System Modeling – Lecture 3
Mechanical Systems and Lagrange

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What is the goal of the lecture?

- Be able to understand, explain and use:
  - Kinetic and potential energy
  - Generalized and minimal coordinates
  - Degrees of freedom
  - Lagrange formalism
  - Holonomic and non-holonomic (more in Lecture 4)
  - Generalized forces

Theory – Potential Energy

The potential energy can always be expressed as a function only of the body's coordinates, i.e.,

$$U = U(x, y)$$

Examples of potential energies for a mechanical system are:

Gravitational

$$U = mgx$$

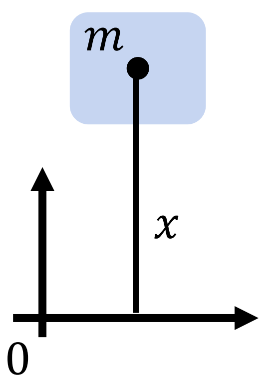
Linear Spring

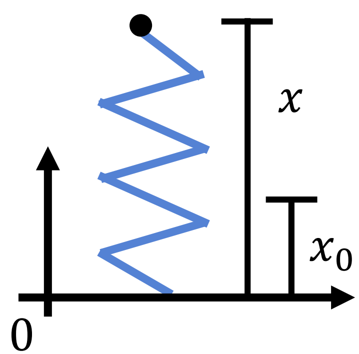
$$U = \frac{1}{2} k_l (x - x_0)^2$$

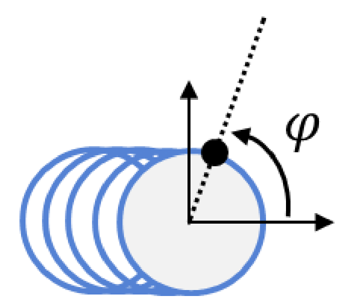
Torsional Spring

$$U = \frac{1}{2} k_r (\varphi - \varphi_0)^2$$

Spring constant

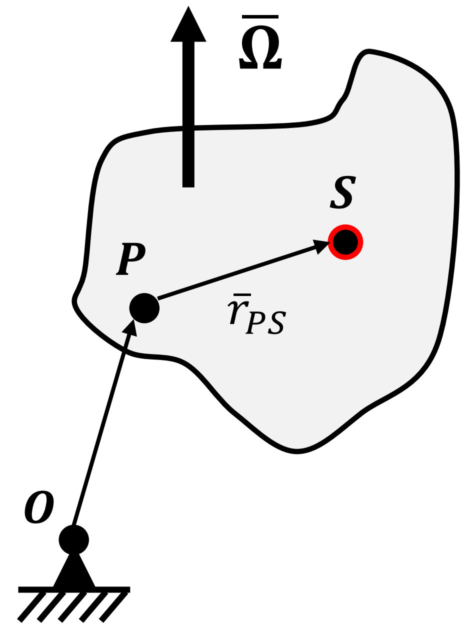






Theory – Kinetic Energy

$$T = \underbrace{\frac{1}{2} m \bar{v}_P^T \bar{v}_P}_{\text{Translational term}} + \underbrace{m \bar{v}_P^T (\bar{\Omega} \times \bar{r}_{PS})}_{\text{Coupling term}} + \underbrace{\frac{1}{2} \bar{\Omega}^T \Theta_P \bar{\Omega}}_{\text{Rotational term}}$$



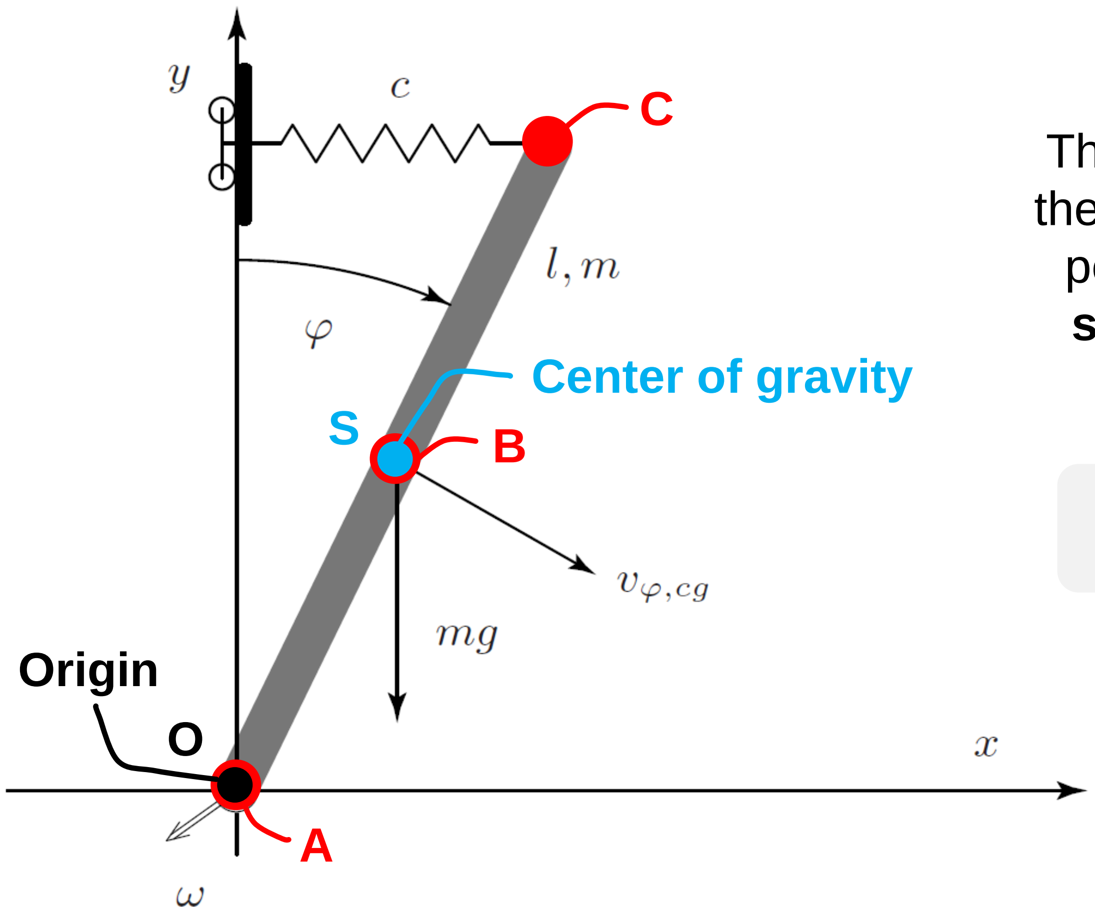
where

- $\bar{v}_P$ is the velocity of the point P
- $\bar{r}_{PS}$ is the position vector from point P to the point of the body's center of gravity S
- $\bar{\Omega}$ is the rotational speed of the body
- m is the mass of the body
- Θ_P is the moment of inertia of the body in the point P

$\bar{\Omega}$ is the same in each point of the body!

