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Algebraic Geometry: A Complete Guide

– In Pursuit of Abstract Nonsense –

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Preface

This text consists of a collection of Algebraic Geometry notes taken at the University of Calgary.

Place(s),   *Firstname Surname*
month year   *Firstname Surname*

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Part I

Preliminaries

Chapter 1

Basic Category Theory

In this chapter we cover the basics of Category Theory necessary to discuss algebraic geometry.

1.1.0 Categories

Recall a **category** consists of a collection of **objects** and for each pair of objects a collection of **morphisms** or **arrows** between them. We denote the collection of objects in a category $\mathscr{C}$ by $\mathrm{obj}(\mathscr{C})$, or simply $\mathscr{C}$. We denote the collection of arrows between $A, B \in \mathscr{C}$ by $\mathbf{Hom}(A, B)$, or simply $\mathscr{C}(A, B)$. We have composition of morphisms with equivalent domains and codomains which is associative and unital. We have a sensible notion of **isomorphism** between objects of a category, simply being arrows with inverses. If the domain and codomain of an isomorphism are the same we say that it is an **automorphism**.

Example 1.1 The prototypical example of a category is the category of all small sets, **Set**, with objects small sets and arrows set-functions.

Example 1.2 Another common example is the category $\mathbf{Vect}_k$ of vector spaces over a given field k. Objects are k-vector spaces and morphisms are linear transformations.

A category in which each morphism is an isomorphism is referred to as a **groupoid**. Consequently, it is simple to see that a group is simply a groupoid of one object. A groupoid which is not a group contains more than one object, so the composition rule is a partial function on the collection of isomorphisms of the category. Generally for $A \in \mathscr{C}$ the subcollection of invertible elements of $\mathbf{Hom}(A, A)$ form a group, the automorphism group of A.

Remark 1.1 If $A, B \in \mathscr{C}$ are isomorphic objects, with $f : A \to B$ such an isomorphism, then $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ with isomorphism mapping $\varphi \mapsto f \circ \varphi \circ f^{-1}$, with inverse $\psi \mapsto f^{-1} \circ \psi \circ f$.

Example 1.3 If X is a topological space, the fundamental groupoid is the category where the objects are points of X and the morphisms $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the pointed fundamental group $\pi_1(X, x_0)$. In the case where X is connected and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoind, the automorphism groups of the objects are all isomorphic but not canonically isomorphic.

Example 1.4 The abelian groups along with group homomorphisms form a category **Ab**.

Example 1.5 If R is a ring, then the (left) R-modules form a category **R-Mod**. Taking $R = k$ we obtain **Vect**$_k$ and taking $R = \mathbb{Z}$ we obtain **Ab**.

Example 1.6 There is a category of Rings, where the objects are rings and the morphisms are ring homomorphisms.

Example 1.7 The topological spaces with continuous maps form a category **Top** with isomorphisms being homeomorphisms.

All of the examples thus far were categories where the objects are sets with additional structure; so-called **concrete categories**. This needn't hold in general.

Example 1.8 A **partially ordered set** or **poset** is a set S with a reflexive, transitive, and antisymmetric binary relation $\leqslant$. The partially ordered set $(S, \leqslant)$ can be interpreted as a category whose objects are the elements of S with a single morphisms from x to y if and only if $x \leqslant y$.

Example 1.9 If X is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Informally, if $U \subseteq V$, then we have exactly one morphism $U \rightarrow V$ in the category, and otherwise none. Similarly, if X is a topological space, then the open sets form a partially ordered set, where the order is again given by inclusion.

Definition 1.1 A **subcategory** $\mathscr{A}$ of a category $\mathscr{B}$ has as its objects some of the objects of $\mathscr{B}$ and some of the morphisms such that the morphisms of $\mathscr{A}$ include all identity morphisms for objects in $\mathscr{A}$, and are closed under operations of composition, tail, and head.

1.1.1 Functors

A **covariant functor** F from a category $\mathscr{A}$ to a category $\mathscr{B}$, denoted $F : \mathscr{A} \rightarrow \mathscr{B}$, consists of the following data:

- We have a map of objects, $F : \mathrm{Obj}\mathscr{A} \rightarrow \mathrm{Obj}\mathscr{B}$

- For each $A, A' \in \mathscr{A}$, a map of arrows $F : \mathscr{A}(A, A') \rightarrow \mathscr{B}(FA, FA')$ which satisfies

$$F(g \circ f) = Fg \circ Ff$$
$$F(\mathrm{Id}_A) = \mathrm{Id}_{FA}$$

for all $f : A \rightarrow A', g : A' \rightarrow A''$ in $\mathscr{A}$. That is the map of arrows preserves composition and identity.

A trivial example is the identity functor $\mathrm{Id}_{\mathscr{A}} : \mathscr{A} \rightarrow \mathscr{A}$. Here are some notable examples.

Example 1.10 Consider the functor from **Vect**$_k$ to **Set** associating to each vector space its underlying set and each linear transformation its underlying set-function. This is an example of a **forgetful functor**, where some additional structure is forgotten. We have similar functors such as **R-Mod** $\rightarrow$ **Ab**.

Example 1.11 The fundamental group functor π_1 is covariant, sending a topological space X with a choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$. This is not simply a functor from **Top** but from **Top**$_*$, the collection of all vector spaces with a distinguished base point and pointed continuous maps as morphisms. The ith homomoly functor **Top** $\rightarrow$ **Ab** sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a continuous morphism of pointed topological spaces $\phi : X \rightarrow Y$ with $\phi(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, and similarly for homology groups.

Example 1.12 Suppose $A \in \mathscr{A}$. Then there is a functor $H^A : \mathscr{C} \to \mathbf{Set}$ sending $B \in \mathscr{C}$ to $\mathbf{Hom}(A, B)$, and sending $f : B \to B'$ to $f_* : \mathbf{Hom}(A, B) \to \mathbf{Hom}(A, B')$ given by post-composition.

Definition 1.2 If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are covariant functors, then we define a functor $G \circ F : \mathscr{A} \to \mathscr{C}$ in the sensible way. Note this operation is associative with identities the identity functors.

A covariant functor $F : \mathscr{A} \to \mathscr{B}$ is **faithful** if for all $A, A' \in \mathscr{A}$, the map $\mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is injective, and **full** if it is surjective. A functor that is full and faithful is **fully faithful**. A subcategory $\iota : \mathscr{A} \hookrightarrow \mathscr{B}$ is a **full subcategory** if ι is full. As an example the forgetful functor $\mathbf{Vect}_k \to \mathbf{Set}$ is faithful but not full. If R is a ring the category of finitely generated R-modules is a full subcategory of the category **R-Mod**.

Definition 1.3 A **contravariant functor** is defined in the same way as a covariant functor, except the arrows switch direction. In other words, $F : \mathscr{A} \to \mathscr{B}$ satisfies for all $A, A' \in \mathscr{A}$ that the arrow map is $F : \mathscr{A}(A, A') \to \mathscr{B}(FA', FA)$, and for all $A \xrightarrow{f} A' \xrightarrow{g} A''$, $F(g \circ f) = Ff \circ Fg$.

Contravariant functors are really covariant functors out of the **opposite category**, $\mathscr{A}^{op}$, which has the same objects as $\mathscr{A}$ but all arrows are reversed.

Example 1.13 A well known example is the contravariant dual functor $(\cdot)^* : \mathbf{Vect}_k \to \mathbf{Vect}_k$, sending each vector space to its dual and each linear transformation $T : V \to W$ to its dual $f^* : W^* \to V^*$ given by pre-composition.

Example 1.14 There is a contravariant functor $\mathbf{Top} \to \mathbf{Ring}$ taking a topological space X to the ring of real-valued continuous functions on X. A morphism of topological spaces $X \to Y$ induces a pullback map from functions on Y to functions on X (pre-composition).

Example 1.15 If $A \in \mathscr{C}$, we have a contravariant functor $H_A : \mathscr{C} \to \mathbf{Set}$ sending $B \in \mathscr{C}$ to $\mathbf{Hom}(B, A)$, and sending $f : B \to B'$ to $f^* : \mathbf{Hom}(B', A) \to \mathbf{Hom}(B, A)$ given by pre-composition.

1.1.2 Natural Transformations

Suppose $F, G : \mathscr{A} \to \mathscr{B}$ are functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of the following data: for each $A \in \mathscr{A}$ a morphism $\eta_A : FA \to GA$ such that for any $f : A \to A'$ in $\mathscr{A}$ the following diagram commutes:

$$
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
\eta_A \downarrow & & \downarrow \eta_{A'} \\
GA & \xrightarrow{Gf} & GA'
\end{array}
$$

A **natural isomorphism** is a natural transformation in which each component map is an isomorphism. Analogous definitions hold for F and G both contravariant.

The data of functors $F : \mathscr{A} \to \mathscr{B}$ and $F' : \mathscr{B} \to \mathscr{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $\mathrm{Id}_{\mathscr{B}}$ on $\mathscr{B}$ and $F' \circ F$ is naturally isomorphic to $\mathrm{Id}_{\mathscr{A}}$ is said to be an **equivalence of catogories**.

Example 1.16 Let $f.d.\mathbf{Vect}_k$ denote the category of finite-dimensional vector spaces over k. Then the double dual functor $(\cdot)^{**} : f.d.\mathbf{Vect}_k \to f.d.\mathbf{Vect}_k$ is naturally isomorphic to the identity functor on $f.d.\mathbf{Vect}_k$. Since in the case of

finite dimensional vector spaces $V^{**} \cong V$, we can define a natural transformation $\eta : \mathrm{Id} \Rightarrow (\cdot)^{**}$ such that for each $V \in f.d.\mathbf{Vect}_k$. $\eta_V : V \to V^{**}$ is defined by sending $v \in V$ to ev_v. This is a linear isomorphism so all we need show is naturality. Let $f : V \to W$. Then we need show $\eta_W \circ f$ is equal to $f^{**} \circ \eta_V$. Indeed, f^{**} is given on $\varphi \in V^{**}$ by $f^{**}(\varphi)(\phi) = \varphi(\phi \circ f)$ for any $\phi \in W^*$. For $v \in V$ we have $\eta_W \circ f(v) = \mathrm{ev}_{f(v)}$ and $f^{**} \circ \eta_V(v) = f^{**}(\mathrm{ev}_v)$. Then for any $\phi \in W^*$, we have

$$f^{**}(\mathrm{ev}_v)(\phi) = \mathrm{ev}_v(\phi \circ f) = \phi(f(v)) = \mathrm{ev}_{f(v)}(\phi)$$

In sum we find that $\eta_W \circ f = f^{**} \circ \eta_V$, so η is indeed a natural isomorphism.

Ignoring set-theoretic issues, we can show that the category $\mathscr{V}$ of k-vector spaces k^n for each $n \geq 0$ and morphisms linear transformations is equivalent to $f.d.\mathbf{Vect}_k$. In particular, if we take the inclusion functor $\iota : \mathscr{V} \to f.d.\mathbf{Vect}_k$ we can construct a suitable inverse like functor. For each $V, W \in f.d.\mathbf{Vect}_k$ we can choose bases β_V, β_W, which induces an isomorphism $[\cdot]_{\beta_V}^{\beta_W} : \mathscr{L}(V, W) \to \mathscr{L}(k^{\dim(V)}, k^{\dim(W)})$. We then define a functor $\dim : f.d.\mathbf{Vect}_k \to \mathscr{V}$ by $\dim(V) = k^{\dim V}$ and with component arrow maps given by the isomorphisms $[\cdot]_{\beta_V}^{\beta_W}$. Then we define a natural isomorphism $\eta : \mathrm{Id}_\mathscr{V} \Rightarrow \dim \circ \iota$ by defining $\eta_{k^n} : k^n \to k^n$ to be the appropriate change of basis for our choice in the previous part. Then $k^n \xrightarrow{A} k^m \xrightarrow{\eta_{k^m}} \dim(\iota(k^n))$ is equivalent to $k^n \xrightarrow{\eta_{k^n}} \dim(\iota(k^n)) \xrightarrow{[A]_{\beta_{k^n}}^{\beta_{k^m}}} \dim(\iota(k^m))$ given the composition properties of basis changes. On the other hand we have a natural isomorphism $\varepsilon : \mathrm{Id}_{f.d.\mathbf{Vect}_k} \to \iota \circ \dim$ such that $\varepsilon_V : V \to k^{\dim(V)}$ is the coordinate map $[\cdot]_{\beta_V}$.

1.2.0 Universal Properties

Universal properties are the ideal method of constructing objects in categories. They give us objects which satisfy some property that characterizes them uniquely up to unique isomorphism. Then as long as we can construct a single example of such an object to show existence we can obfiscate the technical construction moving forward, only caring about its property.

A canonical example of this type of construction is the notion of a product, which as we shall see later is a type of limit. Explicitly, for $A, B \in \mathscr{C}$, a product object, if it exists, is a triple $(A \times B, \pi_1, \pi_2)$ with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ which is universal in this property. That is for any other triple (P, p_1, p_2), there exists a unique arrow from P to $A \times B$ for which the following diagram commutes.

