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1 ODE

1.1 Definitions

- **Autonomous**: $F(u(t))$ is not an *expl.* func. of time.
- **Non-Autonomous:** $F(t, u(t))$
- **Linear:** $F(t, u(t)) = A(t) \cdot u(t) + C(t)$, hom: $C \equiv 0$
- **Non-Linear:** e.g. $u'(t) = \sin(u(t))$
- **Scalar vs. Systems of ODEs**

1.1.1 Non-Autonomous → Autonomous

$$\boxed{u'(t) = F(t, u(t)) \quad \rightarrow \quad w'(t) = G(w(t))}$$

$$w(t) = [u^T(t), \alpha(t)]^T, \quad w(0) = [u^T(0), \alpha(0)]^T$$
$$G(w) = w'(t) = [u'(t)^T, \alpha'(t)]^T$$

1.2 Initial Value Problem (IVP)

1.2.1 Lipschitz Theorem

If $F(t, u(t))$ is **lipschitz-continuous**, there exists a time interval $[0, T^*]$, in which the IVP has a **unique** solution.

$$\boxed{\|F(t, u) - F(t, u^*)\| \le L \cdot \|u - u^*\|}$$

with any vector norm $\|\cdot\|$ and

$$(u, t), (u^*, t) \in [0, T] \times u, \qquad L > 0$$

"An lc function is limited in how fast it can change."

2 Numerical Methods

$$u'(t) = F(t, u(t))$$
$$u(0) = u_0$$

2.1 Discretization

$$\text{Find } \{u^n\}_{n=0}^N \text{ s.t. } u^n \approx u(t^n) \quad \forall n \in \{0, N\}$$

$$\Delta t := \frac{T}{N} \qquad \text{time step}$$
$$t^n := n \cdot \Delta t \qquad \text{time level}$$

2.2 Quadrature Rules (QR)

Left Rectangle QR

$$\int_a^b f(x)\, dx = (b - a) \cdot f(a)$$

Right Rectangle QR

$$\int_a^b f(x)\, dx = (b - a) \cdot f(b)$$

Midpoint QR

$$\int_a^b f(x)\, dx = (b - a) \cdot f\left(\frac{a + b}{2}\right)$$

Trapezoidal QR

$$\int_a^b f(x)\, dx = (b - a) \cdot \frac{f(a) + f(b)}{2}$$

2.3 Discretization Methods

2.3.1 Forward Euler (FE) - explicit $\mathcal{O}(\Delta t^1)$

$$\boxed{u^{n+1} = u^n + \Delta t \cdot F(t^n, u^n)}$$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

⊕ easy, cheap (computation wise)

⊖ limited stability

2.3.2 Backward Euler (BE) - implicit $\mathcal{O}(\Delta t^1)$

$$\boxed{u^{n+1} = u^n + \Delta t \cdot F\left(t^{n+1}, u^{n+1}\right)}$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

⊕ stable, cheap (if invertible)

⊖ expensive (if !invertible) → Newton's M.

2.3.3 Midpoint - explicit $\mathcal{O}(\Delta t^2)$

$$\boxed{u^{n+1} = u^{n-1} + 2\Delta t \cdot F(t^n, u^n)}$$
$$\boxed{u^{n+\frac{1}{2}} = u^{n-\frac{1}{2}} + \Delta t \cdot F(t^n, u^n)}$$

⊕ more accurate

⊖ memory intensive, "jumpstart" (need for two IV)

2.3.4 Trapezoidal - implicit $\mathcal{O}(\Delta t^2)$

$$\boxed{u^{n+1} = u^n + \Delta t \cdot \frac{F(t^n, u^n) + F\left(t^{n+1}, u^{n+1}\right)}{2}}$$

⊕ more accurate

⊖ expensive (if !invertible) → Newton's M.

2.4 Newton's Method

Needed for implementation of *implicit* methods.

Goal: Find z s.t. $G(z) = 0$

Algorithm:

1. Choose "reasonable" z_0, $(G(z_0) \approx 0)$
2. While $(|G(z_k)| > \varepsilon)$

$$\boxed{z_{k+1} = z_k - \frac{G(z_k)}{G'(z_k)}}$$

3. return z_k

$$G'_{i,j}(\boldsymbol{z}) = \frac{\partial G(\boldsymbol{z})_i}{\partial \boldsymbol{z}_j}$$

This method is **not guaranteed to converge**!
Implement maximal number of iterations.

