Control Systems 2 Cheatsheet

Noa Sendlhofer - nsendlhofer@ethz.ch

Version: May 14, 2024
Template by Micha Bosshart

1 Discrete Time

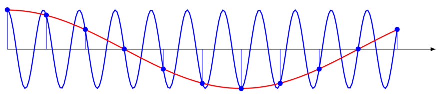
1.1 Sampling

T_s: Sampling Time
ω_s: Sampling frequency

$$\omega_s = \frac{2\pi}{T_s}$$

1.2 Aliasing

$$y_1[k] = cos(\omega k T_s), \qquad k = 0, 1, 2$$
$$y_2[k] = cos((\omega + n\frac{2\pi}{T_s})kT_s), \qquad n = 0, 1, 2$$
$$= cos(\omega k T_s + \cancel{n 2\pi k}) = y_1[k]$$



1.2.1 Nyquist-Shannon Sampling theorem

$$f_N = \frac{1}{2T_s}\ [\text{Hz}] \quad \text{or} \quad \omega_N = \frac{\pi}{T_s}\left[\frac{\text{rad}}{\text{s}}\right]$$

No aliasing if $\omega < \omega_N$!

1.3 DT State Space Representation

$$x[k+1] = A_d x[k] + B_d u[k]$$
$$= e^{AT}x[k] + \left(\int_0^T e^{A(T-\tau)}d\tau\right) Bu[k]$$
$$y[k] = C_d x[k] + D_d u[k]$$
$$= Cx[k] + Du[k]$$

If A is invertible: $B_d = A^{-1}(A_d - I)B$

1.3.1 Homogeneous response

$$x[0] = x_0, \qquad u[k] = 0$$

$$x[k+1] = Ax[k], \Rightarrow x[k] = A^k x_0$$
$$y[k] = Cx[k]$$
$$= CA^k x_0$$

$$A^k = (T\Lambda T^{-1})^k = T\Lambda^k T^{-1}$$

$$\lim_{k\to+\infty} A^k = 0 \implies |\lambda_i| < 1$$

1.3.2 Forced response

$$x[1] = Bu[0],$$
$$x[2] = ABu[0]Bu[1], \ \ldots,$$
$$x[k] = \sum_{i=0}^{k-1} A^{k-i-1}Bu[i],$$

$$y[k] = \underbrace{CA^k x_0}_{\text{Homogeneous}} + \underbrace{C\sum_{i=0}^{k-1} A^{k-i-1}Bu[i] + Du[k]}_{\text{Forced}}$$

1.4 DT Transfer Function

$$u[k] = u_0 z^k = u_0 e^{ksT} = u(kT)$$

$$y[k] = C\sum_{i=0}^{k-1} A^{k-i-1}Bu_0 z^i + Du_0 z^k$$
$$= \underbrace{CA^k(x_0 - C(zI-A)^{-1}Bu_0)}_{\text{Transient}}$$
$$+ \underbrace{C(zI-A)^{-1}Bu_0 z^k + Du_0 z^k}_{\text{Steady-state}}$$

$$\lim_{k\to+\infty} A^k = 0 \quad \Rightarrow \quad y[k] \approx [C(zI-A)^{-1}B + D]u[k]$$

$$y[k] \approx G(z)u[k], \quad G(z) := C(zI-A)^{-1}B + D$$

1.5 Approximation Methods

1.5.1 Emulation

Exact	$s = \frac{1}{T_s}\cdot \ln(z)$	$z = e^{s\cdot T_s}$
Euler forward	$s = \frac{z-1}{T_s}$	$z = s\cdot T_s + 1$
Euler backward	$s = \frac{z-1}{z\cdot T_s}$	$z = \frac{1}{1 - s\cdot T_s}$
Tustin	$s = \frac{2}{T_s}\cdot\frac{z-1}{z+1}$	$z = \frac{1 + s\cdot\frac{T_s}{2}}{1 - s\cdot\frac{T_s}{2}}$

2 System Properties

2.1 Similarity Transformation

$$\begin{cases} x^+ = Ax + Bu \\ y = Cx + Du \end{cases} \implies \begin{cases} \tilde{x}^+ = (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y = (CT)\tilde{x} + Du \end{cases}$$

2.1.1 Modal decomposition

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0) \qquad x(t) = \sum_{i=1}^{n} e^{\lambda_i t}\tilde{x}_i(0)v_i$$

2.2 Reachability

$$\mathcal{R} := \left[A^{n-1}B|\ldots|AB|B\right] \in \mathbb{R}^{n\times n\cdot m} \qquad U := \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-1] \end{bmatrix}$$

$\Rightarrow x[n] = \mathcal{R}U$

The systen is reachable if and only if $\mathcal{R}$ has full row rank n

2.3 Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad Y = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} \qquad Y = \mathcal{O}x[0]$$

The systen is observable if and only if $\mathcal{O}$ has full column rank n

2.4 Controllability

A system is controllable if, for any initial condition x_0, there exists a control input u that brings the state x to 0 in finite time.
For CT Systems: Controllability = Reachability
For DT Systems: A is invertible $\Rightarrow$ Controllability = Reachability

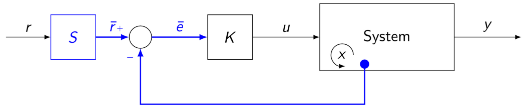
2.5 Kalman Decomposition

$$x^+ = \begin{bmatrix} \Lambda_{r\bar{o}} & 0 & 0 & 0 \\ 0 & \Lambda_{ro} & 0 & 0 \\ 0 & 0 & \Lambda_{\bar{r}\bar{o}} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}o} \end{bmatrix} x + \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix} x + Du$$