We begin with some simple but useful examples of universal objects. An object of $\mathscr{C}$ is an **initial object** if it has precisely one map to every object. It is a **final object** if it has precisely one map from every object. It is a **zero object** if it is both an initial object and a final object. Note these notions are dual in the sense that an initial object in $\mathscr{C}$ is a final object in $\mathscr{C}^{op}$ and vice-versa. Naturally, these constructions are universal in their property, being unique up to unique isomorphisms.

Example 1.17 The initial object in **Set** is the empty set, and the final object is any one-point set $\{*\}$. The initial object in **Ring** is $\mathbb{Z}$, and the final object the zero ring. In **Top** the initial object is the empty set and the final object is any one point space.

1.2.1 Localization of Rings and Modules

Localization of a ring is another important example of a definition by universal property. First for a constructive definition. Recall a **multiplicative subset** S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}A$. The elements of $S^{-1}A$ are of the form a/s where $a \in A$ and $s \in S$, and where $a_1/s_1 = a_2/s_2$ if and only if there exists $u \in S$ such that $u(a_1 s_1 - a_2 s_2) = 0$. We define $(a_1/s_1) + (a_2/s_2) = (a_1 s_2 + s_1 a_2)/(s_1 s_2)$ and $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$. As in the case of the construction of $\mathbb{Q}$ these operations are independent of representative and hence well-defined with properties making $S^{-1}A$ into a ring. We have a canonical map $A \to S^{-1}A$ sending $a \mapsto a/1$. Note that if $0 \in S$, $S^{-1}A$ is the 0-ring (all points are equivalent).

We have two primary examples of multiplicative subsets. First, for $f \in A$, the set $\{1, f, f^2, ...\}$ is multiplicative with localization denoted by A_f. The second is $A - \mathfrak{p}$ for $\mathfrak{p}$ a prime ideal. This localization is denoted by $A_\mathfrak{p}$. Note, if $\mathfrak{p}$ is a prime ideal then $A_\mathfrak{p}$ means you're allowed to divide by elements **not** in $\mathfrak{p}$. However, if $f \in A$, A_f means you're allowed to divide by f, and this is the minimal case where this can be done. In particular, if (f) is a prime ideal, $A_f \neq A_{(f)}$.

The localization map need not be injective. Indeed we have the following result.

Claim The natural map $A \to S^{-1}A$ is injective if and only if S contains no zerodivisors.

Proof To prove the claim first suppose that the map is injective. That is if $a/1 = 0/1$ if and only if $a = 0$. In particular, by construction of S^{-1}, we have that if $a \neq 0$, then $ua \neq 0$ for all $u \in S$. Hence $0 \notin S$, and no element of S is a zero divisor.
Conversely, if S has no zero divisors, then $ua = 0$ only if $a = 0$, for any $u \in S$. In particular, $A \to S^{-1}A$ is injective.□

If A is an integral domain and $S = A - \{0\}$, then $S^{-1}A$ is called the **fraction field** of A, which we denote by $K(A)$. The previous exercise shows that A is a subring of its fractional field $K(A)$. We now turn to the case of A being a general commutative ring.

Theorem 1.1 (Universal Property of Localization) *Let A be a general commutative ring with multiplicative set S. Then $S^{-1}A$ is initial among A-algebras B where every element of S is sent to an invertible element in B. (Recall: the data of an A-algebra B and a ring map $A \to B$ are the same)*

Proof First let B be an A-algebra. Let $\varphi : A \to B$ be the associated ring map. Define a map $f : S^{-1}A \to B$ by $f(a/s) = \varphi(a)\varphi(s)^{-1}$. Note if $a/s = a'/s'$, so we have $u \in S$ such that $u(as' - a's) = 0$, we have $\varphi(u)(\varphi(a)\varphi(s') - \varphi(a')\varphi(s)) = 0$. But $\varphi(u)$ is invertible so in particular it is not a zero-divisor, which means this equality can only hold if $\varphi(a)\varphi(s') = \varphi(a')\varphi(s)$. Then using the invertibility of $\varphi(S)$ and the fact B is commutative we have $\varphi(a)\varphi(s)^{-1} = \varphi(a')\varphi(s')^{-1}$ so the map is well-defined. As φ is a ring map so is f. In particular, $f(aa'/s) = \varphi(aa')\varphi(s)^{-1} = \varphi(a)\varphi(a')\varphi(s)^{-1} = a \cdot f(a'/s)$, so f is an A-algebra homorphism. For uniqueness if we have any other A-algebra homomorphism, g, $g(a/1_A) = a \cdot g(1_{S^{-1}A}) = \varphi(a)$, and for any $s \in S$, $\varphi(s)g(1_A/s) = g(s/s) = g(1_{S^{-1}A}) = 1_B$, so by uniqueness of inverses $g(1_A/s) = \varphi(s)^{-1}$. Thus by multiplicativity of ring maps g is given by $g(a/s) = \varphi(a)\varphi(s)^{-1} = f(a/s)$, so we have uniqueness and hence $S^{-1}A$ is initial. □

This result implies that an $S^{-1}A$-module is the same thing as an A-module for which $s \cdot - : M \to M$ is an A-module isomorphism for all $s \in S$.

We now use this as inspiration to define localization of modules by a universal property. Suppose M is an A-module. We define the A-module map $\phi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N ($s \cdot - : N \to N$ is an isomorphism for all $s \in S$). More precisely, any such map $\alpha : M \to N$ factors uniquely through ϕ.

$$M \xrightarrow{\phi} S^{-1}M$$
$$\searrow^{\alpha} \quad \downarrow^{\exists!}$$
$$N$$

This is known as a **universal arrow** in the language of category theory. Precisely the category here is the

$$S^{-1}A\mathrm{Mod}$$
$$\downarrow^{Res_A}$$
$$1 \xrightarrow{M} A\mathrm{Mod}$$

with objects being pairs (μ, N) where $\mu : M \to Res_A(N)$, N is an $S^{-1}A$-module, and Res_A is the restriction of scalars functor induced by the canonical ring homomorphism $A \to S^{-1}A$.

This determines $(\phi : M \to S^{-1}M, S^{-1}M)$ up to unique isomorphism, and essentially by definition the A-module structure on $S^{-1}M$ extends to an $S^{-1}A$-module structure. Explicitly we can construct $\phi : M \to S^{-1}M$ by defining $S^{-1}M$ to consist of elements m/s for $s \in S$, with $m_1/s_1 = m_2/s_2$ if and only if $u(s_2 m_1 - s_1 m_2) = 0$ for some $u \in S$, and addition and scalar multiplication defined similarly to $S^{-1}A$. Then ϕ is simply the map sending m to $m/1$.

Claim Localization commutes with arbitrary direct sums. If $\{M_i : i \in I\}$ is a family of A-modules, then $S^{-1} \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} S^{-1}M_i$.

Proof We show that $\bigoplus_{i \in I} S^{-1}M_i$ is also initial in the specified comma category. For each $i \in I$ we have a unique map $\varphi_i : M_i \xrightarrow{\phi_i} S^{-1}M_i \xrightarrow{\iota_i'} \bigoplus_{i \in I} S^{-1}M_i$ combining our two universal properties, where $(\phi_i, S^{-1}M_i)$ is initial in the appropriate comma category, and $\iota_i' : S^{-1}M_i \to \bigoplus_{i \in I} S^{-1}M_i$ is the canonical injection. This implies we have a unique map

$$\Phi : \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} S^{-1}M_i$$

such that $\Phi \circ \iota_i = \varphi_i : M_i \to \bigoplus_{i \in I} S^{-1}M_i$ for each $i \in I$.

I claim this pair is initial in our comma category. Let $\mu : \bigoplus_{i \in I} M_i \to N$ be any pair in the comma category (note we drop Res_A here). Then for each $i \in I$ we have $\mu \circ \iota_i : M_i \to N$. As N is an $S^{-1}A$-module, by the universal property there exists a unique map $f_i : S^{-1}M_i \to N$ such that $\mu \circ \iota_i = f_i \circ \phi_i$. Then by the universal property of direct sums we have a unique map $f : \bigoplus_{i \in I} S^{-1}M_i \to N$ such that $f \circ \iota_i' = f_i$. It follows that $f \circ \Phi : \bigoplus_{i \in I} M_i \to N$ satisfies

$$
\begin{aligned}
f \circ \Phi \circ \iota_j &= f \circ \varphi_j & \text{(as } \Phi \circ \iota_j = \varphi_j\text{)} \\
&= f \circ \iota_j' \circ \phi_j & \text{(as } \varphi_j := \iota_j' \circ \phi_j\text{)} \\
&= f_j \circ \phi_j \\
&= \mu \circ \iota_j
\end{aligned}
$$

Then by uniqueness of the map from $\bigoplus_{i \in I} M_i \to N$ making our triangles commute, we must have that $\mu = f \circ \Phi$. Thus, as f had to be unique by the universal property of the direct sum $\bigoplus_{i \in I} S^{-1}M_i$, we have that $\left(\Phi, \bigoplus_{i \in I} S^{-1}M_i\right)$ is indeed initial in the comma category

$$S^{-1}A\mathrm{Mod}$$
$$\downarrow^{Res_A}$$
$$1 \xrightarrow{\bigoplus_{i \in I} M_i} A\mathrm{Mod}$$

Hence, it is uniquely isomorphic to the pair $\left(\Psi, S^{-1} \bigoplus_{i \in I} M_i\right)$ for $\Psi : \bigoplus_{i \in I} M_i \to S^{-1} \bigoplus_{i \in I} M_i$. $\qquad\square$

This result, however, does not hold for infinite products. Consider $\mathbb{Z}^\omega$ as a $\mathbb{Z}$-module and $S = \mathbb{Z} - \{0\}$. Then the map induced by the universal property of localization, $S^{-1}\mathbb{Z}^\omega \to (S^{-1}\mathbb{Z})^\omega$ is not an isomorphism. Indeed, note $S^{-1}\mathbb{Z} = \mathbb{Q}$, so this is a $\mathbb{Q}$-algebra homomorphism $S^{-1}\mathbb{Z}^\omega \to \mathbb{Q}^\omega$ sending $(a_1, ..., a_n, ...)/s \mapsto (a_1/s, ..., a_n/s, ...)$. Consider the element $(1, 1/2, 1/3, ...)$ in $\mathbb{Q}^\omega$. Then at some point we have $1/(s+1)$ in the sequence. However, for all $a \in \mathbb{Z}$, $a/s > 1/(s+1)$, and so cannot be equal to it. Thus the map cannot be surjective, and hence can't be an isomorphism.

Remark 1.2 We will see later that localization also does not always commute with **Hom**, but in situations where the first argument is finitely presented we do get commutivity.

1.2.2 Tensor Products

Another notable example of a universal property construction is the notion of a **tensor product** of A-modules

$$\otimes_A : \text{obj}(\mathbf{Mod}_A) \times \text{obj}(\mathbf{Mod}_A) \to \text{obj}(\mathbf{Mod}_A)$$
$$(M, N) \mapsto M \otimes_A N$$

If M, N are A-modules, then $M \otimes_A N$ is the free A-module generated by $M \times N$ which is quotiented by the sumbodule generated by elements of the from

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n), (m, n_1 + n_2) - (m, n_1) - (m, n_2), a(m, n) - (am, n), a(m, n) - (m, an)$$

The image of (m, n) in this quotient is denoted by $m \otimes n$. If A is a field k, this is the tensor product of vector spaces.

Theorem 1.2 *For any A module N, the covariant function $- \otimes_A N : \mathbf{Mod}_A \to \mathbf{Mod}_A$ is **right-exact**.*

Proof To begin, let

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of A-modules. Then we have the induced sequence

$$M' \otimes_A N \xrightarrow{f \otimes_A \text{Id}_N} M \otimes_A N \xrightarrow{g \otimes_A \text{Id}_N} M'' \otimes_A N \to 0$$

First note that for any $m'' \otimes n \in M'' \otimes_A N$, by exactness of our original sequence there exists $m \in M$ such that $g(m) = m''$ so $g \otimes_A \text{Id}_N(m \otimes n) = m'' \otimes n$. Thus $\text{Im} g$ contains a generating set for $M'' \otimes_A N$, implying that $\text{Im} g = M'' \otimes_A N$.

Next, we observe that $g \otimes_A \text{Id}_N \circ f \otimes_A \text{Id}_N = g \circ f \otimes_A \text{Id}_N = 0 \otimes_A \text{Id}_N = 0 : M' \otimes_A N \to M'' \otimes_A N$ as $\ker g = \text{Im} f$. This implies that $\text{Im} f \otimes_A \text{Id}_N \subseteq \ker g \otimes_A \text{Id}_N$.