2.4.1 Example BE

$$u' = F(t, u) := \sin(u)$$

$$u^{n+1} = u^n + \Delta t \cdot F(t^{n+1}, u^{n+1})$$
$$u^{n+1} = u^n + \Delta t \cdot \sin(u^{n+1})$$
$$z = u^n + \Delta t \cdot \sin(z)$$
$$G(z) = z - u^n - \Delta t \cdot \sin(z) = 0$$

3 Error

3.1 Truncation Error T_n

1. write scheme in *consistent* form as: $Q(\dots) = 0$
2. plug exact solution $u(t^n)$ into $Q \rightarrow Q_{T_n} = T_n$

$$T_n = \mathcal{O}(\Delta t^\alpha) = \mathcal{O}(N^{-\alpha})$$

→ Numerical Method is of order α.

consistent form: $u' = F \rightarrow$ plug in approx for u'

Taylor Expansion

Taylor expansion of $f(x)$ about a:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

$$f(\boldsymbol{x}_0) + \frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}\frac{\Delta x_1}{1!} + \frac{\partial f(\boldsymbol{x}_0)}{\partial x_2}\frac{\Delta x_2}{1!} + \frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1^2}\frac{\Delta x_1^2}{2!}$$
$$+ \frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2^2}\frac{\Delta x_2^2}{2!} + 2 \cdot \frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_2}\frac{\Delta x_1 \Delta x_2}{2!} + \dots$$

3.1.1 Example FE

$$\boxed{u^{n+1} = u^n + \Delta t \cdot F(t^n, u^n)} \text{ (scheme)}$$

1. $Q = \frac{1}{\Delta t} \cdot (u^{n+1} - u^n) - F(t^n, u^n) = 0$
2. $T_n = \frac{1}{\Delta t} \cdot (u(t^{n+1}) - u(t^n)) - F(t^n, u(t^n))$

Taylorexpand $u(t^{n+1})$ *about* t^n:

$$u(t^{n+1}) = u(t^n) + u'(t^n) \cdot \Delta t + \frac{u''(t^n)}{2!} \cdot (\Delta t)^2 + \mathcal{O}(\Delta t^3)$$

$$T_n = \frac{u''(t^n)}{2!} \cdot \Delta t + \mathcal{O}(\Delta t^2) \approx C \cdot \mathcal{O}(\Delta t)$$

The truncation error is of first order. $\mathcal{O}(\Delta t^1)$

3.2 Empirical Error

1. Choose $F(u)$ s.t. $F(u) = u'$ with $F(u)$ known.
2. Choose a wide array of values for N.
3. Compute Empirical Error ε_n:

$$\boxed{\varepsilon_n = \left|u^N - u(T)\right|}$$

$$Q\!: \ \exists \alpha \ \text{ s.t. } \ \varepsilon_n = \mathcal{O}(N^{-\alpha}) \ ?$$

3.3 One Step Error L_n

Insert exact solution into *update form*.
If $u^n = u(t^n)$ exactly:

$$\boxed{L_n := \left|u\left(t^{n+1}\right) - u^{n+1}\right|} = \Delta t \cdot T_n$$

3.4 Global Error E_n

$$\boxed{E_n := \sum_{j=0}^{n} L_j} \quad \le \quad \sum_{j=0}^{N} L_j \approx N \cdot \Delta t \cdot T_n$$

$$\mathcal{O}(E_n) = \mathcal{O}(T_n)$$

4 Higher Order Finite Difference NM

4.1 Runge-Kutta-2 (RK-2) $\mathcal{O}(\Delta t^2)$

Midpoint Rule with $u^{n+\frac{1}{2}}$ approximated with FE.

$$\begin{cases} y_1 &= u^n \\ y_2 &= u^n + \frac{\Delta t}{2} \cdot F(t^n, y_1) \\ u^{n+1} &= u^n + \Delta t \cdot F\left(t^{n+\frac{1}{2}}, y_2\right) \\ u^0 &= u_0 \end{cases}$$

$$\boxed{u^{n+1} = u^n + \Delta t \cdot F\left(t^{n+\frac{1}{2}}, u^n + \frac{\Delta t}{2} \cdot F(t^n, u^n)\right)}$$

