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Matrix Exponential

$$e^{At} := I + At + \frac{(At)^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \qquad \frac{de^{At}}{dt} = Ae^{At}$$

CT to DT

Discretization of CT LTI Systems

Consider the discretization of the following CT system

$$\dot{q}(t) = A_C q(t) + B_C u(t) \qquad q(0) = q[0]$$
$$y(t) = C_C q(t) + D_C u(t) \qquad u(0) = u[0]$$

- **Forward Euler Method** approximates derivatives as:

$$\dot{p}(t) \approx \frac{p(t+T_s) - p(t)}{T_s} \qquad \ddot{p}(t) \approx \frac{p(t+2T_s) - 2p(t+T_s) + p(t)}{T_s^2}$$

- **Exact Discretization** defines the following **quadratic** M:

$$M := \begin{bmatrix} A_c & B_c \\ \underline{0} & \underline{0} \end{bmatrix} \qquad F = \begin{bmatrix} F_{11} & F_{12} \\ \underline{0} & \mathbb{I} \end{bmatrix} := e^{MT_s}$$

$$A_d := F_{11} \qquad B_d := F_{12} \qquad C_d := C_c \qquad D_d := D_c$$

to obtain the exact DT state-space description

$$q[n+1] = A_d q[n] + B_d u[n] \qquad y[n] = C_d q[n] + D_d u[n]$$

Discrete-Time LTI Systems

DT Classification

- **Memoryless (LTI: $h[n] = 0 \ \forall n \neq 0$)** output at n only depends on input at same timestep: $y[n] = f_n(u[n])$
- **Causal (LTI: $h[n] = 0 \ \forall n < 0$)** output $y[n]$ only depends on present and past inputs $u[k], k \leq n$.
  If a system and its input sequence are both causal, the output sequence will also be causal.
- **Linear** $G\{\alpha_1 u_1[n] + \alpha_2 u_2[n]\} = \alpha_1 G\{u_1[n]\} + \alpha_2 G\{u_2[n]\}$ $\forall \ \{u_1[n]\}, \{u_2[n]\}$ and $\forall \ \alpha_1, \alpha_2$.
- **Time-invariant** same output to same input at any time.
  $u_2[n] = u_1[n-k] \Rightarrow y_2[n] = y_1[n-k]$
- **Stable (LTI: $\sum |h[n]| < \infty$, ROC contains unit circle)** if there exists a finite value M, such that for all input sequences u bounded by 1, the output sequence y is bounded by M (BIBO stability).

Definitions of useful DT signals

Unit Impulse Sequence

$$\delta[n] := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Unit Step Sequence

$$s[n] := \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Signal Representation

$$\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k] \cdot \{\delta[n-k]\} \quad \forall n$$

Convolution

$$x * h = \{x[n]\} * \{h[n]\} := \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}$$

commutative, associative and distributive
→ Order in which LTI systems are cascaded does not matter.

Response to Arbitrary Inputs $\{y[n]\}$

$\{h[n]\} = G\{\delta[n]\}$ being output given a unit impulse input.

$$\{y[n]\} = G\{u[n]\} = \{u[n]\} * \{h[n]\} = \sum_{k=-\infty}^{\infty} u[k]\{h[n-k]\}$$

Step response $r[n]$

$$r[n] = G(s[n]) = G\left(\sum \delta[n]\right) \overset{L}{=} \sum_{k=-\infty}^{n} h[n]$$

$$r[n] - r[n-1] = h[n]$$

Finite and Infinite Impulse Response

Causal systems have a **finite impulse response (FIR)** if:

$$\exists N \in \mathbb{Z}, \text{ s.t. } \quad \boxed{h[n] = 0 \quad \forall n \geq N}$$

Otherwise it has an **infinite impulse response (IIR)**.

Linear Constant-Coefficient Difference Equations

Definition

LCCDE systems are by definition **linear** and **time-invariant**.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k], \quad a_k, b_k \in \mathbb{R}$$

Recursive Definition

Assuming the system is causal (& $a_0 \neq 0$) one can find the recursive definition:

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k u[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

LCCDE → State-Space

This class will mainly consider SISO systems, where the following holds true:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad D = \begin{bmatrix} b_0 \end{bmatrix}$$

State-Space → Impulse Response

System with zero initial conditions ($q[n] = 0 \quad \forall n \leq 0$) has impulse response:

$$h = \{D, CB, CAB, \cdots, CA^{n-1}B, \cdots\}$$

Frequency Domain Concepts

Periodicity Constraint

A **periodic CT** signal will result in a **periodic DT** signal iff:

$$\frac{\Omega}{2\pi} = \frac{m}{N} \qquad m, N \in \mathbb{Z}$$

e.g. $x[n] = \cos(\Omega n)$ with $\Omega = \omega T_s$
If $\frac{m}{N}$ is an *irreducible fraction*, N is the **fundamental period**.

