Control Systems 2 Cheatsheet

Noa Sendlhofer - nsendlhofer@ethz.ch

Version: June 17, 2024

Template by Micha Bosshart

1 Discrete Time

1.1 Sampling

T_s: Sampling Time

ω_s: Sampling frequency

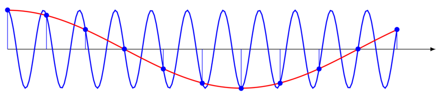
$$\omega_s = \frac{2\pi}{T_s}$$

1.2 Aliasing

$$y_1[k] = cos(\omega k T_s), \qquad k = 0,1,2$$

$$y_2[k] = cos((\omega + n\frac{2\pi}{T_s})kT_s), \qquad n = 0,1,2$$

$$= cos(\omega k T_s + \cancel{n2\pi k}) = y_1[k]$$



1.2.1 Nyquist-Shannon Sampling theorem

$$f_N = \frac{1}{2T_s}\,[\text{Hz}] \quad \text{or} \quad \omega_N = \frac{\pi}{T_s}\left[\frac{\text{rad}}{\text{s}}\right]$$

No aliasing if $\omega < \omega_N$!

1.3 DT State Space Representation

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ &= e^{AT}x[k] + \left(\int_0^T e^{A(T-\tau)}d\tau\right) Bu[k] \\ y[k] &= C_d x[k] + D_d u[k] \\ &= Cx[k] + Du[k] \end{aligned}$$

If A is invertible: $B_d = A^{-1}(A_d - I)B$

1.3.1 Homogeneous response

$$x[0] = x_0, \qquad u[k] = 0$$

$$\begin{aligned} x[k+1] &= Ax[k], \Rightarrow x[k] = A^k x_0 \\ y[k] &= Cx[k] \\ &= CA^k x_0 \end{aligned}$$

$$A^k = (T\Lambda T^{-1})^k = T\Lambda^k T^{-1}$$

$$\lim_{k\to+\infty} A^k = 0 \implies |\lambda_i| < 1$$

1.3.2 Forced response

$$x[1] = Bu[0],$$
$$x[2] = ABu[0]Bu[1], \ \ldots,$$
$$x[k] = \sum_{i=0}^{k-1} A^{k-i-1}Bu[i],$$

$$y[k] = \underbrace{CA^k x_0}_{\text{Homogeneous}} + \underbrace{C\sum_{i=0}^{k-1} A^{k-i-1}Bu[i] + Du[k]}_{\text{Forced}}$$

1.4 DT Transfer Function

$$u[k] = u_0 z^k = u_0 e^{ksT} = u(kT)$$

$$\begin{aligned} y[k] &= C\sum_{i=0}^{k-1} A^{k-i-1}Bu_0 z^i + Du_0 z^k \\ &= \underbrace{CA^k(x_0 - C(zI-A)^{-1}Bu_0)}_{\text{Transient}} \\ &+ \underbrace{C(zI-A)^{-1}Bu_0 z^k + Du_0 z^k}_{\text{Steady-state}} \end{aligned}$$

$$\lim_{k\to+\infty} A^k = 0 \quad \Rightarrow \quad y[k] \approx [C(zI-A)^{-1}B + D]u[k]$$

$$y[k] \approx G(z)u[k], \quad G(z) := C(zI-A)^{-1}B + D$$

1.5 Approximation Methods

1.5.1 Emulation

Exact	$s = \frac{1}{T_s}\cdot \ln(z)$	$z = e^{s\cdot T_s}$
Euler forward	$s = \frac{z-1}{T_s}$	$z = s\cdot T_s + 1$
Euler backward	$s = \frac{z-1}{z\cdot T_s}$	$z = \frac{1}{1 - s\cdot T_s}$
Tustin	$s = \frac{2}{T_s}\cdot\frac{z-1}{z+1}$	$z = \frac{1 + s\cdot\frac{T_s}{2}}{1 - s\cdot\frac{T_s}{2}}$

2 System Properties

2.1 Similarity Transformation

$$\begin{cases} x^+ = Ax + Bu \\ y = Cx + Du \end{cases} \implies \begin{cases} \tilde{x}^+ = (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y = (CT)\tilde{x} + Du \end{cases}$$

2.1.1 Modal decomposition

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0) \qquad x(t) = \sum_{i=1}^n e^{\lambda_i t}\tilde{x}_i(0)v_i$$

