Stochastic Process

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Homework 13

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1. 设 $\xi(t) = X\sin(Yt)$; $t \geq 0$，而随机变量 X, Y 是相互独立且都服从 $[0,1]$ 上的均匀分布，试求此过程的均值函数及相关函数。并问此过程是否是平稳过程，是否连续、可导？

由题可得：$X, Y \sim U[0,1]$，则 $E\{X\} = \frac{1}{2}, E\{X^2\} = \frac{1}{3}$

当 $t = 0$ 时：

$$E\{\xi(0)\} = E\{X\sin(0)\} = 0$$

当 $t > 0$ 时：

$$\begin{aligned}E\{\xi(t)\} = E\{X\sin(Yt)\} &= E\{X\}E\{\sin(Yt)\} \\ &= \frac{1}{2}\int_0^1 E\{\sin(yt)\}f_Y(y)dy \\ &= -\frac{1}{2t}\cos t + \frac{1}{2t}\end{aligned}$$

故：

$$E\{\xi(t)\} = \begin{cases} 0, & t = 0 \\ -\frac{1}{2t}\cos t + \frac{1}{2t}, & t \geq 0 \end{cases}$$

令 $0 < s < t$，有：

$$\begin{aligned}R_Y(s,t) = E\{\xi(s)\overline{\xi(t)}\} &= E\{X\sin(Ys)X\sin(Yt)\} \\ &= E\{X^2\}E\{\sin(Ys)\sin(Yt)\} \\ &= \frac{1}{3}\cdot(-\frac{1}{2})E\big\{\cos[Y(s+t)] - \cos[Y(t-s)]\big\} \\ &= -\frac{1}{6}\bigg(E\big\{\cos\left[Y(s+t)\right]\big\} - E\big\{Cos\left[Y(t-s)\right]\big\}\bigg) \\ &= -\frac{1}{6}\left(\int_0^1 \cos[y(s+t)]f_Y(y)dy - \int_0^1 \cos[y(t-s)]f_Y(y)dy\right) \\ &= \frac{1}{6}\bigg(\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\bigg)\end{aligned}$$

当 $s=t\neq 0$，有：

$$
\begin{aligned}
R_Y(t,t) = E\{\xi(t)\overline{\xi(t)}\} &= E\{X^2\sin^2(Yt)\} = \frac{1}{6}E\{1-\cos(2Yt)\} \\
&= \frac{1}{6} - \frac{1}{6}E\{\cos(2Yt)\} \\
&= \frac{1}{6} - \frac{1}{6}\int_0^1 \cos\left(2yt\right) f_Y\left(y\right) dy \\
&= \frac{1}{6}\left(1-\frac{\sin(2t)}{2t}\right)
\end{aligned}
$$

当 $s=t=0$，有：

$$
R_Y(0,0) = E\{X^2\sin^2(0)\} = 0
$$

故：

$$
R_Y(s,t) = \begin{cases} 0, & s=t=0 \\ \frac{1}{6}\big(1-\frac{\sin(2t)}{2t}\big), & s=t\neq 0 \\ \frac{1}{6}\big(\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\big), & s\neq t, s\neq 0, t\neq 0 \end{cases}
$$

又因为：

$$
\lim_{s,t\to 0}\frac{1}{6}\big(1-\frac{\sin(2t)}{2t}\big) = \lim_{s,t\to 0}\frac{1}{6}\times 0 = 0
$$

因此 $R_Y(s,t)$ 在 $(t_0,t_0), t_o\geq 0$ 上连续，故 $R_Y(s,t)$ 均方连续

又因为：

$$
\frac{\partial^2 R_Y(s,t)}{\partial s\partial t} = \frac{-\sin(2x)+2x\cos(2x)+2x^2\sin(2x)}{6x^3},\ \lim_{s,t\to 0}\frac{\partial^2 R_Y(s,t)}{\partial s\partial t} = 0
$$

因此 $\frac{\partial^2 R_Y(s,t)}{\partial s\partial t}$ 在 $(t_0,t_0), t_o\geq 0$ 上连续，故 $R_Y(s,t)$ 均方可导

2. 设 $\{X(t), t\in R\}$ 是连续平稳过程，均值为 m ，协方差函数为 $C_X(\tau)=ae^{-b|\tau|}$，其中：$\tau\in\mathbb{R}$；$a,b>0$。对固定的 $T>0$，令 $Y=T^{-1}\int_0^T X(s)ds$，证明 $E\{Y\} = m, Var(Y) = 2a[(bT)^{-1}-(bT)^{-2}(1-e^{-bT})]$

