the categories with sets (spaces) with a specified basepoint as objects and basepoint preserving (continuous) functions as morphisms. Note that a basepoint is a distinguished point in the set (space).
4. **Grp** is the category with groups homomorphisms as morphisms. The categories **Ring** of associative and unital rings and ring homomorphisms and **Field** of fields and field homomorphisms are defined similarly.
5. For a fixed unital but not necessarily commutative ring  $R$ , **R-Mod** is the category of left  $R$ -modules and  $R$ -module homomorphisms. This category is denoted by **Vect** <sub>$k$</sub>  when the ring happens to be a field  $k$  and abbreviated as **Ab** (for abelian groups) in the case of **Mod** <sub>$\mathbb{Z}$</sub> , as a  $\mathbb{Z}$ -module is precisely an abelian group.

6. **Graph** is the category with graphs as objects and graph morphisms (functions carrying vertices to vertices and edges to edges, preserving incidence relations) as morphisms. In the variant **DirGraph**, objects are directed graphs, whose edges are now depicted as arrows, and morphisms are directed graph morphisms, which preserve sources and targets.
7. **Man** is the category with smooth (i.e. infinitely differentiable) manifolds as objects and smooth maps as morphisms.
8. **Meas** is the category with measurable spaces as objects and measurable functions as morphisms.
9. **Poset** is the category with partially ordered sets as objects and order-preserving functions as morphisms.
10. **Ch<sub>R</sub>** is the category with chain complexes of  $R$ -modules as objects and chain homomorphisms as morphisms.

**Definition 1.2** A **chain complex**  $C_*$  is a collection  $(C_n)_{n \in \mathbb{Z}}$  of  $R$ -modules equipped with  $R$ -module homomorphisms  $d : C_n \rightarrow C_{n-1}$ , called **boundary homomorphisms**, with the property that  $d^2 = 0$ , i.e., the composite of any two boundary maps is the zero homomorphism. A map of chain complexes  $f : C_* \rightarrow C'_*$  is comprised of a collection of homomorphisms  $f_n : C_n \rightarrow C'_n$  so that  $df_n = f_{n-1}d$  for all  $n \in \mathbb{Z}$ .

11. For any **signature**  $\sigma$ , specifying  $n$ -array relation symbols, and for any collection of well formed sentences  $\mathbb{T}$  in the first order language associated to  $\sigma$ , there is a category **Model<sub>T</sub>** whose objects are  $\sigma$ -structures that **model**  $\mathbb{T}$ , i.e., sets equipped with appropriate  $n$ -array relations satisfying the axioms  $\mathbb{T}$ . Morphisms are functions that preserve the specified  $n$ -array relations in the “usual sense.” (4-6, 9, 10 are special cases of this)

These are all examples of **concrete** categories, those whose objects have underlying sets and whose morphisms are functions between those underlying sets.

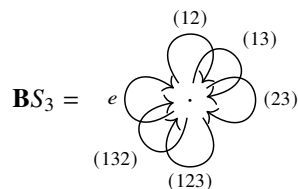
*Example 1.2*

1. For a unital ring  $R$ , **Mat<sub>R</sub>** is the category whose objects are positive integers and in which the set of morphisms from  $n$  to  $m$  is the set  $m \times n$  with values in  $R$ . Composition is by matrix multiplication

$$n \xrightarrow{A} m, \quad m \xrightarrow{B} k, \quad \rightsquigarrow \quad n \xrightarrow{B \cdot A} k$$

with identity matrices serving as the identity morphisms.

2. A group  $G$  (or, more generally, a monoid) defines a category **BG** with a single object. The group elements are its morphisms, each group element representing a distinct endomorphism of the single object, with composition given by multiplication. The identity element  $e \in G$  acts as the identity morphism for the unique object in this category. Example



3. A poset  $(P, \leq)$  (or, more generally, a preorder) may be regarded as a category. The elements of  $P$  are the objects of the category and there exists a unique morphism  $x \rightarrow y$  if and only if  $x \leq y$ . Transitivity of the relation " $\leq$ " implies that the required composite morphisms exist. Reflexivity implies that identity morphisms exist.
4. For any ordinal  $\alpha = \{\beta | \beta < \alpha\}$  defines a category whose objects are the smaller ordinals. For example, **0** is the category with no objects and no morphisms. **1** is the category with a single object and only its identity morphism. **2** is the category with two objects and a single non-identity morphism, conventionally depicted as  $0 \rightarrow 1$ ,  $\omega$  is the category freely generated by the graph

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$$

in the sense that every non-identity morphism can be uniquely factored as a composite of morphisms in the displayed graph.

5. A set may be regarded as a category in which the elements of the set define the objects and the only morphisms are the required identities. A category is discrete if every morphism is an identity.
6. **Htpy** is the category with spaces as its objects and homotopy classes of continuous maps as its morphisms.
7. **Measure** has measure spaces as objects. One reasonable choice for the morphisms is to take equivalence classes of measurable functions, where a parallel pair of functions are equivalent if their domain of difference is contained within a set of measure zero.

**Definition 1.3** A category is small if it has only a set's worth of arrows.

**Corollary 1.1** *By our previous remark we have that a small category has only a set's worth of objects. If  $\mathbf{C}$  is a small category, then there are functions*

$$\mathbf{Hom}_{\mathbf{C}} \begin{array}{c} \xrightarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\text{cod}} \end{array} \mathbf{Ob}_{\mathbf{C}}$$

*that send a morphism to its domain and its codomain and an object to its identity.*

**Definition 1.4** A category is locally small if between any pair of objects there is only a set's worth of morphisms. It is traditional to write  $\mathbf{C}(X, Y)$  or  $\mathbf{Hom}(X, Y)$  for the set of morphisms from  $X$  to  $Y$  in a locally small category  $\mathbf{C}$ . The set of arrows between a pair of fixed objects in a locally small category is typically called a hom-set.



