ETH zürich

IRIS
Institute of Robotics and Intelligent Systems

Screw Theory

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Exam

Date and Time

- End of Semester
  Tuesday, 15.12.2020
- 9:00 – 10:00 (60 minutes)

Format

- Online
- Based on lecture and assignments

Allowed Material

- Open book exam (everything is allowed)

Week	Topics	Practice Sessions
1	Intro	-
2	Mech Design	Intro
3	Spatial Descriptions 1	Mech Design + H-Matrix
4	Spatial Descriptions 2	Screw Theory
5	Forward Kinematics	Forward Kinematics + DH convention
6	Rigid Body Velocity	Velocity
7	Jacobian	Jacobian
8	Inverse Kinematics + Red. and Para. Robots	Inverse Kin and parallel Robots
9	-	Review + **Mock Exam**
10	Force Control + Vision 1	Force Control
11	Vision 2	Computer Vision
12	Vision 3 + Mobile Robots	Computer vision 2 + quick review of CV
13	Q&A	-
14	Written Exam	-

Repetition from last week: Degrees of freedom vs. degrees of mobility

Degrees of freedom (DoF in a d-dimensional space)

- d : translational DoF
- $\frac{d(d-1)}{2}$: rotational DoF

Degrees of freedom of the end effector

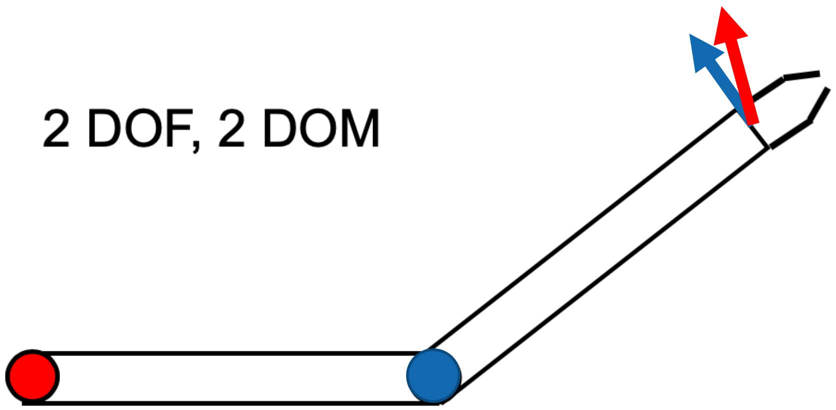
- Number of independent motions of a system
  The number of joints determines the number of DOF
- May depend on robot configuration

