Stochastic Process
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Homework 11

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1. 设 $X_n = \sum_{k=1}^{N} \sigma_k \sqrt{2}\cos(\alpha_k n - U_k)$，其中 σ_k 和 a_k 为正常数，$U_K \sim U(0, 2\pi)$，且相互独立，$k = 1, 2, \cdots, N$，试计算 $\{X_n, n = 0, \pm1, \cdots\}$ 的均值函数和相关函数，并说明其是否是平稳过程。

由题可得:

$$
\begin{aligned}
E\{X_n\} &= E\{\sum_{k=1}^{N} \sigma_k \sqrt{2}\cos(\alpha_k n - U_k)\} = \sqrt{2}\sum_{k=1}^{N} \sigma_k E\left\{\cos(\alpha_k n - u_k)\right\} \\
&= \sqrt{2}\sum_{k=1}^{N} \sigma_k \int_0^{2\pi} E\{\cos(\alpha_k n - u) | U_k = u\} f_{U_k}(u)\ du \\
&= \frac{\sqrt{2}}{2\pi}\sum_{k=1}^{N} \sigma_k \int_0^{2\pi} E\{\cos(\alpha_k n - u)\}\ du = 0
\end{aligned}
$$

假设 $0 < s < t$，则 X_n 相关函数计算如下

$$
\begin{aligned}
R_X(s,t) = E\{X_s \overline{X_t}\} &= E\left\{\left[\sum_{k=1}^{N} \sigma_k \sqrt{2}\cos(\alpha_k s - U_k)\right]\left[\sum_{j=1}^{N} \sigma_j \sqrt{2}\cos(\alpha_j t - U_j)\right]\right\} \\
&= 2 \cdot \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} \sigma_k \sigma_j \underbrace{E\left\{\left[\cos(a_k s + a_j t - U_k - U_j) + \cos(a_k s - a_j t - U_k + U_j)\right]\right\}}_{A}
\end{aligned}
$$

当 $k \neq j$ 时:

$$
\begin{aligned}
A &= E\left\{\left[\cos(a_k s + a_j t - U_k - U_j) + \cos(a_k s - a_j t - U_k + U_j)\right]\right\} \\
&= E\left\{\cos(a_k s + a_j t - U_k - U_j)\right\} + E\left\{\cos(a_k s - a_j t - U_k + U_j)\right\} \\
&= \int_0^{2\pi}\int_0^{2\pi} E\left\{\cos(a_k s + a_j t - u_k - u_j) | U_k = u_k, U_j = u_j\right\} f_{U_k}(u_k) f_{U_j}(u_j) du_k du_j \\
&\quad + \int_0^{2\pi}\int_0^{2\pi} E\left\{\cos(a_k s - a_j t - U_k + U_j) | U_k = u_k, U_j = u_j\right\} f_{U_k}(u_k) f_{U_j}(u_j) du_k du_j \\
&= 0
\end{aligned}
$$

当 $k=j$ 时：

$$\begin{aligned}A&=E\left\{\cos[\alpha_k(s+t)-2U_k]+\cos[\alpha_k(s-t)]\right\}\\&=E\left\{\cos[\alpha_k(s+t)-2U_k]\right\}+E\left\{\cos[\alpha_k(s-t)]\right\}\\&=\int_0^{2\pi}E\{\cos[\alpha_k(s+t)-2U_k]|U_k=u\}f_{U_k}(u)\ du+E\left\{\cos[\alpha_k(s-t)]\right\}\\&=E\left\{\cos[\alpha_k(s-t)]\right\}=\cos[\alpha_k(s-t)]\end{aligned}$$

综上：

$$R_X(s,t)=\sum_{k=1}^{N}\sigma_k^2\cos[\alpha_k(s-t)]=\sum_{k=1}^{N}\sigma_k^2\cos[\alpha_k(t-s)]=\sum_{k=1}^{N}\sigma_k^2\cos[\alpha_k\tau]\ (\tau=t-s)$$