⊕ fast (faster than trapez.), no extra memory

⊖ limited stability (needs small Δt)

4.2 Runge-Kutta-4 (RK-4) $\mathcal{O}(\Delta t^4)$

$$\begin{cases} y_1 &= u^n \\ y_2 &= u^n + \frac{\Delta t}{2} \cdot F(t^n, y_1) \\ y_3 &= u^n + \frac{\Delta t}{2} \cdot F\left(t^{n+\frac{1}{2}}, y_2\right) \\ y_4 &= u^n + \Delta t \cdot F\left(t^{n+\frac{1}{2}}, y_3\right) \\ u^{n+1} &= u^n + \frac{\Delta t}{6} \cdot \Big[F(t^n, y_1) + 2 \cdot F\left(t^{n+\frac{1}{2}}, y_2\right) \\ &\qquad + 2 \cdot F\left(t^{n+\frac{1}{2}}, y_3\right) + F\left(t^{n+1}, y_4\right)\Big] \\ u^0 &= u_0 \end{cases}$$

4.3 Runge-Kutta-N (RK-N) - s-stage

$$\begin{cases} y_1 &= u_n + \Delta t \cdot \sum_{j=1}^{s} a_{1j} \cdot F(t^{n+c_j}, y_j) \\ y_2 &= u_n + \Delta t \cdot \sum_{j=1}^{s} a_{2j} \cdot F(t^{n+c_j}, y_j) \\ \vdots & \\ y_s &= u_n + \Delta t \cdot \sum_{j=1}^{s} a_{sj} \cdot F(t^{n+c_j}, y_j) \\ u_{n+1} &= u_n + \Delta t \cdot \sum_{j=1}^{s} b_j \cdot F(t^{n+c_j}, y_j) \\ u_0 &= u_0 \end{cases}$$

4.3.1 Butcher Tableau

Any RK method is uniquely described by

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

Consistency $(\Delta t \to 0 \Rightarrow L_n \to 0)$

Consistent iff the following conditions hold.

$$\sum_{j=1}^{s} b_j = 1 \qquad \sum_{j=1}^{s} a_{ij} = c_i,\ \forall i$$

Accuracy/ Order

- At *least* **second order** if:

$$\sum_{j=1}^{s} b_j c_j = \frac{1}{2}$$

- At *least* **third order** if:

$$\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3} \qquad \sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$$

- γ-order accuracy

$\to$ $\gamma < 5$ at least γ stages necessary

$\to$ $\gamma \geq 5$ strictly more than γ stages necessary

Explicitness

Explicit iff $A = a_{ij}$ is *strictly lower triangular.*

$$a_{ij} = 0 \quad \forall j \geq i$$

Time marching theme.

Diagonally Implicit RK (DIRK)

DIRK iff $A = a_{ij}$ is *lower triangular.*

$$a_{ij} = 0 \quad \forall j > i$$

5 Poisson Equation (PE)

$$\boxed{-\Delta u = f}$$

5.1 1D PE Dirichlet BC

$$\begin{cases} -u_{xx} &= f(x) \\ u(0) &= u(1) = 0 \end{cases}$$

$$u(x) = \int_0^1 G(x,y)\cdot f(y)\cdot dy$$

$$G(x,y) = \begin{cases} y(1-x) & 0 \leq y \leq x \\ x(1-y) & x \leq y \leq 1 \end{cases}$$

5.2 1D PE FDM Dirichlet BC

$$\Delta x = \frac{1}{N+1}$$

$$x_0 = 0, \quad x_{N+1} = 1, \quad x_i = i\cdot\Delta x, \qquad i = 1,\dots,N$$

$$\boxed{u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2}}$$

Plugging this approximation into the PE, it follows that:

$$-u(x_{i-1}) + 2u(x_i) - u(x_{i+1}) = \Delta x^2 \cdot f(x_i)$$

With BC $u(0) = u(N+1) = 0$ we write:

$$\boxed{A\cdot U = F}$$

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}}_{=:\mathcal{A}} \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ \vdots \\ u(x_N) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_N) \end{bmatrix} \Delta x^2$$