2.2 Reachability

$$\mathcal{R} := \begin{bmatrix} A^{n-1}B|\ldots|AB|B \end{bmatrix} \in \mathbb{R}^{n\times n\cdot m} \qquad U := \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-1] \end{bmatrix}$$

$$\Rightarrow x[n] = \mathcal{R}U$$

The systen is reachable if and only if $\mathcal{R}$ has full row rank n

2.3 Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad Y = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} \qquad Y = \mathcal{O}x[0]$$

The systen is observable if and only if $\mathcal{O}$ has full column rank n

2.4 Controllability

A system is controllable if, for any initial condition x_0, there exists a control input u that brings the state x to 0 in finite time.

For CT Systems: Controllability = Reachability

For DT Systems: A is invertible $\Rightarrow$ Controllability = Reachability

2.5 Kalman Decomposition

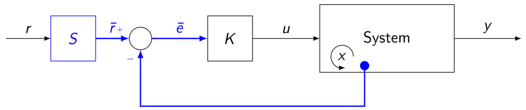
$$x^+ = \begin{bmatrix} \Lambda_{r\bar{o}} & 0 & 0 & 0 \\ 0 & \Lambda_{ro} & 0 & 0 \\ 0 & 0 & \Lambda_{\bar{r}\bar{o}} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}o} \end{bmatrix} x + \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix} x + Du$$

- **Stabilizability**
  A system is said to be stabilizable if all unstable modes are reachable
- **Detectability**
  A system is said to be detectable if all unstable modes are observable

3 State Feedback

3.1 State Feedback



$$\begin{aligned} x^+ &= (A - BK)x + BKSr \\ &= (A - BK)x + B\overline{N}r, \qquad \overline{N} = KS \\ y &= Cx + (D = 0) \end{aligned}$$

$$\begin{aligned} G^{cl}_{yr}(s) &= C(sI - A + BK)^{-1}BKS \\ &= C(sI - A - BK)^{-1}B\overline{N} \end{aligned}$$

K - **Direct Method**

$$p^*_{cl} = \prod_{i=1}^n (s + \lambda_i), \quad K = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$$

$$p_{cl} = det(sI - A + BK) = p^*_{cl}$$

K - **Ackermann's Formula**

$$K = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}\mathcal{R}^{-1}p^*_{cl}(A)$$

S

No steady-state error: $\Rightarrow G^{cl}_{yr}(0) = 1$

$$\implies \overline{N} = -(C(A - BK)^{-1}B)^{-1}, \quad S = K^{-1}\overline{N}$$

3.2 LQR

LQR guarantees:

- phase margin $\geq 60°$
- gain margin $(\frac{1}{2}, +\infty)$

3.2.1 Continuous Time

$$\min_K J(x,u) = \int_0^{+\infty}[x(t)^TQx(t) + u(t)^TRu(t)]dt,$$

s.t.: $\dot{x}(t) = Ax(t) + Bu(t),$
$u(t) = -Kx(t)$

soln.: $0 = A^TX + XA + Q - XBR^{-1}B^TX$
$K = R^{-1}B^TX$

3.2.2 Discrete Time

$$\min_K J(x,u) = \sum_{k=0}^{+\infty}(x[k]^TQx[k] + u[k]^TRu[k]),$$

s.t.: $x[k+1] = Ax[k] + Bu[k],$
$u[k] = -Kx[k]$

soln.: $X = A^TXA - (A^TXB)(B'XB + R)^{-1}(B^TXA) + Q$

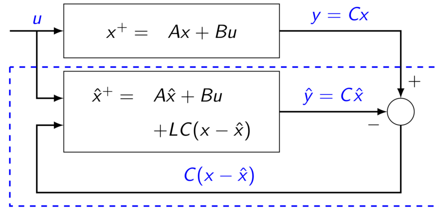
$K = (R + B^TXB)^{-1}B^TXA$

3.2.3 LQR Servo

$$\begin{bmatrix} x \\ \epsilon \end{bmatrix}^+ = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}u + \begin{bmatrix} 0 \\ I \end{bmatrix}r$$

4 State Estimation

4.1 Luenberger Observer



Luenberger Observer

$$\begin{aligned} \hat{x}^+(t) &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned}$$

