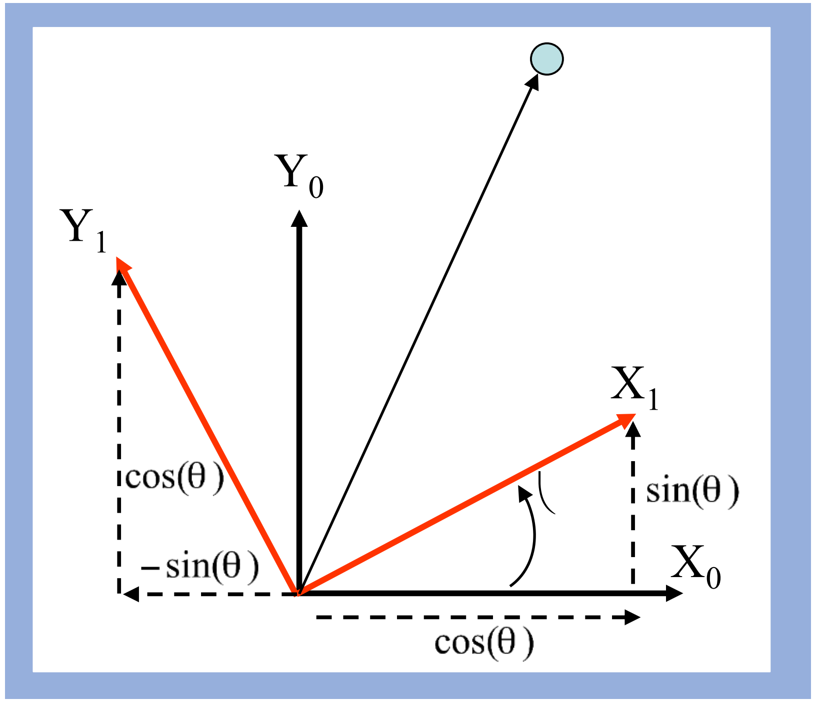
Spatial Descriptions I: Rotation Matrix



- Rotation in the Plane

$$R_1^0 = \begin{bmatrix} X_1^0 & Y_1^0 \end{bmatrix} = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$$

Unit vectors

- Rotation in Three Dimensions

$$R_1^0 = R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}, \quad R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

Spatial Descriptions I: Properties of a Rotation Matrix

- Rotation matrices are **orthogonal** matrices

$$P^0 = R_1^0 P^1$$

- Orthogonal matrices have
  Special orthogonal: SO(3) → $\det R = 1$

$$det R = \pm 1$$

- Each column of R is a **unit vector**

$$R_1^0 = \begin{bmatrix} X_1^0 & Y_1^0 \end{bmatrix}$$

- Rotations do **not change vector norm**

$$\left\| P^0 \right\| = \left\| P^1 \right\|$$

- A rotation matrix is **always invertible**

$$P^1 = (R_1^0)^{-1} P^0$$

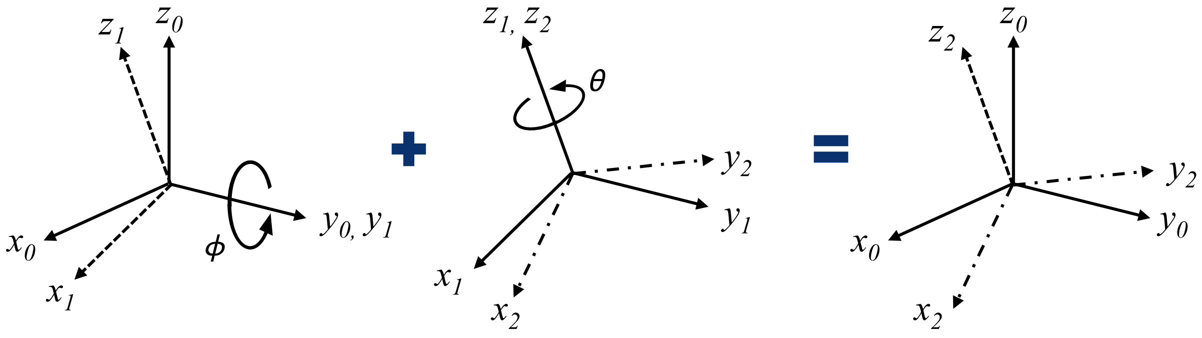
- The inverse is simply the **transpose**