5.3 2D PE FDM Dirichlet BC

$$\Delta x = \frac{1}{N_x+1} \qquad \Delta y = \frac{1}{N_y+1}$$

$$x_0 = 0, \quad x_{N_x+1} = 1, \quad x_i = i\cdot\Delta x, \quad i = 1,\dots,N_x$$

$$y_0 = 0, \quad y_{N_y+1} = 1, \quad y_j = j\cdot\Delta y, \quad j = 1,\dots,N_y$$

Using Midpoint QR twice (central finite difference):

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

Plugging approximation for derivative into PE:

$$-\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} - \frac{u_{1,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = f$$

With $N_x = N_y = N$ and BC $u|_{\partial\Omega} \equiv 0$ we write:

$$\boxed{A\cdot U = F}$$

$$\begin{bmatrix} B & -\mathbb{I} & 0 & 0 & \dots & 0 \\ -\mathbb{I} & B & -\mathbb{I} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\mathbb{I} & B & -\mathbb{I} \\ 0 & \dots & 0 & 0 & -\mathbb{I} & B \end{bmatrix} \cdot U = \begin{bmatrix} f(x_1,y_1) \\ f(x_1,y_2) \\ \vdots \\ \vdots \\ f(x_N,y_N) \end{bmatrix} \Delta x^2$$

$$B := \begin{bmatrix} 4 & -1 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 4 & -1 \\ 0 & \dots & 0 & 0 & -1 & 4 \end{bmatrix} \in \mathbb{R}^{N\times N}$$

5.4 1D PE FEM

5.4.1 Variational Formulation (VF) for 1D PE

$$\begin{cases} u:\Omega \to \mathbb{R} & \Omega := [0,1] \\ u|_{\partial\Omega} \equiv 0 \end{cases}$$

$$J(u) = \frac{1}{2}\int_\Omega |u'(x)|^2 dx - \int_\Omega f(x)u(x)dx$$

Find $u = \arg\min_{v\in H_0^1} J(v)$.

Precise VF for 1D PE

Given $f \in L^2([0,1])$, find u $\in H_0^1([0,1])$ s.t. $\forall v \in H_0^1([0,1])$,

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx.$$

Definitions

$H_0{}^1([0,1]) :=$

$$\left\{ v[0,1] \to \mathbb{R} : v(0) = v(1) = 0 \text{ and } \int_0^1 |v'(x)|^2 dx < \infty \right\}$$

If continuous and piecewise smooth then mostly $u \in H_0^1$

Important Norms:

$$\|u\|_{H_0^1([0,1])} := \left(\int_0^1 |u'(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\|u\|_{L^2([0,1])} := \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}$$

Poincaré Inequality (PI)

$$\|u\|_{L^2([0,1])} \leq \|u\|_{H_0^1([0,1])}$$

PDE $\to$ VF

1. let $v \in H_0^1$
2. multiply PDE with v
3. integrate over $[0,1]$
4. integration by parts + Zero-Dirichlet BC
5. Find $u \in H_0^1(\Omega)$ s.t. $\forall v \in H_0^1(\Omega)$, $(u', v') = (f, v)$.

5.4.2 FEM Formulation

- Discretize domain as in (5.2) with $h = \Delta x$.
- Vectorspace $V^h \subseteq V$ is N-dimensional.

$$V^h = \Big\{ w : [0,1] \to R : w(0) = w(1) = 0, w \text{ continuous}, \; w|_{[x_i, x_{i+1}]}\ i = 1,\dots N \text{ is linear} \Big\}$$

- The Hat Functions form a basis of V^h.

$$\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & \text{otw} \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}) \\ 0 & \text{otw} \end{cases}$$

- Discrete Variational Formulation (DVF)

$$(u_h, v)_{H_0^1(\Omega)} = \boxed{(u_h', v') = (v, f)} \qquad u_h, v \in V^h$$

$$u_h = \sum_{i=1}^N u_i\cdot\phi_i(x) \qquad v = \sum_{j=1}^N v_j\cdot\phi_j(x)$$

Inserting the previous into the DVF we get

$$\sum_{i=1}^N u_i\cdot(\phi_i'(x), \phi_j'(x)) = (f, \phi_j) \qquad \forall j \in [1,\dots,N],$$

which can be written in matrix notation:

$$\boxed{A\cdot U = F}$$

$$F_j = (f, \phi_j) = h\cdot f_j$$

$$A_{ij} = (\phi_i', \phi_j') = A_{ji}$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}$$