**Definition 1.5** An isomorphism in a category is a morphism  $f : X \rightarrow Y$  for which there exists a morphism  $g : Y \rightarrow X$  such that  $g \circ f = \text{Id}_X$  and  $f \circ g = \text{Id}_Y$ . The objects  $X$  and  $Y$  are said to be isomorphic whenever there exists an isomorphism between  $X$  and  $Y$ , in which case one writes  $X \cong Y$ .

**Definition 1.6** An endomorphism, i.e., a morphism whose domain equals its codomain, that is an isomorphism is called an automorphism.

*Example 1.3*

1. The isomorphisms in **Set** are precisely the bijections.
2. The isomorphisms in **Grp**, **Ring**, **Field**, or **Mod<sub>R</sub>** are the bijective homomorphisms.
3. The isomorphisms in the category **Top** are the homeomorphisms, i.e., the continuous functions with continuous inverse, which is a stronger property than merely being a bijective continuous function.
4. The isomorphisms in the category **Htpy** are the homotopy equivalences.
5. In a poset  $(P, \leq)$ , the axiom of antisymmetry asserts that  $x \leq y$  and  $y \leq x$  imply that  $x = y$ . That is, the only isomorphisms in the category  $(P, \leq)$  are identities.

### ? I

In a concrete category, when are the isomorphisms precisely those maps in the category that induce bijections between the underlying sets? This requires some more constructions before we can answer it sufficiently.

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**Definition 1.7** A groupoid is a category in which every morphism is an isomorphism.

**Definition 1.8**

1. A group is a groupoid with one object.
2. For any space  $X$ , its fundamental groupoid  $\prod_1(X)$  is a category whose objects are the points of  $X$  and whose morphisms are endpoint-preserving homotopy classes of paths.

**Definition 1.9** A subcategory **D** of a category **C** is defined by restricting to a subcollection of objects and subcollection of morphisms subject to the requirements that the subcategory **D** contains the domain and codomain of any morphism in **D**, the identity morphism of any object in **D**, and the composite of any composable pair of morphisms in **D**.

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**Lemma 1.1** Any category  $\mathbf{C}$  contains a maximal groupoid, the subcategory containing all of the objects and only those morphisms that are isomorphisms.

**Proof** Let  $\mathbf{G}$  be the collection of isomorphisms with all objects in  $\mathbf{C}$ . First, since  $\mathbf{G}$  contains all objects of  $\mathbf{C}$ , it contains all domains and codomains for its morphisms. Next, observe that for any object  $X$  of  $\mathbf{G}$ ,  $\text{Id}_X \circ \text{Id}_X = \text{Id}_X$ , so  $\text{Id}_X$  is an isomorphism by definition, and is consequently in  $\mathbf{G}$ . Finally, let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be isomorphisms in  $\mathbf{G}$  with inverse morphisms  $f^{-1} : B \rightarrow A$  and  $g^{-1} : C \rightarrow B$  (which are also in  $\mathbf{G}$ ). Then, we consider the composite morphisms  $g \circ f : A \rightarrow C$  and  $f^{-1} \circ g^{-1} : C \rightarrow A$ . It follows that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ \text{Id}_B \circ f = f^{-1} \circ f = \text{Id}_A$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ f \circ f^{-1} \circ g^{-1} = g \circ \text{Id}_B \circ g^{-1} = g \circ g^{-1} = \text{Id}_C$$

so by definition we have that  $g \circ f$  and  $f^{-1} \circ g^{-1}$  are isomorphisms, and hence in  $\mathbf{G}$ . Thus  $\mathbf{G}$  is a subcategory in  $\mathbf{C}$ . Moreover, every isomorphism of  $\mathbf{C}$  is in  $\mathbf{G}$ , so it is indeed the maximal groupoid of  $\mathbf{C}$ .  $\square$

**Definition 1.10** For any category  $\mathbf{C}$  and object  $c$  of  $\mathbf{C}$ :

1. There is a category  $c/\mathbf{C}$  whose objects are morphisms  $f : c \rightarrow x$  with domain  $c$  in which a morphism from  $f : c \rightarrow x$  to  $g : c \rightarrow y$  is a map  $h : x \rightarrow y$  (in  $\mathbf{C}$ ) between the codomains so that the triangle:

$$\begin{array}{ccc} & c & \\ f \swarrow & & \searrow g \\ x & \xrightarrow{h} & y \end{array}$$

commutes, i.e., so that  $g = h \circ f$ . This category is called the slice category of  $\mathbf{C}$  under  $c$ .

2. There is a category  $\mathbf{C}/c$  whose objects are morphisms  $f : x \rightarrow c$  with codomain  $c$  in which a morphism from  $f : x \rightarrow c$  to  $g : y \rightarrow c$  is a map  $h : x \rightarrow y$  (in  $\mathbf{C}$ ) between the domains so that the triangle:

$$\begin{array}{ccc} x & \xrightarrow{h} & y \\ f \searrow & & \swarrow g \\ & c & \end{array}$$

commutes, i.e., so that  $f = g \circ h$ . This category is called the slice category of  $\mathbf{C}$  over  $c$ .

## 1.2 Duality

Let us consider the notion of “reversing the arrows” of a particular category.