To prove the reverse inclusion consider the canonical quotient map $\pi : (M \otimes_A N)/\text{Im} f \circ \text{Id}_N \to M'' \otimes_A N$ which exists since $\text{Im} f \circ \text{Id}_A \subseteq \ker g \otimes_A \text{Id}_N$. We aim to show this map is injective by construction of an inverse. Note that for each $m'' \in M''$, $g^{-1}(m'') \neq \varnothing$. Let $m_1, m_2 \in g^{-1}(m'')$. Then $m_1 - m_2 \in \ker g = \text{Im} f$, so there exists $m' \in M'$ such that $m_1 = f(m') + m_2$. Define $\varphi : M'' \otimes_A N \to (M \otimes_A N)/\text{Im} f \circ \text{Id}_N$ by sending $m'' \otimes n$ to $[m \otimes n]$ for $m \in g^{-1}(m'')$. By our previous remark this is independent of the choice of representative in g^{-1}. Now we need only show it is independent of the choice of representative for $m'' \otimes n$. We can show this by showing that elements of the generating set of our quotient submodule are sent to zero. Indeed, if ρ is one of the generating elements of our quotient submodule, then $\rho = (m_1'' + m_2'', n) - (m_1'', n) - (m_2'', n), (m'', n_1 + n_2) - (m'', n_1) - (m'', n_2), (am'', n) - a(m'', n), (m'', an) - a(m'', n)$. In any case, ρ is again sent to an element of this form with m_1'', m_2'', and m'' being replaced by some m_1, m_2, m in M. Thus the map is indeed well-defined, and by construction an inverse to π. Hence, π is an isomorphism so we must have that $\text{Im} f \circ \text{Id}_N = \ker g \circ \text{Id}_N$. In conclusion $- \otimes_A N$ is indeed a right-exact functor for any N. $\qquad\square$

In contrast, the tensor product is not left-exact: a counter-example is given by tensoring the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

by $\mathbb{Z}/2\mathbb{Z}$.

We now move away from this awkward construction to our universal property.

Theorem 1.3 *If M, N, P are A-modules, then for any A-bilinear mapping $M \times N \to P$, there exists a unique linear map $M \otimes_A N \to P$ which it factors through.*

Proof Let $p : M \times N \to M \otimes_A N$ be the canonical projection, noting that p is bilinear by construction of the quotient. Now, if $f : M \times N \to P$ is bilinear, define $\overline{f} : M \otimes_A N \to P$ by $\overline{f}\left(\sum_i m_i \otimes n_i\right) = \sum_i f(m_i, n_i)$. Note that any bilinear mapping sends the submodule generating the quotient $M \otimes_A N$ to zero, so $\overline{f}$ is well-defined. Additionally, by construction $\overline{f}$ is linear and $\overline{f} \circ p = f$. Finally, if g is any other linear mapping satisfying this property, $\overline{f} \circ p = g \circ p$, so as they will be equal on a generating set of $M \otimes_A N$, they must in fact be equivalent A-linear maps. $\square$

We can of course take this as the definition of the tensor product. That is a tensor of M and N is an A-module T equipped with an A-bilinear map $t : M \times N \to T$ such that for any A-bilinear map $t' : M \times N \to T'$, there is a unique A-linear map $f : T \to T'$ such that $t' = f \circ t$.

Here we use double arrows to simply represent bilinear maps. We can also formulate this in terms of initial objects. In particular, (t, T) is an initial object in the comma category:

 That is to say for any pair (f, L), so $f \in Bilin(M, N; L)$, there exists a unique map

$T \in \mathbf{Hom}(M \otimes_A N, L)$ such that $Bilin(M, N; T) \circ \otimes = f$, where $\otimes : M \times N \to M \otimes_A N$ is the canonical bilinear mapping. Consequently $(M \otimes_A N, \otimes)$ is unique up to unique isomorphism.

1.2.3 Fibered Products

Suppose we have morphisms $\alpha : X \to Z$ and $\beta : Y \to Z$ in some category $\mathscr{C}$. Then the **fibered product** is an object $X \times_Z Y$ along with morphisms $\mathrm{pr}_X : X \times_Z Y \to X$ and $\mathrm{pr}_Y : X \times_Z Y \to Y$, where the two compositions $\alpha \circ \mathrm{pr}_X, \beta \circ \mathrm{pr}_Y : X \times_Z Y \to Z$ agree, such that given any object W with maps to X and Y whose compositions to Z agree, these maps factor through some unique $W \to X \times_Z Y$. This is also known as a pullback along α and β, and can be illustrated as follows.

$$
\begin{array}{c}
W \\
\downarrow \exists! \\
X \times_Z Y \xrightarrow{\ \mathrm{pr}_Y\ } Y \\
\mathrm{pr}_X \downarrow \qquad \downarrow \beta \\
X \xrightarrow{\ \alpha\ } Z
\end{array}
$$
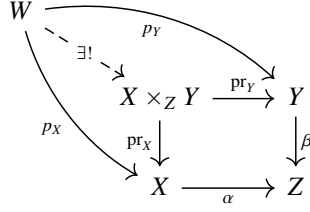
By the usual universal property argument, if it exists, it is unique up to unique isomorphism. Additionally, we know what maps to it are: they are precisely maps to X and maps to Y that agree as maps to Z.

Example 1.18 In **Set**, the fibered product for $\alpha : X \to Z$ and $\beta : Y \to Z$ is

$$
X \times_Z Y = \{(x,y) \in X \times Y : \alpha(x) = \beta(y)\}
$$

Indeed, to show this is true, we consider the right side with its corresponding restricted projections onto X and Y. Let B be any set with associated projection p_X and p_Y that agree as maps to Z. By the universal property of the product we have a unique map $p_X \times p_Y : B \to X \times Y$, such that $p_X = \pi_X \circ p_X \times p_Y$ and $p_Y = \pi_Y \circ p_X \times p_Y$. I claim $p_X \times p_Y(B) \subseteq$ the right hand side of our proposed fibered product. Indeed, if $(x,y) \in p_X \times p_Y(B)$ for some $b \in B$, we have that $\alpha(p_X(b)) = \alpha(x)$ and $\beta(p_Y(b)) = \beta(y)$ are equivalent. In other words $(x,y) \in X \times_Z Y$, so restricting the codomain $p_X \times p_Y : B \to X \times_Z Y$, so by uniqueness of the map $X \times_Z Y$ satisfies the universal property of the fibered product.

This example motivates the interpretation of a fibered product as a sub-object of the product satisfying a particular equation, or in other words the solution to some map equation or diagram.

Example 1.19 If X is a topological space and we consider the category $\mathscr{O}(X)$ of open sets, and $U, V, W \in \mathscr{O}(X)$, with maps $\iota_{U,W} : U \to W, \iota_{V,W} : V \to W$, the fibered product is simply the intersection $U \cap V$, since all maps are unique and exist based on inclusions.

Example 1.20 If Z is the final object in a category $\mathscr{C}$, and $X, Y \in \mathscr{C}$, then $X \times_Z Y$ and $X \times Y$ are equivalent. Indeed, consider $X \times Y$ with its projections π_X, π_Y. Then there exist unique maps from X and Y to Z, f_X and f_Y, and a unique map $f_{X \times Y}$. Additionally, by uniqueness, $f_{X \times Y} = f_X \circ \pi_X = f_Y \circ \pi_Y$, so the projections make the square commute. Suppose (B, p_X, p_Y) is a candidate fibered product. Then by the universal property of products there exists a unique map $g : B \to X \times Y$ such taht $\pi_X \circ g = p_X$ and $\pi_Y \circ g = p_Y$. Thus as it satisfies the universal property, $X \times Y$ is the fibered product of X and Y along Z in $\mathscr{C}$ with respect to the unique maps from X and Y to Z.

Definition 1.4 We define the **coproduct** in a category by reversing all the arrows in the definition of a product:

$$
\begin{array}{c}
Z \\
\nearrow \ \uparrow \exists! \ \nwarrow \\
X \xrightarrow{\ i_X\ } X \times_{co} Y \xleftarrow{\ i_Y\ } Y
\end{array}
$$
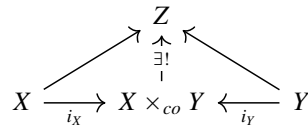
Similarly, the **fibered coproduct** in a category is defined by reversing all the arrows in the definition of a fibered product:

$$
\begin{array}{c}
Z \xrightarrow{\ f_Y\ } Y \\
f_X \downarrow \qquad \downarrow \iota_Y \\
X \xrightarrow{\ \iota_X\ } X \times_Z^{co} Y \\
\searrow \ \exists! \ \searrow \\
M
\end{array}
$$
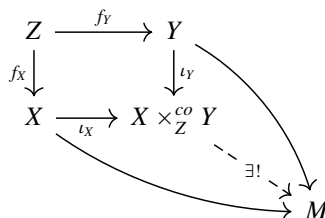
Example 1.21 The coproduct for **Set** is the disjoint union. Let X, Y be sets with disjoint union $X \sqcup Y$. Then we have natural injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$, sending x to x and y to y. We claim the triple $(X \sqcup Y, i_X, i_Y)$ is the coproduct. Indeed, if (N, f_X, f_Y) is another candidate triple, with $f_X : X \to N$ and $f_Y : Y \to N$, we define $F : X \sqcup Y \to N$ by $F(x) = F \circ i_X(x) = f_X(x)$ and $F(y) = F \circ i_Y(y) = f_Y(y)$. As this determines the action of F on all of $X \sqcup Y$ and by definition satisfies commutivity we have existence. Now for uniqueness, if $g : X \sqcup Y \to N$ making the diagram commute was another such map, then for all $x \in X$, $g(x) = g \circ i_X(x) = f_X(x) = F(x)$ and for all $y \in Y$, $g(y) = g \circ i_Y(y) = f_Y(y) = F(y)$. Thus $F = G$ so the map is unique.

Example 1.22 If A, B, C are rings with ring morphisms $A \to B$ and $A \to C$, then the corresponding fibered coproduct is $B \otimes_A C$. Recall $B \otimes_A C$ has a ring structure. Then we define maps $\iota_B : B \to B \otimes_A C$ and $\iota_C : C \to B \otimes_A C$ by $b \mapsto b \otimes 1_C$ and $c \mapsto 1_B \otimes c$. Observe $bb' + b'' \mapsto (bb' + b'') \otimes 1_C = bb' \otimes 1_C + b'' \otimes 1_C = (b \otimes 1_C)(b' \otimes 1_C) + b'' \otimes 1_C$, and similarly for the map $C \to B \otimes_A C$. Thus both are morphisms of rings, and by definition of $B \otimes_A C$ we have $a \mapsto a \cdot 1_B \mapsto (a \cdot 1_B) \otimes 1_C = a \cdot (1_B \otimes 1_C) = 1_B \otimes a \cdot 1_C$, so we having that $B \otimes_A C$ with these canonical maps gives a candidate for the fibered coproduct. Suppose (M, i_B, i_C) is another such candidate. Then $F := i_B \circ f = i_C \circ g : A \to M$ gives M an A module structure with $a \cdot m = F(a)m$. We define a map $B \times C$ to M by $(b, c) \mapsto i_B(b)i_C(c)$. Observe that

$$(f(a)b + b', c) \mapsto i_B(f(a)b + b')i_C(c) = (i_B(f(a))i_B(b) + i_B(b'))i_C(c) = F(a)i_B(b)i_C(c) + i_B(b')i_C(c)$$

and identically for the second component. Hence the map is A-bilinear, and so by the universal property of the tensor product we have a unique map $B \otimes_A C \to M$ acting on generators by $b \otimes c \mapsto i_B(b)i_C(c)$. By construction we have $b \mapsto b \otimes 1_C \mapsto i_B(b)i_C(1_C) = i_B(b)$, and similarly for c, so it makes our diagram commute.

Finally, we need only show uniqueness of this map. Suppose $G : B \otimes_A C \to M$ is another such ring map making our diagram commute; that is $G \circ \iota_B = i_B$ and $G \circ \iota_C = i_C$. Then

$$G(b \otimes c) = G(b \otimes 1_C)G(1_B \otimes c) = G(\iota_B(b))G(\iota_C(c)) = i_B(b)i_C(c)$$

Thus, as they act identically on simple tensors G is equal to our previously defined map, and we have uniqueness.

1.2.4 Monomorphisms and Epimorphisms

Definition 1.5 A morphism $\pi : X \to Y$ is a **monomorphism** if any two morphisms $\mu_1 : Z \to X$ and $\mu_2 : Z \to X$ such that $\pi \circ \mu_1 = \pi \circ \mu_2$ must satisfy $\mu_1 = \mu_2$. In other words, there is at most one way of filling in the dotted arrow so that the diagram

commutes — for any object Z, the natural map $\mathbf{Hom}(Z, X) \to \mathbf{Hom}(Z, Y)$ is an injection.