The z-Transform

Given a sequence $x[n]$, its z-transform $X(z)$ is defined as

$$X(z) := \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad z \in \mathbb{C}$$

The z-transform has the following properties:

Accumulation	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{z}{z-1} X(z)$
Linearity	$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1(z) + \alpha_2 X_2(z)$
Convolution	$\{x_1[n]\} * \{x_2[n]\} \leftrightarrow X_1(z) \cdot X_2(z)$
Time-shifting	$x[n+\alpha] \leftrightarrow z^{\alpha} X(z)$

Common z-Transform Pairs

$$\delta[n] \longleftrightarrow 1 \qquad s[n] \longleftrightarrow \frac{z}{z-1}$$
$$\delta[n-n_0] \longleftrightarrow z^{-n_0} \qquad n \cdot x[n] \longleftrightarrow -z \cdot \frac{d}{dz} X(z)$$
$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right) \qquad x^*[n] \longleftrightarrow X^*[z^*]$$
$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z) \qquad n_0^n x[n] \longleftrightarrow X\left(\frac{z}{n_0}\right)$$

Region of Convergence (ROC)

The ROC must not contain any poles. If the system is stable (no poles with $|p_i| = 1$), the ROC must contain the unit circle.

Transfer Functions

For an LTI system with impulse response $\{h[n]\}$ we have

$$\{y[n]\} = \{u[n]\} * \{h[n]\} \longleftrightarrow Y(z) = H(z)U(z) \qquad H(z) = \frac{Y(z)}{U(z)}$$

and call $H(z)$ the transfer function of the system.
It can also be easily derived from LCCDEs:

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}}$$

Transfer Functions in Discrete-Time

The following relationship between input and output holds true:

$$q = (z\mathbb{I} - A_d)^{-1} \cdot B_d \cdot u$$
$$y = (C_d \cdot (z\mathbb{I} - A_d)^{-1} \cdot B_d + D_d) \cdot u$$

$$\boxed{H(z) = C_d (z\mathbb{I} - A_d)^{-1} B_d + D_d}$$

Causality-Stability Theorem

System with TF $H(z)$ and poles p_i is:

- **stable** iff: p_i not on the unit circle
- **causal and stable** iff: p_i within unit circle

Complex Exponential

$$\boxed{\{y[n]\} = G\{z_0^n\} = H(z_0)\{z_0^n\}}$$

$$y[n] = |H(\Omega_0)| \cdot e^{j(\Omega_0 n + \angle H(\Omega_0))}$$

Discrete-Time Fourier Analysis

signal property	→	analysis tool
infinite, summable	→	Discrete Time FT (DTFT)
periodic	→	Discrete Fourier Series (DFS)
finite length	→	Discrete FT (DFT)

Discrete-Time Fourier Transform (DTFT) $\mathcal{F}$

The Fourier Transform is the z-Transform for $z = e^{j\Omega}$.

Definition

The Fourier Transform X of a DT signal x is defined as

$$X(\Omega) := \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} \qquad x[n] \longleftrightarrow X(\Omega) \qquad X = \mathcal{F}x$$

Inverse Discrete-Time Fourier Transform (IDTFT)

The Fourier Transform operator $\mathcal{F}$ is invertible:

$$\{x[n]\} = \mathcal{F}^{-1} X := \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega\right\}$$

Properties of the FT

Linearity	$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$
Convolution	$\{x_1[n]\} * \{x_2[n]\} \leftrightarrow X_1(\Omega) \cdot X_2(\Omega)$
Parseval	$\sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(\Omega) \rvert^2 d\Omega$

Common DTFT Pairs

$$e^{j\Omega_0 n} \longleftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad \delta[n-n_0] \longleftrightarrow e^{-j\Omega n_0}$$
$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega) \qquad x[-n] \longleftrightarrow X(-\Omega)$$