Remark
the kinetic energy equation simplifies if the point $\boldsymbol{P}$ is chosen to be equal to $\boldsymbol{O}$ or $\boldsymbol{S}$

Example – Kinetic Energy



The equation describing the **kinetic energy** of the pendulum must be the **same in all the three** considered points.

$$T_A = T_B = T_C$$

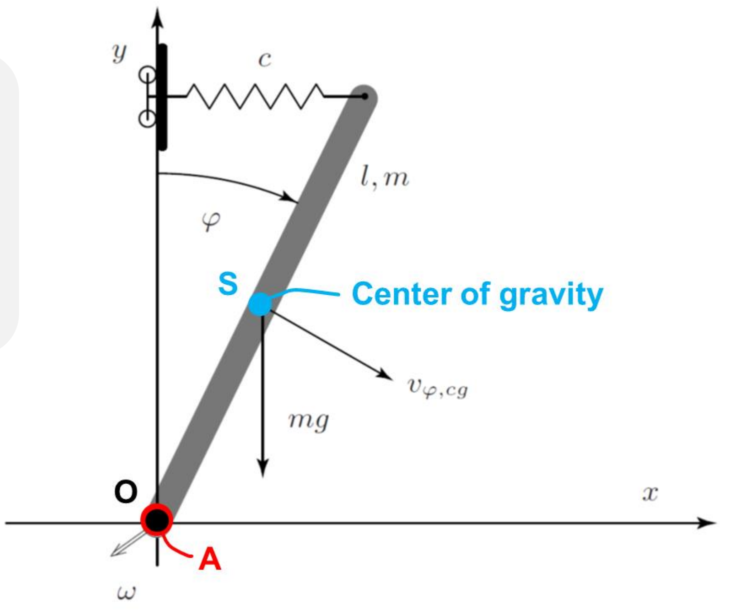
Example – Kinetic Energy – Point A

$$\bar{r}_{AS} = \begin{bmatrix} \frac{l}{2}\sin(\varphi) \\ \frac{l}{2}\cos(\varphi) \\ 0 \end{bmatrix}$$

$$\bar{r}_{OA} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\bar{r}}_{OA} = \bar{v}_A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\Omega} = \begin{bmatrix} 0 \\ 0 \\ -\dot{\varphi} \end{bmatrix}$$



$$\frac{1}{2} m \bar{v}_A^T \bar{v}_A = 0$$

$$m \bar{v}_A^T (\bar{\Omega} \times \bar{r}_{AS}) = 0$$

$$\Theta_A = \frac{1}{3} m l^2$$

$$\frac{1}{2} \bar{\Omega}^T \Theta_A \bar{\Omega} = \frac{1}{2} \Theta_A \dot{\varphi}^2 = \frac{1}{6} m l^2 \dot{\varphi}^2$$

$$T_A = \frac{1}{6} m l^2 \dot{\varphi}^2$$

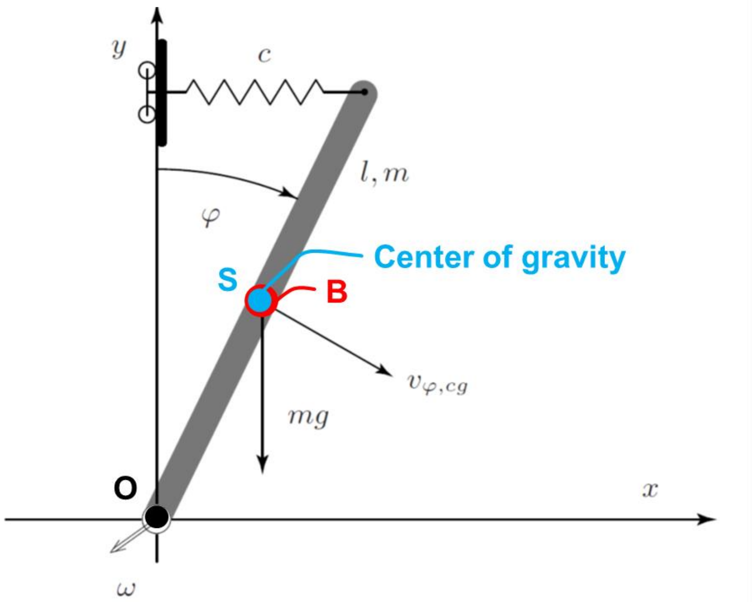
Example – Kinetic Energy – Point B

$$\bar{r}_{BS} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\Omega} = \begin{bmatrix} 0 \\ 0 \\ -\dot{\varphi} \end{bmatrix}$$

$$\bar{r}_{OB} = \begin{bmatrix} \frac{l}{2}\sin(\varphi) \\ \frac{l}{2}\cos(\varphi) \\ 0 \end{bmatrix}$$

$$\dot{\bar{r}}_{OB} = \bar{v}_B = \begin{bmatrix} \frac{l}{2}\cos(\varphi)\dot{\varphi} \\ -\frac{l}{2}\sin(\varphi)\dot{\varphi} \\ 0 \end{bmatrix}$$



$$\frac{1}{2}m\bar{v}_B^T\bar{v}_B = \frac{1}{2}m\frac{l^2}{4}\dot{\varphi}^2\underbrace{(\cos^2(\varphi)+\sin^2(\varphi))}_{1}$$

$$m\bar{v}_B^T(\bar{\Omega}\times\bar{r}_{BS}) = 0$$

$$\Theta_B = \frac{1}{12}ml^2$$

$$\frac{1}{2}\bar{\Omega}^T\Theta_B\bar{\Omega} = \frac{1}{2}\Theta_B\dot{\varphi}^2 = \frac{1}{24}ml^2\dot{\varphi}^2$$

$$T_B = \frac{1}{6}ml^2\dot{\varphi}^2$$

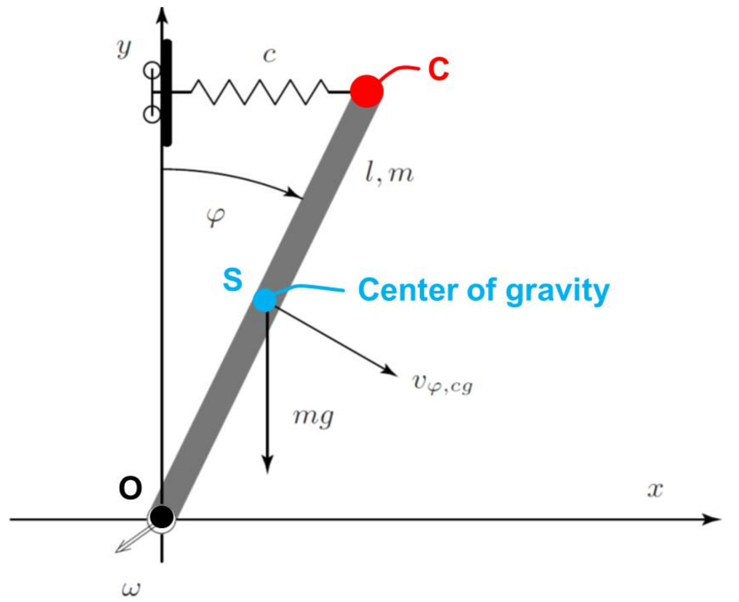
Example – Kinetic Energy – Point C

$$\bar{r}_{CS} = \begin{bmatrix} -\frac{l}{2}\sin(\varphi) \\ -\frac{l}{2}\cos(\varphi) \\ 0 \end{bmatrix}$$

$$\bar{\Omega} = [0 \quad 0 \quad -\dot{\varphi}]^T$$

$$\bar{r}_{OC} = \begin{bmatrix} l\sin(\varphi) \\ l\cos(\varphi) \\ 0 \end{bmatrix}$$

$$\dot{\bar{r}}_{OC} = \bar{v}_C = \begin{bmatrix} l\cos(\varphi)\dot{\varphi} \\ -l\sin(\varphi)\dot{\varphi} \\ 0 \end{bmatrix}$$



$$\frac{1}{2}m\bar{v}_C^T\bar{v}_C = \frac{1}{2}ml^2\dot{\varphi}^2\underbrace{(\cos^2(\varphi)+\sin^2(\varphi))}_{1}$$

$$m\bar{v}_C^T(\bar{\Omega}\times\bar{r}_{CS}) = -\frac{1}{2}ml^2\dot{\varphi}^2$$

