Recursion Formulae for Relative Pose Estimation

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1 Introduction

This document gives recursion formulae to approximately solve the the relative localization problem from introduced in the paper, given by

$$
\begin{aligned}
\boldsymbol{d}_N^*, \psi^* &= \underset{\boldsymbol{d}_N, \psi}{\arg\min} \sum_{k=1}^{N} \mathrm{e}^{-\frac{t_N - t_k}{\tau}} w_k^2 \\
\text{s. t.} \quad w_k &= \left\| \boldsymbol{d}_N - \int_{t_k}^{t_N} \boldsymbol{T}\, \boldsymbol{v}_j(\tau) - \boldsymbol{v}_i(\tau)\, \mathrm{d}\tau \right\| - d_k \\
\boldsymbol{T} &= \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{1}
$$

with a nested optimization strategy

$$
\min_{\psi \in \{0^\circ, \ldots, 359^\circ\}} \left(\min_{\boldsymbol{d}_N} \sum_{k=1}^{N} \mathrm{e}^{-\frac{t_N - t_k}{\tau}} w_k^2 \right) \tag{2}
$$

We derive in the paper an approximation to the problem with a quadratic objective function and give the following recursion of the objective function parameters $\boldsymbol{Q}_{k+1}$, $\boldsymbol{c}_{k+1}^{\mathsf{T}}$ and e_{k+1}:

$$
\boldsymbol{Q}_{k+1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{pmatrix} + \mathrm{e}^{-\frac{t_{k+1} - t_k}{\tau}} \left(\boldsymbol{A}_k^{\mathsf{T}}\, \boldsymbol{Q}_k\, \boldsymbol{A}_k \right) \tag{3}
$$

$$
\boldsymbol{c}_{k+1}^{\mathsf{T}} = \mathrm{e}^{-\frac{t_{k+1} - t_k}{\tau}} \left(\boldsymbol{c}_k^{\mathsf{T}} \boldsymbol{A}_k + \boldsymbol{b}_k^{\mathsf{T}} \boldsymbol{Q}_k\, \boldsymbol{A}_k \right) \tag{4}
$$

$$
e_{k+1} = \mathrm{e}^{-\frac{t_{k+1} - t_k}{\tau}} \left(e_k + \boldsymbol{b}_k^{\mathsf{T}} \boldsymbol{b}_k \right) \tag{5}
$$

where

$$\boldsymbol{A}_k = \begin{pmatrix} \boldsymbol{I} & 0 \\ \frac{\boldsymbol{u}_k(\psi)^\mathsf{T}}{d_k} & \frac{d_{k+1}}{d_k} \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_k = \begin{pmatrix} \boldsymbol{u}_k(\psi) \\ -\frac{\|\boldsymbol{u}_k(\psi)\|^2 + d_{k+1}^2 - d_k^2}{2\, d_k} \end{pmatrix} \tag{6}$$

and

$$\boldsymbol{u}_k = (\boldsymbol{v}_{i,k} - \boldsymbol{T}\boldsymbol{v}_{j,k})\,(t_{k+1} - t_k) \tag{7}$$

2 Shorthand Notations

The following shorthand notations are introduced for a convenient notation. We define for the exponential decay between two consecutive time steps

$$\alpha_k := \mathrm{e}^{\frac{t_{k+1} - t_k}{\tau}} \tag{8}$$

In order to express $\boldsymbol{u}_k = \boldsymbol{\mu}_k + \boldsymbol{T}\,\boldsymbol{\nu}_k$, we define

$$\boldsymbol{\mu}_k := \boldsymbol{v}_{i,k}\,(t_{k+1} - t_k) \quad \text{and} \quad \boldsymbol{\nu}_k = -\boldsymbol{v}_{j,k}\,(t_{k+1} - t_k) \tag{9}$$

We further define

$$\gamma_k := \frac{d_k^2 - d_{k+1}^2 - \|\boldsymbol{\mu}_k\|^2 - \|\boldsymbol{\nu}_k\|^2}{2\, d_k} \tag{10}$$

and we define $\boldsymbol{\delta}_k$ implicitly (and explicitly in Equation 16) such that

$$\boldsymbol{\delta}_k^\mathsf{T}\boldsymbol{T}\,\boldsymbol{1} = \frac{-1}{d_k}\boldsymbol{\mu}_k\boldsymbol{T}\boldsymbol{\nu}_k \tag{11}$$

in order to express

$$-\frac{\|\boldsymbol{u}_k(\psi)\|^2 + d_{k+1}^2 - d_k^2}{2\, d_k} = \gamma_k + \boldsymbol{\delta}_k^\mathsf{T}\boldsymbol{T}\,\boldsymbol{1} \tag{12}$$