Frequency Response of LTI Systems

For an LTI system with impulse response h we have

$$y = u * h \longleftrightarrow Y(\Omega) = H(\Omega)U(\Omega) \qquad \therefore H(\Omega) = \frac{Y(\Omega)}{U(\Omega)}$$

We can obtain $H(\Omega)$ from $H(z)$ or the LCCDE:

$$\boxed{H(\Omega) = H(z)\big|_{z=e^{j\Omega}}}$$

$$\boxed{H(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + \ldots + b_M e^{-Mj\Omega}}{a_0 + a_1 e^{-j\Omega} + \ldots + a_N e^{-Nj\Omega}}}$$

Response to Complex Exponential

If the input u to an LTI system G is a complex exponential:

$$y[n] = |H(\Omega_0)| \cdot e^{j(\Omega_0 n + \angle H(\Omega_0))}$$

This is only valid if the input sequence is applied for all time.

Response to Real Sinusoids

Let $u[n] = A\cos(\Omega_0 n + \phi)$. The real part of the input affects the real part of the output:

$$y[n] = |H(\Omega_0)| A \cos(\Omega_0 n + \phi + \angle H(\Omega_0))$$

This is only valid if the input sequence is applied for all time.

Discrete Fourier Series (DFS) $\mathcal{F}_s$

Definition

The DFS is a different representation of signal x with period N

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \qquad X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$$

where $\Omega_0 := \frac{2\pi}{N}$. *Note:* $X[k]$ is also periodic with period N.

Properties of the DFS

Linearity	$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$
Parseval	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$
Real Signals	$X[N-\alpha] = X^*[\alpha] \quad (x[n] \in \mathbb{R})$

Response to Complex Exponential Sequences

$$\boxed{Y[k] = H\left(e^{jk\Omega_0}\right) \cdot U[k]} \qquad H\left(e^{jk\Omega_0}\right) = H(z)\big|_{z=e^{jk\Omega_0}}$$

DFS ↔ DTFT

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=0}^{N-1} X[k] \cdot \delta(\Omega - k\Omega_0)$$

Discrete Fourier Transform (DFT)

Definition

The DFT coefficients of a signal are the DFS coefficients of the signal's periodic extension. For $k, n = 0, \ldots, N-1$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n} \qquad X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$$

The CT frequency f_k that belongs to $X[k]$ is calculated as $f_k = k \cdot f_s / N$.

Effect of Causal Inputs

$$u[n] = \begin{cases} e^{j\Omega n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Let the **LTI** system G be **stable** and let $y = Gu$. Then

$$y[n] \to H(z = e^{j\Omega}) \cdot e^{j\Omega n} \qquad as \ n \to \infty$$

Aliasing

Aliasing is the effect of multiple CT frequencies mapping to the same DT frequency. It is avoided by ensuring that all the signal's frequency content is below the **Nyquist frequency** / half the sampling frequency:

$$|\omega| < \frac{\pi}{T_s}$$

Filtering Basics

Probability Density Function, Expected Value, Variance

Let $x \in \mathbb{R}$ be a scalar continuous random variable with PDF $p(x)$. Consider the following definitions:

$$\int_{-\infty}^{\infty} p(x)\, dx = 1 \qquad \mathrm{E}(x) := \int_{-\infty}^{\infty} x p(x)\, dx \qquad \mathrm{Var}(x) := \mathrm{E}\left((x - \mathrm{E}(x))^2\right)$$

Definition of White Noise

White noise is essentially a signal $d[n]$ with a flat spectrum. In mathematical terms $d[n]$ has to satisfy for $n = 0, \dots, N-1$:

$$\mathrm{E}(d[n]) = 0 \qquad \mathrm{Var}(d[n]) = 1 \qquad \mathrm{E}(d[n]d[l]) = \begin{cases} 0 & n \neq l \\ 1 & n = l \end{cases}$$

Phase Delay of a Filter

Phase delay $-\angle H(\Omega)/\Omega$ of a filter states how many samples a sinusoid at frequency Ω is delayed by the filter. If $\angle H(\Omega)$ is linear in Ω, the filter is said to have linear phase and the phase delay of the filter is constant.

