Stochastic Process

University of Chinese Academy of Sciences

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Yingfei Sun

Homework 14

Chenkai GUO

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1. 设有一线性系统，其输入为零均值白高斯噪声 $n(t)$,，其功率谱密度为 $\frac{N_0}{2}$ ，系统的冲激响应为：

$$h(t)=\begin{cases} e^{-\alpha t}, & t\geq 0 \\ 0, & t<0 \end{cases}$$

此线性系统的输出为 $\xi(t)$。令：$\eta(t)=\xi(t)-\xi(t-T)$ ，其中 $T>0$ 为一常数，试求过程 $\eta(t)$ 的一维概率密度函数。

由题可得：对 $h(t)$ 作傅里叶变换得到系统的转移函数 $H(j\omega)$

$$H(jw)=\mathcal{F}[h(t)]=\int_0^{+\infty}\mathrm{e}^{-\alpha t}\mathrm{e}^{-jwt}\mathrm{d}t=-\frac{1}{\alpha+jw}\mathrm{e}^{-(\alpha+jw)t}\Big|_0^{\infty}=\frac{1}{\alpha+jw}$$

因此可得输出的功率谱密度函数：

$$S_{\xi(\omega)}=|H(jw)|^2S_X(w)=\frac{N_0}{2(\alpha^2+w^2)}=\frac{N_0}{4\alpha}\cdot\frac{2\alpha}{\alpha^2+w^2}$$

对输出的功率谱密度函数作傅里叶逆变换可得：

$$R_{\xi(\tau)}=\mathcal{F}^{-1}[S_{Y(w)}]=\frac{N_0}{4\alpha}e^{-a|\tau|}$$

因为输入为白高斯噪声，系统为线性系统，因此输出过程为正态过程，现计算：

$$E\left\{\eta(t)\right\}=E\left\{\xi(t)-\xi(t-T)\right\}=E\left\{\xi(t)\right\}-E\left\{\xi(t-T)\right\}=0-0=0$$

$$\begin{aligned}\operatorname{Var}\left\{y(t)\right\}=E\left\{\left(y(t)-\mu_{y(t)}\right)^2\right\}&=E\left\{y^2(t)\right\}\\&=E\left\{y^2(t)-23(t)\frac{3}{3}(t-T)+3^2(t-T)\right\}\\&=E\left\{\xi^2(t)\right\}-2E\left\{\xi(t)\xi(t-T)\right\}+E\left\{\xi^2(t-T)\right\}\\&=2R_\xi^2(0)-2R_\xi^2(T)\end{aligned}$$

$$= \frac{N_0}{2a} - \frac{N_0}{2a}e^{-aT}$$
$$= \frac{N_0}{2a}(1 - e^{-aT})$$

故输出过程 $\eta(t) \sim N\big(0, \frac{N_0}{2a}(1 - e^{-aT})\big)$，其一维概率密度函数为：

$$\eta(t) = \sqrt{\frac{a}{\pi N_0(1 - e^{-aT})}} \cdot e^{-\frac{ax^2}{N_0(1-e^{-aT})}}$$

2. 设 $s(t)$ 为一确定性信号，在 $(0,T)$ 内具有能量 $E_s = \int_0^T s^2(t)dt$ ，$n(t)$ 为一零均值的白高斯过程，其相关函数为：$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$。令：$\eta_1 = \int_0^T s(t)[s(t) + n(t)]dt$，$\eta_2 = \int_0^T s(t)n(t)dt$。试求：
(1) 给定一常数 γ，求概率 $P\{\eta_1 > \gamma\}$；
(2) 给定一常数 γ，求概率 $P\{\eta_2 > \gamma\}$.