**Definition 1.11** Let  $\mathbf{C}$  be any category. The opposite category  $\mathbf{C}^{op}$  has

1. the same objects as in  $\mathbf{C}$ , and
2. a morphism  $f^{op}$  in  $\mathbf{C}^{op}$  for each morphism  $f$  in  $\mathbf{C}$  such that the domain of  $f^{op}$  is defined to be the codomain of  $f$  and the codomain of  $f^{op}$  is defined to be the domain of  $f$ : that is

$$f^{op} : X \rightarrow Y \in \mathbf{C}^{op} \quad \leftrightarrow \quad f : Y \rightarrow X \in \mathbf{C}$$

That is,  $\mathbf{C}^{op}$  has the same objects and morphisms as  $\mathbf{C}$ , except that “each morphism is pointing in the opposite direction.” The remaining structure of the category  $\mathbf{C}^{op}$  is given as follows:

1. For each object  $X$ , the arrow  $\text{Id}_X^{op}$  serves as its identity in  $\mathbf{C}^{op}$
2. To define composition, observe that a pair of morphisms  $f^{op}, g^{op}$  in  $\mathbf{C}^{op}$  is composable precisely when the pair  $g, f$  is composable in  $\mathbf{C}$ , i.e., precisely when the codomain of  $g$  equals the domain of  $f$ . We then define  $g^{op} \circ f^{op}$  to be  $(f \circ g)^{op}$ :

$$\begin{array}{ccc} f^{op} : X \rightarrow Y, g^{op} : Y \rightarrow Z \in \mathbf{C}^{op} & \rightsquigarrow & g^{op} \circ f^{op} : X \rightarrow Z \in \mathbf{C}^{op} \\ \updownarrow & & \updownarrow \\ g : Z \rightarrow Y, f : Y \rightarrow X \in \mathbf{C} & \rightsquigarrow & f \circ g : Z \rightarrow X \in \mathbf{C} \end{array}$$

*Example 1.4*

1.  $\mathbf{Mat}_R^{op}$  is the category whose objects are non-zero natural numbers and in which a morphism from  $m$  to  $n$  is an  $m \times n$  matrix with values in  $R$ .
2. When a preorder  $(P, \leq)$  is regarded as a category, its opposite category is the category that has a morphism  $x \rightarrow y$  if and only if  $y \leq x$ . For example,  $\omega^{op}$  is the category freely generated by the graph

$$\dots \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0.$$

3. If  $G$  is a group, regarded as a one-object groupoid, the category  $(\mathbf{B}G)^{op} \cong \mathbf{B}(G^{op})$  is again a one-object groupoid, and hence a group. The group  $G^{op}$  is called the opposite group and is used to define right actions as a special case of left actions.

*Remark 1.2* Any theorem containing a universal quantification of the form “for all categories  $\mathbf{C}$ ” also necessarily applies to the opposites of these categories. Interpreting the result in the dual context

leads to a **dual theorem**, proven by the dual of the original proof, in which the direction of each arrow appearing in the argument is reversed.

**Lemma 1.2** *The following are equivalent:*

1.  $f : x \rightarrow y$  is an isomorphism in  $\mathbf{C}$
2. For all objects  $c \in \mathbf{C}$ , post-composition with  $f$  defines a bijection

$$f_* : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, y)$$

3. For all objects  $c \in \mathbf{C}$ , pre-composition with  $f$  defines a bijection

$$f^* : \mathbf{C}(y, x) \rightarrow \mathbf{C}(x, c)$$

*The is to say, isomorphisms in a locally small category are defined representably in terms of isomorphisms in the category of sets. However, this also applies to non-locally small categories given certain set theoretical foundations.*

**Proof** First we will prove the equivalence 1.  $\iff$  2.:

Assuming 1., namely that  $f : x \rightarrow y$  is an isomorphism with inverse  $g : y \rightarrow x$ , then, as an immediate application of the associativity and identity laws for composition in a category, post-composition with  $g$  defines an inverse function

$$g_* : \mathbf{C}(c, y) \rightarrow \mathbf{C}(c, x)$$

to  $f_*$  in the sense that the composites

$$g_* \circ f_* : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, x) \quad \text{and} \quad f_* \circ g_* : \mathbf{C}(c, y) \rightarrow \mathbf{C}(c, y)$$

are both the identity function: for any  $h : c \rightarrow x$  and  $k : c \rightarrow y$ ,  $g_* \circ f_*(h) = g \circ f \circ h = h$ , and  $f_* \circ g_*(k) = f \circ g \circ k = k$ .

Conversely, assuming 2., there must be an element  $g \in \mathbf{C}(y, x)$  whose image under  $f_* : \mathbf{C}(y, x) \rightarrow \mathbf{C}(y, y)$  is  $\text{Id}_y$ . By construction,  $\text{Id}_y = f \circ g$ . But, now by associativity of composition, the elements  $g \circ f, \text{Id}_x \in \mathbf{C}(x, x)$  have the common image  $f$  under the function  $f_* : \mathbf{C}(x, x) \rightarrow \mathbf{C}(x, y)$ , whence  $g \circ f = \text{Id}_x$ . Thus,  $f$  and  $g$  are inverse isomorphisms.