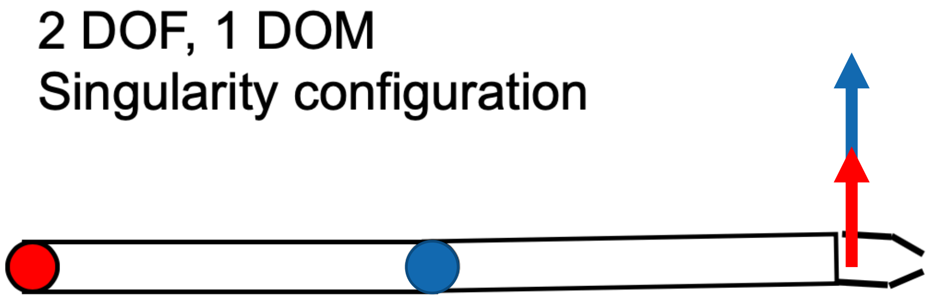
Degrees of Mobility of a robot

- The number of independently controlled joints on a robot

DoF $\neq$ DoM in the case of parallel robots or in singular configurations

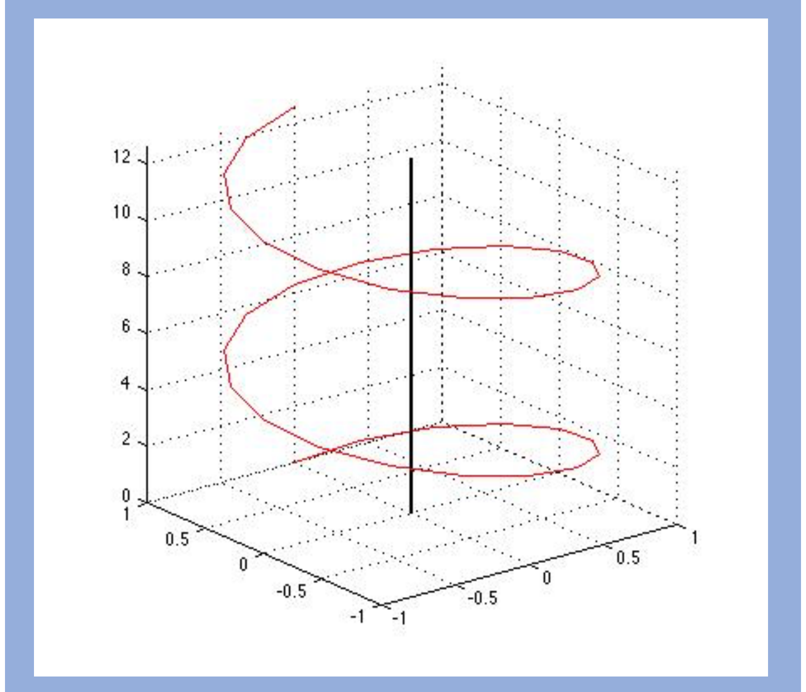
Repetition from last week: Degrees of freedom vs. degrees of mobility





Screw theory

Every rigid body motion can be expressed by a rotation about an axis combined with a translation parallel to that axis. (= screw motion)



Method is based on rigid body **motion** instead of **location!**

→ **Time dependent now!**

Rigid body motions can be described in different ways:
- Screws (geometrical description: screw parameter)
- Twists (mathematical description: abstract)
- Product of exponentials (mathematical description: homogeneous)

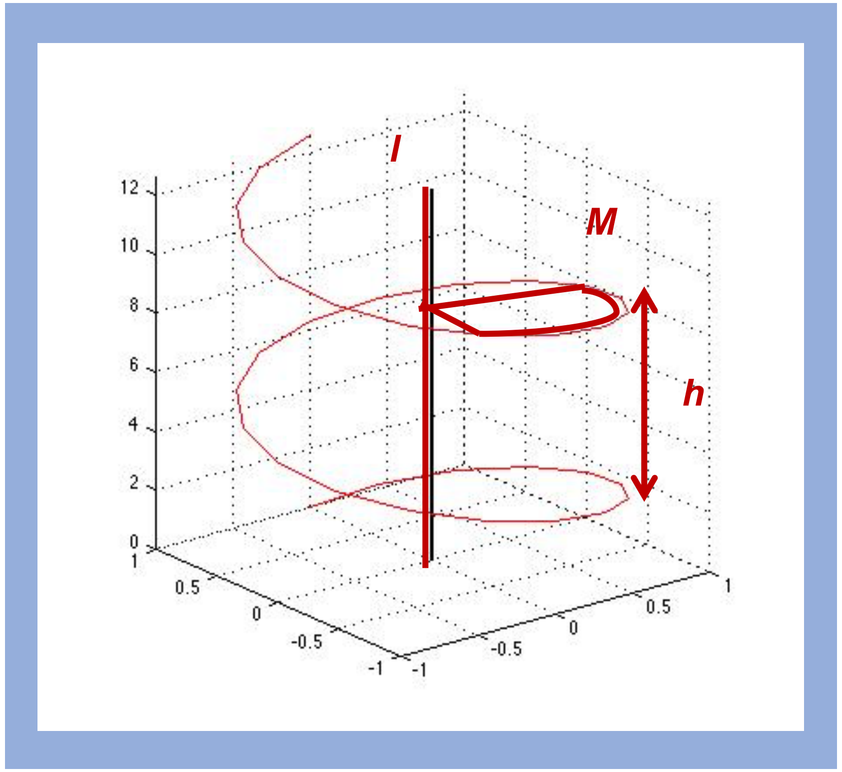
Screw Theory: Geometrical Description

Pitch ***h***
- Ratio of translational motion to rotational motion

$$h = \frac{\omega^T v}{||\omega^2||} = \begin{cases} 0 & \text{pure rotation} \\ \in (0, \infty) & \text{screw motion} \\ \infty & \text{pure translation} \end{cases}$$

Axis ***l***
- Axis of rotation, line through a point
- Direction of translation

$$l = \begin{cases} q + \lambda\omega & \text{pure rotation} \\ \frac{\omega \times v}{||\omega^2||} + \lambda\omega & \text{screw motion} \\ \lambda v & \text{pure translation} \end{cases}$$