Error Analysis (1D & 2D)

$$\mathcal{O}(h^2) = \|u_h - u\|_{L^2} \quad \leq \quad \|u_h - u\|_{H_0^1} = \mathcal{O}(h)$$

1D: $\mathcal{O}(h^2) = \mathcal{O}(N^{-2})$ **2D:** $\mathcal{O}(h^2) = \mathcal{O}(N^{-1})$

higher dimension $\Rightarrow$ slower convergence w.r.t. N

5.5 2D PE FEM

$$(u,v)_{H_0^1(\Omega)} = (f,v)_{L^2(\Omega)}$$

5.5.1 $H_0^1(\Omega)$

$$H_0^1(\Omega) = \left\{ w : \Omega \to \mathbb{R} : w|_{\partial\Omega} = 0, \int_\Omega \|\nabla w(x)\|^2 dx < \infty \right\}$$

$$\|w\|_{H_0^1(\Omega)} := \left(\int_\Omega \|\nabla w(x)\|^2 dx\right)^{\frac{1}{2}}$$

$$(u,v)_{H_0^1(\Omega)} := \int_\Omega \langle \nabla u(x), \nabla v(x)\rangle\, dx$$

5.5.2 $L^2(\Omega)$

$$L^2(\Omega) = \left\{ w : \Omega \to \mathbb{R} : \int_\Omega w^2(x)\, dx < \infty \right\}$$

$$\|w\|_{L^2(\Omega)} := \left(\int_\Omega w^2(x)\, dx\right)^{\frac{1}{2}}$$

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\, dx$$

5.5.3 1^{st} Green Identity

$$\int_\Omega \langle \nabla u, \nabla v\rangle\, \mathrm{d}x = \int_{\partial\Omega} \langle v\nabla u, n\rangle\, \mathrm{d}O - \int_\Omega v\cdot\Delta u\, \mathrm{d}x$$

5.5.4 Divergence Theorem

$$\int_\Omega \nabla u\, \mathrm{d}x = \int_{\partial\Omega} \langle u, n\rangle\, \mathrm{d}O$$

5.5.5 Finite Element Formulation

- $\mathcal{T}^h = \{K_i\}_{i=1}^M$ is a **triangulation** with M triangles.
- $h = \max_{K\in\mathcal{T}^h} \text{diam}(K)$; $\text{diam}(K)$ being longest edge.
- $\widetilde{\mathcal{N}}^h = \{N_i\}_{i=1}^N$ are **interior nodes** of $\mathcal{T}^h$, $= \{x\in\Omega : x \text{ is a vertex for some } K\in\mathcal{T}^h, x\notin\partial\Omega\}$
- $V^h = \{v:\Omega\to\mathbb{R} : v \text{ cont}, v|_K \text{ linear } \forall K, v|_{\partial\Omega}=0\}$
- Hat Functions form a basis of V^h, $\dim(V^h) = N$

$$\phi_i(\mathcal{N}_j) = \delta_{ij} \quad \forall \mathcal{N}_j\in\widetilde{\mathcal{N}}^h$$

$$\to \quad u(x) = \sum_{i=1}^N u_i\phi_i(x), \quad u_i = u(\mathcal{N}_i)$$

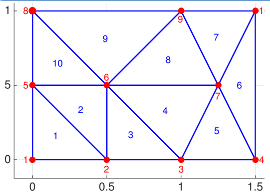
- The VF then becomes

$$\underbrace{\sum_{i=1}^N u_i}_{U_i} \cdot \underbrace{\int_\Omega \langle\nabla\phi_i,\nabla\phi_j\rangle dx}_{A_{ij}} = \underbrace{\int_\Omega f(x)\phi_j(x)dx}_{F_i}$$

which can be written as $A\cdot U = F$, with A as in 5.3.