Proposition 1.1 *The morphism $\pi : X \to Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced morphism $X \to X \times_Y X$ is an isomorphism.*

Proof If π is a monomorphism, we first consider $(X, \mathrm{Id}_X, \mathrm{Id}_X)$ as a candidate fibered product. Indeed, if (M, p_1, p_2) is a candidate fibered product so $\pi \circ p_1 = \pi \circ p_2$, then $p := p_1 = p_2$ is the unique map from M to X for which $p \circ \mathrm{Id}_X = p$. Thus we have existence of the fibered product. Additionally, for any other fibered product $X \times_Y X$ we have that the unique induced map $X \to X \times_Y X$ is an isomorphism.

On the other hand, if the induced map $X \to X \times_Y X$ is an isomorphism, we have that $p_X : X \times_Y X \to X$ is its inverse, and hence also an isomorphism. Thus $(X, \mathrm{Id}_X, \mathrm{Id}_X)$ is the fibered product for $\pi : X \to Y$, which by inverting our previous argument yields that π is a monomorphism. $\qquad\square$

The notion of an **epimorphism** is dual to the definition of monomorphism, where all the arrows are reversed.

1.2.5 Representable Functors and Yoneda's Lemma

Much of our discussion about universal properties can be neatly expressed in terms of representable functors, under the rubric of **Yoneda's Lemma**. Informally speaking, Yoneda's lemma says that you can essentially recover an object in a category by knowing the maps into (dually out of) it. For example, we have seen that the data of maps to $X \times Y$ are naturally (canonically) the data of maps to X and to Y.

Recall the functors $H_A : \mathscr{C}^{op} \to \mathbf{Set}$ and $H^A : \mathscr{C} \to \mathbf{Set}$ for $A \in \mathscr{C}$. Yoneda's Lemma states that the functor H_A determines A up to unique isomorphism.

Theorem 1.4 (Yoneda's Lemma) *Let $\mathscr{C}$ be a locally small category with $A \in \mathscr{C}$. Then $[\mathscr{C}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$ naturally in X and A.*

Naturally in X and A means the functors $\mathscr{C}^{op} \times [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$ given by $(A, X) \mapsto [\mathscr{C}^{op}, \mathbf{Set}](H_A, X)$ and $(A, X) \mapsto X(A)$ are naturally isomorphic. In our proof we shall use the well known fact that we can show naturality in each component individually, while the other is fixed.

Proof (Yoneda's Lemma) We begin by defining for each A and X a bijection between our sets, and then showing the bijection is natural. To begin let $\hat{(\ )} : [\mathscr{C}^{op}, \mathbf{Set}](H_A, X) \to X(A)$ be defined by $\hat{\eta} = \eta_A(\mathrm{Id}_A) \in X(A)$. Then, we construct an inverse for $\hat{(\ )}$ to show it is bijective. We define $\tilde{(\ )} : X(A) \to [\mathscr{C}^{op}, \mathbf{Set}](H_A, X)$ by stating that for each $x \in X(A)$, the natural transformation $\tilde{x}$ is given for each $B \in \mathscr{C}$ by $\tilde{x}_B : \mathbf{Hom}(B, A) \to X(B)$, which sends $(f : B \to A) \mapsto X(f)(x)$.
We now show that $\tilde{x}$ is indeed a natural transformation. Let $B, C \in \mathscr{C}$ with $f : C \to B$. Then we need to show the diagram

$$
\begin{array}{ccc}
H_A(B) & \xrightarrow{H_A(f)} & H_A(C) \\
\tilde{x}_B \downarrow & & \downarrow \tilde{x}_C \\
X(B) & \xrightarrow[X(f)]{} & X(C)
\end{array}
$$

commutes. So we show the maps $\tilde{x}_C \circ H_A(f)$ and $X(f) \circ \tilde{x}_B$ are equivalent. Let $g : B \to A$ by in $H_A(B)$. Then

$$\tilde{x}_C \circ H_A(f)(g) = \tilde{x}_C(g \circ f) = X(g \circ f)(x)$$

while

$$X(f) \circ \tilde{x}_B(g) = X(f)(X(g)(x)) = X(g \circ f)(x)$$

since X is a contravariant functor. Hence $\tilde{x}$ is indeed a natural transformation. Now we show they are bijective. Indeed, if $x \in X$, then $\hat{\tilde{x}} = \tilde{x}_A(\mathrm{Id}_A) = X(\mathrm{Id}_A)(x) = \mathrm{Id}_{X(A)}(x) = x$, and for any $B \in \mathscr{C}$ and $g : B \to A$,

$$\tilde{\hat{\eta}}_B(g) = \eta_A(\tilde{\mathrm{Id}}_A)_B(g) = X(g)(\eta_A(\mathrm{Id}_A))$$

By naturality of η we have

$$X(g)(\eta_A(\mathrm{Id}_A)) = \eta_B(H_A(g))(\mathrm{Id}_A) = \eta_B(\mathrm{Id}_A \circ g) = \eta_B(g)$$

THus $\tilde{\hat{\eta}} = \eta$, and so $\tilde{(\,)}$ and $\hat{(\,)}$ are inverse.

Finally we show naturality of $\tilde{(\,)}$ and $\hat{(\,)}$ in A and X. It is sufficient to show naturality of one since their component maps are inverse. First fix X and consider $f : A' \to A$. We require the commutivity of the diagram

$$
\begin{array}{ccc}
[\mathscr{C}^{op}, \mathbf{Set}](H_A, X) & \xrightarrow{-\circ H_f} & [\mathscr{C}^{op}, \mathbf{Set}](H_{A'}, X) \\
\hat{(\,)} \downarrow & & \downarrow \hat{(\,)} \\
X(A) & \xrightarrow[X(f)]{} & X(A')
\end{array}
$$

where $H_f : H_{A'} \Rightarrow H_A$ is defined by $H_f(B) : H_{A'}(B) \to H_A(B)$ by post composition, $H_f = f \circ -$. Then for any $\eta \in [\mathscr{C}^{op}, \mathbf{Set}](H_A, X)$,

$$
\widehat{- \circ H_f}(\eta) = \widehat{\eta \circ H_f} = \eta_{A'} \circ H_f(A')(\mathrm{Id}_{A'}) = \eta_{A'}(f)
$$

and

$$
X(f)(\hat{\eta}) = X(f)(\eta_A(\mathrm{Id}_A)) = \eta_{A'}(H_A(f)(\mathrm{Id}_A)) = \eta_{A'}(f)
$$

by the naturality of η. Thus the isomorphism is natural in A. Next, fix A and consider $\mu : X \Rightarrow X'$ contravariant functors. We need to show the commutivity of the diagram

$$
\begin{array}{ccc}
[\mathscr{C}^{op}, \mathbf{Set}](H_A, X) & \xrightarrow{\mu \circ -} & [\mathscr{C}^{op}, \mathbf{Set}](H_A, X') \\
\hat{(\,)} \downarrow & & \downarrow \hat{(\,)} \\
X(A) & \xrightarrow[\mu_A]{} & X'(A)
\end{array}
$$

To show commutivity as usual let $\eta : H_A \Rightarrow X$. Then

$$
\widehat{\mu \circ -}(\eta) = \widehat{\mu \circ \eta} = \mu_A \circ \eta_A(\mathrm{Id}_A)
$$

and

$$
\mu_A(\hat{\eta}) = \mu_A(\eta_A(\mathrm{Id}_A)) = \mu_A \circ \eta_A(\mathrm{Id}_A)
$$

Thus again we have commutivity, and hence naturality in X.

Therefore $\hat{(\,)}$ is a natural isomorphism with inverse $\tilde{(\,)}$. $\qquad\square$

There is an analogous statement with all arrows reversed, where instead of maps into A we consider maps from A.

Definition 1.6 A contravariant functor $F : \mathscr{C}^{op} \to \mathbf{Set}$ is said to be **representable** if there is a natural isomorphism $\xi : F \xrightarrow{\sim} H_A$ for some $A \in \mathscr{C}$. A is determined up to isomorphism by (F, ξ). A similar definition holds for covariant functors.

Yoneda's lemma also tells us that the functor H_- sending A to H_A is a fully faithful functor, called the **Yoneda embedding**.

1.3.0   Limits and Colimits

Limits and colimits are important definitions determined by universal properties which generalize a number of the constructions we have seen thus far.

1.3.1 Limits

We say that a category is a **small cateogry** if the objects and the morphisms are sets. Suppose $\mathscr{D}$ is a small category, and $\mathscr{C}$ is any other category. Then a functor $F : \mathscr{D} \to \mathscr{C}$ is said to be a **diagram indexed by** $\mathscr{D}$. We call $\mathscr{D}$ an **index category**. Our index categories will usually be partially ordered sets, in which there is at most one morphism between any two objects. For example if $\square$ is the category

$$
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \longrightarrow & \bullet
\end{array}
$$
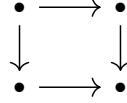
and $\mathscr{A}$ is a category, then a functor $\square \to \mathscr{A}$ is precisely the data of a commuting square in $\mathscr{A}$.

Then the **limit of the diagram** is an object $\varprojlim_{\mathscr{D}} A_i$ of $\mathscr{C}$ along with morphisms $f_j : \varprojlim_{\mathscr{D}} A_i \to A_j$ for each $j \in \mathscr{D}$, such that if $m : j \to k$ is a morphism in $\mathscr{D}$, then

$$
\begin{array}{ccc}
\varprojlim_{\mathscr{D}} A_i & & \\
f_j \downarrow & \searrow^{f_k} & \\
A_j & \xrightarrow[F(m)]{} & A_k
\end{array}
$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property. More precisely, given any other object W along with maps $g_i : W \to A_i$ commuting with the $F(m)$, then there is a unique map

$$
g : W \to \varprojlim_{\mathscr{D}} A_i
$$

so that $g_i = f_i \circ g$ for all i. In some cases this is called the **inverse limit** or **projective limit**. By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

Example 1.23 For example, if $\mathscr{D}$ is the partially ordered set

$$
\begin{array}{ccc}
& & \bullet \\
& & \downarrow \\
\bullet & \longrightarrow & \bullet
\end{array}
$$
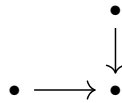
we obtain the fibered product. If $\mathscr{D}$ is

$$
\bullet \qquad \bullet
$$

we obtain the product.

If $\mathscr{D}$ is a set (i.e., the only morphisms are the identity maps), then the limit is called the **product** of the A_i, and is denoted $\prod_{i \in \mathscr{D}} A_i$.

Example 1.24 For a prime number p, the p-**adic integers**, $\mathbb{Z}_p$, are often described informally as being of the form $\mathbb{Z}_p = a_0 + a_1 p + a_2 p^2 + \cdots$, where $0 \leqslant a_i < p$. They are an example of a limit in the category of rings:

$$\mathbb{Z}_p$$

$$\cdots \qquad \mathbb{Z}/p^3\mathbb{Z} \qquad \mathbb{Z}/p^2\mathbb{Z} \qquad \mathbb{Z}/p\mathbb{Z}$$

In particular, they are an example of a **completion** which we shall explore much later.

Limits need not always exist for any index category $\mathscr{D}$. However, you can often check that limits exist if the objects of your category can be interpreted as sets with additional structure, and arbitrary products exist (respecting the set-like structure).

Example 1.25 In the category **Set**, for $F : \mathscr{D} \to$ **Set**, the set

$$\left\{ (a_i)_{i \in \mathscr{D}} \in \prod_{i \in \mathscr{D}} A_i : F(m)(a_j) = a_k \text{ for all } m \in \mathbf{Hom}_{\mathscr{D}}(j,k) \right\}$$

along with the natural projection maps to each A_i, is the limit $\varprojlim_{\mathscr{D}} A_i$.

Indeed, by construction the set with the projections commutes with all $F(m)$, making it a **cone** of the diagram. Additionally, if $(B, f_i; i \in \mathscr{D})$ is another cone, commuting with all $F(m)$, then we can define a map g from B to our candidate, sending $b \in B$ to $(f_i(b))_{i \in \mathscr{D}}$. Since B is a cone this indeed is an element of our set for each $b \in B$, so g is a well-defined map. Additionally, by construction if π_i denote the projections for our map, $\pi_i \circ g = f_i$. Finally, if any other map p existed which satisfied these commuting triangles, then $p(b) = (f_i(b))_{i \in \mathscr{D}}$, and hence be the same map.

The uniqueness of this map gives us that our claimed set is indeed our desired limit.

Note that this construction also works in $\mathbf{Mod}_A$, $\mathbf{Vect}_k$, **Ab**, and **Ring**. Going back to our example on p-adics, a point $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as a sequence $(2, 2 + 3p, 2 + 3p + 2p^2, ...)$.

1.3.2 Colimits

More immediately relevant for our work will be the dual of the notion of limit. We just flip the arrows f_i in our definition and get the notion of a **colimit**, which is denoted $\varinjlim_{\mathscr{D}} A_i$, and is universal (initial) with this property.

$$\varinjlim_{\mathscr{D}} A_i$$

$$\iota_j \uparrow \qquad \nwarrow \iota_k$$

$$A_j \xrightarrow[F(m)]{} A_k$$

Again, if it exists it is unique up to unique isomorphism. The colimit is sometimes referred to as the **direct limit**, **inductive limit**, or **injective limit**.