故该随机过程 X_n 的相关函数只与时间差 $\tau=t-s$ 有关，因此该过程为（宽）平稳过程

2. 设有随机过程 $X(t)=A\cos(\omega t+\pi\eta(t))$，其中 $\omega>0$ 为常数，$\{\eta(t),t\geq 0\}$ 是泊松过程，A 是与 $\eta(t)$ 独立的随机变量，且 $P\{A=-1\}=P\{A=1\}=\frac{1}{2}$。
(1) 试画出此过程的样本函数，并问样本函数是否连续？
(2) 试求此过程的相关函数，并问该过程是否均方连续？

(1) 由题可得：该随机过程的一条样本函数如下：

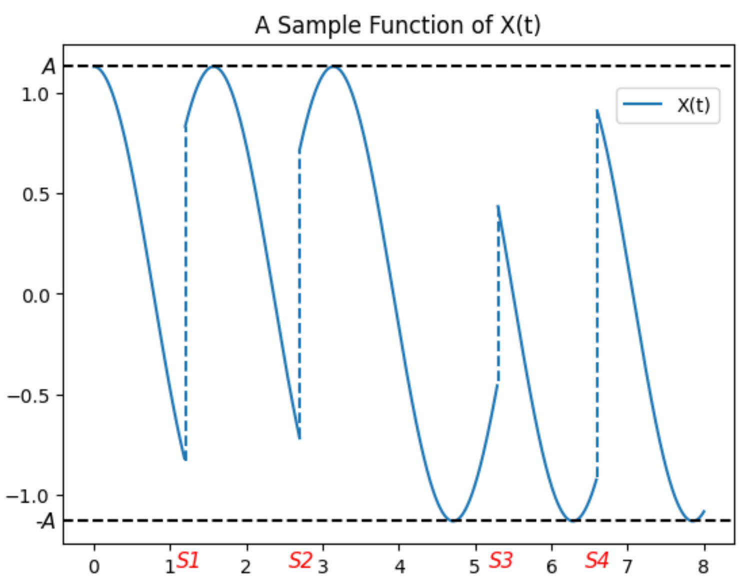

图 1: 随机过程 X(t) 的一条样本函数示意图

显然样本函数不连续

（2）由题可得：$E\{A^2\} = (-1)^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} = 1$

$$
\begin{aligned}
R_X(s,t) = E\{X_s\overline{X_t}\} &== E\left\{A\cos\left[ws + \pi\eta(s)\right] \cdot A\cos\left[wt + \pi\eta\left(t\right)\right]\right\} \\
&= \frac{1}{2}E\{A^2\}E\{\cos[w(s+t) + \pi\left(\eta(s)+\eta(t)\right)] + \cos[w(s-t) + \pi\big(\eta(s)-\eta(t)\big)]\} \\
&= \frac{1}{2}\Big\{ \underbrace{E\Big\{ \cos[w(s+t) + \pi\big(\eta(s)+\eta(t)\big)]\Big\}}_{A} + \underbrace{E\Big\{ \cos[w(s-t) + \pi\big(\eta(s)-\eta(t)\big)]\Big\}}_{B} \Big\}
\end{aligned}
$$

根据三角函数和差化积、积化和差公式可得：

$$
\begin{aligned}
A &= E\Big\{ \cos[w(s+t) + \pi\big(\eta(s)+\eta(t)\big)]\Big\} \\
&= E\Big\{ \cos[w(s+t)]\cos[\pi\big(\eta(s)+\eta(t)\big)] - \sin[w(s+t)]\sin[\pi\big(\eta(s)+\eta(t)\big)]\Big\} \\
&= E\Big\{ \cos[w(s+t)]\Big\}E\Big\{ \cos[\pi\big(\eta(s)+\eta(t)\big)]\Big\} - E\Big\{ \sin[w(s+t)]\Big\}E\Big\{ \sin[\pi\big(\eta(s)+\eta(t)\big)]\Big\} \\
&= E\Big\{ \cos[w(s+t)]\Big\} \underbrace{E\Big\{ \cos\big(\eta(s)\big)\cos\big(\eta(t)\big) - \sin\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}}_{①} \\
&\quad - E\Big\{ \sin[w(s+t)]\Big\} \underbrace{E\Big\{ \sin\big(\eta(s)\big)\cos\big(\eta(t)\big) + \cos\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}}_{②}
\end{aligned}
$$