由题可得：$R_n(0) = \frac{N_0}{2} = E\{n^2(t)\} = Var\{n(t)\}$，$\therefore n(t) \sim N(0, \frac{N_0}{2})$
又 $\because S(t)$ 为确定性信号，输入 $n(t)$ 为白高斯过程，η_1, η_2 为 $S(t), n(t)$ 的线性变换（积分），$\therefore$ 输出信号 η_1, η_2 亦服从高斯分布，且 $\eta_1 = \eta_2 + E_s$ 因此 η_1, η_2 的均值为：

$$E\{\eta_2\} = E\{\int_0^T S(t)n(t)dt\} = \int_0^T E\{s(t)n(t)\}dt = \int_0^T E\{s(t)\}E\{n(t)\}dt = 0$$
$$E\{\eta_1\} = E\{\eta_2 + E_s\} = 0 + E_s = E_s$$

η_1, η_2 的二阶矩为：

$$\begin{aligned} E\{n_2^2\} &= E\{\int_0^T S(s)n(s)ds \int_0^T S(t)n(t)dt\} \\ &= \int_0^T \int_0^T E\left\{S(s)n(s)S(t)n(t)\right\} dsdt \\ &= \int_0^T \int_0^T \Big(E\left\{S(s)S(t)\right\} E\left\{n(s)n(t)\right\} + E\left\{S(s)n(s)\right\} E\left\{S(t)n(t)\right\} \\ &\qquad + E\left\{S(s)n(t)\right\} E\left\{S(t)n(s)\right\} \Big) dsdt \\ &= \int_0^T \int_0^T E\left\{S(s)S(t)\right\} \frac{N_0}{2}\delta(t-s)dsdt \\ &= \int_0^T E\{s^2(t)\}\frac{N_0}{2}dt = \frac{N_0}{2} \cdot E_s \end{aligned}$$

$\therefore Var\{\eta_2^2\} = E\{\eta_2^2\} - E^2\{\eta_2\} = \frac{N_0}{2} \cdot E_s$

$$\begin{aligned} \therefore Var\{\eta_1\} = E\{\eta_1^2\} - E^2\{\eta_1\} &= E\{(\eta_2 + E_s)^2\} - E_s^2 \\ &= E\{\eta_2^2\} + 2E_sE\{\eta_2\} + E_s^2 - E_s^2 \\ &= \frac{N_0}{2} \cdot E_s \end{aligned}$$

故 $\eta_1 \sim N\big(E_s, \frac{N_0}{2a}(1-e^{-aT})\big), \eta_2 \sim N\big(0, \frac{N_0}{2a}(1-e^{-aT})\big)$

综上可得:

(1)

$$P\{\eta_1 > x\} = 1 - P\{\eta_1 \le x\} = 1 - \int_0^{\gamma} \frac{1}{\sqrt{\pi N_0 E_s}} e^{-\frac{(x-E_s)^2}{N_0 E_s}} dx$$

(2)

$$P\{\eta_2 > x\} = 1 - P\{\eta_2 \le x\} = 1 - \int_0^{\gamma} \frac{1}{\sqrt{\pi N_0 E_s}} e^{-\frac{x^2}{N_0 E_s}} dx$$

3. *设有一非线性系统，其输入为零均值平稳实高斯过程，其协方差函数为:

$$C_\xi(\tau) = Pe^{-\alpha|\tau|}$$

其中 $P > 0$ 为一常数。系统的输出为:

$$\zeta = \frac{1}{T}\int_0^T \xi^2(t)dt$$

试求:

(1) 输出均值: $E\{\zeta\}$;

(2) 输出方差: $D\{\zeta\}$;

(3) 设 $y = \frac{D\{\zeta\}}{[E\{\zeta\}]^2}, x = \alpha T$，画出 y 对 x 的关系简图。

(1) 由题可得: $\because \mu_{\xi(t)} = 0$，$\therefore C_\xi(\tau) = R_\xi(\tau) = Pe^{-\alpha|\tau|}$，$\therefore E\left\{\xi^2(t)\right\} = C_\xi(0) = P$

$$E\left\{\zeta\right\} = E\left\{\frac{1}{T}\int_0^T \xi^2(t)dt\right\} = \frac{1}{T}\int_0^T E\left\{\xi^2(t)\right\} dt = P$$

(2) 由题可得:

$$\begin{aligned}
E\{\zeta^2\} &= E\left\{\frac{1}{T^2}\int_0^T \xi^2(s)ds \int_0^T \xi^2(t)dt\right\} \\
&= \frac{1}{T^2}\int_0^T\int_0^T E\left\{\xi^2(s)\xi^2(t)\right\} dsdt \\
&= \frac{1}{T^2}\int_0^T\int_0^T \Big(E\left\{\xi^2(s)\right\} E\left\{\xi^2(t)\right\} + 2E^2\left\{\xi(s)\xi(t)\right\}\Big) dsdt \\
&= \frac{1}{T^2}\int_0^T\int_0^T P^2 + 2R_\xi^2(t-s) dsdt \\
&= P^2 + \underbrace{\frac{2P^2}{T^2}\int_0^T\int_0^T e^{-2a|t-s|} dsdt}_{①}
\end{aligned}$$

