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1 Mechanical Design

1.1 DOF & DOM & DOF EE

DOF: Indep. of Robot Configuration (2D: 3, 3D: 6)

DOM: # Joints

DOF EE: Indep. Instant. Motions of End-Effector

Grübler's Formula

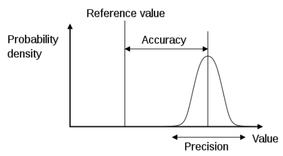
$$DOF = C(N-g) + \sum_{i=1}^{g} f_i$$

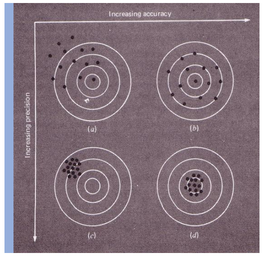
C : 3(2D), 6(3D) f_i : # DOF of i^{th} joint

N : # links w/out baselinks g : # joints

1.2 Precision and Accuracy

Need more than one measurement to be determined.





Precision "Repeatability" of two+ measurements; std(M)

Accuracy "Closeness" to a standard or known value Accuracy mean; mean(M) $- M_R$

1.3 Resolution

Actuator: Smallest commendable distance
Sensor: Smallest measurable interval

2 Spatial Description

$$P^0 = R_1^0 P^1 + T_1^0$$

2.1 Rotations

2.1.1 Rotational Matrix $R_{x_i,\theta}$

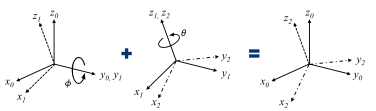
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}_x \quad \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}_y \quad \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_z$$

$$R^{-1} = R^T \qquad \det(R) = \pm 1$$

2.1.2 Composition of Rotations

Post Multiply

About "new"/ current frame of object:

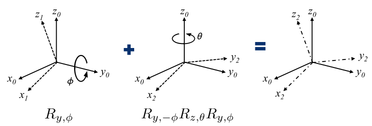


$R_{y,\phi}$ $R_{z,\theta}$

$$R = R_{y,\phi} R_{z,\theta}$$

Pre Multiply "Fixed First"

About original/ fixed frame:



$R_{y,\phi}$ $R_{y,-\phi}R_{z,\theta}R_{y,\phi}$

$$R = R_{y,\phi}[R_{y,-\phi}R_{z,\theta}R_{y,\phi}] = R_{z,\theta}R_{y,\phi}$$

2.2 Homogeneous Transformation

Principle

$$y = ax + b, \qquad y = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Application H_1^0

$$P^0 = R_1^0 P^1 + T_1^0 \longrightarrow P_H^0 = H_1^0 P_H^1$$

$$\begin{bmatrix} P^0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1^0 & T_1^0 \\ [0] & 1 \end{bmatrix}}_{H_1^0} \begin{bmatrix} P^1 \\ 1 \end{bmatrix}$$

T_1^0: Translation from 0 to 1

Inverse $(H_1^0)^{-1}$

$$(H_1^0)^{-1} = H_0^1 = \begin{bmatrix} (R_1^0)^T & -(R_1^0)^T T_1^0 \\ [0] & 1 \end{bmatrix}$$

3 Screw Theory

3.1 Rigid Body Transformation

A mapping g is a rigid body transformation if:

- **length** (distances between pts.) is preserved

$$\|g(q) - g(p)\| = \|q - p\|$$

- **crossproduct** (orientation) is preserved:

$$g(v \times w) = g(v) \times g(w)$$

3.2 Mathematical Remarks

3.2.1 Skew - Symmetric Matrix $\hat{a}$

$$a \times b = \hat{a} b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

3.2.2 Matrix Exponential

$$e^{\hat{\omega}\theta} = \mathbb{I} + \hat{\omega}\theta + \frac{1}{2!}(\hat{\omega}\theta)^2 + \frac{1}{3!}(\hat{\omega}\theta)^3 + \dots$$

$$\stackrel{*}{=} \mathbb{I} + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