由题可得：

$$
\begin{aligned}
E\{Y\} = E\{\frac{1}{T}\int_0^T x(s)ds\} &= \frac{1}{T}\int_0^T E\{X(s)\}ds \\
&= \frac{1}{T}\int_0^T m\ ds = \frac{1}{T}\cdot mT = M
\end{aligned}
$$

又因为：

$$
\begin{aligned}
C_X(\tau) = ae^{-b|\tau|} &= E\big\{\,[X(s)-\mu_x]\,[X(t)-\mu_x]\big\}^2 \\
&= E\{X(s)X(t)\} - \mu_x\bigg(E\left\{X(s)\right\} + E\left\{X(t)\right\}\bigg) + \mu_x^2 \\
&= R_x(\tau) - m^2
\end{aligned}
$$

因此：$R_X(\tau) = ae^{-b|\tau|} + m^2$

现计算过程 $Y(t)$ 的二阶矩：

$$\begin{aligned}
E\{Y^2\} &= E\left\{\frac{1}{T^2}\int_0^T\int_0^T X(s)X(t)dsdt\right\} \\
&= \frac{1}{T^2}\int_0^T\int_0^T \underbrace{E\left\{X(s)X(t)\right\}}_{R_X(\tau)} dsdt \\
&= \frac{1}{T^2}\int_0^T\int_0^T ae^{-b|\tau|} + m^2 dsdt \\
&= \frac{a}{T^2}\left(\int_s^T\int_0^T e^{-b(t-s)}dtds + \int_0^T\int_t^T e^{-b(s-t)}dsdt\right) + m^2 \\
&= \frac{a}{T^2}\left(\int_0^T(-\frac{1}{b}e^{bs-bT} + \frac{1}{b})ds + \int_0^T(-\frac{1}{b}e^{bt-bT} + \frac{1}{b})dt\right) + m^2 \\
&= \frac{a}{T^2}\left(\big(-\frac{1}{b^2}e^{bs-bT} + \frac{1}{b}s\Big|_0^T\big) + \big(-\frac{1}{b^2}e^{bt-bT} + \frac{1}{b}t\Big|_0^T\big)\right) + m^2 \\
&= \frac{a}{T^2}(-\frac{2}{b^2} + \frac{2T}{b} + \frac{2}{b^2}e^{-bT}) + m^2 \\
&= 2a[(bT)^{-1} - (bT)^{-2}(1 - e^{-bT})] + m^2
\end{aligned}$$

故 $Var(Y) = E\{Y^2\} - E^2\{Y\} = 2a[(bT)^{-1} - (bT)^{-2}(1 - e^{-bT})]$

证毕

3. 设 $(X, Y) \sim N(0, 0, \sigma_1^2, \sigma_2^2, \rho)$，令 $X(t) = X + tY$，以及 $Y(t) = \int_0^t X(u)du$，$Z(t) = \int_0^t X^2(u)du$，对于任意 $0 \le s \le t$，

(*1*) 求 $E\{X(t)\}, E\{Y(t)\}, E\{Z(t)\}, Cov(X(s), X(t)), Cov(Y(s), Y(t))$；

(*2*) 证明 $X(t)$ 在 $t > 0$ 上均方连续、均方可导；

(*3*) 求 $Y(t)$ 及 $Z(t)$ 的均方导数。

(*1*) 由题可得：$X \sim N(0, \sigma_1^2), Y \sim N(0, \sigma_2^2)$

$$E\{X(t)\} = E\{X + tY\} = E\{X\} + tE\{Y\} = 0 + t \times 0 = 0$$

$$E\{Y(t)\} = E\{\int_0^t X(u)du\} = \int_0^t E\{X(u)\}du = 0$$

$$\because P = \frac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_x\sigma_y} \iff E\{XY\} = \rho\sigma_1\sigma_2$$

$$\begin{aligned}
E\left\{X^2(t)\right\} = E\left\{X^2 + t^2Y^2 + 2tXY\right\} &= E\left\{X^2\right\} + t^2E\left\{Y^2\right\} + 2tE\left\{XY\right\} \\
&= 0^2 + \sigma_1^2 + t^2(0^2 + \sigma_2^2) + 2t\rho\sigma_1\sigma_2 \\
&= \sigma_1^2 + t^2\sigma_2^2 + 2t\rho\sigma_1\sigma_2
\end{aligned}$$

$$\therefore E\{Z(t)\} = E\{\int_0^t X^2(u)du\} = \int_0^t E\{x^2(u)\}du = \int_0^t \sigma_1^2 + u^2\sigma_2^2 + 2u\rho(\sigma_1\sigma_2)du$$
$$= \sigma_1^2 t + \rho\sigma_1\sigma_2 t^2 + \frac{1}{3}\sigma_2^2 t^3$$

