Control Systems 2 Cheatsheet

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1 Discrete Time

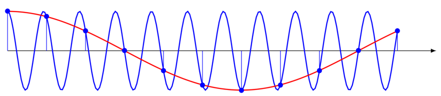
1.1 Sampling

T_s: Sampling Time
ω_s: Sampling frequency

$$\omega_s = \frac{2\pi}{T_s}$$

1.2 Aliasing

$$\begin{aligned} y_1[k] &= cos(\omega k T_s), && k = 0,1,2 \\ y_2[k] &= cos((\omega + n\frac{2\pi}{T_s})kT_s), && n = 0,1,2 \\ &= cos(\omega k T_s + \cancel{n2\pi k}) = y_1[k] \end{aligned}$$



1.2.1 Nyquist-Shannon Sampling theorem

$$f_N = \frac{1}{2T_s}\,[\text{Hz}] \quad \text{or} \quad \omega_N = \frac{\pi}{T_s}\left[\frac{\text{rad}}{\text{s}}\right]$$

No aliasing if $\omega < \omega_N$!

1.3 DT State Space Representation

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ &= e^{AT_s}x[k] + \left(\int_0^{T_s} e^{A\tau}d\tau\right)Bu[k] \\ y[k] &= C_d x[k] + D_d u[k] \\ &= Cx[k] + Du[k] \end{aligned}$$

If A is invertible: $B_d = A^{-1}(A_d - I)B$

1.3.1 Emulation

Exact	$s = \frac{1}{T_s}\cdot \ln(z)$	$z = e^{sT_s}$
Euler forward	$s = \frac{z-1}{T_s}$	$z = s\cdot T_s + 1$
Euler backward	$s = \frac{z-1}{z\cdot T_s}$	$z = \frac{1}{1 - s\cdot T_s}$
Tustin	$s = \frac{2}{T_s}\cdot\frac{z-1}{z+1}$	$z = \frac{1+s\cdot\frac{T_s}{2}}{1-s\cdot\frac{T_s}{2}}$

2 System Properties

2.1 Similarity Transformation

$$\begin{cases} x^+ = Ax + Bu \\ y = Cx + Du \end{cases} \implies \begin{cases} \tilde{x}^+ = (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y = (CT)\tilde{x} + Du \end{cases}$$

2.1.1 Modal decomposition

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0) \qquad x(t) = \sum_{i=1}^{n} e^{\lambda_i t}\tilde{x}_i(0)v_i$$

2.2 Reachability

$$\mathcal{R} := \left[A^{n-1}B|\ldots|AB|B\right] \in \mathbb{R}^{n\times n\cdot m} \qquad U := \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-1] \end{bmatrix}$$

$\Rightarrow x[n] = \mathcal{R}U$

The systen is reachable if and only if $\mathcal{R}$ has full row rank n

2.3 Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad Y = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} \qquad Y = \mathcal{O}x[0]$$

The systen is observable if and only if $\mathcal{O}$ has full column rank n

2.4 Controllability

A system is controllable if, for any initial condition x_0, there exists a control input u that brings the state x to 0 in finite time.
For CT Systems: Controllability = Reachability
For DT Systems: A is invertible $\Rightarrow$ Controllability = Reachability

2.5 Kalman Decomposition

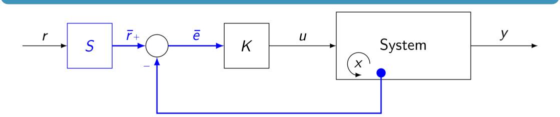
$$x^+ = \begin{bmatrix} \Lambda_{r\bar{o}} & 0 & 0 & 0 \\ 0 & \Lambda_{ro} & 0 & 0 \\ 0 & 0 & \Lambda_{\overline{ro}} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}o} \end{bmatrix} x + \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix} x + Du$$

- **Stabilizability**
  A system is said to be stabilizable if all unstable modes are reachable
- **Detectability**
  A system is said to be detectable if all unstable modes are observable

3 State Feedback

3.1 State Feedback



$$\begin{aligned} \dot{x} &= (A - BK)x + BKSr \\ &= (A - BK)x + B\overline{N}r, \qquad \overline{N} = KS \\ y &= Cx + (D = 0) \end{aligned}$$

$$\begin{aligned} G_{yr}^{cl}(s) &= C(sI - A + BK)^{-1}BKS \\ &= C(sI - A - BK)^{-1}B\overline{N} \end{aligned}$$

3.1.1 K - Direct Method

$$p_{cl}^* = \prod_{i=1}^{n}(s + \lambda_i), \quad K = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$$

$$p_{cl} = det(sI - A + BK) = p_{cl}^*$$

3.1.2 K - Ackermann's Formula

$$K = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \mathcal{R}^{-1}p_{cl}^*(A)$$

3.1.3 S

No steady-state error: $\Rightarrow G_{yr}^{cl}(0) = 1$

$$\implies \overline{N} = -(C(A - BK)^{-1}B)^{-1}, \quad S = K^{-1}\overline{N}$$

3.2 LQR

3.2.1 Continuous Time

$$\min_K J(x,u) = \int_0^{+\infty}[x(t)^TQx(t) + u(t)^TRu(t)]dt,$$

$$\text{s.t.:} \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ u(t) &= -Kx(t) \end{aligned}$$

$$\text{soln.:} \quad \begin{aligned} 0 &= A^TP + PA + Q - PBR^{-1}B^TP \\ K &= R^{-1}B^TP \end{aligned}$$

3.2.2 Discrete Time

$$\min_K J(x,u) = \sum_{k=0}^{+\infty}(x[k]^TQx[k] + u[k]^TRu[k]),$$

$$\text{s.t.:} \quad \begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ u[k] &= -Kx[k] \end{aligned}$$

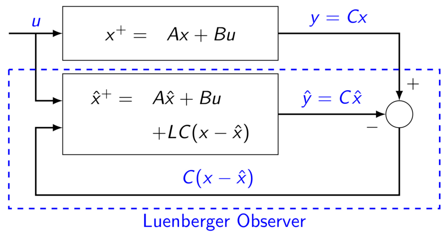
$$\text{soln.:} \quad \begin{aligned} P &= A^TPA - (A^TPB)(B'PB + R)^{-1}(B^TPA) \\ &\quad + Q \\ K &= (R + B^TPB)^{-1}B^TPA \end{aligned}$$

3.2.3 LQR Servo

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}u + \begin{bmatrix} 0 \\ I \end{bmatrix}r$$

4 State Estimation

4.1 Luenberger Observer



$$\begin{aligned} \dot{\hat{x}}(t) &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned}$$

$\Rightarrow$ **Exactely the same as finding a control gain K**

$$L = p_{cl}^*(A)\mathcal{O}^-1[0,\ldots,0,1]^T$$

4.2 LQE

$$Q = \mathbb{E}[w(t)w(t)^T], \quad R = \mathbb{E}[n(t)n(t^T)], \quad \forall t \geq 0$$

Problem: Find L, such that the steady-state covariance of the state error is minimized.
Solution:

$$\begin{aligned} 0 &= AY + YA^T - YC^TR^{-1}CY + Q \\ L &= -YC^TR^{-1} \end{aligned}$$

5 Dynamic Output Feedback