$$
\begin{aligned}
① &= E\Big\{ \cos\big(\eta(s)\big)\cos\big(\eta(t)\big)\Big\} - E\Big\{ \sin\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\} \\
&= E\Big\{ \cos\big(\eta(s)\big)\cos\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} - E\Big\{ \sin\big(\eta(s)\big)\sin\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} \\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\cos\big(\pi\eta(t)-\pi\eta(s)\big) - \sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&\quad - E\left\{\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\sin\big(\pi\eta(t)-\pi\eta(s)\big) - \sin^2\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\right\}E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\} - \frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&\quad - \frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\} + \underbrace{E\left\{\sin^2\big(\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\right\}E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\} \\
&= E\left\{\cos\big(\pi\eta(t-s)\big)\right\} = \sum_{k=0}^{\infty}\cos(\pi k)\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)} \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)} \\
&= e^{-\lambda(t-s)}\sum_{k=0}^{\infty}\frac{[-\lambda(t-s)]^k}{k!} = e^{-2\lambda(t-s)}
\end{aligned}
$$

$$
\begin{aligned}
② &= E\Big\{ \sin\big(\eta(s)\big)\cos\big(\eta(t)\big)\Big\} + E\Big\{ \cos\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\} \\
&= E\Big\{ \sin\big(\eta(s)\big)\cos\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} + E\Big\{ \cos\big(\eta(s)\big)\sin\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} \\
&= E\left\{\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos\big(\pi\eta(t)-\pi\eta(s)\big) - \sin^2\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ E\left\{\cos^2\big(\pi\eta(s)\big)\sin\big(\pi\eta(t)-\pi\eta(s)\big)-\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&= \frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\}-\underbrace{E\left\{\sin^2\big(\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&+\underbrace{E\left\{\cos^2\big(\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 1}E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\}-\frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&= E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\}=E\left\{\sin\big(\pi\eta(t-s)\big)\right\}\\
&= \sum_{k=0}^{\infty}\sin(\pi k)\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)}=0
\end{aligned}
$$

因此 $A=\cos[w(s+t)]e^{-2\lambda(t-s)}$

$$
\begin{aligned}
B &= E\Big\{\cos[w(s-t)+\pi\big(\eta(s)-\eta(t)\big)]\Big\}\\
&= E\Big\{\cos[w(s-t)]\cos[\pi\big(\eta(s)-\eta(t)\big)]-\sin[w(s-t)]\sin[\pi\big(\eta(s)+\eta(t)\big)]\Big\}\\
&= E\Big\{\cos[w(s-t)]\Big\}E\Big\{\cos[\pi\big(\eta(s)-\eta(t)\big)]\Big\}-E\Big\{\sin[w(s-t)]\Big\}E\Big\{\sin[\pi\big(\eta(s)-\eta(t)\big)]\Big\}\\
&= E\Big\{\cos[w(s-t)]\Big\}\underbrace{E\Big\{\cos\big(\eta(s)\big)\cos\big(\eta(t)\big)+\sin\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}}_{③}\\
&\quad -E\Big\{\sin[w(s-t)]\Big\}\underbrace{E\Big\{\sin\big(\eta(s)\big)\cos\big(\eta(t)\big)-\cos\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}}_{④}
\end{aligned}
$$