* 星号题为第五章非作业布置的课后习题

$$
\begin{aligned}
① &= \frac{2P^2}{T^2}\int_0^T\int_0^T e^{-2a|t-s|}dsdt \\
&= \underbrace{\int_0^T\int_s^T e^{-2a(t-s)}dtds}_{if\ \ t\geq s} + \underbrace{\int_0^T\int_t^T e^{2a(t-s)}dsdt}_{if\ \ t<s} \\
&= \int_0^T(-\frac{1}{2a}e^{2a(s-T)}+\frac{1}{2a})ds + \int_0^T(-\frac{1}{2a}e^{2a(t-T)}+\frac{1}{2a})dt \\
&= \left(-\frac{1}{4a^2}e^{2a(s-T)}+\frac{1}{2a}s\Big|_0^T\right) + \left(-\frac{1}{4a^2}e^{2a(t-T)}+\frac{1}{2a}t\Big|_0^T\right) \\
&= \frac{1}{2a^2}(e^{-aT}+2aT-1)
\end{aligned}
$$

代入原式可得：$E\{\zeta^2\} = \frac{P^2}{a^2T^2}(e^{-aT}+2aT-1)+P^2$，因此：

$$
D\{\zeta\} = E\{\zeta^2\} - E^2\{\zeta\} = \frac{P^2}{a^2T^2}(e^{-aT}+2aT-1)
$$

（3）有题可得：

$$
y = \frac{D\{\xi\}}{E^2\{\xi\}} = \frac{1}{a^2T^2}\left(e^{-aT}+2aT-1\right) = \frac{e^{-x}+2x-1}{x^2}
$$

因此 y 对 x 的关系简图如下：

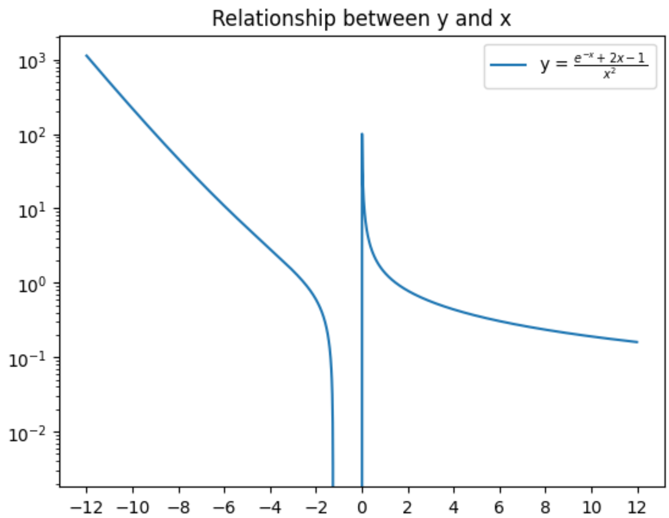

图 1: y 对 x 的关系简图

4. *设有一线性系统，输入输出分别为 $\xi(t)$ 和 $\eta(t)$，其中输入过程 $\xi(t)$ 为零均值平稳实高斯过程，它的相关函数为：$R_\xi(\tau) = \sigma_\xi^2 e^{-a|\tau|}(a>0)$。系统的单位冲激响应为：

$$
h(t) = \begin{cases} e^{-\beta t}, & t\geq 0, \beta>0, \beta\neq\alpha \\ 0, & t<0 \end{cases}
$$

若 $\xi(t)$ 在 $t=-\infty$ 时接入系统，试求：
(1) 在 $t=0$ 时输出 $\eta(0)$ 大于 y 的概率 $P\{\eta(0)>y\}$；
(2) 求条件概率 $P\{\eta(0)>y|\xi(-T)=0\}$，其中 $T>0$；
(3) 求条件概率 $P\{\eta(0)>y|\xi(T)=0\}$，其中 $T>0$。