Finite Impulse Response (FIR) Filters

FIR filters have an LCCDE and impulse response of the form

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k] \qquad \Rightarrow \qquad h = \{b_0, b_1, \dots, b_{M-1}\}$$

and are therefore always stable.

Definitions

M: filter length $\quad M-1$: filter order $\quad b_k$: filter coefficients

Transfer Function and Frequency Response

$$H(z) = \sum_{k=0}^{M-1} h[k] z^{-k} \qquad H(\Omega) = \sum_{k=0}^{M-1} \underbrace{h[k]}_{b_k} e^{-j\Omega k}$$

Moving Average (MA) Filter

LCCDE and Frequency Response

The MA filter averages the current and past inputs to produce its output and is represented by the following LCCDE

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} u[n-k]$$

Its frequency response is

$$H(\Omega) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j\Omega k} = \frac{1}{M} \frac{1 - e^{-j\Omega M}}{1 - e^{-j\Omega}}$$

Zeros therefore occur at $\Omega = 2\pi k/M$ where $k > 0$ is an integer.

Phase Response

For small frequencies Ω, the phase can be approximated by

$$\angle H(\Omega) \approx -\frac{\Omega(M-1)}{2}$$

Magnitude Response

The magnitude response of an MA filter is

$$|H(\Omega)| = \left| \frac{1}{M} \frac{\sin\left(\frac{\Omega M}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} \right| \xrightarrow{M \to \infty} \left| \frac{\sin\left(\frac{\Omega M}{2}\right)}{\frac{\Omega M}{2}} \right| = \left| \mathrm{sinc}\left(\frac{\Omega M}{2}\right) \right|$$

Non-Causal Moving Average (NCMA) Filter

The NCMA filter has the following impulse response

$$h = \{\dots, 0, \frac{1}{M}, \dots, \underset{\uparrow}{\frac{1}{M}}, \dots, \frac{1}{M}, 0, \dots\}$$

And the filter's frequency response is given by

$$H(\Omega) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j\Omega\left(k - \frac{M-1}{2}\right)} = e^{j\Omega\left(\frac{M-1}{2}\right)} H_{\mathsf{MA}}(\Omega)$$

where H_{MA} is the frequency response of the causal MA filter. Therefore, the frequency response of the NCMA filter has an added phase of $\Omega(M-1)/2$ compared to the MA filter. The magnitude, however, stays the same.

Weighted Moving Average (WMA) Filter

The WMA filter places less emphasis on older inputs

$$y[n] = \frac{1}{S} \sum_{k=0}^{M-1} w_k u[n-k] \qquad S = \frac{M(M+1)}{2}$$

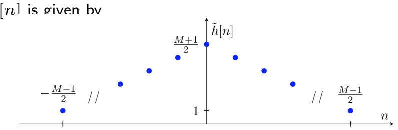
S is the normalization constant chosen such that the sum of all filter coefficients equals one. (often $w_k = (M-k)$)

Non-Causal WMA (NCWMA) Filter

The impulse response of an NCWMA filter is

$$h[n] = \frac{1}{S} \tilde{h}[n] \qquad S = \sum_{k=-\infty}^{\infty} \tilde{h}[n]$$

where $\tilde{h}[n]$ is given by



An NCWMA Filter does **not** add any phase.
In general, if $H(z)$ is the transfer function of a MA filter with M coefficients, then $H(z)H(z^{-1})$ is a non-causal WMA filter with $2M-1$ coefficients.

Infinite Impulse Response (IIR) Filters

IIR filters have an LCCDE of the form

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]$$

The filter order is given by $\max(M-1, N-1)$ and is the size of the state in a state-space description of the system.

Transfer Function and Frequency Response

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} \qquad H(\Omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{N-1} a_k e^{-j\Omega k}}$$

First-Order Low-Pass Filter

First-order, low-pass, causal IIR Filters have the LCCDE

$$y[n] = \alpha y[n-1] + (1-\alpha) u[n]$$

and are stable if $0 \leq \alpha < 1$.

Transfer Function and Frequency Response

$$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} \qquad H(\Omega) = \frac{1-\alpha}{1-\alpha e^{-j\Omega}}$$

Decay Time

Time T_0 to reach the value e^{-1}. Assume $y[0] = 1$ and $u[n] = 0$.

$$\alpha = e^{-\frac{T_s}{T_0}} \approx 1 - \frac{T_s}{T_0} \qquad T_0 \gg T_s$$

Longer Decay Time results in faster magnitude decrease w.r.t. frequency.