$$(R_1^0)^{-1} = (R_1^0)^T$$

We'll make use of these special properties in our calculations

Spatial Descriptions I: Rule of composition of rotations

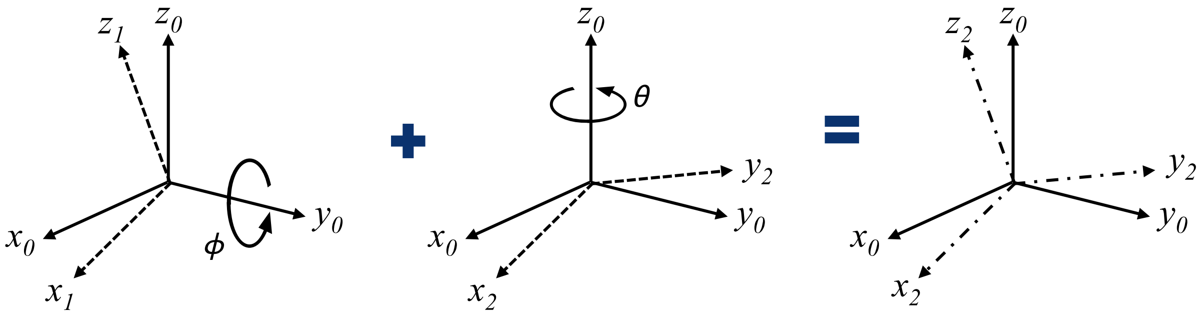
- **Post multiply** for rotations about the current frame



$R_{y,\phi}$ $R_{z,\theta}$

$$R = R_{y,\phi} R_{z,\theta}$$

- **Pre multiply** for rotations about the original/fixed frame



$R_{y,\phi}$ $R_{y,-\phi} R_{z,\theta} R_{y,\phi}$

$$R = R_{y,\phi}[R_{y,-\phi} R_{z,\theta} R_{y,\phi}]$$
$$= R_{z,\theta} R_{y,\phi}$$

Spatial Descriptions I: Homogeneous transformations

$$P^0 = R_1^0 P^1 + T_1^0 = R_2^0 P^2 + T_2^0$$

We can write this as a multiplication of a **4x4 matrix**:

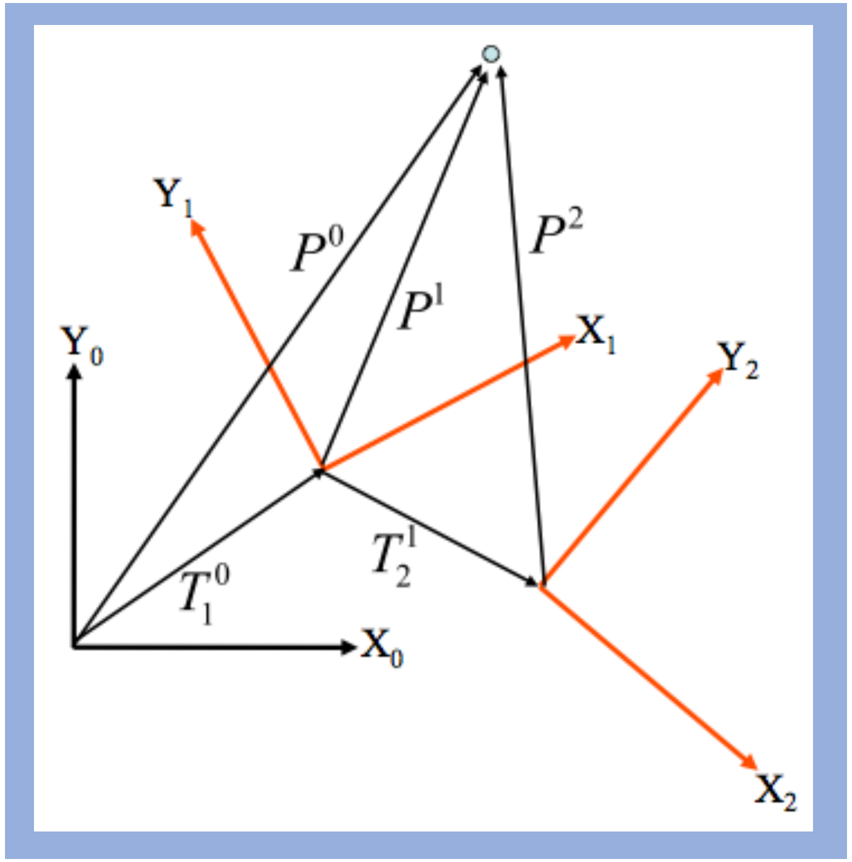
$$P_{\text{H}}^0 = \begin{bmatrix} R_1^0 & T_1^0 \\ [0] & 1 \end{bmatrix} P_{\text{H}}^1$$

$$P^0 = H_1^0 P^1$$

The transformation can be reversed by **inverting** the 4x4 matrix:

$$P_{\text{H}}^1 = (H_1^0)^{-1} P_{\text{H}}^0$$

$$(H_1^0)^{-1} = \begin{bmatrix} (R_1^0)^T & -(R_1^0)^T T_1^0 \\ [0] & 1 \end{bmatrix}$$



Rotation and translation of a point