(∗) Rodrigues' Formula

3.3 Screw Parameters & Twist

All parameters respresented in **reference frame.**

Pitch h : Ratio of translational and rotational motion

Axis l : Axis of rotation / direction of translation

Magnitude M : Amount of rotation/ translation

General Case

$$h = \frac{\omega^T v}{\|\omega^2\|} \qquad l = \frac{\omega \times v}{\|\omega^2\|} + \lambda\omega \qquad M = \|\omega\|$$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q + h\omega \\ 0 \quad 0 \quad 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix} \in \mathbb{R}^{6\times1}$$

Rotation

$$h = 0 \qquad l = q + \lambda\omega \qquad M = \theta$$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 \quad 0 \quad 0 & 0 \end{bmatrix}$$

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

Translation

$$h = \infty \qquad l = \lambda v \qquad M = \theta$$

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathbb{R}^{6\times1}$$

3.4 Matrix Exponentials

$$\boxed{g_{st}(\theta_1, \dots, \theta_n) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_n\theta_n} \cdot g_{st}(0) = H_n^0}$$

$g_{st}(0)$: IC of n^{th} frame w.r.t. 0^{th} frame.

3.4.1 Homogeneous Transformation

The twist matrix exponential yields a Homogeneous Transformation. For $\|\omega\| = \|v\| = 1$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (\mathbb{I} - e^{\hat{\omega}\theta})(\omega \times v) + h\theta\omega \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

Revolute **Prismatic**

$$\begin{bmatrix} e^{\hat{\omega}\theta} & (\mathbb{I} - e^{\hat{\omega}\theta})(\omega \times v) \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbb{I} & \theta v \\ 0 & 1 \end{bmatrix}$$

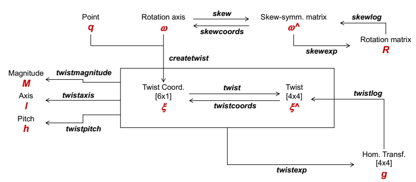
3.4.2 Initial Coniditions

$$\boxed{g_{ab}(\theta) = e^{\hat{\xi}\theta} \cdot g_{ab}(0)}$$

$g_{ab}(0)$ describes transformation from base to toolframe.

$$g_{ab}(0) = \begin{bmatrix} R(0) & q \\ 0 & 1 \end{bmatrix}$$

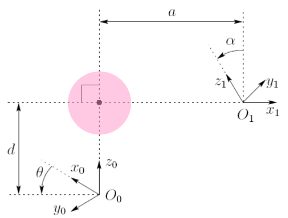
3.5 Matlab - Kinematics Toolbox



4 Forward Kinematics (FK)

4.1 Denavit-Hartenberg (DH) Convention

$\boxed{\text{The } x_i \text{ axis must be \textbf{perpendicular} and \textbf{intersect} the } z_{i-1} \text{ axis.}}$



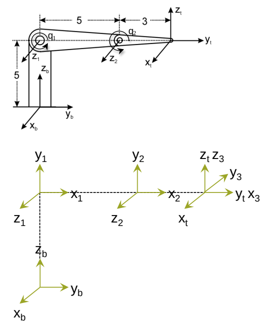
θ: joint angle (about original z-axis)

d: link offset (along original z-axis)

a: link length (along current x-axis)

α: link twist (about current x-axis)

Example



Link	a	α	d	θ
B to 1	0	90	5	90
1 to 2	5	0	0	θ_1
2 to 3	3	-90	0	θ_2
3 to t	0	0	0	-90

Homogeneous Transformation

All operations w.r.t. **current frame!**

$$\boxed{A_i = \text{Rot}_z(\theta)\ \text{Trans}_z(d)\ \text{Trans}_x(a)\ \text{Rot}_x(\alpha) = H_i^{i-1}}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha & a c_\theta \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha & a s_\theta \\ 0 & s_\alpha & c_\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Choosing Axis