$$\Theta_C = \frac{1}{3}ml^2$$

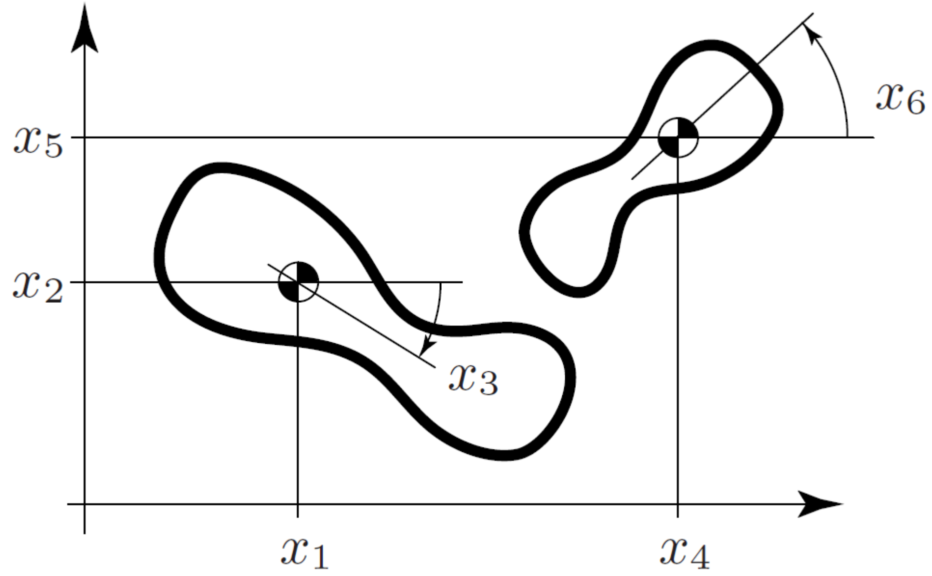
$$\frac{1}{2}\bar{\Omega}^T\Theta_C\bar{\Omega} = \frac{1}{2}\Theta_C\dot{\varphi}^2 = \frac{1}{6}ml^2\dot{\varphi}^2$$

$$T_C = \frac{1}{6}ml^2\dot{\varphi}^2$$

Theory – Degrees of freedom

A planar mechanical system consists of $\boldsymbol{n}$ **rigid bodies**. Each body has 3 **degrees of freedom:**

- Horizontal
- Vertical
- Angular



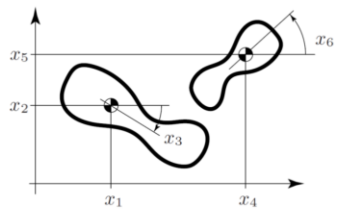
Planar mechanical systems that are constrained in their movement by $\boldsymbol{k}$ **holonomic constraints,** only have $\boldsymbol{3n - k}$ **degrees of freedom**.

There is always a set of independent **generalized coordinates** $q(t) = [q_1(t), \dots, q_{3n-k}(t)]^T$ that describes the behaviour of the constrained system. The choice of the generalized coordinates is **not unique**.

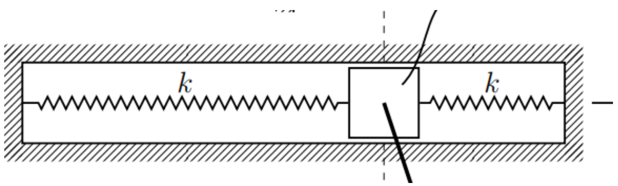
Remarks – Lagrange Formalism

The **Lagrange formalism** is a powerful tool that allows to derive the equations of motion of a mechanical system. This method offers several advantages over the reservoir based approach or the Newton’s law if:

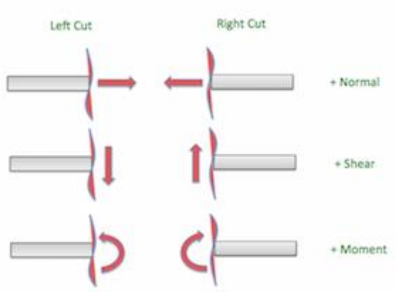
- The system features multiple degrees of freedom and/or bodies.



- Constraint forces do not have to be explicitely accounted.



- No knowledge about internal forces is required.



https://study.com/academy/lesson/internal-force-definition-examples.html

- Direct inclusion of non-holonomic constraints in the equation of motion is sought (more in Lecture 4).

Theory – Lagrange Formalism (Holonomic)

The **Lagrange function** for a *holonomic* mechanical system is defined as the difference between the **total** kinetic and the **total** potential energy.

$$L(q,\dot{q}) = \sum T(q,\dot{q}) - \sum U(q)$$

where the variables q represent the system's generalized coordinates completely describing the $\boldsymbol{3n-k}$ **degrees of freedom** available, and $\dot{q}$ their corresponding generalized velocities. According to the theory of Lagrangian mechanics, the system dynamics can be described by:

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_i}\right\} - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, \dots, 3n-k$$

where $\boldsymbol{Q_i}$ represents the i^{th} **generalized force** (or torque) acting on the i^{th} generalized coordinate q_i.

Theory – Lagrange «Approach»

- Choose/identify a set of generalized coordinates q.
- Define the total kinetic and potential energy of the system as a function of the generalized coordinates.
- Define the Lagrange function: $$L(q,\dot{q}) = \sum T(q,\dot{q}) - \sum U(q)$$
- If there are non-conservative forces and/or torques acting on the system, compute the generalized forces Q_i.
- Compute the system's equations using the Lagrange formalism for each generalized coordinate q_i:

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_i}\right\} - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, \ldots, 3n - k$$

Theory – Holonomic & Non-holonomic

Constraints establish a mathematical relation between the generalized coordinates and their time derivatives. In general, non-holonomic constraints can be written as:

$$f(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$$

A **non-holonomic** constraint can be thought as a restriction of the trajectory that the system takes to reach a certain configuration. When no dependency on $\dot{\boldsymbol{q}}$ is present, the constraint is called holonomic and reads:

$$f(\boldsymbol{q}, t) = 0$$

Remark: if the constraint depends on $\dot{\boldsymbol{q}}$ it is not necessarily non-holonomic.

A **holonomic** constraint can be interpreted as a restriction of the reachable configurations that the system can take.

Example – Holonomic & Minimal Coordinates

One body → $n = 1$

$$q = \begin{bmatrix} x_P \\ y_P \\ \varphi \end{bmatrix}$$

Holonomic constraints

But ...

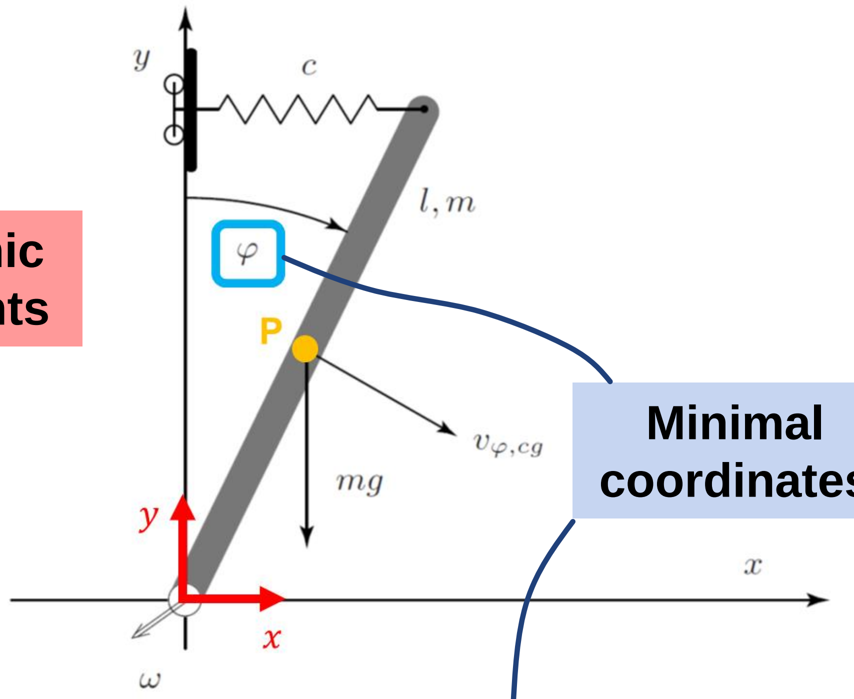
- The coordinates x and y are geometrically coupled. In fact the position of the pendulum's CoG can be written as:

