Differential Equations: A Complete Guide

Differential Equations

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Solo Pursuit of Learning

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Where do PDEs Come From?

1.1.0 What is a PDE: Notation and Definitions

Partial Derivatives

Consider a function u of several variables:

$$u = u(x, y, z) \text{ or more generally } u = u(x_1, x_2, ..., x_n)$$

for $(x, y, z) \in U \subset \mathbb{R}^3$ or $(x_1, x_2, ..., x_n) \in U \subset \mathbb{R}^n$. We also write $\mathbf{x} = \vec{x} = (x_1, x_2, ..., x_n)$. The $x_1, ..., x_n$ are called ***independent variables***

Definition 1.1.1. *We say that $U \subseteq \mathbb{R}^n$ is a domain if and only if U is connected, $U^\circ \neq \emptyset$, and ∂U is smooth.*

Notation 1.1.1. Let $u : U \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function (e.g. $u \in C^1(U)$). We denote the partial derivatives with

$$\lim_{h \to 0} \frac{u(\mathbf{x} + he_i) - u(\mathbf{x})}{h} = \frac{\partial u}{\partial x_i}(\mathbf{x}) = u_{x_i}(\mathbf{x}), \quad i = 1, 2, ..., n$$

where e_i is the ith standard basis vector in $\mathbb{R}^n$. For partial derivatives of order $k \in \mathbb{N}$, we write

$$\frac{\partial^k u}{\partial x_{i_1} ... \partial x_{i_k}}(\mathbf{x}) = u_{x_{i_1}, ..., x_{i_k}}(\mathbf{x}), \quad i_1, ..., i_k \in \{1, ..., n\}$$

For the collection of all partial derivatives of order $k \in \mathbb{N}$ we write

$$D^k u := \{u_{x_{i_1}, ..., x_{i_k}} : i_1, ..., i_k \in \{1, ..., n\}\}$$

Differential Operators

Definition 1.1.2. *For $u : U \subset \mathbb{R}^n \to \mathbb{R}$ smooth, we define the* **gradient** *of u to be*

$$\nabla u := (u_{x_1}, ..., u_{x_n})$$

where ∇ is the Nabla differential operator.

Definition 1.1.3. *Given a vector* $v = (v_1, ..., v_n) \in \mathbb{R}^n$ *and* $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ *smooth, the directional derivative of* u *in the direction of* v *is given by (by application of the chain rule):*

$$D_v u = \nabla u \cdot v = \sum_{i=1}^{n} u_{x_i} v_i =: \frac{\partial u}{\partial v}$$

In particular, $\nabla u \cdot e_i = u_{x_i}$.

Definition 1.1.4. *For* $V : U \subset \mathbb{R}^n \to \mathbb{R}^n$ *a smooth vector field, we write* $V(\mathbf{x}) = (V^1(\mathbf{x}), ..., V^n(\mathbf{x}))$, *and we have the differential of* V, *or Jacobian,*

$$J_V = DV = \begin{bmatrix} V^1_{x_1} & \dots & V^1_{x_n} \\ \vdots & \ddots & \vdots \\ V^n_{x_1} & \dots & V^n_{x_n} \end{bmatrix}$$

We then define the **divergence** *of* V *to be*

$$DivV := \nabla \cdot V = trDV = \sum_{i=1}^{n} V^i_{x_i}$$

Definition 1.1.5. *Let* $u(\mathbf{x}) = u(x_1, ..., x_n)$ *be a smooth real-valued function,* $\curvearrowleft \in U$. *Then* $\nabla u : U \subset \mathbb{R}^n \to \mathbb{R}^n$ *and as above we have Hessian matrix for* u, *i.e. the Jacobian of the gradient of* u*:*

$$H_u = D\nabla u = \begin{bmatrix} u_{x_1,x_1} & \dots & u_{x_1,x_n} \\ \vdots & \ddots & \vdots \\ u_{x_n,x_1} & \dots & u_{x_n,x_n} \end{bmatrix}$$

and we then define the **laplacian** *of* u *to be the divergence of* ∇u*:*

$$\Delta u := trD\nabla u = \sum_{i=1}^{n} u_{x_i,x_i}$$

1.2.0 What is a Partial Differential Equation (PDE)?

Definition 1.2.1. *A PDE is an equation which relates an unknown function* u, *its partial derivatives, and its independent variables.*