To prove the equivalence 1.  $\iff$  3. for all categories, we use the principle of duality. Indeed, since we have proven 1.  $\iff$  2. for all categories, it applies to the category  $\mathbf{C}^{op}$ : i.e., a morphism  $f^{op} : y \rightarrow x$  in  $\mathbf{C}^{op}$  is an isomorphism if and only if

$$f_*^{op} : \mathbf{C}^{op}(c, y) \rightarrow \mathbf{C}^{op}(c, x)$$

is an isomorphism for all  $c \in \mathbf{C}^{op}$ . Interpreting the data of  $\mathbf{C}^{op}$  in its opposite category, the previous statement expresses the same mathematical content as

$$f^* : \mathbf{C}(y, c) \rightarrow \mathbf{C}(x, c)$$

is an isomorphism for all  $c \in \mathbf{C}$ . That is:  $\mathbf{C}^{op}(c, x) = \mathbf{C}(x, c)$ , post composition with  $f^{op}$  in  $\mathbf{C}^{op}$  translates to pre-composition with  $f$  in the opposite category  $\mathbf{C}$ . The notion of isomorphism is self-dual:  $f^{op} : y \rightarrow x$  is an isomorphism in  $\mathbf{C}^{op}$  if and only if  $f : x \rightarrow y$  is an isomorphism in  $\mathbf{C}$ . So the equivalence 1.  $\iff$  2. in  $\mathbf{C}^{op}$  expresses the equivalence 1.  $\iff$  3. in  $\mathbf{C}$ .  $\square$

**Definition 1.12** A morphism  $f : x \rightarrow y$  in a category is

1. a **monomorphism** if for any parallel morphisms  $h, k : w \rightrightarrows x$ ,  $f \circ h = f \circ k$  implies that  $h = k$  (left cancellable); or
2. an **epimorphism** if for any parallel morphisms  $h, k : y \rightrightarrows z$ ,  $h \circ f = k \circ f$  implies that  $h = k$  (right cancellable)

*Remark 1.3* Note that a monomorphism or epimorphism in  $\mathbf{C}$  is, respectively, an epimorphism or monomorphism in  $\mathbf{C}^{op}$ .

*Note 1.1* In adjectival form, a monomorphism is **monic**, or in shorthand **mono**, and is denoted by " $\rightarrow$ ," while an epimorphism is **epic**, or in shorthand **epi**, and is denoted by " $\rightarrow$ ."

**Definition 1.13 (Alternative Mono and Epi Definitions)** A morphism  $f : x \rightarrow y$ :

1. is a monomorphism in  $\mathbf{C}$  if and only if for all objects  $c \in \mathbf{C}$ , post-composition with  $f$  defines an injection  $f_* : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, y)$ .
2. is an epimorphism in  $\mathbf{C}$  if and only if for all objects  $c \in \mathbf{C}$ , pre-composition with  $f$  defines an injection  $f^* : \mathbf{C}(y, c) \rightarrow \mathbf{C}(x, c)$ .

*Example 1.5* Suppose  $f : X \rightarrow Y$  is a monomorphism in the category of sets. Then in particular, given any two maps  $x, x' : \mathbf{1} \rightrightarrows X$ , whose domain is the singleton set, if  $f \circ x = f \circ x'$  then  $x = x'$ . Thus, monomorphisms are injective functions. Conversely, any injective function can easily be seen to be a monomorphism.

Similarly, a function  $f : X \rightarrow Y$  is an epimorphism in the category of sets if and only if it is surjective. Given functions  $h, k : Y \rightrightarrows Z$ , the equation  $h \circ f = k \circ f$  says exactly that  $h$  is equal to  $k$  on the image of  $f$ . This implies that  $h = k$  in the case where the image is all of  $Y$ .

*Example 1.6* Suppose that  $x \xrightarrow{s} y \xrightarrow{r} x$  are morphisms so that  $r \circ s = \text{Id}_x$ . The map  $s$  is a **section** or **right inverse** to  $r$ , while the map  $r$  defines a **retraction** or **left inverse** to  $s$ . The maps  $s$  and  $r$  express the object  $x$  as a **retract** of the object  $y$ .

In this case,  $s$  is always a monomorphism and, dually,  $r$  is always an epimorphism. To acknowledge these one-sided inverses,  $s$  is said to be a **split monomorphism** and  $r$  is said to be a **split epimorphism**.

**Lemma 1.3**

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- (i) If  $f : x \rightarrowtail y$  and  $g : y \rightarrowtail z$  are monomorphisms, then so is  $g \circ f : x \rightarrowtail z$ .
- (ii) If  $f : x \rightarrow y$  and  $g : y \rightarrow z$  are morphisms so that  $g \circ f$  is monic, then  $f$  is monic.

*Dually:*

- (i') If  $f : x \twoheadrightarrow y$  and  $g : y \twoheadrightarrow z$  are epimorphisms, then so is  $g \circ f : x \twoheadrightarrow z$ .
- (ii') If  $f : x \rightarrow y$  and  $g : y \rightarrow z$  are morphisms so that  $g \circ f$  is epic, then  $g$  is epic.

## Chapter 2

# Functors and Natural Transformations

### 2.1 Functoriality

Following the principles of Category Theory, we note that Categories are themselves mathematical objects, so what is a morphism between categories?

**Definition 2.1** A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ , between categories  $\mathbf{C}$  and  $\mathbf{D}$ , consists of the following data:

1. An object  $Fc \in \mathbf{D}$ , for each object  $c \in \mathbf{C}$ .
2. A morphism  $Ff : Fc \rightarrow Fc' \in \mathbf{D}$ , for each morphism  $f : c \rightarrow c' \in \mathbf{C}$ , so that the domain and the codomain of  $Ff$  are, respectively, equal to  $F$  applied to the domain or codomain of  $f$ .

The assignments are required to satisfy the following two functoriality axioms:

1. For any composable pair  $f, g$  in  $\mathbf{C}$ ,  $Fg, Ff$  are composable and  $Fg \circ Ff = F(g \circ f)$ .
2. For each object  $c$  in  $\mathbf{C}$ ,  $F(1_c) = 1_{Fc}$ .

Concisely a functor consists of a mapping on objects and a mapping on morphisms that preserves all of the structure of a category, namely domains and codomains, composition, and identities.