6 FEM Implementation

6.1 Generate Mesh



$$Z = \begin{bmatrix} 0 & \frac{1}{2} & 1 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & 2 & \cdots \\ 2 & 5 & 3 & \cdots \\ 5 & 6 & 6 & \cdots \end{bmatrix}$$

$Z\in\mathbb{R}^{2\times N}$: $\quad Z(:,j)=\mathcal{N}_j$ (vertex coords)

$T\in\mathbb{R}^{3\times M}$: $\quad T(:,i) =$ vertices of triangle i

6.2 Stiffness Matrices and Load Vectors

Using **Parametric Elements**, $\widehat{K}$ can be mapped to K using the mapping $\Phi_K:\widehat K\to K$.

$$x = \Phi_K(\widehat x) = (\mathcal{N}_b-\mathcal{N}_a \quad \mathcal{N}_c-\mathcal{N}_a)\cdot\widehat x + \mathcal{N}_a = J_K\cdot\widehat x+\mathcal{N}_a$$

J_K being the Jacobian of Φ_K.

Local Element Stiffness Matrix

$$A^K_{\alpha,\beta} = \int_K \langle\nabla\phi_\alpha,\nabla\phi_\beta\rangle \, dx = \int_{\widehat K}\langle J_K^{-T}\widehat\nabla\widehat\phi_{\widehat\alpha} \, , \, J_K^{-T}\widehat\nabla\widehat\phi_{\widehat\beta}\rangle \, |\det(J_K)| \, d\widehat x$$

$$\boxed{\nabla\phi_\alpha(x) = J_K^{-T}\widehat\nabla\widehat\phi_\alpha(\widehat x)}$$

Local Element Load Vector

$$F^K_\alpha = \int_K f(x)\cdot\phi_\alpha(x)dx = \int_{\widehat K} f(\Phi_K(\widehat x))\cdot\widehat\phi_\alpha(\widehat x) \, |\det(J_K)| \, d\widehat x$$

Local Shape Functions for linear approach

$$\widehat\phi_1(\widehat x) = 1-\widehat x-\widehat y \qquad \widehat\phi_2(\widehat x) = \widehat x \qquad \widehat\phi_3(\widehat x) = \widehat y$$

6.3 Assembly

```
A = zeros(N,N); F = zeros(N);
for m = 1 : M,
  Approximate A^Km , F^Km as above
  for alpha = 1 : 3,
    F(T(alpha,m)) += F_alpha^Km
    for beta = 1:3,
     A(T(alpha,m),T(beta,m)) += A_alpha,beta^Km
    end
  end
end
```

6.3.1 Homogeneous BC

In case of homogeneous boundary conditions, all contributions from *boundary nodes* should be *skipped.*

```
compute A and F from above
compute InteriorNodes
A = A(InteriorNodes,InteriorNodes)
F = F(InteriorNodes)
```

6.4 Solving $A\cdot U = F$

6.4.1 Inhomogeneous BC

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

Let $\tilde g:\Omega\to\mathbb{R}$ s.t. $\tilde g(\partial\Omega) = g$ and

$$v = u - \tilde g \quad \Rightarrow \quad v|_{\partial\Omega} = 0.$$

Solve

$$\begin{cases} -\Delta v = f - \Delta g \\ v|_{\partial\Omega} \equiv 0. \end{cases}$$

$$\boxed{u = v + g}$$

7 Heat Equation (HE) / Parabolic PDE

$$\begin{cases} u_t - u_{xx} = 0, & (0,1)\times(0,T) \\ u(x,0) = u_0(x), & (0,1) \quad \text{IC} \\ u(0,t) = u(1,t) = 0, & (0,T) \quad \text{BC} \end{cases}$$

7.1 Exact Solution

By *separation of variables* with the ansatz

$$u(x,t) = \mathcal{T}(t)\cdot\mathcal{X}(x)$$

we obtain the exact solution

$$u(x,t) = \sum_{k=1}^\infty u_k^0\cdot\sin(k\pi x)\cdot e^{-(k\pi)^2 t}$$

$$u_k^0 = 2\int_0^1 u_0(x)\sin(k\pi x)\,dx, \quad k\in\mathbb{N}$$

Error due to finite truncation of Fourier sine series.
Error due to use of QR to approximate integrals.