$$\begin{aligned}\mathrm{Cov}\{X(s), X(t)\} = R_X(s,t) = E\{X(s)X(t)\} &= E\{(X+sY)(X+tY)\}\\ &= E\{X^2 + (s+t)XY + stY^2\}\\ &= \sigma_1^2 + (s+t)\rho\sigma_1\sigma_2 + st\sigma_2^2\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}\{Y(s), Y(t)\} = R_Y(s,t) &= E\{Y(s)Y(t)\}\\ &= E\{\int_0^t X(u)du \int_0^t X(v)dv\}\\ &= E\left\{\int_0^t \int_0^s X(u)X(v)dudv\right\}\\ &= \int_0^t \int_0^s E\{X(u)X(v)\}dudv\\ &= \int_0^t \int_0^s \left[\sigma_1^2 + (u+v)\rho\sigma_1\sigma_2 + uv\sigma_2^2\right]dudv\\ &= \sigma_1^2 st + \frac{1}{2}\rho\sigma_1\sigma_2 s^2 t + \frac{1}{2}\rho\sigma_1\sigma_2 st^2 + \frac{1}{4}\rho_2^2 s^2 t^2\\ &= \sigma_1^2 st + \frac{1}{2}\rho\sigma_1 s_2 st(s+t) + \frac{1}{4}\sigma_2^2 s^2 t^2\end{aligned}$$

(*2*) 由（*1*）可得，显然 $X(t)$ 和 $Y(t)$ 均不为（宽）平稳过程，因此当 $t > 0$ 时：

$$R_X(t_0, t_0) = \sigma_1^2 + 2t_0\rho\sigma_1\sigma_2 + t_0^2\sigma_2^2$$
$$\frac{\partial^2 R_X(s,t)}{\partial s \partial t} = \sigma^2\delta(t-s) + \rho\sigma_1\sigma_2 + \sigma_2^2 \Longleftrightarrow \left.\frac{\partial^2 R_X(s,t)}{\partial s\partial t}\right|_{s=t=t_0} = \rho\sigma_1\sigma_2 + \sigma_2^2$$

故 $R_X(t_0, t_0)$ 和 $\frac{\partial^2 R_X(s,t)}{\partial s\partial t}$ 在 (t_0, t_0) 处均连续，因此 $X(t)$ 在 $t > 0$ 上均方连续、均方可导。

(*3*) 由题可得：$Y(t)$ 的均方导数为：

$$\begin{aligned}\frac{dY(t)}{dt} = \underset{h\to 0}{l.i.m}\,\frac{Y(t+h) - Y(t)}{h} &= \underset{h\to 0}{l.i.m}\,\frac{\int_0^{t+h} X(u)du - \int_0^t X(v)dv}{h}\\ &= \underset{h\to 0}{l.i.m}\,\frac{\int_0^{t+h}(X+uY)du - \int_0^t (X+vY)dv}{h}\\ &= \underset{h\to 0}{l.i.m}\,\frac{\left((t+h)X + \frac{1}{2}Y(t+h)^2\right) - \left(tX + \frac{1}{2}Y(t)^2\right)}{h}\\ &= \underset{h\to 0}{l.i.m}(X + tY + \frac{1}{2}hY)\\ &= X + tY = X(t)\end{aligned}$$

$Z(t)$ 的均方导数为：

$$\begin{aligned}
\frac{dZ(t)}{dt} &= \underset{h\to 0}{l.i.m}\,\frac{Z(t+h)-Z(t)}{h} = \underset{h\to 0}{l.i.m}\,\frac{\int_0^{t+h}X^2(u)du-\int_0^t X^2(v)dv}{h} \\
&= \underset{h\to 0}{l.i.m}\,\frac{\int_0^{t+h}(X+uY)^2du-\int_0^t(X+vY)^2dv}{h} \\
&= \underset{h\to 0}{l.i.m}\,\frac{\left((t+h)X^2+(t+h)^2XY+\frac{1}{3}(t+h)^3Y^2\right)-\left(tX^2+t^2XY+\frac{1}{3}t^3Y^2\right)}{h} \\
&= \underset{h\to 0}{l.i.m}(X^2+2tXY+hXY+t^2Y^2+thY^2+\frac{1}{3}h^2) \\
&= X^2+2tXY+t^2Y^2 = X^2(t)
\end{aligned}$$