**Definition 2.2** An endofunctor is a functor whose domain is equal to its codomain.

*Example 2.1*

- (i) There is an endofunctor  $P : \mathbf{Set} \rightarrow \mathbf{Set}$  that sends a set  $A$  to its power set  $PA = \{A' \subseteq A\}$  and a function  $f : A \rightarrow B$  to the direct image function  $f_* : PA \rightarrow PB$  that sends  $A' \subseteq A$  to  $f(A') \subseteq B$ .
- (ii) Many categories have a forgetful functor, a general term that is used for any functor that forgets structure, whose codomain is the category of sets. For example,  $U : \mathbf{Grp} \rightarrow \mathbf{Set}$  sends a group to its underlying set and a group homomorphism to its underlying function.

- (iii) There are intermediate forgetful functors  $\mathbf{Mod}_R \rightarrow \mathbf{Ab}$  and  $\mathbf{Ring} \rightarrow \mathbf{Ab}$  that forget some but not all of the algebraic structure. The inclusion functors  $\mathbf{Ab} \hookrightarrow \mathbf{Group}$  and  $\mathbf{Field} \hookrightarrow \mathbf{Ring}$  may also be regarded as “forgetful.”
- (iv) Similarly, there are forgetful functors  $\mathbf{Group} \rightarrow \mathbf{Set}_*$  and  $\mathbf{Ring} \rightarrow \mathbf{Set}_*$  that take the basepoint to be the identity and zero elements, respectively. These assignments are functorial because group and ring homomorphisms necessarily preserve these elements.
- (v) The *fundamental group* defines a functor  $\pi_1 : \mathbf{Top}_* \rightarrow \mathbf{Group}$ ; a continuous function  $f : (X, x) \rightarrow (Y, y)$  of based spaces induces a group homomorphism  $f_* : \pi_1(X, x) \rightarrow \pi_1(Y, y)$  and this assignment is functorial, satisfying the two functoriality axioms.
- (vi) There is a functor  $F : \mathbf{Set} \rightarrow \mathbf{Group}$  that sends a set  $X$  to the *free group* on  $X$ . This is the group whose elements are finite “words” whose letters are elements of  $x \in X$  or their formal inverses  $x^{-1}$ , modulo an equivalence relation that equates the words  $xx^{-1}$  and  $x^{-1}x$  with the empty word. Multiplication is by concatenation, with the empty word serving as the identity.
- (vii) The chain rule expresses the functoriality of the derivative. Let  $\mathbf{Euclid}_*$  denote the category whose objects are pointed finite-dimensional Euclidean spaces  $(\mathbb{R}^n, a)$ , or, better, open subsets thereof, and whose morphisms are pointed differentiable functions. The *total derivate* of  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , evaluated at the designated basepoint  $a \in \mathbb{R}^n$ , gives rise to a matrix called the *Jacobian matrix* defining the directional derivatives of  $f$  at the point  $a$ . If  $f$  is given by component functions  $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$ , the  $(i, j)$ -entry of this matrix is  $\frac{\partial}{\partial x_j} f_i(a)$ . This defines the action on morphisms of a functor  $D : \mathbf{Euclid}_* \rightarrow \mathbf{Mat}_{\mathbb{R}}$ ; on objects,  $D$  assigns a pointed Euclidean space its dimension. Given  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  carrying the designated basepoint  $f(a) \in \mathbb{R}^m$  to  $gf(a) \in \mathbb{R}^k$ , functoriality of  $D$  asserts that the product of the Jacobian of  $f$  at  $a$  with the Jacobian of  $g$  at  $f(a)$  equals the jacobian of  $gf$  at  $a$ .

To illustrate the usefulness of functoriality, we will apply the functoriality of the fundamental group construction  $\pi_1 : \mathbf{Top}_* \rightarrow \mathbf{Group}$  to prove the following theorem:

**Theorem 2.1 (Brouwer Fixed Point Theorem)** *Any continuous endomorphisms of a 2-dimensional disk  $D^2$  has a fixed point.*

**Proof** Assuming  $f : D^2 \rightarrow D^2$  is such that  $f(x) \neq x$  for all  $x \in D^2$ , there is a continuous function  $r : D^2 \rightarrow S^1$  that carries a point  $x \in D^2$  to the intersection of the ray from  $f(x)$  to  $x$  with the boundary circle  $S^1$ . Note that  $r$  fixes the points on the boundary circle  $S^1 \subseteq D^2$ . Thus,  $r$  defines a retraction of the inclusion  $\iota : S^1 \hookrightarrow D^2$ , which is to say, the composite  $S^1 \xrightarrow{\iota} D^2 \xrightarrow{r} S^1$  is the identity.

Pick any basepoint on the boundary circle  $S^1$  and apply the functor  $\pi_1$  to obtain a composable pair of group homomorphisms:

$$\pi_1(S^1) \xrightarrow{\pi_1(\iota)} \pi_1(D^2) \xrightarrow{\pi_1(r)} \pi_1(S^1)$$

By the functoriality axioms, we must have

$$\pi_1(r) \circ \pi_1(\iota) = \pi_1(r \circ \iota) = \pi_1(1_{S^1}) = 1_{\pi_1(S^1)}$$



However, a computation involving covering spaces reveals that  $\pi_1(S^1) = \mathbb{Z}$ , while  $\pi_1(D^2) = 0$ , the trivial group. The composite endomorphism  $\pi_1(r) \circ \pi_1(\iota)$  of  $\mathbb{Z}$  must be zero, since it factors through the trivial group. Thus, it cannot equal the identity homomorphism, which carries the generator  $1 \in \mathbb{Z}$  to itself ( $0 \neq 1$ ). This contradiction proves that the retraction  $r$  cannot exist, and so  $f$  must have a fixed point.  $\square$

The functors defined thus far are called **covariant** so as to distinguish them from another variety of functor now introduced:

**Definition 2.3** A **contravariant functor**  $F$  from  $\mathbf{C}$  to  $\mathbf{D}$  is a functor  $F : \mathbf{C}^{op} \rightarrow \mathbf{D}$ . Explicitly, this consists of the following data:

1. An object  $Fc \in \mathbf{D}$ , for each object  $c \in \mathbf{C}$ .
2. A morphism  $Ff : Fc' \rightarrow Fc \in \mathbf{D}$ , for each morphism  $f : c \rightarrow c' \in \mathbf{C}$ , so that the domain and codomain of  $Ff$  are, respectively, equal to  $F$  applied to the codomain or domain of  $f$ .