$\Rightarrow$ **Exactely the same as finding a control gain K**

$$L = p^*_{cl}(A)\mathcal{O}^{-}1[0,\ldots,0,1]^T$$

4.2 LQE

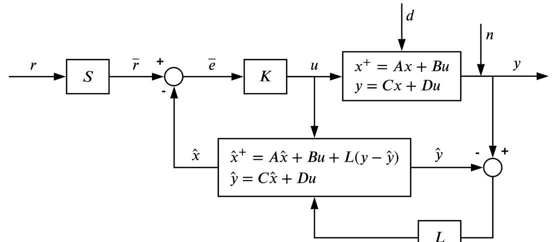
$$Q = \mathbb{E}[w(t)w(t)^T], \quad R = \mathbb{E}[n(t)n(t^T)], \quad \forall t \geq 0$$

Problem: Find L, such that the steady-state covariance of the state error is minimized.

Solution:

$$\begin{aligned} 0 &= AY + YA^T - YC^TR^{-1}CY + Q \\ L &= -YC^TR^{-1} \end{aligned}$$

5 Dynamic Output Feedback



5.1 LQG

$$\begin{bmatrix} x \\ \eta \end{bmatrix}^+ = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} BKS \\ 0 \end{bmatrix}r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ \eta \end{bmatrix}$$

5.1.1 LQG Servo

$$\begin{bmatrix}x\\x_I\\\eta\end{bmatrix}^+ = \begin{bmatrix}A-BK & -BK_I & BK\\ -C & 0 & 0\\ 0 & 0 & A-LC\end{bmatrix}\begin{bmatrix}x\\x_I\\\eta\end{bmatrix} + \begin{bmatrix}BKS\\I\\0\end{bmatrix} r$$

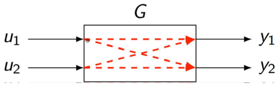
$$y = \begin{bmatrix}C & 0 & 0\end{bmatrix}\begin{bmatrix}x\\x_I\\\eta\end{bmatrix}$$

5.1.2 LQG Stability

LQG guarantees closed loop stability, but the margins can be arbitrarily small.

6 MIMO

6.1 Transfer Function



$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} \qquad G(s) = \begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}$$

Push through identity

$$G_1(I+G_2G_1)^{-1} = (I+G_1G_2)^{-1}G_1$$

MIMO state space to tf

$$G(s) = C(sI-A)^{-1}B + D$$

6.2 State Space

$$x^+ = \underbrace{A}_{n\times n}\underbrace{x}_{n\times 1} + \underbrace{B}_{n\times m}\underbrace{u}_{m\times 1}$$

$$y = \underbrace{C}_{l\times n}\underbrace{x}_{n\times 1} + \underbrace{D}_{l\times m}\underbrace{u}_{m\times 1}$$

- $x \in \mathbb{R}^n$, where n is the order of the system
- $u \in \mathbb{R}^m$, where m is the number of inputs
- $y \in \mathbb{R}^l$, where l is the number of outputs

6.3 MIMO Poles

MIMO poles are the roots of the pole polynomial of a minimal realization of the transfer function matrix. It is equal to the least common multiple of the denominators of all possible minors of the transfer function.

6.4 MIMO Zeros

6.4.1 Transmission Zeros

$H(s)$ has a transmission zero at frequency ζ_0 if $H(s)$ drops rank at $s = \zeta_0$. In this case the null space of the matrix is non-zero, which means a zero has to exist. $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

$$H(s) = \begin{bmatrix}1 & \frac{1}{s-3}\\ 0 & 1\end{bmatrix}$$

$$\lim_{s\to 3}\begin{bmatrix}1 & \frac{1}{s-3}\\ 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & \infty\\ 0 & 1\end{bmatrix} \approx \begin{bmatrix}1 & \infty\\ 0 & 0\end{bmatrix}$$

$\Longrightarrow \zeta_0$ is a pole and a zero!

$$\lim_{s\to\zeta_0} H(s)u_0(s) = 0, \quad u_0(s) \text{ is called "direction"}$$

6.4.2 Invariant Zeros

A non-zero input frequency, that doesn't show up in the output. The output can still be non-zero. The invariant zeros correspont to the values s_i for which the matrix

$$\begin{bmatrix}sI-A & -B\\ C & D\end{bmatrix}$$

becomes singular. (not full rank)
If the system realization is minimal: invariant zeros $\hat{=}$ transmission zeros.