3 Converting Linear Terms to a Standard Form

We would like to express a linear term in $\boldsymbol{T}$ involving the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in a standard form with the vector $\boldsymbol{w}$ and and a vector $\boldsymbol{1}$ that has all coordinates equal to one, i. e.,

$$\boldsymbol{u}^\mathsf{T}\boldsymbol{T}\boldsymbol{v} = \boldsymbol{w}^\mathsf{T}\boldsymbol{T}\,\boldsymbol{1} \tag{13}$$

and after expanding

$$\begin{aligned} &\sin(w_y - w_x) + \cos(w_x + w_y) + w_z = \\ &\qquad (\cos(v_x u_y - v_y u_x) + \cos(u_x v_x + u_y v_y) + u_z v_z) \end{aligned} \tag{14}$$

We solve for $\boldsymbol{w}$ and define the function $h : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$\boldsymbol{w}^\top = h(\boldsymbol{u}^\top, \boldsymbol{v}) = \begin{pmatrix} \frac{1}{2}\left(u_x v_x + u_y v_y - v_x u_y + v_y u_x\right) \\ \frac{1}{2}\left(u_x v_x + u_y v_y + v_x u_y - v_y u_x\right) \\ u_z v_z \end{pmatrix}^\top \tag{15}$$

The previously implicit definition of δ_k in Equation 11 can now be written explicitly as

$$\boldsymbol{\delta}_k^\top := \frac{-h(\boldsymbol{\mu}_k^\top, \boldsymbol{\nu}_k)}{d_k} \tag{16}$$

The function h is also applicable to term that are linear in the transpose of $\boldsymbol{T}$,

$$\boldsymbol{u}^\top \boldsymbol{T}^\top \boldsymbol{v} = \boldsymbol{v}^\top \boldsymbol{T}\,\boldsymbol{u} = h(\boldsymbol{v}^\top, \boldsymbol{u})\,\boldsymbol{T}\,\mathbf{1} \tag{17}$$

and row-wise to matrices.

$$\boldsymbol{M}\,\boldsymbol{T}\,\boldsymbol{v} = \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{T}\boldsymbol{v} = \begin{pmatrix} h(\boldsymbol{m}_1, \boldsymbol{v}) \\ h(\boldsymbol{m}_2, \boldsymbol{v}) \\ h(\boldsymbol{m}_3, \boldsymbol{v}) \end{pmatrix} \boldsymbol{T}\,\mathbf{1} \tag{18}$$

4 Recursion of Objective Function Parameters

We partition $\boldsymbol{Q}_k$ and $\boldsymbol{c}_k$ with

$$\boldsymbol{Q}_k = \begin{pmatrix} \boldsymbol{Q}_{11,k} & \boldsymbol{Q}_{12,k} \\ \boldsymbol{Q}_{21,k} & Q_{22,k} \end{pmatrix}, \quad \boldsymbol{Q}_{21,k} = \boldsymbol{Q}_{12,k}^\top$$

$$\boldsymbol{c}_k = \begin{pmatrix} \boldsymbol{c}_{1,k} \\ c_{2,k} \end{pmatrix}$$

For the partitions, we derive constant, linear and quadratic terms, implicitly defined by

$$\boldsymbol{Q}_{11,k} = \boldsymbol{Q}_{11C,k} + \boldsymbol{T}\,\boldsymbol{Q}_{11L,k} + \boldsymbol{Q}_{11L,k}^\top \boldsymbol{T}^\top + \boldsymbol{T}\,\boldsymbol{Q}_{11M,k}\boldsymbol{T}^\top \tag{19}$$

$$\boldsymbol{Q}_{12,k} = \boldsymbol{Q}_{12C,k} + \boldsymbol{T}\,\boldsymbol{Q}_{12L,k} \tag{20}$$

$$\boldsymbol{Q}_{22,k} = \boldsymbol{Q}_{22C,k} \tag{21}$$

$$\boldsymbol{c}_{1,k} = \boldsymbol{c}_{1C,k} + \boldsymbol{c}_{1L,k}\boldsymbol{T}\,\mathbf{1} + \boldsymbol{T}\,\boldsymbol{c}_{1M,k}\boldsymbol{T}\,\mathbf{1} \tag{22}$$

$$c_{2,k} = c_{2C,k} + \boldsymbol{c}_{2L,k}\boldsymbol{T}\,\mathbf{1} \tag{23}$$

$$e_k = e_{C,k} + \boldsymbol{e}_{L,k}\boldsymbol{T}\mathbf{1} + \mathbf{1}^\top \boldsymbol{T}^\top \boldsymbol{e}_{M,k}\boldsymbol{T}\,\mathbf{1} \tag{24}$$

by expanding Equations 3 – 5. For the sake of better overview, we give the results as from the expansion without bringing them into a standard form of Equation 13.

