Stochastic Process

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Homework 1

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1. 设随机变量 X 服从参数为 1 的指数分布，随机变量 $Y \sim N(0,1)$，且 X 与 Y 独立，试求随机变量 $Z = \sqrt{2X}|Y|$ 的分布密度函数

   由题可得：$X \sim Exp(1), Y \sim N(0,1)$
   设随机变量 $U = \sqrt{2X}$，则 $f_U(u) = ue^{-\frac{u^2}{2}}(u \geq 0)$；设随机变量 $V = |Y|$，则 $f_V(v) = \frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{v^2}{2}}(v \geq 0)$，显然，$U$ 和 V 也是独立的，此时 $Z = \sqrt{2X}|Y| = UV$
   当 $z < 0$ 时:$F_Z(z) = 0, f_Z(z) = 0$
   当 $z \geq 0$ 时:

$$
\begin{aligned}
F_Z(z) &= \int_0^\infty \frac{1}{u} f_{U,V}(u, \frac{z}{u}) du \\
&= \int_0^\infty \frac{1}{u} f_U(u) f_V(\frac{z}{u}) du \\
&= \int_0^\infty e^{-\frac{u^2}{2}} \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{z^2}{2u^2}} du \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}(u^2 + \frac{z^2}{u^2})} du \\
&= \frac{\sqrt{2z}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{z}{2}(t^2 + \frac{1}{t^2})} dt \ (Let\ u = \sqrt{z}t) \\
&= \frac{\sqrt{2z}e^{-z}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{z}{2}(t - \frac{1}{t})^2} dt
\end{aligned}
$$

   又因为:

$$
\begin{aligned}
\int_0^\infty e^{-\frac{z}{2}(t-\frac{1}{t})^2} dt &= \int_0^\infty \frac{t^2}{t^2+1} e^{-\frac{z}{2}(t-\frac{1}{t})^2} d(t - \frac{1}{t}) \\
&= \int_0^\infty e^{-\frac{z}{2}(t-\frac{1}{t})^2} d(t - \frac{1}{t}) - \int_0^\infty \frac{1}{t^2+1} e^{-\frac{z}{2}(t-\frac{1}{t})^2} d(t - \frac{1}{t}) \\
&= \frac{\sqrt{2\pi}}{\sqrt{z}} - \int_0^\infty \frac{1}{t^2+1} e^{-\frac{z}{2}(t-\frac{1}{t})^2} d(t - \frac{1}{t}) \\
&= \frac{\sqrt{2\pi}}{\sqrt{z}} - \int_0^\infty \frac{1}{t^2} e^{-\frac{z}{2}(t-\frac{1}{t})^2} dt
\end{aligned}
$$

$$= \frac{\sqrt{2\pi}}{\sqrt{z}} - \int_0^\infty e^{-\frac{z}{2}(k-\frac{1}{k})^2} dk \ (Let\ k = \frac{1}{t})$$

所以：

$$\int_0^\infty e^{-\frac{z}{2}(t-\frac{1}{t})^2} dt = \frac{\sqrt{\pi}}{\sqrt{2z}}$$

综上：

$$f_Z(z) = \frac{\sqrt{2z}e^{-z}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{2z}} = e^{-z}$$

因此 Z 的分布密度函数为：

$$f_Z(z) = \begin{cases} e^{-z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

2. 设随机变量 X_1, X_2 独立同分布，服从参数为 $\lambda > 0$ 的指数分布，试证明随机变量 $\frac{X_1}{X_1+X_2} \sim U[0,1]$。

由题可得：$X_1 \sim Exp(\lambda), X_2 \sim Exp(\lambda)$，假设随机变量 $Z = \frac{X_1}{X_1+X_2}$

因此：

$$F_Z(z) = P(Z \leq z) = P(\frac{X_1}{X_1+X_2} \leq z) = P(\frac{1}{1+\frac{X_2}{X_1}} \leq z) = 1 - P(\frac{X_2}{X_1} \leq \frac{1}{z} - 1)$$

假设随机变量 $U = \frac{X_2}{X_1}$，则 U 的概率密度函数为：

$$\begin{aligned}
f_U(u) &= \int_0^\infty x_1 f_{X_1,X_2}(x_1, ux_1) dx_1 \\
&= \int_0^\infty x_1 f_{X_1}(x_1) f_{X_2}(ux_1) dx_1 \\
&= \lambda^2 \int_0^\infty x_1 e^{-\lambda x_1} e^{-\lambda u x_1} dx_1 \\
&= -\frac{\lambda}{1+u} \int_0^\infty x_1 de^{-(\lambda+\lambda u)x_1} \\
&= -\frac{\lambda}{1+u} (x_1 e^{-(\lambda+\lambda u)x_1}|_0^\infty - \int_0^\infty e^{-(\lambda+\lambda u)x_1} dx_1) \quad (Eq.1) \\
&= -\frac{\lambda}{1+u} (\frac{1}{\lambda(1+u)} e^{-(\lambda+\lambda u)x_1}|_0^\infty) \\
&= \frac{1}{(1+u)^2}
\end{aligned}$$