$$y_P = f(x_P) = \sqrt{\left(\frac{l}{2}\right)^2 - x_P^2}$$

$$x_P = f(\varphi) = \frac{l}{2}\sin(\varphi)$$

Therefore

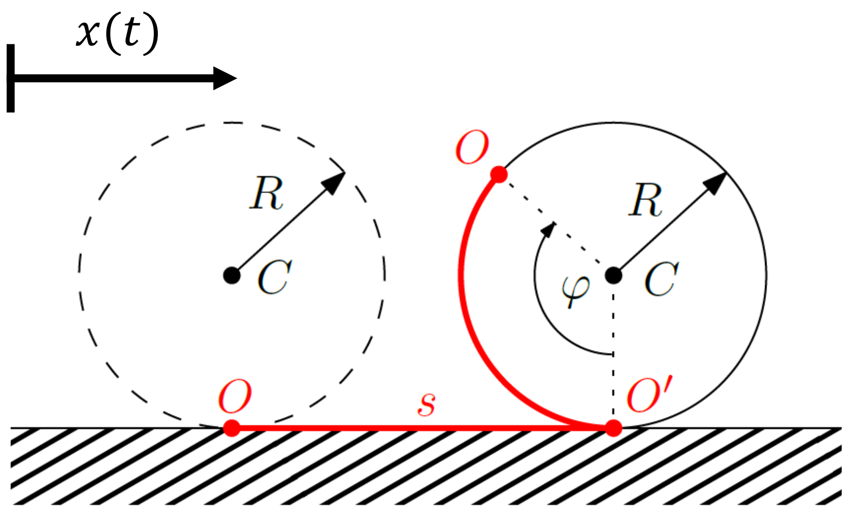
$k = 2$ and the #DOF is $3n - k = 3 - 2 = 1$



Minimal coordinates

Due to the nature of the considered body (pendulum), an appropriate generalized coordinate is the angle φ

Example – Holonomic & Non-holonomic



No slip condition

$$\dot{x}(t) = R \cdot \dot{\varphi}(t)$$

Is this a **holonomic** or a **non-holonomic** constraint?
→ Holonomic, why?

The constraint contains the time derivative of the generalized coordinates. For this reason it can be both: holonomic or non-holonomic (recall remark slide 13).

If you integrate the no slip condition:

$$\Delta x(t) = R \cdot \Delta\varphi(t)$$

And rearranging the terms one can get:

$$x(t) = x_0 + R \cdot (\varphi(t) - \varphi_0)$$

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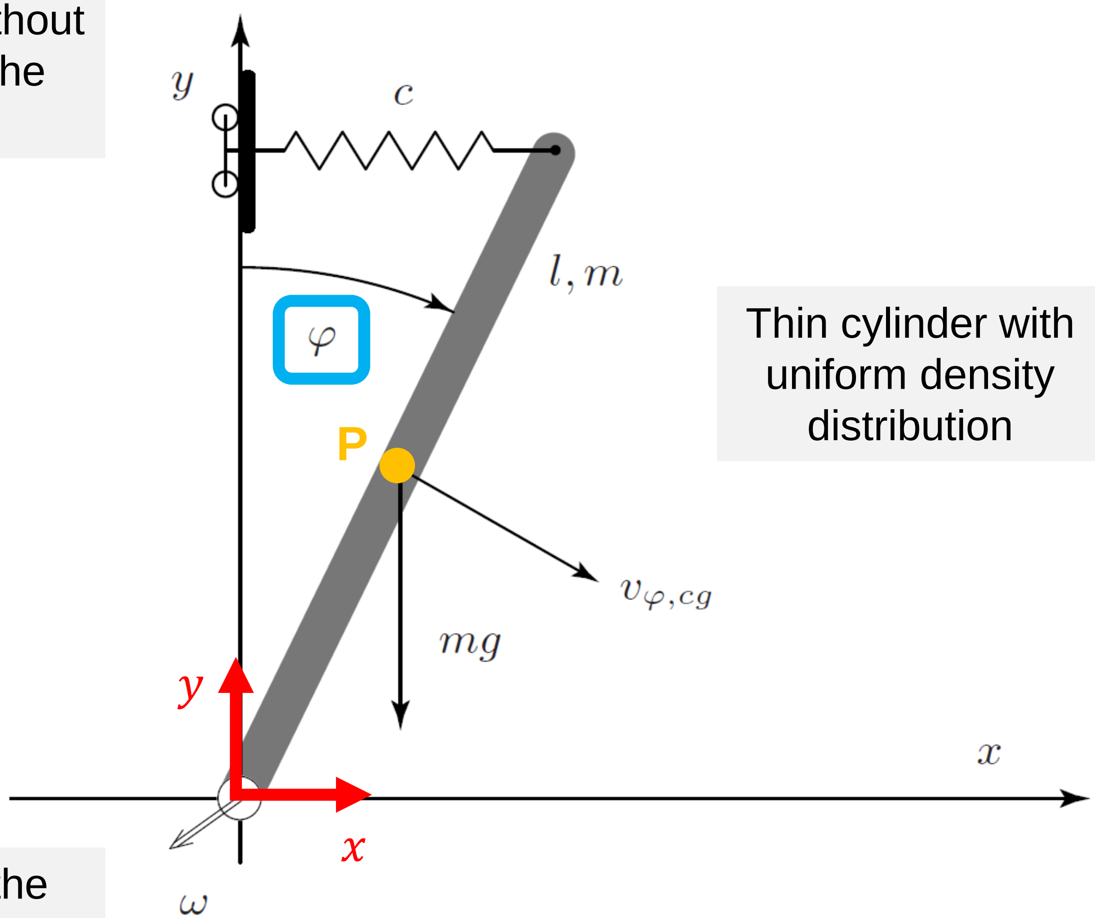
Example – Lagrange (Pendulum with Spring)

Spring slides without friction along the vertical axis



Thin cylinder with uniform density distribution

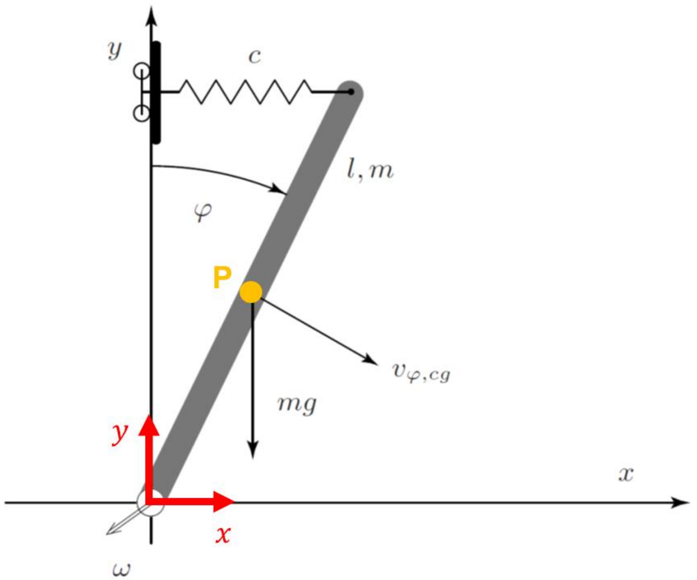
No friction in the pendulum's bearing

Example – Lagrange (Pendulum with Spring)

Pendulum

$$\bar{r}_{OP} = \begin{bmatrix} \frac{l}{2}\sin(\varphi) \\ \frac{l}{2}\cos(\varphi) \end{bmatrix} \qquad \bar{v}_P = \begin{bmatrix} \frac{l}{2}\cos(\varphi)\,\dot{\varphi} \\ -\frac{l}{2}\sin(\varphi)\,\dot{\varphi} \end{bmatrix}$$

$$\bar{\Omega}_P = \omega \cdot \bar{e}_z = -\dot{\varphi} \cdot \bar{e}_z$$



$$T_{pendulum} = \frac{1}{2} m \frac{l^2}{4}\,\dot{\varphi}^2 + \frac{1}{2}\frac{1}{12} m l^2 \dot{\varphi}^2 = \frac{1}{6} m l^2 \dot{\varphi}^2$$

