Reynolds Stress Model with Elliptic Relaxation

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1 Reynolds Averaged Navier Stokes Equations

General RANS equations:

$$\frac{\partial U_i}{\partial t} + \frac{\partial \left(U_i U_j\right)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} \tag{1a}$$

$$\text{and} \quad \frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial^2}{\partial x_i \partial x_j}\left(U_i U_j + \tau_{ij}\right), \tag{1b}$$

where t and x_i are time and space coordinates, respectively; ν is the kinematic viscosity, ρ is the constant fluid density, and p is the pressure. U_i, p, and τ_{ij} represent Reynolds-averaged velocity $\langle U_i \rangle$, Reynolds-averaged pressure $\langle p \rangle$, and Reynolds stresses τ_{ij}, respectively.

2 Reynolds Stress Transport Model

In the Reynolds stress transport model by Durbin (1993), the following equations for the Reynolds stress and rate of dissipation are solved:

$$\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} = P_{ij} - \varepsilon_{ij} + D_{ij} + \mathcal{R}_{ij} \tag{2}$$

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{C^*_{\varepsilon_1} P - C_{\varepsilon_2}\varepsilon}{T} + \frac{\partial}{\partial x_i}\left[\left(C_\varepsilon T \tau_{ij} + \nu \delta_{ij}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{3}$$

where U_k is the Reynolds averaged velocity; τ_{ij} is the Reynolds stress tensor; and $\varepsilon = \frac{1}{2}\varepsilon_{ii}$ is the rate of dissipation of turbulent kinetic energy; δ_{ij} is the Kronecker delta.

The turbulence stress production term

$$P_{ij} = -\tau_{ik}\frac{\partial U_j}{\partial x_k} - \tau_{jk}\frac{\partial U_i}{\partial x_k} \tag{4}$$

is in closed form, and $P = \frac{1}{2}P_{ij}$. The rate of dissipation ε_{ij} and the diffusion D_{ij} of the turbulent stresses are modeled as:

$$\varepsilon_{ij} = \frac{\varepsilon\tau_{ij}}{k} \tag{5}$$

$$D_{ij} = \frac{\partial}{\partial x_k}\left(\frac{C_\mu T}{\sigma^{(k)}}\tau_{km}\frac{\partial \tau_{ij}}{\partial x_m}\right) + \nu\frac{\partial^2 \tau_{ij}}{\partial x_k} \tag{6}$$

with the time scale T defined by:

$$T = \max\left(\frac{k}{\varepsilon},\ C_T\left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}\right). \tag{7}$$

The pressure-rate-of-strain is obtained from:

$$\mathcal{R}_{ij} = kf_{ij}, \tag{8a}$$

$$L^2\nabla^2 f_{ij} - f_{ij} = -\Pi_{ij}. \tag{8b}$$

where the length scale L is formulated as

$$L = C_L \max\left(\frac{k^{\frac{3}{2}}}{\varepsilon},\ C_\eta\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}\right). \tag{9}$$

The source term Π_{ij} in Equation (8b) is given by Rotta's return to isotropy and isotropization of production terms:

$$\Pi_{ij} = \frac{1 - C_1}{kT}\left(\tau_{ij} - \frac{2}{3}k\delta_{ij}\right) + \frac{C_2}{k}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) \tag{10}$$

Compared to the standard ε models, the coefficient $C^*_{\varepsilon_1}$ in Equation (3) is modified as follows $C^*_{\varepsilon_1} = C_{\varepsilon_1}(1 + a_1 P/\varepsilon)$ to account for anisotropic production terms the wall. The value for all other coefficients in the models above are shown in Table 1.

The wall boundary conditions for f_{ij} are imposed according to Manceau and Hanjalić (2002) as

Table 1: Model constants used in the Reynolds stress model.

C_1	C_2	C_{ε_1}	C_{ε_2}	C_η	C_L	C_μ	C_T	$\sigma^{(\varepsilon)}$	$\sigma^{(k)}$	a_1
1.22	0.6	1.44	1.9	80	0.25	0.23	6	1.65	1.2	0.1

follows:

$$\begin{aligned}
f_{ij}^w &= -\frac{20\nu^2}{\varepsilon}\frac{\tau_{ij}}{y^4} \text{ for } f_{22}^w,\ f_{12}^w \text{ and } f_{23}^w, \\
f_{ij}^w &= -\frac{1}{2} f_{22}^w \text{ for } f_{11}^w,\ f_{33}^w, \\
f_{13}^w &= 0,
\end{aligned} \tag{11}$$

where y is the wall distance of the first cell; indexes 1, 2, and 3 indicate streamwise, wall-normal, and spanwise directions, respectively, based on local coordinate aligned with the wall. The boundary conditions for τ_{ij} and ε are $\tau_{ij} = 0$ and $\varepsilon = 2\nu k/y^2$.

3 Implementation Notes

The following items were observed:

1. Imposing wall boundary conditions for f_{ij} in a local coordinate system with y coordinate (or direction 2) aligned with the wall-normal direction, and the x coordinate aligned with mean flow direction at the cell nearest to wall. (Equation (11)).
2. When estimating $y^+ = y/(\nu/\sqrt{\tau_w})$, a "propagation" is used, where y is wall-distance, and τ_w is the wall-shear stress. The functionality is available in OpenFOAM as class `wallDistData`, which is used in van Driest damping.

The following measures are taken during the implementation to ensure stability:

1. Removed off-diagonal terms for the diffusion coefficient $C_\mu T \tau_{ij}$. This effectively prevents inverse diffusion. (Equation (6))
2. Harmonic interpolation of diffusion coefficient for D_{ij} (effects not fully validated).