The formulae for component-wise recursion of $\boldsymbol{Q}_{11,k+1}$ are

$$\begin{aligned}
\boldsymbol{Q}_{11,k+1} = \alpha\, \boldsymbol{Q}_{11,k} \\
&+ \frac{\alpha}{d_k}\left(-\boldsymbol{Q}_{12C,k}\boldsymbol{\mu}_k^{\mathsf{T}} - \boldsymbol{\mu}_k\boldsymbol{Q}_{12C,k}^{\mathsf{T}} + \frac{Q_{22,k}}{d_k}\boldsymbol{\mu}_k\boldsymbol{\mu}_k^{\mathsf{T}}\right) \\
&+ \boldsymbol{T}\frac{\alpha}{d_k}\left(-\boldsymbol{Q}_{12L,k}\boldsymbol{\mu}_k^{\mathsf{T}} - \boldsymbol{\nu}_k\boldsymbol{Q}_{12C,k}^{\mathsf{T}} + \frac{Q_{22,k}}{d_k}\boldsymbol{\nu}_k\boldsymbol{\mu}_k^{\mathsf{T}}\right) \\
&+ \frac{\alpha}{d_k}\left(-\boldsymbol{Q}_{12C,k}\boldsymbol{\nu}_k^{\mathsf{T}} - \boldsymbol{\mu}_k\boldsymbol{Q}_{12L,k}^{\mathsf{T}} + \frac{Q_{22,k}}{d_k}\boldsymbol{\mu}_k\boldsymbol{\nu}_k^{\mathsf{T}}\right)\boldsymbol{T}^{\mathsf{T}} \\
&+ \boldsymbol{T}\frac{\alpha}{d_k}\left(-\boldsymbol{Q}_{12L,k}\boldsymbol{\nu}_k^{\mathsf{T}} - \boldsymbol{\nu}_k\boldsymbol{Q}_{12L,k}^{\mathsf{T}} + \frac{Q_{22,k}}{d_k}\boldsymbol{\nu}_k\boldsymbol{\nu}_k^{\mathsf{T}}\right)\boldsymbol{T}^{\mathsf{T}}
\end{aligned} \tag{25}$$

for $\boldsymbol{Q}_{12,k+1}$,

$$\boldsymbol{Q}_{12,k+1} = \frac{\alpha\, d_{k+1}}{d_k}\boldsymbol{Q}_{12,k} - \frac{\alpha\, d_{k+1}Q_{22,k}}{d_k^2}\boldsymbol{\mu}_k - \frac{\alpha\, d_{k+1}Q_{22,k}}{d_k^2}\boldsymbol{T}\boldsymbol{\nu}_k \tag{26}$$

and for $\boldsymbol{Q}_{22,k+1}$,

$$\boldsymbol{Q}_{22,k+1} = 1 + \frac{\alpha\, d_{k+1}}{d_k}\boldsymbol{Q}_{22,k+1} \tag{27}$$

We calculate for $c_{2,k+1}$

$$\begin{aligned}
c_{2,k+1} = \frac{\alpha\, d_{k+1}}{d_k}c_{2,k} \\
&+ \frac{\alpha\, d_{k+1}}{d_k}\left(-\boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{\mu}_k - \boldsymbol{Q}_{12L,k}^{\mathsf{T}}\boldsymbol{\nu}_k - \gamma_k\, Q_{22,k}\right) \\
&+ \frac{\alpha\, d_{k+1}}{d_k}\left(-\boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{\nu}_k - \boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{Q}_{12L,k} - Q_{22,k}\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{1}\right)
\end{aligned} \tag{28}$$