- **Stabilizability**
  A system is said to be stabilizable if all unstable modes are reachable
- **Detectability**
  A system is said to be detectable if all unstable modes are observable

3 State Feedback

3.1 State Feedback



$$x^+ = (A - BK)x + BKSr$$
$$= (A - BK)x + B\bar{N}r, \qquad \bar{N} = KS$$
$$y = Cx + (D = 0)$$

$$G^{cl}_{yr}(s) = C(sI - A + BK)^{-1}BKS$$
$$= C(sI - A - BK)^{-1}B\bar{N}$$

3.1.1 K - Direct Method

$$p^*_{cl} = \prod_{i=1}^{n}(s + \lambda_i), \quad K = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$$
$$p_{cl} = det(sI - A + BK) = p^*_{cl}$$

3.1.2 K - Ackermann's Formula

$$K = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \mathcal{R}^{-1} p^*_{cl}(A)$$

3.1.3 S

No steady-state error: $\Rightarrow G^{cl}_{yr}(0) = 1$

$$\Longrightarrow \bar{N} = -(C(A - BK)^{-1}B)^{-1}, \quad S = K^{-1}\bar{N}$$

3.2 LQR

LQR guarantees:

- phase margin $\geq 60°$
- gain margin $(\frac{1}{2}, +\infty)$

3.2.1 Continuous Time

$$\min_K J(x, u) = \int_0^{+\infty} [x(t)^T Q x(t) + u(t)^T R u(t)]dt,$$

s.t.: $\dot{x}(t) = Ax(t) + Bu(t),$
$u(t) = -Kx(t)$

soln.: $0 = A^TP + PA + Q - PBR^{-1}B^TP$
$K = R^{-1}B^TP$

3.2.2 Discrete Time

$$\min_K J(x, u) = \sum_{k=0}^{+\infty}(x[k]^T Q x[k] + u[k]^T R u[k]),$$

s.t.: $x[k+1] = Ax[k] + Bu[k],$
$u[k] = -Kx[k]$

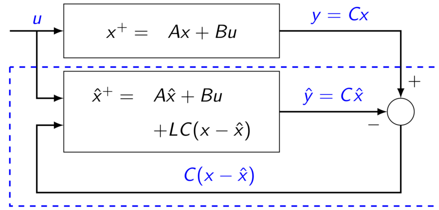
soln.: $P = A^TPA - (A^TPB)(B'PB + R)^{-1}(B^TPA) + Q$
$K = (R + B^TPB)^{-1}B^TPA$

3.2.3 LQR Servo

$$\begin{bmatrix} x \\ \epsilon \end{bmatrix}^+ = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

4 State Estimation

4.1 Luenberger Observer



Luenberger Observer

$$\hat{x}^+(t) = (A - LC)\hat{x}(t) + Bu(t) + Ly(t)$$
$$\hat{y}(t) = C\hat{x}(t)$$

$\Rightarrow$ Exactly the same as finding a control gain K

$$L = p^*_{cl}(A)\mathcal{O}^{-}1[0, \ldots, 0, 1]^T$$

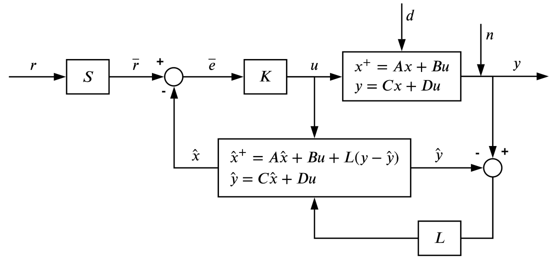
4.2 LQE

$$Q = \mathbb{E}[w(t)w(t)^T], \quad R = \mathbb{E}[n(t)n(t^T)], \quad \forall t \geq 0$$

Problem: Find L, such that the steady-state covariance of the state error is minimized.
Solution:

$$0 = AY + YA^T - YC^TR^{-1}CY + Q$$
$$L = -YC^TR^{-1}$$

5 Dynamic Output Feedback



5.1 LQG

$$\begin{bmatrix} x \\ \eta \end{bmatrix}^+ = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} BKS \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ \eta \end{bmatrix}$$

5.1.1 LQG Servo

$$\begin{bmatrix} x \\ x_I \\ \eta \end{bmatrix}^+ = \begin{bmatrix} A - BK & -BK_I & BK \\ -C & 0 & 0 \\ 0 & 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ x_I \\ \eta \end{bmatrix} + \begin{bmatrix} BKS \\ I \\ 0 \end{bmatrix} r$$

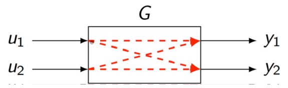
$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \\ \eta \end{bmatrix}$$

5.1.2 LQG Stability

LQG guarantees closed loop stability, but the margins can be arbitrarily small.

6 MIMO

6.1 Transfer Function



$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad G(s) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

Push through identity

$$G_1(I + G_2 G_1)^{-1} = (I + G_1 G_2)^{-1} G_1$$

MIMO state space to tf

$$G(s) = C(sI - A)^{-1} B + D$$

6.2 State Space

$$x^+ = \underbrace{A}_{n\times n} \underbrace{x}_{n\times 1} + \underbrace{B}_{n\times m} \underbrace{u}_{m\times 1}$$

$$y = \underbrace{C}_{l\times n} \underbrace{x}_{n\times 1} + \underbrace{D}_{l\times m} \underbrace{u}_{m\times 1}$$

- $x \in \mathbb{R}^n$, where n is the order of the system
- $u \in \mathbb{R}^m$, where m is the number of inputs
- $y \in \mathbb{R}^l$, where l is the number of outputs