Example 1.26 The set $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of the diagram

$$\mathbb{Z} \to 5^{-1}\mathbb{Z} \to 5^{-2}\mathbb{Z} \to \cdots$$

16

The colimit over an index set I is called the **coproduct**, denoted $\amalg_{i \in I} A_i$, and is the dual notion to the product.

Colimits do not always exist, but we have two useful large classes of examples for which they do.

Definition 1.7 A nonempty partially ordered set $(S, \leqslant)$ is **filtered** or said to be a **filtered set** if for each $x, y \in S$, there is a z such that $x \leqslant z$ and $y \leqslant z$. More generally, a nonempty category $\mathscr{D}$ is **filtered** if:

(i) for each $x, y \in \mathscr{D}$, there is $z \in \mathscr{D}$ and arrows $x \to z$ and $y \to z$, and

(ii) for every two arrows $u : x \to y$ and $v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$

The second condition is known as the existence of a cofork.

Example 1.27 Suppose $\mathscr{D}$ is filtered. Then any diagram in **Set** indexed by $\mathscr{D}$ has the following as a colimit:

$$L = \{(a_i, i) \in \amalg_{i \in \mathscr{D}} A_i\} \setminus \left(\begin{array}{l} (a_i, i) \sim (a_j, j) \text{ if and only if there are } f : A_i \to A_k \text{ and} \\ g : A_j \to A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k \end{array} \right)$$

where for each $j \in \mathscr{D}$ we have the map $\iota_j : A_j \to L$ given sending $a_j \in A_j$ to the equivalence class of (a_j, j) in L. First we show that $\sim$ is indeed an equivalence relation. First, choosing any $f : A_i \to A_k$ in the diagram, we observe that $(a_i, i) \sim (a_i, i)$ for all $a_i \in A_i$ and all i. By definition the relation is independent of the ordering of the terms, so it is symmetric. Finally, suppose $(a_i, i) \sim (a_j, j) \sim (a_k, k)$ with maps $f_i : A_i \to A_l, f_j : A_j \to A_l$ and $g_j : A_j \to A_r, g_k : A_k \to A_r$ satisfying the relations. As $\mathscr{D}$ is filtered there exists $t \in \mathscr{D}$ and maps $h_l : A_l \to A_t$ and $h_r : A_r \to A_t$ in the diagram. Then, by the filtered condition we have that for $h_l \circ f_j : A_j \to A_t$ and $h_r \circ g_j : A_j \to A_t$ we have a map $w : A_t \to A_r$ in the diagram such that $w \circ h_l \circ f_j = w \circ h_r \circ g_j$. It follows that

$$w \circ h_l \circ f_i(a_i) = w \circ h_l \circ f_j(a_j) = w \circ h_r \circ g_j(a_j) = w \circ h_r \circ g_k(a_k)$$

so $(a_i, i) \sim (a_k, k)$. Thus $\sim$ is indeed an equivalence relation on $\amalg_{i \in \mathscr{D}} A_i$ and the quotient is well-defined. If $f : A_i \to A_k$ is a map in the diagram and $a_i \in A_i$, then for any other map $g : A_k \to A_l$ in the diagram, which exists by the fact that the $\mathscr{D}$ is filtered, we have $g \circ f(a_i) = g(f(a_i))$, so a_i and $f(a_i)$ are in the same equivalence class. Thus $\iota_i(a_i) = \iota_k(f(a_i))$. Thus L and its associated maps form a cocone of the diagram. Let B with associated maps $\rho_j : A_j \to B$ be another cocone of our diagram. We define a map $F : L \to B$ by $L([(a_i, i)]) = \rho_i(a_i)$. We must show this is independent of representative. Suppose $(a_i, i) \sim (a_j, j)$. Then there exist maps $f : A_i \to A_k$ and $g : A_j \to A_k$ in the diagram such that $f(a_i) = g(a_j)$. Then since B is a cocone, $\rho_i(a_i) = \rho_k(f(a_i))$ and $\rho_j(a_j) = \rho_k(g(a_j))$. Thus, as $\rho_k(f(a_i)) = \rho_k(g(a_j))$, the map F is well-defined. Additionally, by definition $F \circ \iota_i(a_i) = \rho_i(a_i)$ so F is a map of cocones.

Finally, suppose $G : L \to B$ is another map of cocones. Then $G([(a_i, i)]) = G \circ f_i(a_i) = \rho_i(a_i)$ for all $i \in \mathscr{D}$ and $a_i \in A_i$. Hence $G = F$, so the map is unique and L with its associated component maps is the colimit.

1.4.0   Adjoints

Just as a universal property "essentially" determines an object in a category, "adjoint" essentially determine a functor (assuming it exists). Two **covariant** functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are **adjoint** if there is a natural bijection for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$

$$\tau_{AB} : \mathbf{Hom}_{\mathscr{B}}(F(A), B) \to \mathbf{Hom}_{\mathscr{A}}(A, G(B))$$

We say (F, G) form an **adjoint pair**, and that F is **left-adjoint** to G, and G is **right-adjoint** to F. We say F is **left adjoint**, and G is **right adjoint**. By natural we mean $\tau : \mathbf{Hom}_{\mathscr{B}}(F(-), -) \Rightarrow \mathbf{Hom}_{\mathscr{A}}(-, G(-))$ is a natural isomorphism. In other words, for all $f : A \to A'$ in $\mathscr{A}$ and $g : B \to B'$ in $\mathscr{B}$, the diagrams

$$\mathbf{Hom}(F(A'), B) \xrightarrow{-\circ F(f)} \mathbf{Hom}(F(A), B)$$

$$\tau_{A'B} \downarrow \qquad\qquad \downarrow \tau_{AB}$$

$$\mathbf{Hom}(A', G(B)) \xrightarrow[-\circ f]{} \mathbf{Hom}(A, G(B))$$

and

$$\mathbf{Hom}(F(A), B) \xrightarrow{f \circ -} \mathbf{Hom}(F(A), B')$$

$$\tau_{AB} \downarrow \qquad\qquad \downarrow \tau_{AB'}$$

$$\mathbf{Hom}(A, G(B)) \xrightarrow[G(f) \circ -]{} \mathbf{Hom}(A, G(B'))$$

commute.

Proposition 1.2 *There exist natural transformations $\eta : \mathrm{Id}_{\mathscr{A}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{Id}_{\mathscr{B}}$, called the **unit** and **counit** respectively, such that for any $g : F(A) \to B$ and $f : A \to G(B)$, $\tau_{AB}(g) = G(g) \circ \eta_A : A \to G(B)$, and $\tau_{AB}^{-1}(f) = \epsilon_B \circ F(f) : F(A) \to B$.*

Proof For this set $\eta_A = \tau_{AF(A)}(\mathrm{Id}_{F(A)})$ and $\epsilon_B = \tau_{G(B)B}^{-1}(\mathrm{Id}_{G(B)})$. Now let $f : A \to G(B)$ and $g : F(A) \to B$. It follows by naturality of τ that

$$\tau_{AB}(g) = \tau_{AB}(g \circ \mathrm{Id}_{F(A)}) = G(g) \circ \tau_{AF(A)}(\mathrm{Id}_{F(A)}) = G(g) \circ \eta_A$$

and

$$\tau_{AB}^{-1}(f) = \tau_{AB}^{-1}(\mathrm{Id}_{G(B)} \circ f) = \tau_{G(B)B}^{-1}(\mathrm{Id}_{G(B)}) \circ F(f) = \epsilon_B \circ F(f)$$

Thus η and ϵ satisfy our needed conditions, so now we need only show naturality.
Let now $f : A \to A'$ and $g : B \to B'$. Then we have

$$GF(f) \circ \eta_A = \tau_{AF(A')}(F(f)) = \tau_{AF(A')}(\mathrm{Id}_{F(A')} \circ F(f)) = \tau_{A'F(A')}(\mathrm{Id}_{F(A')}) \circ f = \eta_{A'} \circ f$$

using our property for η and naturality of τ. Similarly,

$$\epsilon_{B'} \circ FG(g) = \tau_{G(B)B'}^{-1}(G(g)) = g \circ \tau_{G(B)B}^{-1}(\mathrm{Id}_{G(B)}) = g \circ \epsilon_B$$

as desired. □

The left adjoint determines the right adjoint up to natural isomorphis, and vice-versa. If we endow additional conditions on the unit and counit, in particular $G\epsilon \circ \eta G = \mathrm{Id}_G$ and $\epsilon F \circ F\eta = \mathrm{Id}_F$ (the triangle identities), then we can recover the adjunction.

1.5.0 Abelian Categories

We now define the notion of an **abelian category**, which is the right general setting in which one can do "homological algebra", in which kernels, cokernels, etcetera are defined, and one can work with complexes and exact sequences.

Two central examples of abelian categories are the category of abelian groups, **Ab**, and the category of A-modules, **Mod**$_A$. The first is a special case of the second with $A = \mathbb{Z}$.

We first define what an **additive category** is.

Definition 1.8 A category $\mathscr{C}$ is said to be **additive** if it satisfies the following properties
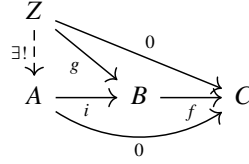
Ad1. For each $A, B \in \mathscr{C}$, **Hom**(A, B) is an abelian group such that composition of morphisms distributes over addition

Ad2. $\mathscr{C}$ has a zero object, denoted 0. (this is an object that is simultaneously an initial object and a final object)

Ad3. It has products of two objects, and hence by induction, products of any finite number of objects.

A functor between additive categories preserving the additive structure of the hom set is called an **additive functor**.

Definition 1.9 Let $\mathscr{C}$ be a category with a 0-object (and thus 0-morphisms). A **kernel** of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. Diagramatically:

$$
\begin{array}{ccccc}
Z & & & & \\
{\scriptstyle \exists!} \downarrow & \searrow {\scriptstyle g} & & {\scriptstyle 0} & \\
A & \xrightarrow[i]{} & B & \xrightarrow[f]{} & C \\
& & & & \\
& {\scriptstyle 0} & & &
\end{array}
$$

Hence it is unique up to unique isomorphism. THe kernal is writte $\ker f \to B$. A **cokernal** is defined dually by reversing the arrows.

If $i : A \to B$ is a monomorphism, then we say that A is a **subobject** of B, where the map i is implicit. There is also the notion of a **quotient object**, defined dually to subobject.

An **abelian category** is an additive category satisfyingn three additional properties.

(1) Every map has a kernel and cokernel

(2) Every monomorphism is the kernel of its cokernel

(3) Every epimorphism is the cokernel of its kernel

The **image** of a morphism $f : A \to B$ is defined as $\operatorname{Im} f = \ker(\operatorname{coker} f)$, whenever it exists. The morphism $f : A \to B$ factors uniquely through $\operatorname{Im} f \to B$ whenever $\operatorname{Im} f$ exists, and $A \to \operatorname{Im} f$ is an epimorphism and a cokernel of $\ker f \to A$ in every abelian category.

The cokernel of a monomorphism is called the **quotient**. The quotient of a monomorphism $A \to B$ is often denoted B/A.

1.5.1 Complexes, Exactness, and Homology

Throughout we consider an abelian category. We say a sequence

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

is a **complex at** B if $g \circ f = 0$, and is **exact at** B if $\ker g = \operatorname{Im} f$.**To be finished**

1.6.0   Spectral Sequences

Chapter 2

Sheaves

Recall that geometric spaces are often best understood in terms of structure preserving functions on them. For example, a differentiable manifold that is a subset of $\mathbb{R}^n$ can be studied in terms of its differentiable functions. Since geometric spaces can have few everywhere-defined functions, a more precise insight is that the structure of the space can be well understood by considering all functions on all open subsets of the space.

2.1.0 Motivation: The Sheaf of Differentiable Functions

Consider differentiable functions on a differentiable manifold M. The sheaf of differentiable functions on M is the data of all differentiable functions on all open subsets on M. On each open set $U \subseteq M$, we have a ring of differentiable functions, denoted by $\mathcal{O}(U)$. Given a differentiable function on an open set, you can restrict it to a smaller open set, obtaining a differentiable function there. That is, if $U \subseteq V$ is an inclusion of open sets, we have a "restriction map" $\text{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U)$.

Note if we have $U \subseteq V \subseteq W$, we can restrict functions step-wise or all at once, yielding the same result. This is codified in the commutivity of the following diagram:

$$
\begin{array}{ccc}
\mathcal{O}(W) & \xrightarrow{\text{res}_{W,V}} & \mathcal{O}(V) \\
& {\scriptstyle \text{res}_{W,U}} \searrow \quad \swarrow {\scriptstyle \text{res}_{V,U}} & \\
& \mathcal{O}(U) &
\end{array}
$$

Another property is that if we have two differentiable functions f_1 and f_2 on an open set U, and an open cover of U, $\{U_i\}_{i \in I}$ such that $\text{res}_{U,U_i} f_1 = \text{res}_{U,U_i} f_2$ for all i, then we must have that $f_1 = f_2$. Thus we can identify functions on an open set by looking at them on a covering by small open sets.

