

Control Systems 2 Cheatsheet

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1 Discrete Time

1.1 Sampling

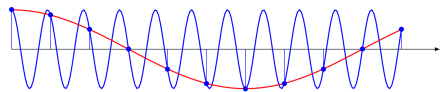
$$T_s: \text{Sampling Time} \quad \omega_s = \frac{2\pi}{T_s}$$
$$\omega_s: \text{Sampling frequency}$$

1.2 Aliasing

$$y_1[k] = \cos(\omega k T_s), \quad k = 0, 1, 2$$

$$y_2[k] = \cos((\omega + n \frac{2\pi}{T_s})k T_s), \quad n = 0, 1, 2$$

$$= \cos(\omega k T_s + 2\pi n k) = y_1[k]$$



1.2.1 Nyquist-Shannon Sampling theorem

$$f_N = \frac{1}{2T_s} [\text{Hz}] \quad \text{or} \quad \omega_N = \frac{\pi}{T_s} \left[\frac{\text{rad}}{\text{s}} \right]$$

No aliasing if $\omega < \omega_N$!

1.3 DT State Space Representation

$$x[k+1] = A_d x[k] + B_d u[k]$$
$$= e^{A T_s} x[k] + \left(\int_0^{T_s} e^{A\tau} d\tau \right) B u[k]$$
$$y[k] = C_d x[k] + D_d u[k]$$
$$= C x[k] + D u[k]$$

If A is invertible: $B_d = A^{-1}(A_d - I)B$

1.3.1 Emulation

Exact	$s = \frac{1}{T_s} \cdot \ln(z)$	$z = e^{s \cdot T_s}$
Euler forward	$s = \frac{z-1}{T_s}$	$z = s \cdot T_s + 1$
Euler backward	$s = \frac{z-1}{z \cdot T_s}$	$z = \frac{1}{1-s \cdot T_s}$
Tustin	$s = \frac{2}{T_s} \cdot \frac{z-1}{z+1}$	$z = \frac{1+s \cdot \frac{T_s}{2}}{1-s \cdot \frac{T_s}{2}}$

2 System Properties

2.1 Similarity Transformation

$$\begin{cases} x^+ = Ax + Bu \\ y = Cx + Du \end{cases} \implies \begin{cases} \tilde{x}^+ = (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y = (CT)\tilde{x} + Du \end{cases}$$

2.1.1 Modal decomposition

$$\tilde{x}_i(t) = e^{\lambda_i t} \tilde{x}_i(0) \quad x(t) = \sum_{i=1}^n e^{\lambda_i t} \tilde{x}_i(0) v_i$$

2.2 Reachability

$$\mathbf{R} := \begin{bmatrix} A^{n-1}B | \dots | AB | B \end{bmatrix} \in \mathbb{R}^{n \times n \cdot m} \quad \mathbf{U} := \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-1] \end{bmatrix}$$

$$\Rightarrow x[n] = RU$$

The system is reachable if and only if **R** has full row rank **n**

2.3 Observability

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} \quad \mathbf{Y} = \mathbf{O}x[0]$$

The system is observable if and only if **O** has full column rank **n**

2.4 Controllability

A system is controllable if, for any initial condition x_0 , there exists a control input u that brings the state x to 0 in finite time.

For CT Systems: Controllability = Reachability

For DT Systems: A is invertible \Rightarrow Controllability = Reachability

2.5 Kalman Decomposition

$$x^+ = \begin{bmatrix} \Lambda_{r\bar{o}} & 0 & 0 & 0 \\ 0 & \Lambda_{r\bar{o}} & 0 & 0 \\ 0 & 0 & \Lambda_{\bar{r}\bar{o}} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}\bar{o}} \end{bmatrix} x + \begin{bmatrix} B_{r\bar{o}} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & C_{r\bar{o}} & 0 & C_{\bar{r}\bar{o}} \end{bmatrix} x + Du$$

• Stabilizability

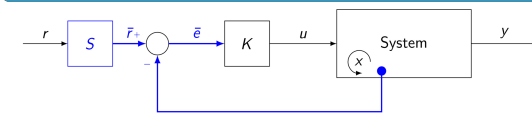
A system is said to be stabilizable if all unstable modes are reachable

• Detectability

A system is said to be detectable if all unstable modes are observable

3 State Feedback

3.1 State Feedback



$$\dot{x} = (A - BK)x + BKSr$$
$$= (A - BK)x + B\bar{N}r, \quad \bar{N} = KS$$
$$y = Cx + (D = 0)$$

$$G_{yr}^{cl}(s) = C(sI - A + BK)^{-1}BKS$$
$$= C(sI - A - BK)^{-1}B\bar{N}$$

3.1.1 K - Direct Method

$$p_{cl}^* = \prod_{i=1}^n (s + \lambda_i), \quad K = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$
$$p_{cl} = \det(sI - A + BK) = p_{cl}^*$$

3.1.2 K - Ackermann's Formula

$$K = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} R^{-1} p_{cl}^*(A)$$

3.1.3 S

$$\text{No steady-state error: } \Rightarrow G_{yr}^{cl}(0) = 1$$
$$\implies \bar{N} = -(C(A - BK)^{-1}B)^{-1}, \quad S = K^{-1}\bar{N}$$

3.2 LQR

3.2.1 Continuous Time

$$\min_K J(x, u) = \int_0^{+\infty} [x(t)^T Q x(t) + u(t)^T R u(t)] dt,$$

$$\text{s.t.: } \dot{x}(t) = Ax(t) + Bu(t),$$
$$u(t) = -Kx(t)$$

$$\text{soln.: } 0 = A^T P + PA + Q - PBR^{-1}B^T P$$
$$K = R^{-1}B^T P$$

3.2.2 Discrete Time

$$\min_K J(x, u) = \sum_{k=0}^{+\infty} (x[k]^T Q x[k] + u[k]^T R u[k]),$$

$$\text{s.t.: } x[k+1] = Ax[k] + Bu[k],$$
$$u[k] = -Kx[k]$$

$$\text{soln.: } P = A^T P A - (A^T P B)(B^T P B + R)^{-1}(B^T P A)$$
$$+ Q$$
$$K = (R + B^T P B)^{-1} B^T P A$$

3.2.3 LQR Servo

$$\frac{d}{dt} \begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$