7.2 Energy Estimates / Props of HE Sols

Let u be a solution to the HE. We define the energy function $\mathcal{E}$:

$$\mathcal{E}(t) := \frac{1}{2}\int_0^1 |u(x,t)|^2\,dx$$

7.2.1 Stability

It can be shown that $\mathcal{E}(t)\le\mathcal{E}(0)$, since

$$\frac{d\mathcal{E}}{dt} = -\int_0^1 u_x^2\,dx \le 0.$$

Therefore solutions to the HE are stable.

7.2.2 Uniqueness

Let $u,\bar u$ be solutions to the HE, and let $w := u-\bar u$. Using the knowledge from 7.2.1 it follows, that

$$\int_0^1 w^2(x,t)\,dx \le \int_0^1 w^2(x,0)\,dx$$

$$\Rightarrow \int_0^1 w^2(x,t)\,dx \le 0 \qquad \text{(Using IC)}$$

$$\Rightarrow w(x,t)\equiv 0 \quad \iff \quad u(x,t) = \bar u(x,t)$$

Hence, the HE has a unique solution.

7.3 Maximum Principle

Let u be a solution of the HE. Then for all $x\in[0,1]$ and for all $t\in[0,T]$ it holds that

$$\min\left(0, \min_{\tilde x}(u_0(\tilde x))\right) \le u(x,t) \le \max\left(0, \max_{\tilde x}(u_0(\tilde x))\right)$$

The max/min must occur at boundaries (0) or at $t=0$.

7.4 Explicit FD (EFD) $\mathcal{O}(\Delta t^1 + \Delta x^2)$

Discretize domain into $(N+2)\times(M+2)$ equally spaced points.

$$\Delta x = \frac{1}{N+1} \qquad \Delta t = \frac{T}{M+1}$$

$$x_0 = 0, \quad x_{N+1} = 1, \quad x_i = i\cdot\Delta x, \quad i = 1,\dots,N$$
$$t^0 = 0, \quad t^{M+1} = T, \quad t^n = n\cdot\Delta t, \quad n = 1,\dots,M$$

Discretize Solution: $\quad u_i^n = u(x_i,t^n)$

Discretizing the derivatives, we get the FD Scheme:

$$\underbrace{\frac{u_i^{n+1}-u_i^n}{\Delta t}}_{\text{FE}} - \underbrace{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}_{\text{central finite difference}} = 0$$

Defining $\lambda = \frac{\Delta t}{\Delta x^2}$, delivers $U^{n+1} = (\mathbb{I}-\lambda\cdot\mathcal{A})\cdot U^n$

$$\boxed{u_i^{n+1} = (1-2\lambda)\cdot u_i^n + \lambda\cdot u_{i+1}^n + \lambda\cdot u_{i-1}^n}$$

7.4.1 Stability EFD

Courant-Friedrichs-Lewy (CFL) condition:

$$\frac{\Delta t}{\Delta x^2} =: \lambda \le \frac{1}{2}$$

EFD scheme is *stable* iff CFL condition is satisfied.

7.4.2 Discrete Maximum Principle EFD

$$\boxed{\min(0,\min_j u_j^0) \le u_j^{n+1} \le \max(0,\max_j u_j^0)}$$

Consider the FD scheme with $\lambda\le\frac{1}{2}$.

$$u_i^{n+1} = (1-2\lambda)\cdot u_i^n + \lambda\cdot u_{i+1}^n + \lambda\cdot u_{i-1}^n$$

Let $\bar u_i^n = \max(u_{i-1}^n, u_i^n, u_{i+1}^n)$, and by definition:

$$u_{j-1}^n \le \bar u_j^n, \quad u_j^n \le \bar u_j^n, \quad u_{j+1}^n \le \bar u_j^n$$

Therefore one can write:

$$u_i^{n+1} \le (1-2\lambda)\cdot\bar u_i^n + \lambda\cdot\bar u_i^n + \lambda\cdot\bar u_i^n \le \bar u_i^n = \max(u_{i-1}^n, u_i^n, u_{i+1}^n)$$

7.4.3 Truncation Error EFD T_j^n

$$T_j^n = \frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

$$\boxed{|T_j^n| \le C\left(\Delta t + \Delta x^2\right)}$$

7.5 Implicit FD (IFD) $\mathcal{O}(\Delta t^1 + \Delta x^2)$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

The given implicit FD scheme reduces to a matrix eq.:

$$(\mathbb{I}+\lambda\cdot\mathcal{A})\cdot U^{n+1} = \boxed{A\cdot U^{n+1} = U^n}$$