A general PDE on a domain $U \subseteq \mathbb{R}^n$ can be written as

$$F(\mathbf{x}, u, D^1 u, ..., D^k u) = F(\mathbf{x}, u(\mathbf{x}), D^1 u(\mathbf{x}), ..., D^k u(\mathbf{x})) = g(\mathbf{x}), \quad \mathbf{x} \in U \tag{1.2.1}$$

for functions $g(\mathbf{x})$ and $F(\mathbf{x}, \theta, \theta^1, ..., \theta^k)$, where $(x_1, ..., x_n) = \mathbf{x} \in U, \theta \in \mathbb{R}$, and $\theta^i = (\theta^i_1, ..., \theta^i_{n^i}) \in \mathbb{R}^{n^i}$ and $i = 0, 1, ..., k$, representing that for a potential function $u : U \subset \mathbb{R}^n \to \mathbb{R}$, there are n^i partial derivatives of order i. u and $D^1 u, ..., D^k u$ are also called dependent variables.

When we study a PDE often the domain U is not specified yet in the beginning.

Definition 1.2.2. *The* **order** *of a PDE is the highest order of a partial derivative that appears in the equation.*

The most general form of a first order PDE for 2 independent variables is

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = f(x, y, u, u_x, u_y) = g(x, y)$$

Linear PDEs

Definition 1.2.3. *A PDE of the form*

$$F(\mathbf{x}, u, D^1u, ..., D^ku) = g(\mathbf{x}) \tag{1.2.2}$$

is called a **linear** *if the function*

$$(\theta, \theta^1, ..., \theta^k) \in \mathbb{R} \times \mathbb{R}^n \times ... \times \mathbb{R}^{n^k} \mapsto F(\mathbf{x}, \theta, \theta^1, ..., \theta^k) \in \mathbb{R}$$

is multilinear (linear in each $\mathbb{R}$ *component).*

A linear PDE of order 2 in n independent variables can always be written in the form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x})u_{x_i,x_j} + \sum_{k=1}^{n} b_k(\mathbf{x})u_{x_k} + c(\mathbf{x})u = g(\mathbf{x})$$

with coefficients $(a_{i,j}(\mathbf{x}))_{1 \leqslant i,j \leqslant n}, (b_k(\mathbf{x}))_{1 \leqslant k \leqslant n}, c(\mathbf{x})$ that are functions in $\mathbf{x}$.

Example 1.2.1 (Poisson Equation)**.** The Poisson PDE for a function $g : U \subset \mathbb{R}^n \to \mathbb{R}$ is given by

$$\Delta u = \sum_{i=1}^{n} u_{x_i,x_i} = g(\mathbf{x})$$

where in the form of the above equation we have $a_{i,j} = \delta_{i,j}$, the Kronecker delta, $b_i = 0$ and $c = 0$.

Nonlinear PDEs

Definition 1.2.4. *A PDE of the form* $F(\mathbf{x}, u, D^1u, ..., D^ku) = g(\mathbf{x})$ *is called*

- **semi linear** *if we can write*

  $$F(\overline{x}, \theta, \theta^1, ..., \theta^k) = L(\mathbf{x}, \theta^k) + G(\mathbf{x}, \theta, \theta^1, ..., \theta^{k-1})$$

  and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta^k)$ *is multilinear.*

- **quasi linear** *if we can write*

$$F(\overline{x}, \theta, \theta^1, ..., \theta^k) = L(\mathbf{x}, \theta, \theta^1, ..., \theta^k) + G(\mathbf{x}, \theta, \theta^1, ..., \theta^{k-1})$$

and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta, \theta^1, ..., \theta^k)$ *is multilinear.*

- **fully nonlinear** *if the PDE is not linear, semilinear, or quasilinear.*

The following implications are clear: linear implies semi-linear implies quasi-linear.