The assignments are required to satisfy the following two **functoriality axioms**:

1. For any composable pair  $f, g$  in  $\mathbf{C}$ ,  $Fg, Ff$  are composable in  $\mathbf{D}$  and  $Ff \circ Fg = F(g \circ f)$ .
2. For each object  $c$  in  $\mathbf{C}$ ,  $F(1_c) = 1_{Fc}$ .

Pictorially we draw:

$$F : \mathbf{C}^{op} \rightarrow \mathbf{D}$$

$$\begin{array}{ccc} c & \xrightarrow{\quad} & Fc \\ f \downarrow & \xrightarrow{\quad} & \uparrow Ff \\ c' & \xrightarrow{\quad} & Fc' \end{array}$$

*Example 2.2*

- (i) There is a functor  $(-)^* : \mathbf{Vect}_{\mathbb{R}}^{op} \rightarrow \mathbf{Vect}_{\mathbb{R}}$  that carries a vector space to its **dual vector space**  $V^* = \mathbf{Hom}(V, \mathbb{R})$ . A vector in  $V^*$  is a **linear functional** on  $V$ , i.e., a linear map  $V \rightarrow \mathbb{R}$ . This functor is contravariant, with a linear map  $\phi : V \rightarrow W$  sent to the linear map  $\phi^* : W^* \rightarrow V^*$  that pre-composes a linear functional  $W \xrightarrow{\omega} \mathbb{R}$  with  $\phi$  to obtain a linear functional  $V \xrightarrow{\phi^* \omega} \mathbb{R}$ .
- (ii) The functor  $O : \mathbf{Top}^{op} \rightarrow \mathbf{Poset}$  that carries a space  $X$  to its poset  $O(X)$  of open subsets is contravariant on the category of spaces: a continuous map  $f : X \rightarrow Y$  gives rise to a function  $f^{-1} : O(Y) \rightarrow O(X)$  that carries an open subset  $U \subseteq Y$  to its preimage  $f^{-1}(U)$ , which is open in  $X$ . A similar functor  $C : \mathbf{Top}^{op} \rightarrow \mathbf{Poset}$  carries a space to its poset of closed subsets.

- (iii) For a generic small category  $\mathbf{C}$ , a functor  $\mathbf{C}^{op} \rightarrow \mathbf{Set}$  is called a (set-valued) **presheaf** on  $\mathbf{C}$ . A typical example is the functor  $O(X)^{op} \rightarrow \mathbf{Set}$  whose domain is the poset  $O(X)$  of open subset of a topological space  $X$  and whose value at  $U \subseteq X$  is the set of continuous real-valued functions on  $U$ . The action on morphisms is by restriction. This presheaf is a *sheaf*, if it satisfies an axiom to be introduced later.
- (vi) Presheaves on the category  $\Delta$ , of finite non-empty ordinals and order preserving maps, are called **simplicial sets**.  $\Delta$  is also called the **simplex category**. The ordinal  $n + 1 = \{0, 1, \dots, n\}$  may be thought of as a direct version of the topological  $n$ -simplex and, with this interpretation, is typically denoted by “[ $n$ ]” by algebraic topologists.

**Lemma 2.1** *Functors preserve isomorphisms.*

**Proof** Consider a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  between categories  $\mathbf{C}$  and  $\mathbf{D}$ . Suppose  $f : c \rightarrow c'$  is an isomorphism in  $\mathbf{C}$  with inverse isomorphisms  $f^{-1} : c' \rightarrow c$ . Then consider the image morphisms  $Ff : Fc \rightarrow Fc'$  and  $Ff^{-1} : Fc' \rightarrow Fc$ . It follows by functoriality of  $F$  that

$$Ff \circ Ff^{-1} = F(f \circ f^{-1}) = F(1_{c'}) = 1_{Fc'}$$

and

$$Ff^{-1} \circ Ff = F(f^{-1} \circ f) = F(1_c) = 1_{Fc}$$

Thus  $Ff$  is indeed an isomorphism in  $\mathbf{D}$  with inverse  $Ff^{-1}$ . □

**Example 2.3** Let  $G$  be a group, regarded as a one-object category  $\mathbf{BG}$ . A functor  $X : \mathbf{BG} \rightarrow \mathbf{C}$  specifies an object  $X \in \mathbf{C}$  (the unique object in its image) together with an endomorphism  $g_* : X \rightarrow X$  for each  $g \in G$ . This assignment must satisfy two conditions:

- (i)  $h_*g_* = (hg)_*$  for all  $g, h \in G$ .
- (ii)  $e_* = 1_X$ , where  $e \in G$  is the identity element.

In summary, the functor  $\mathbf{BG} \rightarrow \mathbf{C}$  defines an **action** of the group  $G$  on the object  $X \in \mathbf{C}$ . When  $\mathbf{C} = \mathbf{Set}$ , the object  $X$  endowed with such an action is called a  **$G$ -set**. When  $\mathbf{C} = \mathbf{Vect}_{\mathbb{F}}$ , the object  $X$  is called a  **$G$ -representation**. When  $\mathbf{C} = \mathbf{Top}$ , the object  $X$  is called a  **$G$ -space**.