1. Set z axis along rotational or translational axis
2. Set x axis according to DH convention
3. Set y axis using right hand rule

4.2 Definitions

4.2.1 Workspaces

Reachable Workspace: EE Origin can reach with at least 1 orientation
Dexterous Workspace: EE Origin can reach with multiple orientations

4.2.2 Joint- & Taskspace

Jointspace: Independent, actuated parameters
Taskspace: Position & orientation of EE

5 Rigid Body Velocity

5.1 Angular Velocity

$$q_a(t) = R_{ab}(t) \cdot q_b \qquad v_{q_a}(t) = \dot{R}_{ab}(t) \cdot q_b$$

Spatial Angular Vel. **Body Angular Vel.**

$$\hat{\omega}^s_{ab} = \dot{R}_{ab} \cdot R^{-1}_{ab} \qquad \hat{\omega}^b_{ab} = R^{-1}_{ab} \cdot \dot{R}_{ab}$$

Transformation

$$\hat{\omega}^b_{ab} = R^{-1}_{ab} \cdot \hat{\omega}^s_{ab} \cdot R_{ab}$$

5.2 Velocity

Spatial Vel. **Body Vel.**

$$v_{q_a} = \hat{\omega}^s_{ab} \times q_a + v^s_{ab} \qquad v_{q_b} = \hat{\omega}^b_{ab} \times q_b + v^b_{ab}$$

$\boldsymbol{\omega^s_{ab}}$ is the instantaneous angular velocity of the body as viewed in the spatial frame.

$\boldsymbol{\omega^b_{ab}}$ is the angular velocity of the body frame, written in the body coordinates.

$\boldsymbol{v^s_{ab}}$ is the velocity of a point attached to the body frame and passing through the origin of the spatial frame, written in spatial coordinates.

$\boldsymbol{v^b_{ab}}$ is the velocity of the origin of the body frame (relative to the spatial frame) written in the body coordinates.

5.3 Twist Velocities

Homogeneous Transformation $g_{ab}(t)$

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} = e^{\hat{\xi}\theta} \cdot g(0)$$

$$\dot{\boldsymbol{g}} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{g^{-1}} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$= \hat{\xi}\dot{\theta} \cdot e^{\hat{\xi}\theta} \cdot g(0) \qquad = g(0)^{-1} \cdot e^{-\hat{\xi}\theta}$$

$$= \hat{\xi}\dot{\theta} \cdot g(t)$$

$$\dot{\boldsymbol{R}} = \hat{\omega} \cdot R$$

Spatial Velocity

Twist Form:

$$\hat{V}^s_{ab} = \dot{g}_{ab} \cdot g^{-1}_{ab} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T p + \dot{p} \\ 0 & 0 \end{bmatrix} = \hat{\xi}\dot{\theta}$$

Twist Coordinates:

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ \left(\dot{R}R^T\right)^{\vee} \end{bmatrix} = \xi\dot{\theta}$$

Body Velocity

Twist Form:

$$\hat{V}^b_{ab} = g^{-1}_{ab} \cdot \dot{g}_{ab} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} = g^{-1}\hat{V}^s g$$

Twist Coordinates:

$$V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T\dot{p} \\ \left(R^T\dot{R}\right)^{\vee} \end{bmatrix}$$

5.3.1 Transformations

$$\boxed{V^s_{ab} = \mathrm{Adj}_g \cdot V^b_{ab}}$$

$$V^s_{ac} = V^s_{ab} + \mathrm{Adj}_{g_{ab}} \cdot V^s_{bc}$$

$$V^b_{ac} = V^b_{bc} + \mathrm{Adj}_{g^{-1}_{bc}} \cdot V^b_{ab}$$

5.3.2 Adjoint Adj

$$\mathrm{Adj}_g = \begin{bmatrix} R & \hat{p}R \\ 0^{3\times3} & R \end{bmatrix} \in \mathbb{R}^{6\times6}$$

$$\mathrm{Adj}^{-1}_g = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0^{3\times3} & R^T \end{bmatrix} = \mathrm{Adj}_{g^{-1}}$$