Consider a quasi linear PDE $F(\mathbf{x}, u, D^1 u) = g(\mathbf{x})$. Hence, F has the form

$$F(\mathbf{x}, \theta, \theta^1) = \sum_{i=1}^{n} a_i(\mathbf{x}, \theta)\theta^1 + G(\mathbf{x}, \theta)$$

The coefficients $(a_i)_{1 \leqslant i \leqslant n}$ are functions in $\mathbf{x}$ and θ. The PDE takes the form

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u)u_{x_i} + G(\mathbf{x}, u) = g(\mathbf{x})$$

Example 1.2.2 (Inviscid (or Non-viscous) Burger's Equations)**.** The PDE

$$u_t + (u^2)_x = 0 \implies u_t + 2uu_x = 0$$

is a quasi-linear PDE of order 1 in 2 independent variables: $t = x_1$ and $x = x_2$. Here we have $a_1(\mathbf{x}, u) = 1, a_2(\mathbf{x}, u) = u$ and $G = g \equiv= 0$.

Consider a PDE of order 2 $F(\mathbf{x}, u, D^1 u, D^2 u) = g(\mathbf{x})$. If the PDE is quasi-linear, it can be written in the general form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, u, D^1 u)u_{x_i,x_j} + G(\mathbf{x}, u, D^1 u) = g(\mathbf{x})$$

$(a_{i,j})_{1 \leqslant i,j \leqslant n}, G$ are functions in $\mathbf{x}, \theta$, and θ^1.

Solutions

Definition 1.2.5. *Consider a PDE of order k:*

$$F(\mathbf{x}, u, D^1 u, ..., D^k u) = g(\mathbf{x})$$

A classical solution of the PDE on a domain $\Omega \subset \mathbb{R}^n$ where n is the number of independent variables, is a sufficiently smooth function $u(\mathbf{x})$ that satisfies it. By sufficiently smooth, we mean in this case that $u \in C^k(\Omega)$.

Example 1.2.3. The function $u(x, t) = \frac{x}{t}$ solves

$$u_t + uu_x = 0$$

on $\mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$, the upper half plane.

Homogeneous/Inhomogeneous Linear PDEs

Definition 1.2.6. *Consider a linear PDE of order k:* $L(\mathbf{x}, u, D^1 u, ..., D^k u) = \mathcal{L}[u] = g(\mathbf{x})$. *If* $g(\mathbf{x}) \equiv 0$, *the PDE is called* **homogeneous**. *Otherwise, the PDE is called* **inhomogeneous**.

Theorem 1.2.1 (Superposition Principle)**.** *If u and v are solutions to the homogeneous linear PDE* $L(\bar{x}, u, D^1 u, ..., D^k u) = 0$ *on a domain* $\Omega \subset \mathbb{R}^n$, *then for all* $\alpha, \beta \in \mathbb{R}$, $\alpha u + \beta v$ *is a solution to the same homogeneous linear PDE on the domain* Ω.

Theorem 1.2.2 (Particular Solution)**.** *If u solves the homogenous linear PDE* $\mathcal{L}[u] = 0$ *and w solves the inhomogeneous linear PDE* $L[w] = g(\mathbf{x})$, *then* $u + w$ *solves the same inhomogeneous linear PDE.*

It follows that $\mathcal{L}$, defined by $\mathcal{L}[u] = L(\mathbf{x}, u, D^1 u, ..., D^k u)$ is a linear differential operator. Hence, it makes sense to specify appropraite function vector spaces V and W such that $u \in V$ and $\mathcal{L}u \in W$. For instance: For a PDE of order 2, we can choose $V = C^2(\Omega)$ and $W = C^0(\Omega)$. In particular, for a linear PDE of order one for independent variables x and y, we could set $V = C^1(\mathbb{R}^2)$ and $W = C^0(\mathbb{R}^2)$.

Appendices