因此

$$
\begin{aligned}
F_Z(z) = 1 - P(U \leq \frac{1}{z} - 1) &= 1 - \int_0^{\frac{1}{z}-1} \frac{1}{(1+u)^2} du \\
&= 1 - (-\frac{1}{1+u}|_0^{\frac{1}{z}-1}) \\
&= 1 - (1 - z) = z
\end{aligned}
$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = 1$$

因此可得 $Z \sim U[0,1]$

3. 设随机变量 (X,Y) 的两个分量相互独立，且均服从标准正态分布 $N(0,1)$。
(a) 分别写出随机变量 $X+Y$ 和 $X-Y$ 的分布密度
(b) 试问：$X+Y$ 和 $X-Y$ 是否独立？说明理由

(a) 设随机变量 $Z_1 = X+Y, Z_2 = X-Y$，则：

$$f_{Z_1}(z_1) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z_1^2}{4}}, f_{Z_2}(z_2) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z_2^2}{4}}$$

(b) 由 *(a)* 可得：$Z_1 \sim N(0,2), Z_2 \sim N(0,2)$
因此 $M_{Z_1}(s_1) = E(e^{s_1 Z_1}) = e^{\frac{\sigma^2 s_1^2}{2} + \mu s} = e^{s_1^2}, M_{Z_2}(s_2) = e^{s_2^2}$
则：

$$
\begin{aligned}
M_{Z_1,Z_2}(s_1,s_2) = E[e^{s_1 Z_1 + s_2 Z_2}] &= E[e^{s_1(x+y)+s_2(x-y)}] \\
&= E[e^{(s_1+s_2)x} e^{(s_1-s_2)y}] \\
&= E[e^{(s_1+s_2)x}] E[e^{(s_1-s_2)y}] \\
&= e^{\frac{(s_1+s_2)^2}{2}} e^{\frac{(s_1-s_2)^2}{2}} \\
&= e^{s_1^2+s_2^2} = e^{s_1^2} e^{s_2^2} \\
&= M_{Z_1}(s_1) M_{Z_2}(s_2)
\end{aligned}
$$

由此可得 $X+Y$ 和 $X-Y$ 相互独立

4. 设二维随机变量 (X,Y) 的联合密度函数为*:*

$$
\begin{cases}
24(1-x)y, & 0 < y < x < 1 \\
0, & otherwise
\end{cases}
$$

试求：
(a) 边缘密度函数 $f_X(x)$ 和 $f_Y(y)$，以及条件密度函数 $f_{X|Y}(x|y)$ 和 $f_{Y|X}(y|x)$

(b) 当 $0 < y < 1$ 时，确定 $E\{X|Y = y\}$，以及 $E\{X|Y\}$ 的分布密度函数

(a) 由题可得：

$$\begin{aligned} f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 24(1-x)ydy \\ = 12(1-x)y^2|_0^x \\ = 12(x^2 - x^3) \quad (0 < x < 1) \end{aligned}$$

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 24(1-x)ydx \\ = 24y(x - \frac{1}{2}x^2)|_y^1 \\ = 12(y^3 - y^2 + y) \quad (0 < y < 1) \end{aligned}$$

$$\begin{aligned} f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}dx = \frac{24(1-x)y}{12(y^3 - y^2 + y)} \\ = \frac{2(1-x)}{y^2 - y + 1} \quad (0 < y < x < 1) \end{aligned}$$

$$\begin{aligned} f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24(1-x)y}{12(x^2 - x^3)} \\ = \frac{2(1-x)y}{(x^2 - x^3)} \quad (0 < y < x < 1) \end{aligned}$$

(b) 由题可得：

$$\begin{aligned} E\{X|Y = y\} = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_y^1 \frac{2(x - x^2)}{y^2 - y + 1}dx \\ = \frac{1}{y^2 - y + 1}(x^2 - \frac{2}{3}x^3)|_y^1 \\ = \frac{2y^3 - 3y^2 + 1}{3y^2 - 3y + 3} \end{aligned}$$

$$E\{X|Y\} = \begin{cases} \frac{2y^3 - 3y^2 + 1}{3y^2 - 3y + 3}, & 0 < y < 1 \\ 0, & otherwise \end{cases}$$