The action specified by a functor  $\mathbf{BG} \rightarrow \mathbf{C}$  is sometimes called a **left action**. A **right action** is a contravariant functor  $\mathbf{BG}^{op} \rightarrow \mathbf{C}$ . As before, each  $g \in G$  determines an endomorphism  $g^* : X \rightarrow X$  in  $\mathbf{C}$  and the identity element must act trivially. But now, for a pair of elements  $g, h \in G$ , these actions must satisfy the composition rule  $(hg)^* = g^*h^*$ .

Because the elements  $g \in G$  are isomorphisms when regarded as morphisms in the 1-object category  $\mathbf{BG}$  that represents the group, their images under any such functor must also be isomorphisms in the target category. In particular, in the case of a  $G$ -representation  $V : \mathbf{BG} \rightarrow \mathbf{Vect}_{\mathbb{F}}$ , the linear map  $g_* : V \rightarrow V$  must be an *automorphism* of the vector space  $V$ .

**Corollary 2.1** *When a group  $G$  acts functorially on an object  $X$  in a category  $\mathbf{C}$ , its elements  $g$  must act by automorphisms  $g_* : X \rightarrow X$  and, moreover,  $(g_*)^{-1} = (g^{-1})_*$ .*

A functor may or may not preserve monomorphisms or epimorphisms, but an argument similar to that employed previously shows that a functor necessarily preserves split monomorphisms (sections) and split epimorphisms (retracts).

**Definition 2.4** If  $\mathbf{C}$  is locally small, then for any object  $c \in \mathbf{C}$  we may define a pair of covariant and contravariant functors represented by  $c$ :

$$\mathbf{C} \xrightarrow{\mathbf{C}(c, -)} \mathbf{Set} \quad \mathbf{C}^{op} \xrightarrow{\mathbf{C}(-, c)} \mathbf{C}$$

$$\begin{array}{ccccc} x & \longmapsto & \mathbf{C}(c, x) & x & \longmapsto & \mathbf{C}(x, c) \\ \downarrow f & \longrightarrow & \downarrow f_* & \downarrow f & \longrightarrow & \uparrow f^* \\ y & \longmapsto & \mathbf{C}(c, y) & y & \longmapsto & \mathbf{C}(y, c) \end{array}$$

The functor  $\mathbf{C}(c, -)$  carries a morphism  $f : x \rightarrow y$  to the post-composition function  $f_* : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, y)$  while, dually, the functor  $\mathbf{C}(-, c)$  carries  $f$  to the pre-composition function  $f^* : \mathbf{C}(y, c) \rightarrow \mathbf{C}(x, c)$ .

A **bifunctor** is the name for a functor of two variables. Its domain is given by the product of a pair of categories:

**Definition 2.5** For any categories  $\mathbf{C}$  and  $\mathbf{D}$ , there is a category  $\mathbf{C} \times \mathbf{D}$ , their **product**, whose

1. objects are ordered pairs  $(c, d)$ , where  $c$  is an object of  $\mathbf{C}$  and  $d$  is an object of  $\mathbf{D}$ ,
2. morphisms are ordered pairs  $(f, g) : (c, d) \rightarrow (c', d')$ , where  $f : c \rightarrow c' \in \mathbf{C}$  and  $g : d \rightarrow d' \in \mathbf{D}$ , and
3. in which composition and identities are defined componentwise.

**Definition 2.6** If  $\mathbf{C}$  is locally small, then there is a **two-sided represented functor**

$$\mathbf{C}(-, -) : \mathbf{C}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$$

defined in the evident manner. A pair of objects  $(x, y)$  is mapped to the hom-set  $\mathbf{C}(x, y)$ . A pair of morphisms  $f : w \rightarrow x$  and  $h : y \rightarrow z$  is sent to the function

$$\begin{array}{ccc} \mathbf{C}(x, y) & \xrightarrow{(f^*, h_*)} & \mathbf{C}(w, z) \\ g & \mapsto & hgf \end{array}$$

that takes an arrow  $g : x \rightarrow y$  and then pre-composes with  $f$  and post-composes with  $h$  to obtain  $hgf : w \rightarrow z$ .

We denote the category of small categories by **Cat**, and the category of locally small categories by **CAT**. Now, the notion of an isomorphism of categories arises naturally as a pair of inverse functors  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{C}$  so that the composites  $GF$  and  $FG$ , respectively, equal the identity functors on **C** and **D**.

*Example 2.4* For instance:

- (i) The functor  $(-)^{op} : \mathbf{CAT} \rightarrow \mathbf{CAT}$  defines a non-trivial automorphism of the category of categories. Note that a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  also defines a functor  $F : \mathbf{C}^{op} \rightarrow \mathbf{D}^{op}$ .
- (ii) For any group  $G$ , the categories **BG** and **BG**<sup>op</sup> are isomorphic via the functor  $(-)^{-1}$  that sends each morphism  $g \in G$  to its inverse. Any right action can be converted into a left action by precomposing with this isomorphism, which has the effect of “inserting inverses in the formula” defining the endomorphism associated to a particular group element.
- (iii) Any ring  $R$  has an opposite ring  $R^{op}$  with the same underlying abelian group but with the product of elements  $r$  and  $s$  in  $R^{op}$  defined to be the product  $s \cdot r$  of the elements  $s$  and  $r$  in  $R$ . A left  $R$ -module is the same thing as a right  $R^{op}$ -module, which is to say there is a covariant isomorphism of categories  $\mathbf{Mod}_R \cong_{R^{op}} \mathbf{Mod}$  between the category of left  $R$ -modules and the category of right  $R^{op}$ -modules.