Magnitude ***M***
- Amount of displacement
- Net rotation and/or translation

$$M = \begin{cases} \theta & \text{pure rotation} \\ ||\omega|| & \text{screw motion} \\ d & \text{pure translation} \end{cases}$$

Screw Theory: Geometrical Description

Skew-symmetric matrix

$$a \times b = \hat{a} b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} b$$

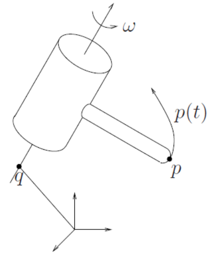
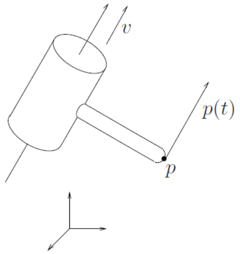
	Twist Coordinates 6x1 vector	**Twist** 4x4 matrix
General description	$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$	$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q + h\omega \\ 0 & 0 \end{bmatrix}$

with $-\omega \times q + h\omega = v$

Derivation: page 39 – A Mathematical Introduction to Robotic Manipulation 1994

Screw Theory: Geometrical Description

Skew-symmetric matrix	$a \times b = \hat{a}b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} b$

	Twist Coordinates 6x1 vector	**Twist** 4x4 matrix
Revolute joint 	$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$	$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}$
Prismatic joint 	$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$	$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$

Derivation: page 39 – A Mathematical Introduction to Robotic Manipulation 1994

Rodrigues' Formula

All **rotation matrices** can be written as a matrix exponential of a **skew-symmetric** matrix!

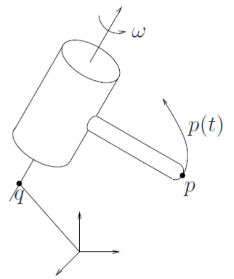
Rodrigues' Formula:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2\left(1-\cos\theta\right)$$

Homogeneous Transformation
4x4 matrix

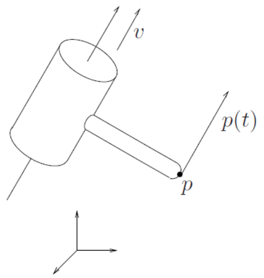
$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)\left(\omega \times v\right) + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

Revolute joint



$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)\left(\omega \times v\right) \\ 0 & 1 \end{bmatrix}$$

Prismatic joint



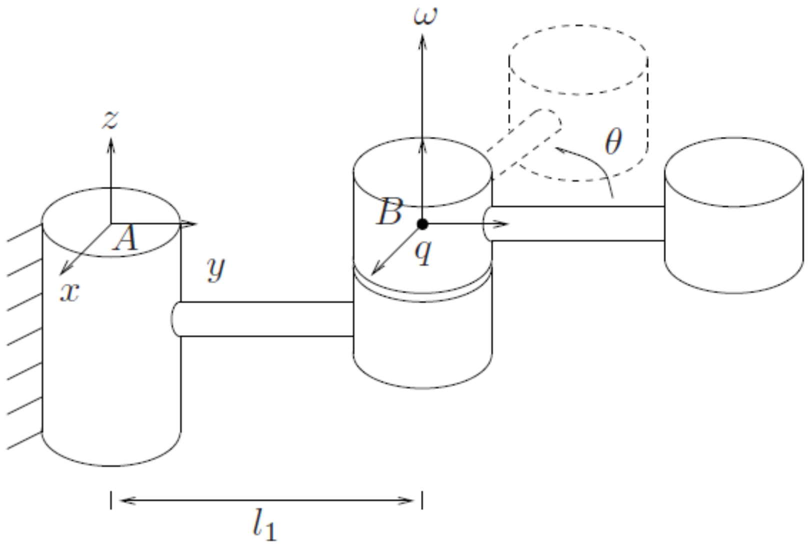
$$e^{\hat{\xi}\theta} = \begin{bmatrix} I & \theta v \\ 0 & 1 \end{bmatrix}$$

Screw Theory: Mathematical Description

If a coordinate frame B is attached to a **rigid body** undergoing **a screw motion**, the **instantaneous** configuration of the coordinate frame B, relative to a fixed frame A, is given by

$$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$$

g(0):
- All joint angles defined as being **zero**
- Describes transformation from the **base frame to tool frame**



This transformation can be **interpreted** as follows:
- Multiplication by $g_{ab}(0)$ maps the coordinates of a point relative to the B frame into A's coordinates
- The exponential map transforms the point to its final location (still in A coordinates).

Kinematics Toolbox