Butterworth Filter Design (Low-Pass)

The CT frequency response with cutoff frequency at 1 rad/sec

$$R(\omega) = \frac{1}{\sqrt{1 + \omega^{2K}}}$$

where K is the order of the filter, serves as starting point.

Transfer Function

The only stable TF that has the above frequency response is

$$H(s) = \left(\prod_{k=1}^{K} (s - s_k) \right)^{-1} \qquad s_k = e^{\frac{j(2k+K-1)\pi}{2K}}$$

Cutoff Frequency Specification

Get desired cutoff frequency ω_c by substitution:

$$s \to \frac{s}{\omega_c}$$

Second-Order Butterworth Low-Pass Filter

A second-order ($K = 2$) Butterworth filter yields

$$s_1 = e^{j3\pi/4} = \frac{-1+j}{\sqrt{2}} \qquad s_2 = e^{j5\pi/4} = \frac{-1-j}{\sqrt{2}}$$

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Applied-Concepts

Median Filter

The median filter of even order M is defined as

$$y[n] = \text{median}(u[n-M/2], \dots, u[n], \dots, u[n+M/2])$$

High-Pass Filter Design

CT

Goal: High-Pass with cutoff frequency ω_c
1. Design low-pass $H_{\mathsf{LP}}(s)$ with cutoff frequency $1/\omega_c$.
2. $H_{\mathsf{HP}}(s) = H_{\mathsf{LP}}(s^{-1})$

DT

Goal: High-Pass with cutoff frequency Ω_c
1. Design low-pass $H_{\mathsf{LP}}(z)$ with cutoff frequency $\pi - \Omega_c$.
2. $H_{\mathsf{HP}}(z) = H_{\mathsf{LP}}(-z)$

Band-Pass Filter Design

CT $\qquad \omega_0 \ll \omega_1$

$$H_{\mathsf{BP}}(s) = H_{\mathsf{LP}}(s) \cdot H_{\mathsf{HP}}(s)$$

CT

Goal: Band-Pass with corner frequencies $\omega_0 < \omega_1$
1. Design low-pass filter $H_{\mathsf{LP}}(s)$ with $\omega_c = \omega_1 - \omega_0$.
2. Transform $H_{\mathsf{LP}}(s)$ using:

$$s \to \frac{s^2 + \omega_0\omega_1}{s}$$

Band-Stop Filter Design

CT $\qquad \omega_0 \ll \omega_1$

$$H_{\mathsf{BS}}(s) = H_{\mathsf{LP}}(s) + H_{\mathsf{HP}}(s)$$

CT

Goal: Band-Stop with corner frequencies $\omega_0 < \omega_1$
1. Design low-pass filter $H_{\mathsf{LP}}(s)$ with $\omega_c = 1/(\omega_1 - \omega_0)$.
2. Transform $H_{\mathsf{LP}}(s)$ using:

$$s \to \frac{s}{s^2 + \omega_0\omega_1}$$

Second Order Notch Filter

$$H_{\mathsf{NO}}(s) = \frac{s^2 + \omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Bilinear Transform (BT)

The BT converts CT to DT filter and vice versa.

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) \qquad z = \frac{1 + s\frac{T_s}{2}}{1 - s\frac{T_s}{2}}$$

Maps s-plane im. axis to z-plane unit circle. (Infinities at -1)
BT frequency mapping: $\quad \omega \in (-\infty, \infty) \to \Omega \in (-\pi, \pi)$

$$\Omega = 2\arctan\left(\omega \frac{T_s}{2}\right) \underset{\omega T_s \lesssim 0.5}{\approx} \omega T_s$$

Frequency Prewarping

BT frequency mapping only yields desired cutoff frequencies for small ω. $\to$ Prewarp CT frequencies before applying BT.

1. Let ω_c be the desired CT corner frequency.
2. $\bar{\omega}_c = \frac{2}{T_s} \tan\left(\frac{\omega_c T_s}{2}\right)$
3. Design CT filter with $\bar{\omega}_c$ and obtain $H(s)$.
4. Apply BT to obtain DT filter $H(z)$ with $\Omega_c \approx \omega_c$.