(1) 由题可得：$\because E\left\{\xi(t)\right\}=0 \quad \therefore Var\left\{\xi(t)\right\}=E\left\{\xi^2(t)\right\}=R_\xi(0)=\sigma_\xi^2$
$\therefore \xi(t)\sim N(0,\sigma_\xi^2)$
对 $h(t)$ 作傅里叶变换得到系统的转移函数 $H(j\omega)$：

$$H(jw)=\mathcal{F}[h(t)]=\int_0^{+\infty}\mathrm{e}^{-\beta t}\mathrm{e}^{-jwt}dt=\frac{1}{\beta+jw}$$

根据 *wienner-khinchine* 定理，有：

$$\begin{aligned}
S_\xi(\omega)=\mathcal{F}[R_\xi(t)]&=\int_{-\infty}^{+\infty}R_\xi(t)\cdot e^{-jw\tau}d\tau\\
&=\int_{-\infty}^{+\infty}\sigma_\xi^2e^{-\alpha|\tau|}e^{-jw\tau}d\tau\\
&=\int_{-\infty}^{0}\sigma_\xi^2\cdot e^{\alpha\tau}e^{-jw\tau}d\tau+\int_0^{\infty}\sigma_\xi^2\cdot e^{-\alpha\tau}e^{-jw\tau}d\tau\\
&=\sigma_\xi^2\left(\frac{1}{\alpha-jw}e^{(\alpha-jw)\tau}\Big|_{-\infty}^{0}-\frac{1}{\alpha+jw}e^{-(\alpha+jw)\tau}\Big|_0^{\infty}\right)\\
&=\sigma_\xi^2\left(\frac{1}{\alpha-jw}+\frac{1}{\alpha+jw}\right)=\frac{2\alpha\sigma_\xi^2}{\alpha^2+w^2}
\end{aligned}$$

根据输入、输出信号的功率谱密度函数关系，有：

$$\begin{aligned}
S_\eta(\omega)=|H(j\omega)|^2S_\xi(\omega)&=\frac{1}{\beta^2+\omega^2}\cdot\frac{2\alpha\sigma_\xi^2}{\alpha^2+\omega^2}\\
&=\frac{\alpha\sigma_\xi^2}{\alpha^2-\beta^2}\left(\frac{1}{\beta}\cdot\frac{2\beta}{\beta^2+\omega^2}-\frac{1}{\alpha}\cdot\frac{2\alpha}{\alpha^2+\omega^2}\right)
\end{aligned}$$

根据 *wienner-khinchine* 定理，有：

$$R_\eta(t)=\mathcal{F}^{-1}[S_\eta(\omega)]=\frac{\alpha\sigma_\xi^2}{\alpha^2-\beta^2}\left(\frac{1}{\beta}\cdot e^{-\beta|\tau|}-\frac{1}{\alpha}\cdot e^{-\alpha|\tau|}\right)$$

因此，有：$R_\eta(0)=\frac{\sigma_\xi^2}{\beta(\alpha+\beta)}=E\{\eta^2(t)\}$
现计算该线性系统的直流增益，即：

$$H(0)=\int_0^{+\infty}\mathrm{e}^{-\beta t}dt=-\frac{1}{\beta}\cdot e^{-\beta t}\Big|_o^{+\infty}=\frac{1}{\beta}$$

因此，有：$E\{\eta(t)\}=E\{\eta(t)\}\cdot H(0)=0\cdot\frac{1}{\beta}=0$

综上，$\eta(t) \sim N\big(0, \frac{\sigma_\xi^2}{\beta(\alpha+\beta)}\big)$，故：

$$p\{\eta(0) > y\} = 1 - p\{\eta(0) \le y\} = 1 - \int_0^y \sqrt{\frac{\beta(\alpha+\beta)}{2\pi}}\sigma_\xi^2 \cdot e^{-\frac{\beta(\alpha+\beta)x^2}{2\sigma_\xi^2}} dx$$

（2）由题可得：

$$P\{\eta(0) > y|\xi(-T) = 0\} = \frac{f\big(\eta(0) > y, \xi(-T) = 0\big)}{f\big(\xi(-T) = 0\big)}$$