4. 设随机过程 $\{X(t), -\infty < t < +\infty\}$ 是均值为零、自相关函数为 $R_X(\tau)$ 的实平稳正态过程。设 $X(t)$ 通过线性全波检波器后，其输出为 $Y(t) = |X(t)|$，试求：

(*1*) 随机过程 $Y(t)$ 的相关函数 $R_Y(\tau)$，并说明其是否为平稳过程；

(*2*) 随机过程 $Y(t)$ 的均值和方差；

(*3*) 随机过程 $Y(t)$ 的一维概率分布密度函数 $f_Y(y)$。

(*1*) 由题可得：$X(t)$ 通过线性全波检波器的输出 $Y(t)$ 为

$$Y(t)=\begin{cases} X(t), & X(t)\geq 0 \\ -X(t), & X(t)<0 \end{cases}$$

根据条件数学期望公式，有：

$$\begin{aligned}
R_Y(\tau) &= E\big\{Y(t)Y(t+\tau)\big\} = E\Big\{E\{Y(t)Y(t+\tau)\Big| X(t)X(t+\tau)\}\Big\} \\
&= \iint E\big\{Y(t)Y(t+\tau)\Big| X(t)X(t+\tau)\big\} f_{(X(t),X(t+\tau))}(x,y)dxdy \\
&= \iint\limits_{xy\geq 0} xyf_{(X(t),X(t+\tau))}(x,y)dxdy - \iint\limits_{xy<0} xyf_{(X(t),X(t+\tau))}(x,y)dxdy
\end{aligned}$$

其中：

$$f_{(X(t),X(t+\tau))}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\right\}$$
$$\sigma_1^2=\sigma_2^2=R_X(0),\quad r=\frac{R_X(\tau)}{R_X(0)}$$

由此可得:

$$
\begin{aligned}
R_Y(\tau) = E\big\{Y(t)Y(t+\tau)\big\} &= E\big\{E\{Y(t)Y(t+\tau)\Big|X(t)X(t+\tau)\}\big\} \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{xy}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Bigg\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\Bigg\}dxdy \\
&+ \int_{-\infty}^0\int_{-\infty}^0 \frac{xy}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Bigg\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\Bigg\}dxdy \\
&- \int_{-\infty}^0\int_0^{+\infty} \frac{xy}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Bigg\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\Bigg\}dxdy \\
&- \int_0^{+\infty}\int_{-\infty}^0 \frac{xy}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Bigg\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right]\Bigg\}dxdy
\end{aligned}
$$

令 $u=\frac{x}{\sigma_1}$,$v=\frac{y}{\sigma_2}$: 则:

$$
|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} = \sigma_1\sigma_2
$$

$$
\begin{aligned}
R_Y(\tau) &= \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_0^{+\infty}\int_0^{+\infty} uv\exp\big\{-\frac{1}{2}[u^2-2ruv+v^2]\big\}dudv \\
&- \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_{-\infty}^0\int_{-\infty}^0 uv\exp\big\{-\frac{1}{2}[u^2-2ruv+v^2]\big\}dudv \\
&- \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_{-\infty}^0\int_0^{+\infty} uv\exp\big\{-\frac{1}{2}[u^2-2ruv+v^2]\big\}dudv \\
&+ \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_0^{+\infty}\int_{-\infty}^0 uv\exp\big\{-\frac{1}{2}[u^2-2ruv+v^2]\big\}dudv \\
&= \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_0^{+\infty}\int_0^{+\infty} uv\exp\Big\{-\frac{1}{2}\left[\left(\frac{u-rv}{\sqrt{1-r^2}}\right)^2+v^2\right]\Big\}dudv \\
&+ \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_{-\infty}^0\int_{-\infty}^0 uv\exp\Big\{-\frac{1}{2}\left[\left(\frac{u-rv}{\sqrt{1-r^2}}\right)^2+v^2\right]\Big\}dudv \\
&- \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_{-\infty}^0\int_0^{+\infty} uv\exp\Big\{-\frac{1}{2}\left[\left(\frac{u-rv}{\sqrt{1-r^2}}\right)^2+v^2\right]\Big\}dudv \\
&- \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-r^2}}\int_0^{+\infty}\int_{-\infty}^0 uv\exp\Big\{-\frac{1}{2}\left[\left(\frac{u-rv}{\sqrt{1-r^2}}\right)^2+v^2\right]\Big\}dudv
\end{aligned}
$$

令 $\frac{u-rv}{\sqrt{1-r^2}} = R\cos\theta$, $v=R\sin\theta$: 则:

$$
|J| = \frac{\partial(u,v)}{\partial(R,\theta)} = \begin{vmatrix} \sqrt{1-r^2}\cos\theta + r\sin\theta & -\sqrt{1-r^2}R\sin\theta + rR\cos\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R\sqrt{1-r^2}
$$