$$
\begin{aligned}
③ &= E\Big\{\cos\big(\eta(s)\big)\cos\big(\eta(t)\big)\Big\}+E\Big\{\sin\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}\\
&= E\Big\{\cos\big(\eta(s)\big)\cos\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\}+E\Big\{\sin\big(\eta(s)\big)\sin\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\}\\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\cos\big(\pi\eta(t)-\pi\eta(s)\big)-\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&\quad + E\left\{\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\sin\big(\pi\eta(t)-\pi\eta(s)\big)-\sin^2\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\right\}E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\}-\frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&\quad +\frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\}-\underbrace{E\left\{\sin^2\big(\pi\eta(s)\big)\right\}}_{equal\ \ to\ \ 0}E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\}\\
&= E\left\{\cos^2\big(\pi\eta(s)\big)\right\}E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\}\\
&= E\left\{\cos\big(\pi\eta(t-s)\big)\right\}=\sum_{k=0}^{\infty}\cos(\pi k)\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)}\\
&= e^{-\lambda(t-s)}\sum_{k=0}^{\infty}\frac{[-\lambda(t-s)]^k}{k!}=e^{-2\lambda(t-s)}
\end{aligned}
$$

$$
④ = E\Big\{\sin\big(\eta(s)\big)\cos\big(\eta(t)\big)\Big\}-E\Big\{\cos\big(\eta(s)\big)\sin\big(\eta(t)\big)\Big\}
$$

$$
\begin{aligned}
&= E\Big\{ \sin\big(\eta(s)\big)\cos\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} - E\Big\{ \cos\big(\eta(s)\big)\sin\big([\eta(t)-\eta(s)]+\eta(s)\big)\Big\} \\
&= E\left\{\sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos\big(\pi\eta(t)-\pi\eta(s)\big) - \sin^2\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&\quad - E\left\{\cos^2\big(\pi\eta(s)\big)\sin\big(\pi\eta(t)-\pi\eta(s)\big) - \sin\big(\pi\eta(s)\big)\cos\big(\pi\eta(s)\big)\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&= \frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ to\ 0} E\left\{\cos\big(\pi\eta(t)-\pi\eta(s)\big)\right\} - \underbrace{E\left\{\sin^2\big(\pi\eta(s)\big)\right\}}_{equal\ to\ 0} E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&\quad - \underbrace{E\left\{\cos^2\big(\pi\eta(s)\big)\right\}}_{equal\ to\ 1} E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\} + \frac{1}{2}\underbrace{E\left\{\sin\big(2\pi\eta(s)\big)\right\}}_{equal\ to\ 0} E\left\{\cos[\pi\eta(t)-\pi\eta(s)]\right\} \\
&= -E\left\{\sin\big(\pi\eta(t)-\pi\eta(s)\big)\right\} = -E\left\{\sin\big(\pi\eta(t-s)\big)\right\} \\
&= \sum_{k=0}^{\infty}\sin(\pi k)\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)} = 0
\end{aligned}
$$

因此 $B = \cos[w(s-t)]e^{-2\lambda(t-s)}$

$$
\begin{aligned}
R_X(s,t) = \frac{1}{2}(A+B) &= \frac{1}{2}\Big(\cos[w(s+t)]e^{-2\lambda(t-s)} + \cos[w(s-t)]e^{-2\lambda(t-s)}\Big) \\
&= \cos(ws)\cos(wt)e^{-2\lambda(t-s)}
\end{aligned}
$$

综上：随机过程 $X(t)$ 的相关函数 $R_X(s,t) = \cos(ws)\cos(wt)e^{-2\lambda(\max(t,s)-\min(t,s))}$，因此该过程显然不是（宽）平稳过程，接下来计算该过程的二阶矩：

$$
\begin{aligned}
E\{X^2(t)\} = E\{A^2\cos^2(\omega t+\pi\eta(t))\} &= E\{A^2\}E\{\frac{1}{2}+\frac{1}{2}\cos\big(2wt+2\pi\eta(t)\big)\} \\
&= \frac{1}{2}+\frac{1}{2}E\{\cos\big(2wt+2\pi\eta(t)\big)\} \\
&= \frac{1}{2}
\end{aligned}
$$

该随机过程二阶矩存在，故为二阶矩过程

$$
E\{X(t)\} = E\{A\cos(\omega t+\pi\eta(t))\} = E\{A\}E\{\cos(\omega t+\pi\eta(t))\} = 0
$$

则有对 $\forall t\in T, \lim_{h\to 0} E\{X(t+h)\} = E\{X(t)\}$，因此该过程具有均方连续性。