6.5 Gilbert's realization

7 Norms

7.1 Definitions

- $U \in \mathbb{R}^{n\times n}$ is **orthogonal** if $U^T U = I$
- $U \in \mathbb{C}^{n\times n}$ is **unitary** if $U'U = I$
  ($U' :=$ complex conjugate transpose)
- S is **hermitian** if $S = S'$
  For any hermitian S, there exists a unitary matrix U s.t. $U'SU$ is diagonal.

$$\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

$$\|A\|_{p,\text{ind}} = \sup_{x\neq 0}\frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1}\|Ax\|_p$$

$$\|A\|_F = \left(\mathrm{Trace}(A'A)\right)^{\frac{1}{p}}$$

7.2 Singular Value Decomposition

$$A = U\Sigma V' \quad \begin{cases} U \in \mathbb{C}^{m\times m}\\ V \in \mathbb{C}^{n\times n}\\ \Sigma \in \mathbb{R}^{m\times n}\end{cases}$$

- U, V are unitary matrices
- Σ is diagonal with non-zero entries

$$A = \overbrace{\begin{pmatrix}\underline{u}_1 & \underline{u}_2\end{pmatrix}}^{U}\overbrace{\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}}^{\Sigma}\overbrace{\begin{pmatrix}\underline{v}_1 & \underline{v}_2\end{pmatrix}'}^{V'}$$

- u_i are the left singular vectors (**normalized** $eig(AA')$)
- v_i are the right singular vectors (**normalized** $eig(A'A)$)
- $\sigma_i = \sqrt{\lambda_i}$
- v_i is the specific input that generates the extremal output u_i with amplification σ_i

$$\|A\|_{2,\text{ind}} = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\max}(A), \quad \inf_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\min}(A)$$

7.3 Signal Norms

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty |u(t)|^p\, dt\right)^{\frac{1}{p}}, \quad p \ge 1$$

$$\|u\|_{\mathcal{L}_\infty} = \sup_t |u(t)|$$

$$\|G\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2\, d\omega\right)^{\frac{1}{2}}$$

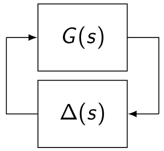
$$\|G\|_{\mathcal{H}_\infty} = \sup_{u\neq 0}\frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} = \sup_{\omega\in\mathbb{R}}\sigma_{\max}[G(j\omega)]$$

$$\|u(t)\|^2_{\mathcal{L}_2} = \int_0^\infty \|u(t)\|_2^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\|U(s)\|_2^2\, d\omega = \|U(s)\|^2_{\mathcal{H}_2}$$

7.4 Solving the Norms

8 Stability and Performance Robustness

8.1 Small Gain Theorem



- $G(s)$ is a known, stable system
- $\Delta(s)$ is unknown, assumed to be stable and $\|\Delta\|_{\mathcal{H}_\infty} < 1$
- Interconnection is stable iff $\|G\|_{\mathcal{H}_\infty} < 1$

8.2 Uncertainty

Additive uncertainty:
$\widetilde{P} = P + W\Delta$

Multiplicative uncertainty:
$\widetilde{P} = P(\mathbb{I} + W\Delta)$

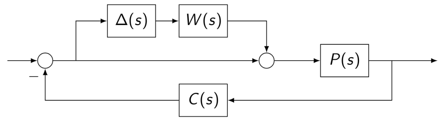
Feedback uncertainty:
$\widetilde{P} = (\mathbb{I} - PW\Delta)^{-1}P$

$\widetilde{P}$: The model of the real System

P : Our best guess of the system through mathematical modeling

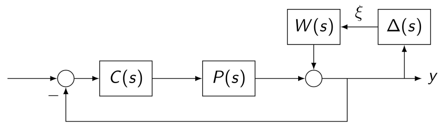
$W\Delta$: The uncertainty

8.3 Robust stability



$$G(s) = -(\mathbb{I}+CP)^{-1}CW \;\Rightarrow\; \left\|-(\mathbb{I}+CP)^{-1}CW\right\|_{\mathcal{H}_\infty} \overset{!}{<} 1$$

8.4 Robust performance



$$\|G_{y\zeta}\|_{\mathbb{H}_\infty} = \left\|(\mathbb{I}+PC)^{-1}W\right\|_{\mathcal{H}_\infty} \overset{!}{<} 1$$

9 Modern Control

9.1 Youla-Parametrization

9.2 Standard Form

9.3 $\mathcal{H}_2$ Control

9.4 $\mathcal{H}_\infty$ Control