Finally, suppose you are given the same U and cover $\{U_i\}_{i \in I}$ with associated differentiable functions f_i. If the functions agree on their overlap, $\text{res}_{U_i, U_i \cap U_j} f_i = \text{res}_{U_j, U_i \cap U_j} f_j$, for all i and j, then they can be glued together to make one differentiable function on all of U. In other words, there exists $f \in \mathcal{O}(U)$ such that $\text{res}_{U,U_i} f = f_i$ for all i.

2.1.1 The Germ of a Differentiable Function

We now define the germ of a differentiable function at a point $p \in M$. Intuitively it is a "shred" of a differentiable function at p.

Definition 2.1 The germ of a function f on M is an object of the form

$$\{(f, U) : p \in U, f \in \mathscr{O}(U), U \in \tau_M\}/\{(f, U) \sim (g, U) \text{ if there exists } W \subseteq U \cap V \text{ containing } p \text{ such that } \mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g\}$$

In other words, two functions that are the same in an open neighborhood of p have the same germ at that point. We call the set of germs at the point p the **stalk** at p, and denote it $\mathscr{O}_p$.

Notice that the stalk is a ring and you can add two germs to get another germ. Additionally, notice that if $p \in U$ and you get a map $\mathscr{O}(U) \to \mathscr{O}_p$ sending a function to its germ. Note that this is given germs as colimits.

We can see that $\mathscr{O}_p$ is a local ring as follows. Consider those germs vanashing at p which we denote by $\mathfrak{m}_p \subseteq \mathscr{O}_p$. They certainly form an ideal. This is in fact maximal, as we have the exact sequence

$$0 \to \mathfrak{m}_p \to \mathscr{O}_p \to \mathbb{R} \to 0$$

Now, if $f \in \mathscr{O}_p \backslash \mathfrak{m}_p$. Then f doesn't vanish at p, so by continuity there is some neighborhood U of p for which f is non-zero. In particular its inverse function $1/f$ is defined and differentiable on U since f is non-zero. Then the product of the germ of f with the germ of $1/f$ will be the germ of the constant function 1. Hence the germ of f is invertible, so $\mathfrak{m}_p$ is indeed a maximal ideal.

Note that we can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal, $\mathscr{O}_p/\mathfrak{m}_p$.

2.2.0   Sheaf and Presheaf

Definition 2.2 A **presheaf** $\mathscr{F}$ of sets on a topological space X is the following data.

- To each open set $U \subseteq X$, we have a set $\mathscr{F}(U)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over** U. Sections of $\mathscr{F}$ mean sections of $\mathscr{F}$ over X, which are called **global sections**.

- For each inclusion $U \subseteq V$ of open sets, we have a **restriction map**, $\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$.

- The map $\mathrm{res}_{U,U}$ is the identity $\mathrm{Id}_{\mathscr{F}(U)}$.

- If $U \subseteq V \subseteq W$ are inclusions of open sets, then the restriction maps commute, i.e.

$$
\begin{array}{ccc}
\mathscr{F}(W) & \xrightarrow{\mathrm{res}_{W,V}} & \mathscr{F}(V) \\
& \searrow{\scriptstyle \mathrm{res}_{W,U}} \quad \swarrow{\scriptstyle \mathrm{res}_{V,U}} & \\
& \mathscr{F}(U) &
\end{array}
$$

commutes.

In particular, if $O(X)$ is the category of open sets of X, then a presheaf of objects of type $\mathscr{C}$ is simply a contravariant functor $\mathscr{F} : O(X)^{op} \to \mathscr{C}$.

We now look to the **stalk** of a presheaf.

Definition 2.3 Let $\mathscr{F}$ be a presheaf of X. We define the **stalk** of $\mathscr{F}$ at $p \in X$ to be the set of **germs** of $\mathscr{F}$ at p, denoted $\mathscr{F}_p$. Germs correspond to sections over some open set containing p, and two of these sections are considered equivalent if they agree on some smaller open set. More precisely, the stalk can be defined as

$$\{(f, U) : p \in U, f \in \mathscr{F}(U), U \in \tau_X\}/\{(f, U) \sim (g, V) \iff \exists W \subseteq U \cap V \text{ where } pinW \text{ and } \text{res}_{U,W} f = \text{res}_{V,W}$$

A useful equivalent categorical definition of a stalk is as a colimit of all $\mathscr{F}(U)$ over all open sets U containing p.

$$\mathscr{F}_p := \varinjlim_{p \in U} \mathscr{F}(U)$$

As the index category is a filtered set, our previously defined stalk is also a colimit, and they are hence naturally isomorphic. If $p \in U$ and $f \in \mathscr{F}(U)$, then the image of f in $\mathscr{F}_p$ is called the **germ of f at p**.

Definition 2.4 A presheaf is a **sheaf** if it satisfies two more axioms. Let $\{U_i\}_{i \in I}$ be an open cover of U.

- (**identity axiom**) If $f_1, f_2 \in \mathscr{F}(U)$ and $\text{res}_{U,U_i} f_1 = \text{res}_{U,U_i} f_2$ for all i, then $f_1 = f_2$.

- (**Gluability axiom**) If $f_i \in \mathscr{F}(U_i)$ for all i, such that $\text{res}_{U_i, U_i \cap U_j} f_i = \text{res}_{U_j, U_i \cap U_j}$ for all i, j, then there exists $f \in \mathscr{F}(U)$ such that $\text{res}_{U,U_i} f = f_i$ for all i.

The two axioms for a presheaf to be a sheaf can be interpreted as "exactness" of the "equalizer exact sequence":

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

Identity is exactness at $\mathscr{F}(U)$, and gluability is exactness at $\prod_i \mathscr{F}(U_i)$.

Example 2.1 Suppose $\mathscr{F}$ is a sheaf on X and U is an open subset of X. Define the **restriction of $\mathscr{F}$ to U**, denoted $\mathscr{F}|_U$, to be the functor $\mathscr{F}|_U : \mathcal{U}^{op} \to \textbf{Set}$ defined by $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for $V \subseteq U$ open, and restrictions don't change.

Example 2.2 Suppose X is a topological space, with $p \in X$, and S is a set. Let $i_p : p \to X$ be the inclusion. Then $i_{p,*} S$ defined by

$$i_{p,*} S(U) = \begin{cases} S & \text{if } p \in U, \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf. Here $\{e\}$ is any one-element set. This is called the **skyscraper sheaf**. There is an analogous definition for sheaves of abelian groups, except $i_{p,*}(S)(U) = \{0\}$ if $p \notin U$; and for sheaves with values in a more general category, $i_{p,*} S(U)$ should be a final object if $p \notin U$.

Example 2.3 Let X be a topological space and S a set. Define $\underline{S}_{pre}(U) = S$ for all open sets U. Then $\underline{S}_{pre}$ is a pre-sheaf with restriction maps simply the identity. This is called the **constant presheaf associated to S**. This isn't in general a sheaf.

Another important perspective of sheaves is in terms of their space of section (**espace étalé**). Suppose $\mathscr{F}$ is a presheaf on a topological space X. Construct a topological space F along with a continuous map $\pi : F \to X$ as

follows: as a set, F is the disjoint union of all the stalks of $\mathscr{F}$. This naturally gives a map of sets $\pi : F \rightarrow X$. We topologize F as follows. Each s in $\mathscr{F}(U)$ determines a subset $\{(x, s_x) : x \in U\}$ of F ($s_x \in \mathscr{F}_x$ is the image of s in the stalk). The topology on F is the weakest topology such that these subsets are open (i.e. they form a base for the topology). For each $y \in F$, there is an open neighborhood V of y and an open neighborhood U of $\pi(y)$ such that $\pi|_V$ is a homeomorphism from V to U. In particular, if $y = (x, s_x)$ and U is an open neighborhood of x, then for $V = \{(x, s_x) : x \in U\}$ we have that $\pi|_V$ is a homeomorphism from V to U. Its inverse is given by sending $x \in U$ to (x, s_x) in V. Then $F = \amalg_{x \in X} \mathscr{F}_x$ is the **space of sections** of $\mathscr{F}$, and in French is the **espace étalé** of $\mathscr{F}$.

Definition 2.5 Suppose $\pi : X \rightarrow Y$ is a continuous map, and $\mathscr{F}$ is a presheaf on X. Then define $\pi_* \mathscr{F}$ by $\pi_* \mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$, where V is an open subset of Y. Then $\pi_* \mathscr{F}$ is a presheaf on Y, and is a sheaf if $\mathscr{F}$ is. This is called the **pushforward** or **direct image** of $\mathscr{F}$ by π.

As the notation suggests, the skyscraper sheaf can be interpreted as the pushforward of the constant sheaf on a one-point spcae p, under the inclusion morphism $i_p : \{p\} \rightarrow X$.

Example 2.4 (Ringed Spaces) Suppose $\mathscr{O}_X$ is a sheaf of rings on a topological space X. Then $(X, \mathscr{O}_X)$ is called a **ringed space**. The sheaf of rings is often denoted by $\mathscr{O}_X$. This sheaf is called the **structure sheaf** of the ringed space. Sections of the structure sheaf $\mathscr{O}_X$ over an open subset U are called **regular functions on** U.

We denote the restriction $\mathscr{O}_X|_U$ of $\mathscr{O}_X$ to an open subset $U \subseteq X$ by $\mathscr{O}_U$. The stalk of $\mathscr{O}_X$ at a point p is written $\mathscr{O}_{X,p}$. Now, just as we have modules over a ring, we have $\mathscr{O}_X$-modules over a sheaf of rings $\mathscr{O}_X$. The definition associated with the name $\mathscr{O}_X$**-module** is a sheaf of abelian groups $\mathscr{F}$ with the following additional structure. For each U, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps: if $U \subseteq V$, then

$$
\begin{array}{ccc}
\mathscr{O}_X(V) \times \mathscr{F}(V) & \xrightarrow{\ action\ } & \mathscr{F}(V) \\
{\scriptstyle \mathrm{res}_{V,U} \times \mathrm{res}_{V,U}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{V,U}} \\
\mathscr{O}_X(U) \times \mathscr{F}(U) & \xrightarrow[\ action\ ]{} & \mathscr{F}(U)
\end{array}
$$

commutes

Note a sheaf of abelian groups is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. Hence, when we are proving things about $\mathscr{O}_X$-modules, we are also proving things about sheaves of abelian groups.

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ an $\mathscr{O}_X$-module. For each $p \in X$, we have that $\mathscr{F}_p$ is an abelian group and $\mathscr{O}_{X,p}$ is a ring. We act $\mathscr{O}_{X,p}$ on $\mathscr{F}_p$ by $[(o, U)] \cdot [(f, V)] := [(\mathrm{res}_{U,U \cap V} o \cdot \mathrm{res}_{V,U \cap V} f, U \cap V)]$. If well defined this will turn $\mathscr{F}_p$ into a $\mathscr{O}_{X,p}$-module. Indeed, if $(o, U) \sim (s, W)$ and $(f, V) \sim (g, Y)$, then we have $E \subseteq U \cap W$ and $E' \subseteq V \cap Y$ containing p for which the restrictions of o and s, and f and g agree, respectively. Set $K = E \cap E'$ which again contains p. Then
$$(\mathrm{res}_{U,U \cap V} o \cdot \mathrm{res}_{V,U \cap V} f, U \cap V) \sim (\mathrm{res}_{U,K} o \cdot \mathrm{res}_{V,K} f, K) = (\mathrm{res}_{W,K} s \cdot \mathrm{res}_{Y,K} g, K)$$
so the action is well-defined.

2.3.0 Morphisms of Presheaves and Sheaves

We now define morphisms of presheaves and similarly for sheaves. A **morphism of presheaves** of value in a category $\mathscr{C}$ on X, $\Phi : \mathscr{F} \Rightarrow \mathscr{G}$, is the data of maps $\Phi(U) : \mathscr{F}(U) \rightarrow \mathscr{G}(U)$ for all U behaving well with respect to restriction: If $U \hookrightarrow V$, then

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\text{res}_{V,U}} & \mathscr{F}(U) \\ {\scriptstyle\Phi_V}\downarrow & & \downarrow{\scriptstyle\Phi_U} \\ \mathscr{G}(V) & \xrightarrow[\text{res}_{V,U}]{} & \mathscr{G}(U) \end{array}$$

commutes. **Morphisms of sheaves** are identical to morphisms of the underlying presheavers. That is the category of sheaves on X is a full subcategory of the category of presheaves on X. Note that a morphism of sheaves is simply a natural transformation between the underlying functors.

If $(X, \mathscr{O}_X)$ is a ringed space, then morphisms of $\mathscr{O}_X$-modules are given by functors between the underlying sheaves of abelian groups which preserves the module structure.