6 Jacobian

$$\begin{bmatrix} v_0 \\ \omega_0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}$$

6.1 DH-Convention

$$\boxed{J = [J_1, J_2, \ldots, J_n] \in \mathbb{R}^{6\times n}}$$

$$H^0_i = A_1 A_2 \cdots A_n = \begin{bmatrix} X^0_i & Y^0_i & Z^0_i & T^0_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

Revolute **Prismatic**

$$J_i = \begin{bmatrix} Z^0_{i-1} \times \left(T^0_n - T^0_{i-1}\right) \\ Z^0_{i-1} \end{bmatrix} \qquad J_i = \begin{bmatrix} Z^0_{i-1} \\ 0^{3\times1} \end{bmatrix}$$

6.2 Manipulator Jacobian

$$V^s_{st} = J^s_{st}(\theta) \cdot \dot{\theta}$$

$$V^b_{st} = J^b_{st}(\theta) \cdot \dot{\theta}$$

$$J^s_{st}(\theta) = \mathrm{Adj}_{g_{st}(\theta)} \cdot J^b_{st}(\theta)$$

Spatial Frame

$$J^s_{st}(\theta) = [\xi_1 \quad \xi'_2 \quad \cdots \quad \xi'_n]$$

$$\xi'_i = \mathrm{Adj}_{\left(e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}\right)} \cdot \xi_i = \mathrm{Adj}_{(g_{i-1})} \cdot \xi_i$$

$$\hat{\xi}'_i = g_{i-1} \cdot \hat{\xi}_i \cdot g^{-1}_{i-1}$$

Body Frame

$$J^b_{st}(\theta) = [\xi^+_1 \quad \xi^+_2 \quad \cdots \quad \xi^+_n]$$

$$\xi^+_i = \mathrm{Adj}^{-1}_{\left(e^{\hat{\xi}_i\theta_i}\cdots e^{\hat{\xi}_n\theta_n}\cdot g_{st}(0)\right)} \cdot \xi_i$$

6.2.1 Singularities $\det(J_{st}) = 0$

Find Singularities with $\det(J_{st}) = 0$.

```
sings = solve(detJ==0,[th1, th2, th3]);
```

Linearly Dependent Joints have nonzero entries in ker(J).

```
th1 = singularity.th1(3);
th2 = singularity.th2(3);
th3 = singularity.th3(3);
JSingular = subs(Jnum); % substitute syms
nullspace = null(JSingular)
```

6.2.2 Velocities / SVD

$$J = U \cdot \Sigma \cdot V^T \qquad \begin{bmatrix} v^0 \\ \omega^0 \end{bmatrix} = J \cdot \dot{\theta}$$

Singular Values of Jacobian correspond to amplification of joint velocities to workspace velocities.

Input Directions correspond to **rows of** V^T

Output Directions correspond to **columns of** U

6.3 Inverse Kinematics

$$\dot{\theta} = J^{-1} \cdot \begin{bmatrix} v^0 \\ \omega^0 \end{bmatrix}$$

6.4 Manipulability μ

Closeness to singularity.

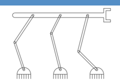
$$\mu = \prod_i \sigma_i$$

σ : Singular values of manipulator Jacobian.

7 Parallel and Redundant Robots

7.1 Parallel Robot

Two or more chains connect EE to base.



7.2 Redundant Robot

Has more DOFs than required to reach certain pos.

Self Motion Manifold

EE fixed → robot can still move

8 Force Control

$$\tau = J^T \cdot F, \qquad F = [F_x, F_y, F_z, \tau_x, \tau_y, \tau_z]^T$$

F : Force at EE

τ : Joint Torque

8.1 Stiffness k

$$F = k \cdot \Delta x$$

8.2 Compliance

Passive Compliance

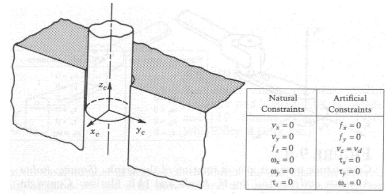
Non-actuated tendency of a body, displaced due to external forces. (Spring)

Active Compliance

Controlled compliance in response to an external force.