接下来计算 $\eta(t)$ 和 $\xi(t)$ 的互相关函数 $R_{\eta\xi}(\tau)$:

$$R_{\eta\xi}(\tau) = \int_{-\infty}^{+\infty} h(u)R_{\xi\xi}(\tau-u)du = \int_0^{+\infty} e^{-\beta u} \cdot \sigma_\xi^2 e^{-\alpha|\tau-u|}du$$

当 $\tau < 0$:

$$\begin{aligned}
R_{\eta\xi}(\tau) = \sigma_\xi^2 \int_0^{+\infty} e^{-\beta u} \cdot e^{-\alpha(u-\tau)}du &= \sigma_\xi^2 \int_0^{+\infty} e^{-(\alpha+\beta)u+\alpha\tau}du \\
&= -\frac{\sigma_\xi^2}{\alpha+\beta} e^{-(\alpha+\beta)u+\alpha\tau}\Big|_0^{+\infty} \\
&= \frac{\sigma_\xi^2}{\alpha+\beta} e^{\alpha\tau}
\end{aligned}$$

当 $\tau \ge 0$:

$$\begin{aligned}
R_{\eta\xi}(\tau) &= \sigma_\xi^2 \Big(\int_\tau^{+\infty} e^{-\beta u} \cdot e^{-\alpha(u-\tau)}du + \int_0^{\tau} e^{-\beta u} \cdot e^{\alpha(u-\tau)}du \Big) \\
&= \sigma_\xi^2 \Big(-\frac{1}{\alpha+\beta} e^{-(\alpha+\beta)u+\alpha\tau}\Big|_\tau^{+\infty} + \frac{1}{\alpha-\beta} e^{(\alpha-\beta)u-\alpha\tau}\Big|_0^{\tau} \Big) \\
&= \sigma_\xi^2 (\frac{1}{\alpha+\beta}e^{-\beta\tau} + \frac{1}{\alpha-\beta}e^{-\beta\tau} - \frac{1}{\alpha-\beta}e^{-\alpha\tau}) \\
&= \frac{2\alpha\sigma_\xi^2}{\alpha^2-\beta^2}e^{-\beta\tau} - \frac{1}{\alpha-\beta}e^{-\alpha\tau}
\end{aligned}$$

$\because T > 0$，因此 $\eta(0)$ 和 $\xi(-T)$ 的互相关函数 $R_{\eta\xi}(T) = \frac{2\alpha\sigma_\xi^2}{\alpha^2-\beta^2}e^{-\beta T} - \frac{1}{\alpha-\beta}e^{-\alpha T}$

又因为 $\xi(t) \sim N(0, \sigma_\xi^2), \eta(t) \sim N\big(0, \frac{\sigma_\xi^2}{\beta(\alpha+\beta)}\big)$，故可得 $\eta(0)$ 和 $\xi(-T)$ 的二维概率密度函数如下：

$$f\big(\xi(-T), \eta(0)\big) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Big\{-\frac{1}{2(1-r^2)}\Big[\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\Big]\Big\}$$

其中，$\sigma_1 = \sigma_\xi, \sigma_2 = \frac{\sigma_\xi}{\sqrt{\beta(\alpha+\beta)}}$

$r = \frac{Cov\big(\xi(-T),\eta(0)\big)}{\sigma_1\sigma_2} = \frac{R_{\eta\xi}(\tau)}{\sigma_1\sigma_2} = \frac{2\alpha\sigma_\xi^2}{\sigma_1\sigma_2(\alpha^2-\beta^2)}e^{-\beta T} - \frac{1}{\sigma_1\sigma_2(\alpha-\beta)}e^{-\alpha T}$

因此：

$$\frac{f\big(\eta(0),\xi(-T)\big)}{f\big(\xi(-T)\big)}=\frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{r-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x^2}{2\sigma_1^2}\right\}}$$
$$=\frac{1}{\sqrt{2\pi(1-r^2)}\sigma_2}\exp\left\{-\frac{1}{2(1-r^2)}(\frac{rx}{\sigma_1}-\frac{y}{\sigma_2})^2\right\}$$