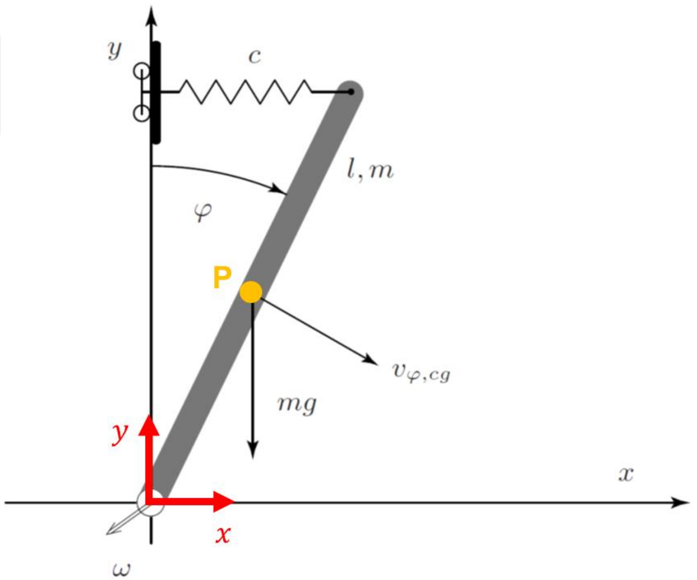
$$U_{pendulum} = mg\frac{l}{2}cos(\varphi)$$

$$U_{spring} = \frac{1}{2} c(l\sin(\varphi))^2$$

Example – Lagrange (Pendulum with Spring)

$$L = \left(T_{pendulum}\right) - \left(U_{pendulum} + U_{spring}\right)$$

$$L = \frac{1}{6}ml^2\dot{\varphi}^2 - mg\frac{l}{2}cos(\varphi) - \frac{1}{2}c(l\sin(\varphi))^2$$



<u>Generalized coordinate</u> φ

$$\frac{\partial L}{\partial \varphi} = \frac{1}{2}mgl\sin(\varphi) \; - cl^2\sin(\varphi)\cos(\varphi)$$

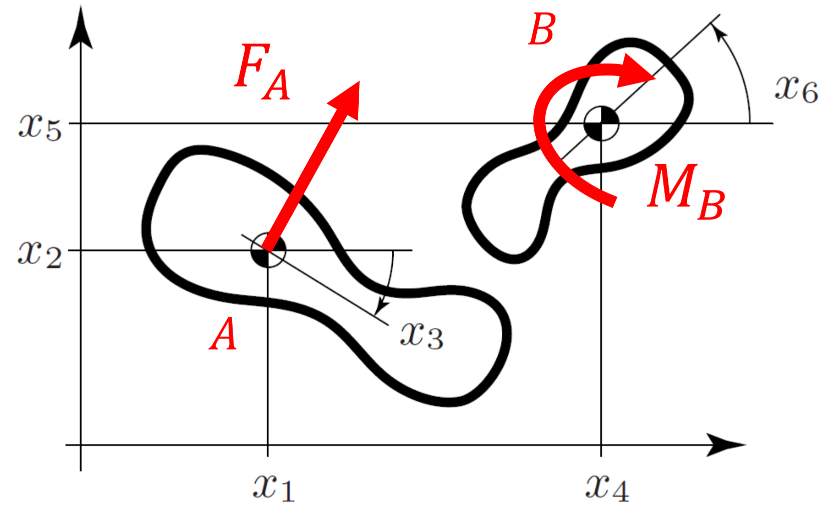
$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{3}ml^2\dot{\varphi} \longrightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}}\right\} = \frac{1}{3}ml^2\ddot{\varphi}$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}}\right\} - \frac{\partial L}{\partial \varphi} = Q_\varphi \longrightarrow \frac{1}{3}ml^2\ddot{\varphi} - \frac{1}{2}mgl\sin(\varphi) \; - cl^2\sin(\varphi)\cos(\varphi) = 0$$

Theory – Generalized Forces

Suppose to have a **force** F_A acting on body 1 in point A and/or a **torque** M_B acting on body 2 in point B.

How do we compute the **generalized forces** $\boldsymbol{Q_1}$, $\boldsymbol{Q_2}$, i.e., the forces/torques written as a function of the generalized coordinates?



Force

$$v_A = J_A \cdot \dot{q} + \xi_A$$
$$Q_A = J_A^T F_A$$

v_A velocity in A

ξ_A offset term

Torque

$$\omega_B = J_B \cdot \dot{q} + \xi_B$$
$$Q_B = J_B^T M_B$$

ω_B angular velocity in B

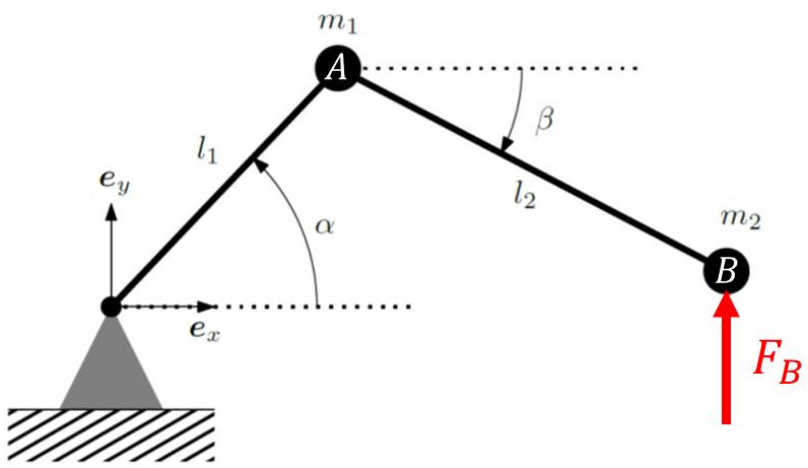
ξ_B offset term

Similarly to the coordinate transformation (recall Analysis), the **Jacobian matrix** $\boldsymbol{J}$ is used to transform the forces/torques written in $(\boldsymbol{x}, \boldsymbol{y})$ to $(\boldsymbol{q_1}, \boldsymbol{q_2})$ system of coordinates.

Example – Generalized Forces

<u>Position</u>

$$\bar{r}_B = \begin{bmatrix} l_1\cos(\alpha) + l_2\cos(\beta) \\ l_1\sin(\alpha) - l_2\sin(\beta) \end{bmatrix}$$



<u>Velocity</u>

$$\bar{v}_B = \begin{bmatrix} -l_1\sin(\alpha)\,\dot{\alpha} - l_2\sin(\beta)\dot{\beta} \\ l_1\cos(\alpha)\dot{\alpha} - l_2\cos(\beta)\,\dot{\beta} \end{bmatrix}$$

$$\bar{v}_B = \underbrace{\begin{bmatrix} -l_1\sin(\alpha) & -l_2\sin(\beta) \\ l_1\cos(\alpha) & -l_2\cos(\beta) \end{bmatrix}}_{J_B} \cdot \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