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 & \dots & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\lambda & 1+2\lambda & -\lambda \\ 0 & \dots & 0 & 0 & -\lambda & 1+2\lambda \end{bmatrix}$$

7.5.1 Stability IFD

The IFD scheme is termed *unconditionally stable.*

7.6 Crank-Nicolson (CN) $\mathcal{O}(\Delta t^2 + \Delta x^2)$

$$u_{xx} \approx \frac{u_{xx}(x_j,t^n) + u_{xx}(x_j,t^{n+1})}{2}$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{2\Delta x^2} + \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{2\Delta x^2}$$

Using $\lambda = \frac{\Delta t}{\Delta x^2}$ we can write the matrix form:

$$\left(\mathbb{I}+\frac{\lambda}{2}\mathcal{A}\right)U^{n+1} = \boxed{A\cdot U^{n+1} = B\cdot U^n} = \left(\mathbb{I}-\frac{\lambda}{2}\mathcal{A}\right)U^n$$

$$F_i^n = \frac{\lambda}{2}\cdot u_{i-1}^n + (1-\lambda)\cdot u_i^n + \frac{\lambda}{2}\cdot u_{i+1}^n$$

$$A = \begin{bmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & 0 & \dots & 0 \\ -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} \\ 0 & \dots & 0 & 0 & -\frac{\lambda}{2} & 1+\lambda \end{bmatrix}$$

Flip written signs in A to obtain B.
Unconditionally stable
Conditionally max/min principle verifying

8 Linear Transp. Eq. (LTE) / Hyperb. PDE

$$\begin{cases} u_t + a(x,t)\cdot u_x = 0 & \forall (x,t) \in \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R} \end{cases}$$

8.1 Method of Characteristics

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + \frac{dx(t)}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0$$

Assume given $x(t)$ along which $u(x,t)$ is constant.

$$\frac{d}{dt}u(x(t),t) = 0$$
$$u_t + \dot{x}(t)\cdot u_x = 0$$

Since $u(x,t)$ also satisfies the LTE:

$$\begin{cases} \dot{x}(t) & = a(x,t) \\ x(0) & = x_0. \end{cases}$$

$x(t)$ is called a *characteristic curve.*
From the initial assumption it follows, that:

$$u(x(t),t) = u(x(0),0) = u(x_0,0) = u_0(x_0).$$

Example

$$\begin{cases} \dot{x}(t) = a, & a \in \mathbb{R} \\ x(0) = x_0 \end{cases}$$

$$x(t) = at + x_0 \qquad \to x_0 = x(t) - at$$

$$u(x,t) = u_0(x_0) = u_0(x - at)$$

8.2 Centered Finite Difference (C-FD)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a(x_i,t^n)\cdot\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

The C-FD is *unconditionally unstable* since for $a > 0$ (flowing to the right), we take information from left and right.

8.3 Upwind Method

$$a^+ := \max\{a,0\} \quad \textit{positive part}$$
$$a^- := \min\{a,0\} \quad \textit{negative part}$$

$$\underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t}}_{\text{FE}} + a^+\cdot\underbrace{\frac{u_i^n - u_{i-1}^n}{\Delta x}}_{\text{BE}} + a^-\cdot\underbrace{\frac{u_{i+1}^n - u_i^n}{\Delta x}}_{\text{FE}} = 0$$

Using $a^+ = |a| + a$ and $a^- = a - |a|$ we rewrite:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{|a|}{2\Delta x}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

8.3.1 Stability - CFL Condition

The upwind scheme is *conditionally stable.*

$$|a|\frac{\Delta t}{\Delta x} \le 1$$

8.3.2 Trapezoidal Timestepping

Trapezoidal timestepping from (2.3.4):

$$\frac{u_t(t^n) + u_t(t^{n+1})}{2} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

Using the linearity of the LTE the corresponding upwind method follows through superposition as:

$$\frac{u_t(t^n) + u_t(t^{n+1})}{2} + a\cdot\frac{u_x(t^n) + u_x(t^{n+1})}{2} = 0.$$

Approximate $u_x(t^n)$ and $u_x(t^{n+1})$ with FE/BE depending on a:
For positive (negative) a, use BE (FE).