We shall write $\mathbf{Set}_X, \mathbf{Ab}_X$, etcetera, for the category of sheaves of sets, abelian groups, etcetera on a topological space X. We also write $\mathbf{Mod}_{\mathscr{O}_X}$ for the category of $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$. Let $\mathbf{Set}_X^{pre}$, etcetera, denote the category of presheaves on sets, etcetera, on X.

Definition 2.6 Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X. Let $\mathbf{Hom}(\mathscr{F}, \mathscr{G})$ be the collection of data

$$\mathbf{Hom}(\mathscr{F}, \mathscr{G})(U) := \mathbf{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

This is again a sheaf of sets on X

If $\mathscr{F}$ and $\mathscr{G}$ are sheavers of abelian groups, then $\mathbf{Hom}_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})$ is defined by taking $\mathbf{Hom}_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})(U)$ to be the maps as sheaves of abelian groups, $\mathscr{F}|_U \to \mathscr{G}|_U$. Similarly, if $\mathscr{F}$ nad $\mathscr{G}$ are $\mathscr{O}_X$-modules, we define $\mathbf{Hom}_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{F}, \mathscr{G})$ in the analogous way.

We can make module-like constructions using presheavers of abelian groups on a topological space X. For example, we can add maps of presheaves nad get another map of presheaves: if $\phi, \psi : \mathscr{F} \Rightarrow \mathscr{G}$, then we define the map $\phi + \psi : \mathscr{F} \to \mathscr{G}$ by $(\phi + \psi)_V = \phi_V + \psi_V$. In this way, presheaves of abelian groups form an additive category.

If $\phi : \mathscr{F} \Rightarrow \mathscr{G}$ is a morphism of presheaves, define the **presheaf kernel** $\ker_{pre} \phi$ by $(\ker_{pre} \phi)_U := \ker \phi_U$. Similarly, we define the **presheaf cokernel** $\text{coker}_{pre}\phi$ similarly. Both are presheafs. It then follows that presheaves of abelian groups form an abelian category. In particular we can define terms such as **subpresheaf**, **image presheaf**, and **quotient presheaf**, and they behave as one would expect.

2.4.0   Properties at the Level of Stalks and Sheafification

2.4.1 Properties at the Level of Stalks

We shall now see that numerous facts about sheaves can be checked using the language of stalks. This reflects the local nature of sheaves.

Claim A section of a sheaf of sets is determined by its germs. That is, the natural map

$$\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$$

is injective.

Proof The natural map is given by $s \in \mathscr{F}(U) \mapsto ([s]_p)_{p \in U}$. To show injectivity we suppose that there is another section t which is mapped to the same point. In other words, $[s]_p = [t]_p$ for all $p \in U$. That is, there exist $p \in V_p \subseteq U$ such that $s|_{V_p} = t|_{V_p}$. However, $\{V_p\}_{p \in U}$ is an open cover of U. Hence, as they agree on all restrictions of the open cover, we have by the identity axiom for sheaves that $s = t$. $\qquad\square$

Definition 2.7 Suppose $\mathscr{F}$ is a sheaf of abelian groups on X, and s is a global section of $\mathscr{F}$. Then let the **support of** s, denoted $\mathrm{Supp}(s)$, be the points p of X where s has a nonzero germ:

$$\mathrm{Supp}(s) := \{p \in X : [s]_p \neq 0 \text{ in } \mathscr{F}_p\}$$

We think of this as the subset of X where the section "lives"— the complement is the locus where s is the 0-section.

The previous claim assures us that all non-zero section "live somewhere," or in other words, are not sent to the zero product under the map $\mathscr{F}(X) \to \prod_{x \in X} \mathscr{F}_x$. For any global section $s \in \mathscr{F}(X)$, $\mathrm{Supp}(s)$ is a closed subset. First, consider the map $X \to \amalg_{p \in X}\mathscr{F}_p$ given by $p \mapsto [s]_p$. Then the support of s is the inverse image of $\amalg_{p \in X}\mathscr{F}_p \backslash \{[0]_p : p \in X\}$. But $\{[x]_p : p \in X\}$ is an open set by definition of the topology on the disjoint union of $\mathscr{F}_p$, so as the map is a homeomorphism the preimage of this closed set is closed.

Definition 2.8 We say that an element $(s_p)_{p \in U}$ of $\prod_{p \in U} \mathscr{F}_p$ consists of **compatible germs** if for all $p \in U$, there is some representative $(p \in U_p \subseteq U, \tilde{s}_p \in \mathscr{F}(U_p))$ for s_p such that the germ of $\tilde{s}_p$ at all $q \in U_p$ is s_q. Equivalently, there is an open cover $\{U_i\}_{i \in I}$ of U, and sections $f_i \in \mathscr{F}(U_i)$, such that if $p \in U_i$, then s_p is the germ of f_i at p. Note any section s of $\mathscr{F}$ over U gives a choice of compatible germs for U.

The image of our previous map consists precisely of elements which consist of compatible germs.

2.4.2 Sheafification

Recall every sheaf is a presheaf. Just as groupification gives an abelian group that best approximates an abelian semigroup, sheafification gives the sheaf that best approximates a presheaf, with an analogous universal property.

Definition 2.9 If $\mathscr{F}$ is a presheaf on X, then a morphism of presheaves $sh : \mathscr{F} \to \mathscr{F}^{sh}$ on X is a **sheafification of** $\mathscr{F}$ if $\mathscr{F}^{sh}$ is a sheaf, and for any sheaf $\mathscr{G}$, and any presheaf morphism $g : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $f : \mathscr{F}^{sh} \to \mathscr{G}$ making the diagram:

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\ sh\ } & \mathscr{F}^{sh} \\
 & {}_{g}\searrow & \downarrow{}^{f} \\
 & & \mathscr{G}
\end{array}
$$

commute. This is an initial object in the comma category

$$
\begin{array}{c}
\mathrm{Shf}(X) \\
\downarrow{}^{\iota} \\
1 \xrightarrow[\mathscr{F}]{} \mathrm{PrShf}(X)
\end{array}
$$

We still need to show that initial objects exist in this category. A consequence of this definition is that sheafification is unique up to unique isomorphism, as initial objects in a category are unique up to unique isomorphism. In particular,

this implies that sheafification is functorial. Indeed, for any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$, we have the following commuting diagram, with the unique map be a naturally induced morphism of sheaves $\phi^{sh} : \mathscr{F}^{sh} \to \mathscr{G}^{sh}$.

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\ \mathrm{sh}_{\mathscr{F}}\ } & \mathscr{F}^{sh} \\
{\scriptstyle \phi}\searrow & & \Big\updownarrow{\scriptstyle \exists!\phi^{sh}} \\
& \mathscr{G} & \\
& {\scriptstyle \mathrm{sh}_{\mathscr{G}}}\searrow & \\
& & \mathscr{G}^{sh}
\end{array}
$$

This is functorial since uniqueness implies that compositions for $\Phi : \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ satisfy $\Phi^{sh} = \psi^{sh} \circ \phi^{sh}$ since both sides will make the appropriate diagram commute.

Remark:

Construction Suppose $\mathscr{F}$ is a presheaf. Define $\mathscr{F}^{sh}$ by defining $\mathscr{F}^{sh}(U)$ as the set of compatible germs of the presheaf $\mathscr{F}$ over U. Explicitly,

$$
\mathscr{F}^{sh}(U) := \Big\{ (f_p \in \mathscr{F}_p)_{p \in U} : \text{for all } p \in U, \text{ there exists an open}
$$

$$
\text{neighborhood } V \text{ of } p, \text{ contained in } U,
$$
$$
\text{and } s \in \mathscr{F}(V) \text{ with } s_q = f_q \text{ for all } q \in V
$$

Here s_q means the germ of s at q.

Remark:

The espace étalé construction yields a different description of sheafification. The main idea is identical: if $\mathscr{F}$ is a presheaf, let F be the space of sections, or espace étalé, of $\mathscr{F}$. Then the sheafification of $\mathscr{F}$ is the sheaf of sections of $F \to X$.

2.4.3 Subsheaves and Quotient Sheaves

Definition 2.10 Suppose $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. If any of the following equivalent conditions hold, then we say that $\mathscr{F}$ is a **subsheaf** of $\mathscr{G}$:

- Φ is a monomorphism in the category of sheaves

- Φ is injective on the level of stalks

- Φ is injective on the level of open sets

Definition 2.11 Suppose $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. If any of the following equivalent conditions hold, then we say that $\mathscr{G}$ is a **quotient sheaf** of $\mathscr{F}$:

- Φ is a epimorphism in the category of sheaves

- Φ is surjective on the level of stalks

Example 2.5 Let $X = \mathbb{C}$ with the classical topology, and define $\mathscr{O}_X$ to be the sheaf of holomorphic functions, and $\mathscr{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \to 1$$

where $\mathbb{Z}$ is the constant sheaf associated to $\mathbb{Z}$.

2.5.0 Recovering Sheaves from a Sheaf on a Base

Definition 2.12 Suppose we have a topological space (X, τ). Then a base of τ is a subcollection $\{B_j\}_{j \in J} \subseteq \tau$ such that for all $U \in \tau$, $U = \bigcup_{B_j \subseteq U} B_j$.

Suppose $X = \mathbb{R}^n$. Then we have the classical countable base of rationally centered open balls, $\{B_r(q) : q \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\}$. Such bases are useful as we need only check continuity at base open sets to find continuity of a map. Now, suppose we have a sheaf $\mathscr{F}$ on a topological space X, and a base $\{B_i\}_{i \in I}$ of open sets on X. Then consider the information

$$(\{\mathscr{F}(B_i)\}, \{\mathrm{res}_{B_i, B_j} : \mathscr{F}(B_i) \to \mathscr{F}(B_j) : B_j \subseteq B_i\})$$

which is a subset of the information contained in the sheaf.

We can recover the entire sheaf from this information. This follows from the fact that we can determine the stalks from this information, and we can determine when germs are compatible.

Definition 2.13 A sheaf of a concrete category on a base $\{B_i\}_{i \in I}$ is the following. For each B_i in the base we have a set $F(B_i)$. If $B_i \subseteq B_j$, we have maps $\mathrm{res}_{B_j, B_i} : F(B_j) \to F(B_i)$, with $\mathrm{res}_{B_i, B_i} = \mathrm{id}_{F(B_i)}$. If $B_i \subseteq B_j \subseteq B_k$, then $\mathrm{res}_{B_k, B_i} = \mathrm{res}_{B_j, B_i} \circ \mathrm{res}_{B_k, B_j}$. So far this is a **presheaf on a base** $\{B_i\}_{i \in I}$. We also require two axioms:

- **Base Identity**: If $B = \bigcup B_i$ and $f, g \in F(B)$ are such that $\mathrm{res}_{B, B_i} f = \mathrm{res}_{B, B_i} g$ for all i, then $f = g$.

- **Base Gluability**: If $B = \bigcup B_i$, and we have $f_i \in F(B_i)$ such that f_i agrees with f_j on any basic open set contained in $B_i \cap B_j$, then there exists $f \in F(B)$ such that $\mathrm{res}_{B, B_i} f = f_i$ for all i.

Theorem 2.1 *Suppose $\{B_i\}_{i \in I}$ is a base on X, and F is a sheaf of sets on this base. Then there is a sheaf $\mathscr{F}$ extending F (with isomorphisms $\mathscr{F}(B_i) \cong F(B_i)$ agreeng with restriction maps). This sheaf $\mathscr{F}$ is unique up to unique isomorphism.*

Proof We will define $\mathscr{F}$ as the sheaf of compatible germs of F. Define the **stalk** of a presheaf F on a base at $p \in X$ by

$$F_p = \varinjlim F(B_i)$$

where the colimit is over all B_i containing p.
We say a family of germs in an open set U is compatible near p if there is a section s of F over some B_i containing p such that the germs over B_i are precisely the germs of s. More formally, define

$$\mathscr{F}(U) := \{(f_p \in F_p)_{p \in U} : \text{for all } p \in U, \text{ there exists } B \text{ with } p \in B \subseteq U$$
$$s \in F(B) \text{ with } s_q = F_q \text{ for all } q \in B\}$$

where each B is in our base. This is a sheaf with the natural map $F(B) \to \mathscr{F}(B)$ being an isomorphism. Indeed, first to show injectivity suppose $f, g \in F(B)$ with $(f_p)_{p \in B} = (g_p)_{p \in B}$. Then $f_p = g_p$ for all $p \in B$. Thus we have a cover $B = \bigcup_{p \in B} B_p$ with B_p in the base for which $\mathrm{res}_{B,B_p} f = \mathrm{res}_{B,B_p} g$. By the base identity axiom we have that $f = g$.