<u>Force</u>

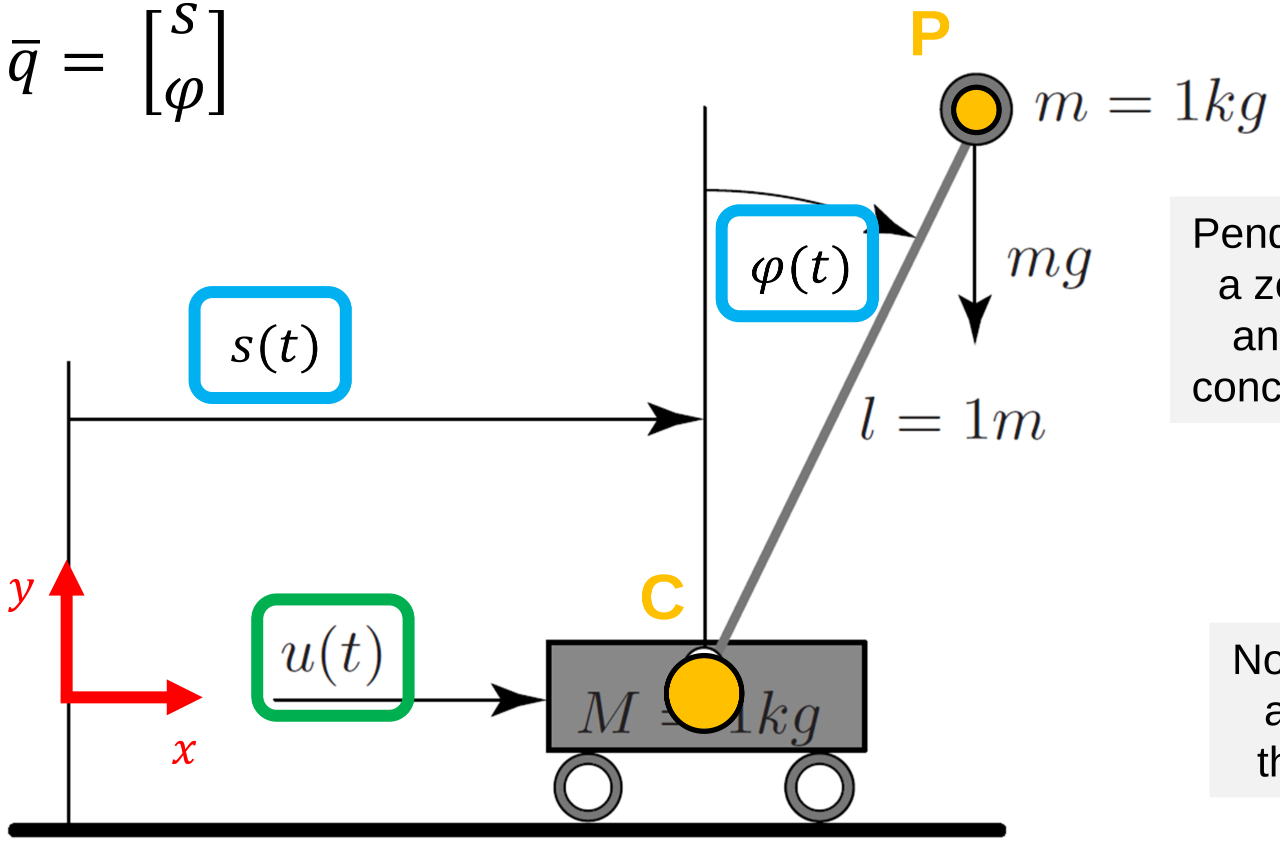
$$\bar{F} = \underbrace{\begin{bmatrix} 0 \\ F_B \end{bmatrix}}_{F}$$

<u>Generalized Forces</u>

$$\boxed{Q_B = J_B^T F = \begin{bmatrix} F_B l_1\cos(\alpha) \\ -F_B l_2 cos(\beta) \end{bmatrix} = \begin{bmatrix} Q_\alpha \\ Q_\beta \end{bmatrix}}$$

Example – Lagrange (Pendulum on a Cart)

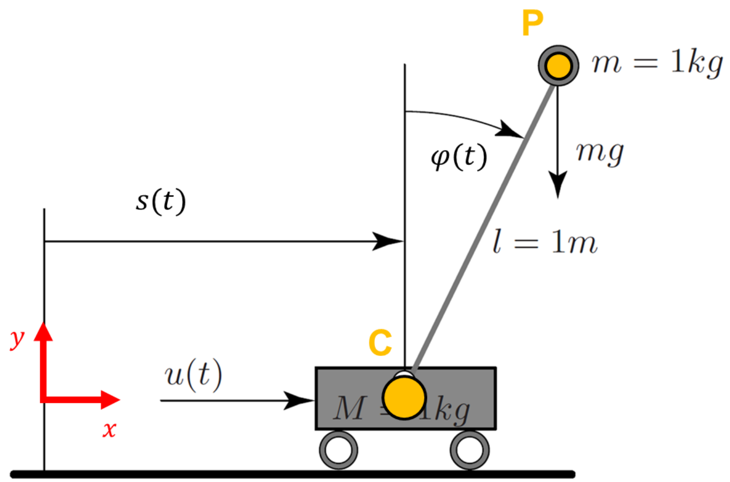
$$\bar{q} = \begin{bmatrix} s \\ \varphi \end{bmatrix}$$



Pendulum consists of a zero-mass beam and a point mass concentrated at its tip

No friction losses are present in the two bodies

Example – Lagrange (Pendulum on a Cart)



Cart

$$\bar{r}_{OC} = \begin{bmatrix} s \\ 0 \end{bmatrix} \qquad \bar{v}_C = \begin{bmatrix} \dot{s} \\ 0 \end{bmatrix}$$

$$T_{cart} = \frac{1}{2} M \dot{s}^2 + 0$$

$$U_{cart} = 0$$

Pendulum

$$\bar{r}_{OP} = \begin{bmatrix} s + l\sin(\varphi) \\ l\cos(\varphi) \end{bmatrix} \qquad \bar{v}_P = \begin{bmatrix} \dot{s} + l\cos(\varphi)\dot{\varphi} \\ -l\sin(\varphi)\dot{\varphi} \end{bmatrix}$$

$$T_{pendulum} = \frac{1}{2} m \left(\dot{s}^2 + 2l\cos(\varphi)\dot{s}\,\dot{\varphi} + l^2\dot{\varphi}^2\right) + \boxed{0}$$

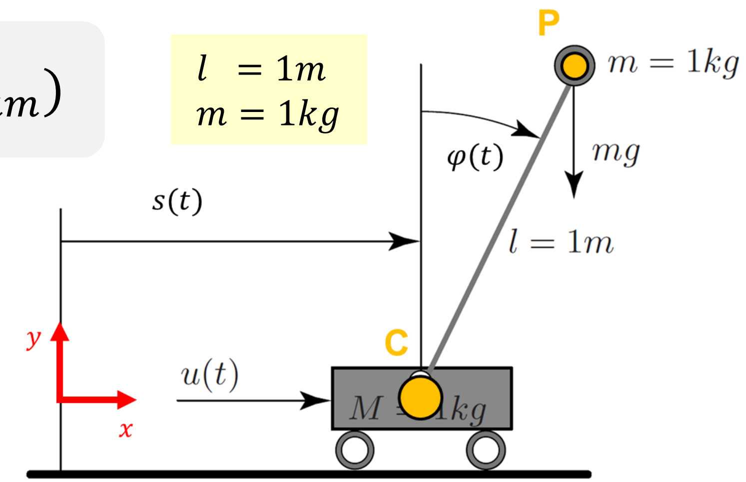
$$U_{pendulum} = mgl\cos(\varphi)$$

The moment of inertia of a point mass is equal to zero.