System Identification

- u_e is a known input, G is causal, stable and LTI
- u_d is an unknown process noise, assumed to be white
- y_d is an unknown measurement noise, assumed to be white
- y_m is the measurement of the systems output, which is corrupted by process and measurement noise:

$$y_m = Gu_e + y_d + Gu_d$$

Identification based on Impulse Responses

Without Noise

In the absence of noise and with $\{u_e[n]\} = \{\delta[n]\}$ we have that

$$\{y[n]\} = \{h[n]\} \qquad H(\Omega) = \sum_{n=0}^{\infty} y_m[n] e^{-j\Omega n}$$

Since most systems have an infinite impulse response, we collect N pieces of data and then take the DFT:

$$Y_m[k] = \sum_{n=0}^{N-1} y_m[n] e^{-j\Omega_k n}$$

At the discrete frequency $\Omega_k = 2\pi k/N$ where $k = 0, 1, \dots, N-1$, the frequency response estimate $\widehat{H}(\Omega_k)$ then becomes

$$\widehat{H}(\Omega_k) := Y_m[k] = H(\Omega_k) - \underbrace{\sum_{n=N}^{\infty} h[n] e^{-j\Omega_k n}}_{H_N(\Omega_k)}$$

Note that the error $H_N(\Omega_k) \to 0$ as $N \to \infty$ since G is stable.

With Noise

Using an impulse as input yields unsatisfactory results if noise is present since the mean squared error of the estimate approaches infinity as the length of the sample increases.

Identification using Sinusoidal Inputs

Consider the case where measurement noise corrupts the output

$$y_m = Gu_e + y_d$$

Let $u_e[n] = \exp(j(2\pi/N)nl)$ for $n = 0, 1, \dots, N_T + N - 1$, with l integer, be a sinusoid with frequency $\Omega_l = 2\pi l/N$, let $y_e = Gu_e$, and let N_T be sufficiently large such that the transient has decayed adequately. Since G is stable and u_e is an eigenfunction of any LTI system

$$y_e[n] = H(\Omega_l) u_e[n] + e_e[n], \qquad n \geq N_T$$

where, for a fixed value of N, $e_e[n] \to 0$ as $N_T \to \infty$. Therefore the transient approaches 0 and the output of the system converges to a shifted and scaled version of the input sinusoid.

Frequency Response

Taking the DFT of y_e, e_e and u_e yields

$$Y_e[l] = H(\Omega_l) U_e[l] + E_e[l]$$

where $E_e[l] \to 0$ as $N_T \to \infty$. Note that $U_e[l] = N$ since all the energy is concentrated at one frequency. The frequency response can be estimated

$$\widehat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]} = H(\Omega_l) + \frac{E_e[l]}{N} + \frac{Y_d[l]}{N}$$

where

$$Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n] e^{-j\frac{2\pi}{N}ln} \qquad Y_d[l] = \sum_{n=N_T}^{N_T+N-1} y_d[n] e^{-j\frac{2\pi}{N}ln}$$

The same results apply to closed loop systems.

Identification of the Transfer Function

Given the design parameters A and B (number of respective coefficients) one can identify the unknown parameters a_k and b_k of the system's transfer function

$$H(z) = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1 + \sum_{k=1}^{A-1} a_k z^{-k}} \qquad H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{A-1} a_k e^{-j\Omega k}}$$

which results in a system of $2L$ equations

$$\begin{aligned} R_l[\cos(\theta_l) + a_1\cos(\theta_l - \Omega_l) + \dots + a_{A-1}\cos(\theta_l - (A-1)\Omega_l)] \\ = b_0 + b_1\cos(\Omega_l) + \dots + b_{B-1}\cos((B-1)\Omega_l) \\ R_l[\sin(\theta_l) + a_1\sin(\theta_l - \Omega_l) + \dots + a_{A-1}\sin(\theta_l - (A-1)\Omega_l)] \\ = -b_1\sin(\Omega_l) - \dots - b_{B-1}\sin((B-1)\Omega_l) \end{aligned}$$

This system of equations can be converted to the least squares problem of minimizing

$$(F\Theta - G)^\top (F\Theta - G)$$

where $\Theta = [a_1 \ a_2 \ \dots \ a_{A-1} \ b_0 \ b_1 \ \dots \ b_{B-1}]^\top$. The LS solution yields the estimated coefficients Θ^*.

$$\Theta^* = (F^\top F)^{-1} F^\top G \qquad F \text{ must have full rank}$$