Next, for surjectivity suppose $(f_p)_{p \in B} \in \mathscr{F}(B)$. Then for each $p \in B$ there exists B_p in the base containing p and contained in B, as well as $s^p \in F(B_p)$ such that $s_q^p = f_q$ for all $q \in B_p$. Then for all $q \in B_p \cap B_{p'}$ there exists B_q' in the base such that $s^p|_{B_q'} = s^{p'}|_{B_q'}$. By the gluability axiom of the sheaf on a base, there exists a unique $s \in F(B)$ such that $s|_{B_p} = s^p$ for all $p \in B$. In particular, $(s_p)_{p \in B} = (f_p)_{p \in B}$, so the map is surjective. $\qquad\square$

Theorem 2.2 *Suppose $\{B_i\}_{i \in I}$ is a base for the topology of X. A morphism $F \to G$ of sheaves on the base is a collection of map $F(B_k) \to G(B_k)$ commuting with restrictions. A morphism of sheaves is determined by the induced morphism of sheaves on the base. Further, a morphism of sheaves on the base gives a morphism of the induced sheaves.*

The above constructions and arguments describe an equivalence of categories between sheaves on X and sheaves on a given base of X. Further, it will be useful to extend these notions to $\mathscr{O}_X$-modules. We will obtain the analogous result of an equivalence of categories between $\mathscr{O}_X$-modules and $\mathscr{O}_X$-modules on a base.

2.6.0 Sheaves of Abelian Groups and Modules Form Abelian Categories

We shall see that kernels, cokernels, exactness, and so forth can be understood at the level of stalks, which are just abelian groups, and that compatibility of germs will come for free.

Note that the category of sheaves of abelian groups on a topological space X is evidently an additive category (its enriched over **Ab** with zero objects and finite products). We now need to show that any morphism $\Phi : \mathscr{F} \to \mathscr{G}$ has a kernel and a cokernel. Recall that the presheaf kernel is a sheaf (since the forgetful functor from sheaves to presheaves is a right adjoint and hence commutes with limits), and is a kernel in the cateogry of sheaves, so Φ has a kernel.

Next, recall that $\Phi : \mathscr{F} \to \mathscr{G}$ has a cokernel in the category of presheaves; call it $\mathscr{H}_{pre}$. Let sh $: \mathscr{H}_{pre} \to \mathscr{H}$ be its sheafification.

Proposition 2.1 *The composition $\mathscr{G} \to \mathscr{H}$ is the cokernel of Φ in the category of sheaves.*

Proof We show that it satisfies the universal property. Given any sheaf $\mathscr{E}$ and a commutative diagram

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\;\Phi\;} & \mathscr{G} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{E}
\end{array}
$$

we construct

Then there exists a unique morphism of presheaves $\Psi : \mathscr{H}_{pre} \to \mathscr{E}$ making the diagram commute. Further, by the universality of sheafification there exists a unique map $\varphi : \mathscr{H} \to \mathscr{E}$ for which $\Psi = \varphi \circ$ sh. $\qquad\square$

Recall that monomorphisms and epimorphisms can be checked at the level of stalks, which is in an abelian category, so we obtain the result:

Theorem 2.3 *Sheaves of abelian groups on a topological space X form an abelian category.*

Part II

Scheme Theory

Chapter 3

Toward Affine Schemes

3.1.0   Toward Schemes

A **scheme** is a type of geometric space which is central to algebraic geometry. We hope to abstract the notion of a manifold to nonsmooth and arithmetic objects. A key insight for this endeavour is that we can understand a geometric space well by understanding the functions on this space. More precisely we will use the sheaf of functions on the space.

Thus, a scheme will consist of the following data:

- The set: the points of the scheme

- The topology: the open sets of the scheme

- The structure sheaf: the sheaf of "algebraic functions" (a sheaf of rings) on the scheme

Recall that a topological space with a sheaf of rings is called a **ringed space**. In the key example of complex affine varieties, the points are the "traditional points" cut out by polynomials in $\mathbb{C}^n$, plus some extra points. The topology is given by specifying that "the subset where an algebraic function vanishes must be closed," and require nothing else. For the sheaf of algebraic functions, we will expect that in the complex plane, $(3x^2 + y^2)/(2x + 4xy + 1)$ should be an algebraic function on the open set consisting of points where the denominator doesn't vanish, and this will largely motivate our definition.

Example 3.1 As motivation we return to the example of differentiable manifolds. Suppose X is a differentiable manifold. It is a topological space, and has a sheaf of differentiable functions $\mathscr{O}_X$. This gives X the structure of a ringed space. Recall the evaluation at a point $p \in X$ gives a surjective map from the stalk to $\mathbb{R}$, $\mathscr{O}_{X,p} \twoheadrightarrow \mathbb{R}$, so the kernel, the germs of functinos vanishing at p, is a maximal ideal $\mathfrak{m}_{X,p}$.

We could define a differentiable real manifold as a topological space X with a sheaf of rings. We would require that there is a cover of X by open sets such that on each open set the ringed space is isomorphic to a ball around the origin in $\mathbb{R}^n$ (with the sheaf of differentiable functions on that ball). With this definition, the ball is the basic patch, and a general manifold is obtained by gluing these patches together.

In the algebraic setting, the basic patch is the notion of an **affine scheme**. Note that classically, functions are determined by their values at points. This won't be true for schemes in general.

Now, how can we describe differentiable maps of manifolds $\pi : X \rightarrow Y$? They are certainly continuous maps, but which ones? Recall we can pull back functions along continuous maps. Differentiable functions pull back to

differentiable functions. More formally, we have a map $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Inverse image is left-adjoint to pushforward, so we also get a map $\pi^\# : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$.

Note that manifolds are covered by disks that are all isomorphic. This isn't true for schemes in general. Informally, this is because in the Zariski topology on schemes, all nonempty open sets are huge and have more "structure."

3.2.0   The Underlying Set of Affine Schemes

For any ring A we are going to define something called Spec A, the **spectrum of** A. Together with a suitable topology and sheaf of rings such an object is called an **affine scheme**. The set Spec A is the set of prime ideals of A. The prime ideal $\mathfrak{p}$ of A when considered as an element of Spec A will be denoted $[\mathfrak{p}]$, to avoid confusion. Elements $a \in A$ will be called **regular functions on Spec** A, and their **value** at the point $[\mathfrak{p}]$ will be $a \mod \mathfrak{p}$.

"An element a of the ring lying in a prime ideal $\mathfrak{p}$" translates to "a function a that is 0 at the point $[\mathfrak{p}]$" or "a function a vanishing at the point $[\mathfrak{p}]$." Note that addition and multiplication of functions correspond to addition and multiplication of their values at points. This is simply a translation of the fact that $A \to A/\mathfrak{p}$ is a ring morphism.

If A is generated over a base field (or base ring) by elements $x_1, ..., x_r$, the elements $x_1, ..., x_r$ are often called **coordinates**, because we will later be able to reinterpret them as restrictions of "coordiantes on r-space."

Example 3.2 (The Complex Affine Line) Define $\mathbb{A}^1_{\mathbb{C}} := \mathrm{Spec}\,\mathbb{C}[x]$. Let's find the prime ideals of $\mathbb{C}[x]$. As $\mathbb{C}[x]$ is an integral domain, 0 is prime. Also, $(x - a)$ is prime for any $a \in \mathbb{C}$ as it is a maximal ideal since its quotient is a field:

$$0 \to (x - a) \to \mathbb{C}[x] \to \mathbb{C} \to 0$$

Now, suppose $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p} \neq (0)$, then $\mathfrak{p} = (f(x))$ for some $f(x) \in \mathbb{C}[x]$ non-zero since $\mathbb{C}[x]$ is a PID. We can write $f(x) = (x - a_1)^{k_1} \cdots (x - a_n)^{k_n}$ as $\mathbb{C}$ is a algebraically closed field. Then as $\mathfrak{p}$ is prime $(x - a_i)$ must be in $\mathfrak{p}$ for some $1 \leqslant i \leqslant n$. However $f(x)$ is the generator of $\mathfrak{p}$, so this implies $k_j = 0$ for $j \neq i$ and $k_i = 1$, so $f(x) = (x - a_i)$.

Thus we can make a picture of $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[x]$. We have one "traditional" point for each complex number, plus one extra point $[(0)]$. We can mostly picture $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$, associating $[(x - a)]$ with $a \in \mathbb{C}$. But, where should $[(0)]$ go? As (0) is contained in all of these prime ideals, we will somehow associate it with this line passing through all the other points. This new point $[(0)]$ is called the "generic point" of the line. It is generically on the line, but it is not at any particular place on the line.

The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. Its value at a point $[(x - 1)]$ is $f(1)$, or equivalently $f(x) \mod (x - 1)$. Its value at $[(0)]$ is just $f(x)$ as $\mathbb{C}[x]$ is already an integral domain.

Example 3.3 (The affine line over k) Let $\mathbb{A}^1_k := \mathrm{Spec}\,k[x]$ where k is an algebraically closed field. This is called the affine line over k. All the arguments in the previous example carry over without change, so we have the same picture.

Example 3.4 Consider Spec$\mathbb{Z}$. Amazingly this will look a lot like the affine line over an algebraically closed field. The integers, like $\bar{k}[x]$, form a PID. The prime ideals are $0\mathbb{Z}$ and $p\mathbb{Z}$ where p is prime.

Consider the function 100 on Spec$\mathbb{Z}$. Its value at (3) is "1 mod 3". Its value at (2) is "0 mod 2".

Example 3.5 The set Spec$k$ where k is any field is one point, $[(0)]$. Spec0, where 0 is the zero-ring, is the empty set, as 0 has no prime ideals.

Example 3.6 Consider $\mathbb{A}^1_{\mathbb{R}} = \text{Spec}\mathbb{R}[x]$. Using the fact that $\mathbb{R}[x]$ is a ED, we have that (0) and $(x-a)$ where $a \in \mathbb{R}$, and $(x^2 + ax + b)$, where $x^2 + ax + b$ is an irreducible quadratic are the prime ideals of $\mathbb{R}[x]$. The latter two are maximal ideals, i.e. their quotients are fields. For example $\mathbb{R}[x]/(x-3) \cong \mathbb{R}, \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

Thus the points of $\mathbb{A}^1_{\mathbb{R}}$ can be pictured as the complex plane folded along the real axis (since we identify conjugate pairs), where we "glue" Galois-conjugate pairs. Now, consider the function $f(x) = x^3 - 1$. Its value at the point $[(x-2)]$ is $f(x) = 7$, or perhaps more accurately, $7(\mod x - 2)$. At $(x^2 + 1)$ we have that $f(x)$ s

$$x^3 - 1 \equiv -x - 1(\mod x^2 + 1)$$

which may be interpreted as $-i - 1$.

Example 3.7 Consider $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec}\mathbb{F}_p[x]$. As with the previous examples, $\mathbb{F}_p[x]$ is a Euclidean domain, so the prime ideals are of the form (0) or $(f(x))$ where $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$.

Note that $\text{Spec}\mathbb{F}_p[x]$ has p points corresponding elements of $\mathbb{F}_p$, but also infinitely many more. This makes this space much richer than simply p points. A polynomial $f(x)$ is not determined by its values at the p elements of $\mathbb{F}_p$, but it is determined by its values at the points of $\text{Spec}\mathbb{F}_p[x]$. (This is not true for all schemes)

Example 3.8 Consider $\mathbb{A}^2_{\mathbb{C}} := \text{Spec}\mathbb{C}[x,y]$. Sadly $\mathbb{C}[x,y]$ is not a PID, for (x,y) is not principal. Some prime ideals of $\mathbb{C}[x,y]$ are (0) and $(x-a, y-b)$ for $(a,b) \in \mathbb{C}^2$ as it is maximal. Also if $f(x,y)$ is an irreducible polynomial, then $(f(x,y))$ is prime. Further, any $\mathfrak{p}$ prime is of these forms.

Example 3.9 Let $\mathbb{A}^n_{\mathbb{C}} := \text{Spec}\mathbb{C}[x_1,...,x_n]$. More generally, let $\mathbb{A}^n_{\mathcal{A}}$ be defined as $\text{Spec}\mathcal{A}[x_1,...,x_n]$, where $\mathcal{A}$ is an arbitrary ring. Analogous to before, $(x_1 - a_1,...,x_n - a_n)$ is a prime ideal, being maximal with residue field $\mathbb{C}$; these are thought of as "0-dimensional points." There are no more maximal ideals by the following result.

Theorem 3.1 (Hilbert's Weak Nullstenelsatz) *If k is an algebraically closed field, then the maximal ideals of $k[x_1,...,x_n]$ are precisely those ideals of the form $(x_1 - a_1,...,x_n - a_n)$, where $a_i \in k$.*

Theorem 3.2 (Hilbert's Nullstellensatz) *If k is any field, every maximal ideal of $k[x_1,...,x_n]$ has residue field a finite extension of k.*

This can be translated as the fact that any field extension of k that is finitely generated as a ring is necessarily also finitely generated as a module.

3.2.1 Quotients and Localizations

We can interpret quotients and localizations of rings in terms of spectra. If A is a ring with ideal I, $\text{Spec}A/I$ can be interpreted as a subset of $\text{Spec}A$, as we have a correspondence between prime ideals of A/I and prime ideals of A containing I.

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