Example – Lagrange (Pendulum on a Cart)

$$L = (T_{cart} + T_{pendulum}) - (U_{cart} + U_{pendulum})$$

$$L = \dot{s}^2 + \cos(\varphi)\dot{s}\,\dot{\varphi} + \frac{1}{2}\dot{\varphi}^2 - \cos(\varphi)\,g$$



Generalized coordinate s

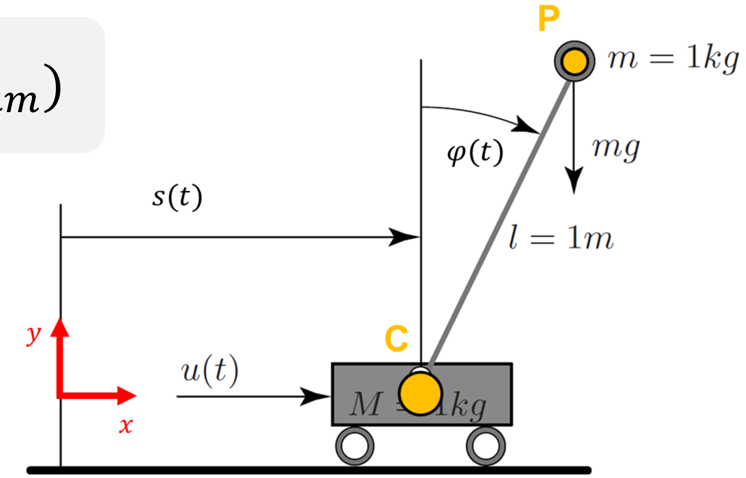
$$\frac{\partial L}{\partial s} = 0$$

$$\frac{\partial L}{\partial \dot{s}} = 2\dot{s} + \cos(\varphi)\,\dot{\varphi} \longrightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{s}}\right\} = 2\ddot{s} - \sin(\varphi)\dot{\varphi}^2 + \cos(\varphi)\,\ddot{\varphi}$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{s}}\right\} - \frac{\partial L}{\partial s} = Q_s \longrightarrow 2\ddot{s} - \sin(\varphi)\dot{\varphi}^2 + \cos(\varphi)\,\ddot{\varphi} = u(t)$$

Example – Lagrange (Pendulum on a Cart)

$$L = (T_{cart} + T_{pendulum}) - (U_{cart} + U_{pendulum})$$



$$L = \dot{s}^2 + \cos(\varphi)\dot{s}\,\dot{\varphi} + \frac{1}{2}\dot{\varphi}^2 - \cos(\varphi)\,g$$

Generalized coordinate φ

$$\frac{\partial L}{\partial \varphi} = -\sin(\varphi)\dot{\varphi}\,\dot{s} + \sin(\varphi)\,g$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \cos(\varphi)\,\dot{s} + \dot{\varphi} \longrightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}}\right\} = -\sin(\varphi)\dot{\varphi}\dot{s} + \cos(\varphi)\,\ddot{s} + \ddot{\varphi}$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}}\right\} - \frac{\partial L}{\partial \varphi} = Q_{\varphi} \longrightarrow \ddot{\varphi} + \cos(\varphi)\,\ddot{s} - \sin(\varphi)\,g = 0$$

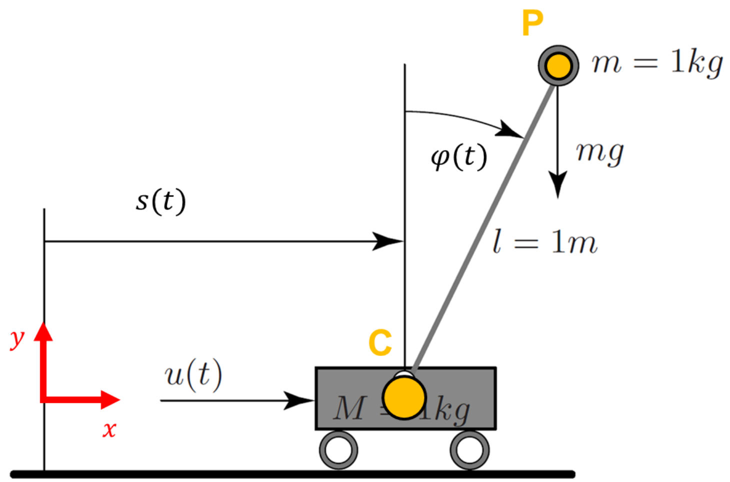
Example – Lagrange (Pendulum on a Cart)

Generalized coordinate s

$$2\ddot{s} - \sin(\varphi)\dot{\varphi}^2 + \cos(\varphi)\,\ddot{\varphi} = u(t)$$

Generalized coordinate φ

$$\ddot{\varphi} + \cos(\varphi)\,\ddot{s} - \sin(\varphi)\,g = 0$$



Solving for the two generalized coordinates s and φ yields

$$\ddot{s} = \frac{\sin(\varphi)\dot{\varphi}^2 - \cos(\varphi)\sin(\varphi)\,g + u}{2 - \cos^2(\varphi)}$$

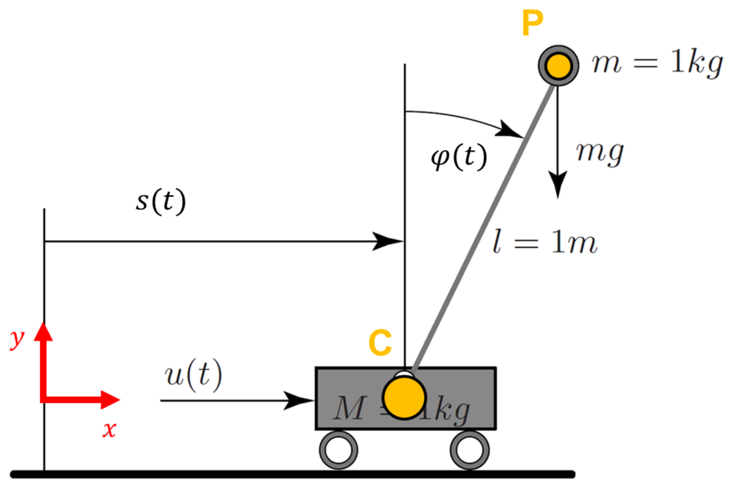
$$\ddot{\varphi} = \frac{2g\sin(\varphi) - \cos(\varphi)\sin(\varphi)\dot{\varphi}^2 - \cos(\varphi)\,u}{2 - \cos^2(\varphi)}$$

Remarks – Lagrange (Pendulum on a Cart)

Equation of motion

$$\ddot{s} = \frac{\sin(\varphi)\dot{\varphi}^2 - \cos(\varphi)\sin(\varphi)\, g + u}{2 - \cos^2(\varphi)}$$

$$\ddot{\varphi} = \frac{2g\sin(\varphi) - \cos(\varphi)\sin(\varphi)\dot{\varphi}^2 - \cos(\varphi)\, u}{2 - \cos^2(\varphi)}$$



The Langrangian formalism produces *in general* equations of the form:

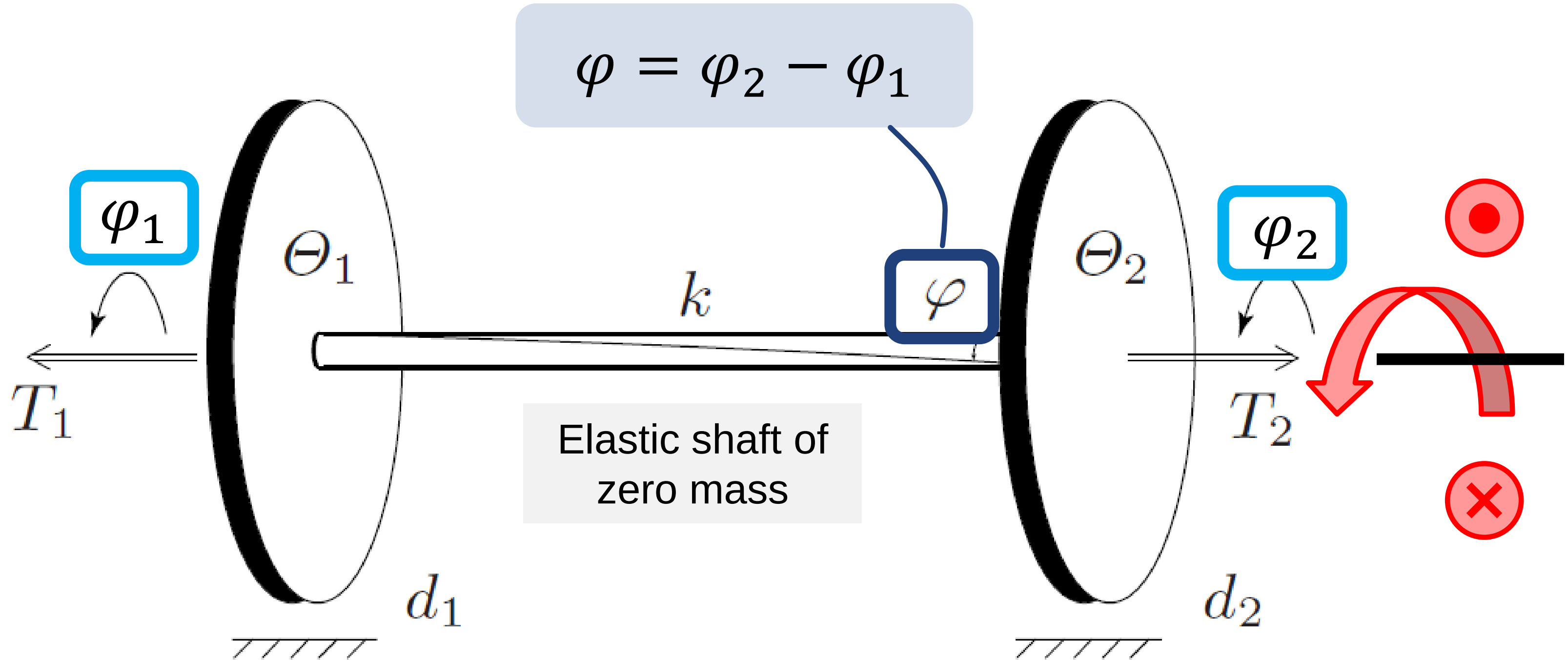
$$M\big(q(t)\big) \cdot \ddot{q}(t) = f\big(q(t), \dot{q}(t), u(t)\big)$$