For $\boldsymbol{c}_{1,k+1}$, we calculate for the constant terms

$$\begin{aligned}
\boldsymbol{c}_{1C,k+1} = \alpha\, \boldsymbol{c}_{1C,k} \\
&- \frac{\alpha}{d_k}\boldsymbol{\mu}_k c_{2C,k} \\
&- \alpha\left(\boldsymbol{Q}_{11C,k}\boldsymbol{\mu}_k + \boldsymbol{Q}_{11L,k}^{\mathsf{T}}\boldsymbol{\nu}_k\right) \\
&+ \frac{\alpha}{d_k}\boldsymbol{\mu}_k\left(\boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{\mu}_k + \boldsymbol{Q}_{12L,k}^{\mathsf{T}}\boldsymbol{\nu}_k\right) \\
&- \alpha\,\boldsymbol{Q}_{12C,k}\gamma \\
&+ \frac{\alpha\, Q_{22,k}}{d_k}\boldsymbol{\mu}_k\gamma
\end{aligned} \tag{29}$$

for the linear terms in $\boldsymbol{T}$,

$$\begin{aligned}
\boldsymbol{c}_{1L,k+1}\boldsymbol{T}\,\mathbf{1} = {} & \alpha\,\boldsymbol{c}_{1L,k}\boldsymbol{T}\,\mathbf{1} \\
& - \frac{\alpha}{d_k}\left(\boldsymbol{\mu}_k\boldsymbol{c}_{2L,k}\boldsymbol{T}\,\mathbf{1} + \boldsymbol{T}\,\boldsymbol{\nu}_k c_{2C,k}\right) \\
& - \alpha\left(\boldsymbol{Q}_{11C,k}\boldsymbol{T}\,\boldsymbol{\nu}_k + \boldsymbol{T}\,\boldsymbol{Q}_{11L,k}\boldsymbol{\mu}_k + \boldsymbol{Q}_{11L,k}^{\mathsf{T}}\boldsymbol{T}^{\mathsf{T}}\boldsymbol{\mu}_k + \boldsymbol{T}\,\boldsymbol{Q}_{11M,k}\boldsymbol{\nu}_k\right) \\
& + \frac{\alpha}{d_k}\left(\boldsymbol{T}\,\boldsymbol{\nu}_k\left(\boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{\mu}_k + \boldsymbol{Q}_{12L,k}^{\mathsf{T}}\boldsymbol{\nu}_k\right) + \boldsymbol{\mu}_k\left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{Q}_{12L,k} + \boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{\nu}_k\right)\right) \\
& - \alpha\left(\boldsymbol{T}\,\boldsymbol{Q}_{12L,k}\gamma + \boldsymbol{Q}_{12C,k}\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1}\right) \\
& + \frac{\alpha\,Q_{22,k}}{d_k}\left(\boldsymbol{\mu}_k\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1} + \boldsymbol{T}\,\boldsymbol{\nu}_k\gamma_k\right) \qquad (30)
\end{aligned}$$

and for the quadratic terms

$$\begin{aligned}
\boldsymbol{T}\,\boldsymbol{c}_{1M,k+1}\boldsymbol{T}\,\mathbf{1} = {} & \alpha\,\boldsymbol{T}\,\boldsymbol{c}_{1M,k}\boldsymbol{T}\,\mathbf{1} \\
& - \frac{\alpha}{d_k}\left(\boldsymbol{T}\,\boldsymbol{\nu}_k\mathbf{1}^{\mathsf{T}}\boldsymbol{T}^{\mathsf{T}}\boldsymbol{c}_{2L,k}\right) \\
& - \alpha\left(\boldsymbol{T}\,\boldsymbol{Q}_{11L,k}\boldsymbol{T}\,\boldsymbol{\nu}_k + \boldsymbol{T}\,\boldsymbol{Q}_{11M,k}\boldsymbol{T}^{\mathsf{T}}\boldsymbol{\mu}_k\right) \\
& + \frac{\alpha}{d_k}\left(\boldsymbol{T}\,\boldsymbol{\nu}_k\left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{Q}_{12L,k} + \boldsymbol{Q}_{12C,k}^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{\nu}_k\right)\right) \\
& - \alpha\,\boldsymbol{T}\,\boldsymbol{Q}_{12L,k}\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1} \\
& + \frac{\alpha\,Q_{22,k}}{d_k}\boldsymbol{T}\,\boldsymbol{\nu}_k\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1} \qquad (31)
\end{aligned}$$

Finally, we calculate for the constant terms of $e_{1,k+1}$,

$$\begin{aligned}
e_{1C,k+1} = {} & \alpha\,e_{1C,k} + \alpha\,Q_{22,k}\gamma_k^2 - 2\,\alpha\,\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{c}_{1C,k} - 2\,\alpha\,\gamma_k c_{2C,k} \\
& + \alpha\left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{Q}_{11C,k}\boldsymbol{\mu}_k + 2\,\boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{Q}_{11L,k}\boldsymbol{\mu}_k + \boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{Q}_{11M,k}\boldsymbol{\nu}_k\right) \\
& + 2\,\alpha\,\gamma\left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{Q}_{12C,k} + \boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{Q}_{12L,k}\right) \qquad (32)
\end{aligned}$$