综上：

$$P\left\{\eta(0)>y\mid\xi(-T)=0\right\}=1-P\left\{\eta(0)\leq y\mid\xi(-T)=0\right\}$$
$$=1-\int_0^y\frac{1}{\sqrt{2\pi(1-r^2)}\sigma_2}\exp\left\{-\frac{1}{2(1-r^2)}\frac{u^2}{\sigma_2^2}\right\}du$$

其中，$\sigma_1=\sigma_\xi,\sigma_2=\frac{\sigma_\xi}{\sqrt{\beta(\alpha+\beta)}},r=\frac{2\alpha\sigma_\xi^2}{\sigma_1\sigma_2(\alpha^2-\beta^2)}e^{-\beta T}-\frac{1}{\sigma_1\sigma_2(\alpha-\beta)}e^{-\alpha T}$

(*3*) 与 (*2*) 同理，可得：

$$P\left\{\eta(0)>y\mid\xi(T)=0\right\}=1-P\left\{\eta(0)\leq y\mid\xi(T)=0\right\}$$
$$=1-\int_0^y\frac{1}{\sqrt{2\pi(1-r^2)}\sigma_2}\exp\left\{-\frac{1}{2(1-r^2)}\frac{u^2}{\sigma_2^2}\right\}du$$

其中，$\sigma_1=\sigma_\xi,\sigma_2=\frac{\sigma_\xi}{\sqrt{\beta(\alpha+\beta)}},r=\frac{Cov\big(\xi(-T),\eta(0)\big)}{\sigma_1\sigma_2}=\frac{R_{\eta\xi}(\tau)}{\sigma_1\sigma_2}=\frac{\sigma_\xi^2}{\sigma_1\sigma_2(\alpha+\beta)}e^{\alpha\tau}$

5. *设实平稳过程 $\{X(t);-\infty<t<\infty\}$ 的自相关函数和功率谱密度分别为 $R_X(\tau)$ 和 $S_X(\omega)$，令随机过程 $Y(t)=X(t+a)-X(t-a)$ 的相关函数和功率谱密度分别为 $R_Y(\tau)$ 和 $S_Y(\omega)$，其中 a 是常数。

(*1*) 试证明；$R_Y(\tau)=2R_X(\tau)-R_X(\tau+2a)-R_X(\tau-2a)$

(*2*) 试证明；$S_Y(\omega)=4S_X(\omega)\sin^2(a\omega)$。

(*1*) 由题可得：

$$\begin{aligned}R_Y(\tau)&=E\{Y(s)Y(t)\}\\&=E\big\{[X(s+a)-X(s-a)]\,[X(t+a)-X(t-a)]\,\big\}\\&=E\big\{X(s+a)X(t+a)\big\}-E\big\{X(s+a)X(t-a)\big\}-E\big\{X(s-a)X(t+a)\big\}\\&\quad+E\big\{X(s-a)X(t-a)\big\}\\&=R_X(\tau)-R_X(\tau-2a)-R_X(\tau+2a)+R_X(\tau)\\&=2R_X(\tau)-R_X(\tau+2a)-R_X(\tau-2a)\end{aligned}$$

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（*2*）由题可得，根据 *wienner-khinchine* 定理，有：

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-jw\tau} d\tau$$

根据傅里叶变换的性质，因此有：

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{+\infty} R_Y(\tau) e^{-jw\tau} d\tau \\
&= \int_{-\infty}^{+\infty} \big[2R_X(\tau) - R_X(\tau + 2a) - R_X(\tau - 2a)\big] \cdot e^{-jw\tau} d\tau \\
&= 2S_X(\omega) - S_X(\omega) e^{jw(2a)} - S_X(\omega) e^{jw(-2a)} \\
&= S_X(\omega)\big[2 - e^{jw(2a)} - e^{jw(-2a)}\big] \\
&= S_X(\omega)\Big[2 - \big(\cos(2a\omega) + j\sin(2a\omega)\big) - \big(\cos(2a\omega) + j\sin(2a\omega)\big)\Big] \\
&= 2S_X(\omega)\big[1 - \cos(2a\omega)\big] \\
&= 2S_X(\omega)\big[1 - \big(1 - 2sin^2(a\omega)\big)\big] \\
&= 4S_X(\omega)\sin^2(a\omega)
\end{aligned}$$

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