8.2.1 Compliance Frame

Allows task-decomposition into pure position or force commands.



Natural Constraints: imposed by environment

→ $f_z = 0$ env won't exceed force on robot in z dir

Artificial Constraints: how we want robot to act

→ $f_x = 0$ don't want to exceed force in x dir on env

- # constrains = # DOF of task space (usually 6)
- constraints usually come in pairs (nat & art)

8.3 Control

Compliance Control Measure actual force; adjust in order to fulfill compliance constraints

Impedance Control Similar to compliance control. System is made to behave like mass-spring-damper sys.

Hybrid Position-Force Control Apply position or force control along different DOFs of compliance frame.

9 Computer Vision

9.1 Thresholding / Binarization

Pixels with intensity *below* threshold → black
Pixels with intensity *above* threshold → white
Threshold usually between fore- and background peak.

9.1.1 Adaptive Thresholding

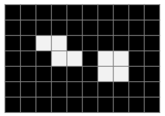
Compute threshold seperately for smaller regions. (useful if multiple objects present)

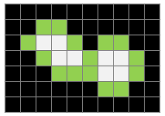
9.1.2 Dilation/ Erosion

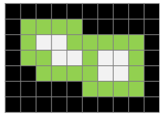
After Thresholding, regions can be distorted by noise and texture.

Dilation: Bright regions grow

Erosion: Bright regions shrink

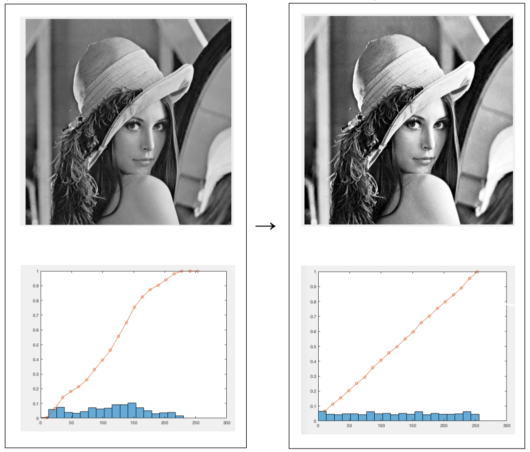



4-connectivity


8-connectivity

9.2 Histogram Equalization

Increase global contrast, create flat histogram.
(Spread intensities to whole spectrum)



9.3 Image Filtering

Replace pixel value with:

Mean: *mean* of neighbouring pixels.

Gaussian: *weighted mean* of neighbouring pixels.

Median: *median intensity* in the window. (sorting)

Mean Filter:

$$\frac{1}{9}\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$$

Gaussian Filter:

$$\frac{1}{16}\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

9.4 Edge Detection

9.4.1 Canny Edge Detector

1. Gaussian Filter to remove noise
2. Find Gradient, Edge Strength and Orientation
3. Non-Maxima Suppression
4. Hysteresis Thresholding

Hysteresis Thresholding

1. Start with Gradient and Direction Map, compare neighbours along edge direction → Non-Maxima Suppression map
2. Mark values above T_H (strong edge), set values below T_L to zero (weak edge)
3. Compare neighbors along edge direction; if neighbour to strong edge is above T_L → strong edge
4. Repeat 3. ("Chain reaction")

9.5 Hough Transform

Feature Extraction technique

1. Use normal representation of line:
$$x\cos(\theta) + y\sin(\theta) = \rho$$
2. For each edge point (x, y), plot normal representation for all θ → Hough Space
3. Intensities in Hough plot accumulate → overlapping points get brighter; peak values describe lines in the image
4. Extract (ρ, θ) of points with higher intensities