则有:

$$
\begin{aligned}
R_Y(\tau) = &\frac{\sigma_1\sigma_2}{2\pi}\int_0^{\arccos(-r)}\int_0^{+\infty} R\cdot R\sin\theta\Big[\sqrt{1-r^2}R\cos\theta + rR\sin\theta\Big]\exp\Big\{-\frac{R^2}{2}\Big\}dRd\theta \\
&+\frac{\sigma_1\sigma_2}{2\pi}\int_{-\arccos r}^{0}\int_{-\infty}^{0} R\cdot R\sin\theta\Big[\sqrt{1-r^2}R\cos\theta + rR\sin\theta\Big]\exp\Big\{-\frac{R^2}{2}\Big\}dRd\theta \\
&-\frac{\sigma_1\sigma_2}{\pi}\int_0^{\arccos(-r)}\int_{-\infty}^{0} R\cdot R\sin\theta\Big[\sqrt{1-r^2}R\cos\theta + rR\sin\theta\Big]\exp\Big\{-\frac{R^2}{2}\Big\}dRd\theta \\
= &\frac{3\sigma_1\sigma_2}{\pi}\int_0^{\arccos(-r)} \sin\theta\Big[\sqrt{1-r^2}\cos\theta + r\sin\theta\Big]d\theta \\
&-\frac{\sigma_1\sigma_2}{\pi}\int_{-\arccos r}^{0} \sin\theta\Big[\sqrt{1-r^2}\cos\theta + r\sin\theta\Big]d\theta \\
= &\frac{3\sigma_1\sigma_2}{\pi}\left\{\left[\frac{1}{2}\sqrt{1-r^2}\sin^2\theta + \frac{1}{2}r\theta - \frac{1}{4}r\sin 2\theta\right]_0^{\arccos(-r)}\right\} \\
&-\frac{\sigma_1\sigma_2}{\pi}\left\{\left[\frac{1}{2}\sqrt{1-r^2}\sin^2\theta + \frac{1}{2}r\theta - \frac{1}{4}r\sin 2\theta\right]_{-\arccos r}^{0}\right\}
\end{aligned}
$$

令 $\sin\varphi = r = \frac{R_X(\tau)}{R_X(0)}$, $|\varphi| \le \frac{\pi}{2}$,则可得:

$$
\begin{aligned}
R_Y(\tau) &= \frac{3R_X(0)}{4\pi}(\pi + 2\varphi + 1)\sin\varphi - \frac{R_X(0)}{4\pi}(\pi - 2\varphi + 1)\sin\varphi \\
&= \frac{R_X(0)}{2\pi}\left[(\pi + 4\varphi)\sin\varphi + 4\cos\varphi\right]
\end{aligned}
$$

(2) 根据条件数学期望公式可得:

$$
\begin{aligned}
m_Y(t) &= E\{Y(t)\} = E\{E\{Y(t)\big|X(t)\}\} \\
&= \int_0^{+\infty}|x|f_{X(t)}(x)dx = 2\int_0^{+\infty}xf_{X(t)}(x)dx \\
&= 2\int_0^{+\infty}\frac{x}{\sqrt{2\pi R_X(0)}}\exp\Big\{-\frac{x^2}{2R_X(0)}\Big\}dx \\
&= 2\sqrt{\frac{R_X(0)}{2\pi}}
\end{aligned}
$$

又 $\because E\{Y^2(t)\} = R_Y(0) = \frac{3R_X(0)}{2}$

$$
\therefore Var\{Y(t)\} = E\{Y^2(t)\} - E^2\{Y(t)\} = \left(3 - \frac{1}{\pi}\right)\cdot\frac{R_X(0)}{2}
$$

（3）由题可得：

$$
\begin{aligned}
\because F_{Y(t)}(y) &= P\{Y(t) \le y\} = P\{|X(t)| \le y\} = P\{-y \le X(t) \le y\} \\
&= \int_{-y}^{y} f_{X(t)}(x)dx = 2\int_{0}^{y} f_{X(t)}(x)dx
\end{aligned}
$$

$$
\therefore f_{Y(t)}(y) = F_{Y(t)}^{'}(y) = \sqrt{\frac{2}{\pi R_X(0)}} \exp\left\{-\frac{y^2}{2R_X(0)}\right\}
$$

上式即为随机过程 $Y(t)$ 的一维概率分布密度函数 $f_Y(y)$