Although we have seen some examples of it, a category is *not* typically isomorphic to its opposite category.

Now, although we have defined the definition of isomorphisms of categories, it is often far too restrictive. Instead we note that the collections **Hom**(**C**, **C**) and **Hom**(**D**, **D**) are more than just collections, but have higher-dimensional structure. Indeed, **Hom**(**C**, **D**) defines a category of functors, as studied next.

## 2.2 Naturality

**Definition 2.7** Given categories **C** and **D** and functors  $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ , a natural transformation  $\alpha : F \Rightarrow G$  consists of:

- an arrow  $\alpha_c : Fc \rightarrow Gc$  in **D** for each object  $c \in \mathbf{C}$ , the collection of which define the components of the natural transformation,

so that, for any morphism  $f : c \rightarrow c'$  in **C**, the following square of morphisms in **D**

$$\begin{array}{ccc}
 Fc & \xrightarrow{Ff} & Fc' \\
 \alpha_c \downarrow & \circlearrowright & \downarrow \alpha_{c'} \\
 Gc & \xrightarrow{Gf} & Gc'
 \end{array}$$

commutes, i.e.,  $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$ .

A **natural isomorphism** is a natural transformation  $\alpha : F \Rightarrow G$  in which every component  $\alpha_c$  is an isomorphism. In this case, the natural isomorphism may be depicted as  $\alpha : F \cong G$ .

*Example 2.5*

- (i) For vector spaces of any dimension, the map  $\mathbf{ev} : V \rightarrow V^{**}$  that sends  $v \in V$  to the linear function  $\mathbf{ev}_v : V^* \rightarrow \mathbb{F}$  defines the components of a natural transformation from the identity endofunctor on  $\mathbf{Vect}_{\mathbb{F}}$  to the double dual functor. To check that the naturality square

$$\begin{array}{ccc}
 V & \xrightarrow{\phi} & W \\
 \mathbf{ev} \downarrow & \circlearrowright & \downarrow \mathbf{ev} \\
 V^{**} & \xrightarrow{\phi^{**}} & W^{**}
 \end{array}$$

commutes for any linear map  $\phi : V \rightarrow W$ , it suffices to consider the image of a generic vector  $v \in V$ . By definition,  $\mathbf{ev}_{\phi v} : W^* \rightarrow \mathbb{F}$  carries a functional  $f : W \rightarrow \mathbb{F}$  to  $f(\phi v)$ . Recalling the definition of the action of the dual functor on morphisms, we see that  $\phi^{**}(\mathbf{ev}_v) : W^* \rightarrow \mathbb{F}$  carries a functional  $f : W \rightarrow \mathbb{F}$  to  $f\phi(v)$ , which amounts to the same thing.

- (ii) By contrast, the identity functor and the single dual functor on finite-dimensional vector spaces are not naturally isomorphic. One technical obstruction is that the identity functor is covariant while the dual functor is contravariant (though this can be accommodated with *extranatural transformations*). More significant is the essential failure of naturality. The isomorphisms  $V \cong V^*$  that can be defined whenever  $V$  is finite dimensional require a choice of basis, which is preserved by essentially no linear maps, indeed by no non-identity linear endomorphism.
- (iii) For a group  $G$ , the functor  $X : \mathbf{BG} \rightarrow \mathbf{C}$  corresponds to an object  $X \in \mathbf{C}$  equipped with a left action of  $G$ . A natural transformation between a pair  $X, Y : \mathbf{BG} \rightrightarrows \mathbf{C}$ , say  $\alpha : X \Rightarrow Y$  consists of a single morphism  $\alpha : X \rightarrow Y$  in  $\mathbf{C}$  that is *G-equivariant*, meaning that for each  $g \in G$ , the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{g_*} & X \\
 \alpha \downarrow & \circlearrowleft & \downarrow \alpha \\
 Y & \xrightarrow{g_*} & Y
 \end{array}$$

commutes.

- (iv) The construction of the opposite group defines a covariant endofunctor  $(-)^{op} : \mathbf{Group} \rightarrow \mathbf{Group}$  of the category of groups: a homomorphism  $\phi : G \rightarrow H$  induces a homomorphism  $\phi^{op} : G^{op} \rightarrow H^{op}$  defined by  $\phi^{op}(g) = \phi(g)$ . This functor is naturally isomorphic to the identity. Define  $\eta_G : G \rightarrow G^{op}$  to be the homomorphism that sends  $g \in G$  to its inverse  $g^{-1} \in G^{op}$ ; this mapping does not define an automorphism of  $G$ , because it fails to commute with the group multiplication, but it does define a homomorphism  $G \rightarrow G^{op}$ . Now given any homomorphism  $\phi : G \rightarrow H$ , the diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\phi} & H \\
 \eta_G \downarrow & \circlearrowleft & \downarrow \eta_H \\
 G^{op} & \xrightarrow{\phi^{op}} & H^{op}
 \end{array}$$

commutes because  $\phi^{op}(g^{-1}) = \phi(g^{-1}) = \phi(g)^{-1}$ .

- (v) Define an endofunctor of  $\mathbf{Vect}_{\mathbb{F}}$  by  $V \mapsto V \otimes V$ . There is a natural transformation from the identity functor to this endofunctor whose components are the zero maps, but this is the only such natural transformation: there is no basis-independent way to define a linear map  $V \rightarrow V \otimes V$ .

## **Chapter 3**

### **Universality**





## **Chapter 4**

### **Cones and Limits**



## **Chapter 5**

### **Adjoint**



## **Chapter 6**

### **Monads**



## **Chapter 7**

### **Kan Extensions**