for the linear terms

$$\begin{aligned}
\boldsymbol{e}_{1L,k+1}\boldsymbol{T}\,\mathbf{1} = {} & \alpha\,\boldsymbol{e}_{1L,k+1}\boldsymbol{T}\,\mathbf{1} + 2\,\alpha\,\gamma_k Q_{22,k}\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1} \\
& + \alpha\left(\boldsymbol{\mu}_k^{\mathsf{T}}\left(\boldsymbol{Q}_{11C,k} + \boldsymbol{Q}_{11C,k}^{\mathsf{T}}\right)\boldsymbol{T}\,\boldsymbol{\nu}_k + \boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\left(\boldsymbol{Q}_{11M,k} + \boldsymbol{Q}_{11M,k}^{\mathsf{T}}\right)\boldsymbol{\nu}_k\right) \\
& + \alpha\left(2\,\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{Q}_{11L,k}\boldsymbol{\mu}_k + 2\,\boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{Q}_{11L,k}\boldsymbol{T}\,\boldsymbol{\nu}_k\right) \\
& + 2\,\alpha\left(\gamma_k\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{Q}_{12L,k} + \gamma_k\boldsymbol{Q}_{12C,k}^{T}\boldsymbol{T}\,\boldsymbol{\nu}_k + \left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{Q}_{12C,k} + \boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{Q}_{12L,k}\right)\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1}\right) \\
& - 2\,\alpha\left(\boldsymbol{\mu}_k^{\mathsf{T}}\boldsymbol{c}_{1L,k}\boldsymbol{T}\,\mathbf{1} + \boldsymbol{c}_{1C,k}^{\mathsf{T}}\boldsymbol{T}\,\boldsymbol{\nu}_k + \boldsymbol{\nu}_k^{\mathsf{T}}\boldsymbol{c}_{1M,k}\boldsymbol{T}\,\mathbf{1}\right) \\
& - 2\,\alpha\left(\gamma_k\boldsymbol{c}_{2L,k}\boldsymbol{T}\,\mathbf{1} + c_{2C,k}\boldsymbol{\delta}_k^{\mathsf{T}}\boldsymbol{T}\,\mathbf{1}\right) \qquad (33)
\end{aligned}$$

and for the quadratic terms

$$
\begin{aligned}
\mathbf{1}^{\top}\boldsymbol{T}^{\top}\boldsymbol{e}_{1M,k+1}\boldsymbol{T}\,\mathbf{1} = {} & \alpha\,\mathbf{1}^{\top}\boldsymbol{T}^{\top}\boldsymbol{e}_{1M,k+1}\boldsymbol{T}\,\mathbf{1} \\
& + \alpha\left(\boldsymbol{\mu}_k^{\top}\boldsymbol{T}\,\boldsymbol{Q}_{11M,k}\boldsymbol{T}^{\top}\boldsymbol{\mu}_k + 2\,\boldsymbol{\mu}_k^{\top}\boldsymbol{T}\,\boldsymbol{Q}_{11L,k}\,\boldsymbol{T}\boldsymbol{\nu}_k + \boldsymbol{\nu}_k^{\top}\boldsymbol{T}^{\top}\boldsymbol{Q}_{11C,k}\boldsymbol{T}\,\boldsymbol{\nu}_k\right) \\
& + 2\,\alpha\,\mathbf{1}^{\top}\boldsymbol{T}^{\top}\boldsymbol{\delta}_k\left(\boldsymbol{\mu}_k^{\top}\boldsymbol{T}\,\boldsymbol{Q}_{12L,k} + \boldsymbol{Q}_{12C,k}^{\top}\boldsymbol{T}\,\boldsymbol{\nu}_k\right) \\
& + \alpha\,Q_{22,k}\mathbf{1}^{\top}\boldsymbol{T}^{\top}\boldsymbol{\delta}_k\boldsymbol{\delta}_k^{\top}\boldsymbol{T}\,\mathbf{1} \\
& - 2\,\alpha\,\mathbf{1}^{\top}\boldsymbol{T}^{\top}\left(\boldsymbol{c}_{1M,k}^{\top}\boldsymbol{T}^{\top}\boldsymbol{\mu}_k + \boldsymbol{c}_{1L,k}^{\top}\boldsymbol{T}\boldsymbol{\nu}_k\right) \\
& - 2\,\alpha\,\mathbf{1}^{\top}\boldsymbol{T}^{\top}\boldsymbol{\delta}_k\boldsymbol{c}_{2L,k}\boldsymbol{T}\,\mathbf{1}
\end{aligned} \tag{34}
$$