Number of generalized coordinates

where M is an $n_q \times n_q$ *mass matrix*. For the class of mechanical systems analyzed, the matrix M is **regular** and therefore **invertible**. Hence, an **explicit formulation** of the system's dynamic equations is always possible:

$$\ddot{q}(t) = M^{-1}\big(q(t)\big) \cdot f\big(q(t), \dot{q}(t), u(t)\big)$$

Example – Lagrange (Elastic Rotor)



Elastic shaft of zero mass

Friction losses are present in both disks/masses

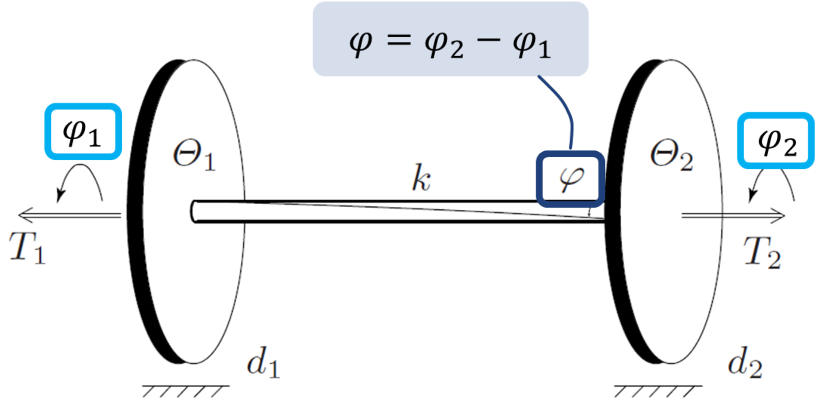
Example – Lagrange (Elastic Rotor)

Body 1

$$T_1 = \frac{1}{2}\Theta_1 \dot{\varphi}_1^{\ 2}$$

$$U_1 = 0$$

$$Q_1 = -T_1 - d_1\,\dot{\varphi}_1$$



Body 2

$$T_2 = \frac{1}{2}\Theta_2 \dot{\varphi}_2^{\ 2}$$

$$U_2 = 0$$

$$Q_2 = T_2 - d_2\,\dot{\varphi}_2$$

Elastic Body

$$T_e = 0$$

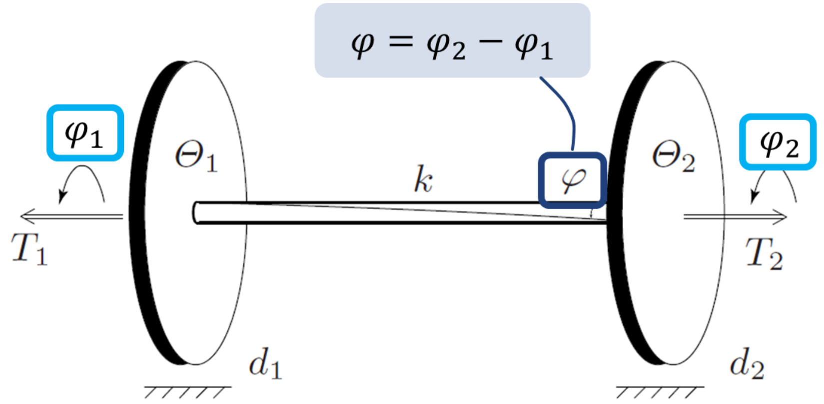
$$U_e = \frac{1}{2}k(\varphi_2 - \varphi_1)^2$$

$$Q_e = 0$$

Example – Lagrange (Elastic Rotor)

$$L = (T_1 + T_2) - (U_e)$$

$$L = \frac{1}{2}\Theta_1\dot{\varphi}_1{}^2 + \frac{1}{2}\Theta_2\dot{\varphi}_2{}^2 - \frac{1}{2}k(\varphi_2 - \varphi_1)^2$$



Generalized coordinate φ_1

$$\frac{\partial L}{\partial \varphi_1} = k(\varphi_2 - \varphi_1)$$

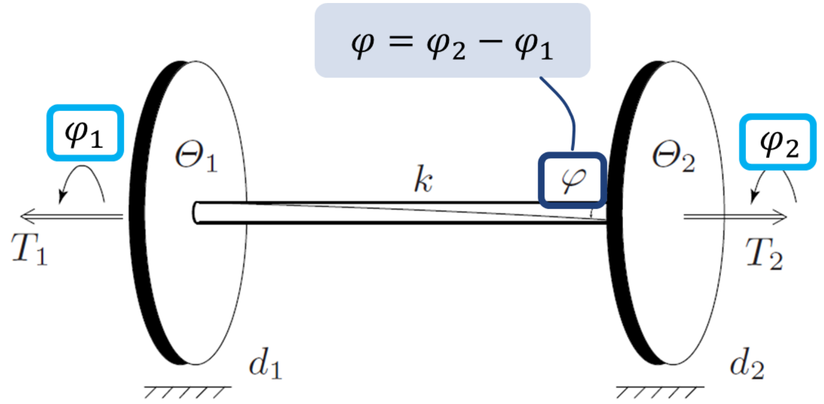
$$\frac{\partial L}{\partial \dot{\varphi}_1} = \Theta_1\dot{\varphi}_1 \longrightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}_1}\right\} = \Theta_1\ddot{\varphi}_1$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}_1}\right\} - \frac{\partial L}{\partial \varphi_1} = Q_{\varphi_1} \longrightarrow \Theta_1\ddot{\varphi}_1 = -T_1 - d_1\dot{\varphi}_1 + k(\varphi_2 - \varphi_1)$$

Example – Lagrange (Elastic Rotor)

$$L = (T_1 + T_2) - (U_e)$$

$$L = \frac{1}{2}\Theta_1\dot{\varphi}_1{}^2 + \frac{1}{2}\Theta_2\dot{\varphi}_2{}^2 - \frac{1}{2}k(\varphi_2 - \varphi_1)^2$$



Generalized coordinate φ_2

$$\frac{\partial L}{\partial \varphi_2} = -k(\varphi_2 - \varphi_1)$$

$$\frac{\partial L}{\partial \dot{\varphi}_2} = \Theta_2\dot{\varphi}_2 \longrightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}_2}\right\} = \Theta_2\ddot{\varphi}_2$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\varphi}_2}\right\} - \frac{\partial L}{\partial \varphi_2} = Q_{\varphi_2} \longrightarrow \Theta_2\ddot{\varphi}_2 = T_2 - d_2\dot{\varphi}_2 - k(\varphi_2 - \varphi_1)$$