

APPLIED MATHEMATICS FOR ELECTRICAL ENGINEERSunit-I Matrix Theory

The cholesky decomposition - Generalized Eigen vectors, canonical basis - QR factorization - Least squares method - singular value decomposition.

ING,

unit-II calculus of variations

Concept of variation and its properties - Euler's equation - Functionals dependent on first and higher order derivatives - Functionals dependent on functions of several independent variables - Variational problems with moving boundaries - Direct methods: Ritz and Kantorovich methods.

unit-III one dimensional Random Variables.

Random variables - probability function - moments - moment generating functions and their properties - Binomial, Poisson, Geometric, uniform, Exponential, Gamma and Normal distributions - Function of a Random variable.

unit-IV Linear programming

Formulation - Graphical solution - Simplex method - Two phase method - Transportation and Assignment Models.

unit-V Fourier series

Fourier Trigonometric series : periodic function as power signals convergence of series - Even and odd function : cosine and sine series - Non-periodic function : Extension to other intervals - Power signals : Exponential Fourier series - Parseval's theorem and power spectrum.

1.

Matrix Theory

12/09/2023

Matrix:

A system of $m \times n$ numbers arranged in the form of rectangular array having m rows and n columns is called a matrix of order $m \times n$.

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

Types of matrix:

1. square matrix ($m=n$)

2. Diagonal matrix (All off-diagonal elements are zero)

3. Scalar matrix (diagonal matrix with all diagonal elements being equal).

4. Upper triangular matrix (lower off-diagonal elements are 0)

5. Lower triangular matrix (upper off-diagonal elements are 0)

6. Unit matrix or Identity matrix (I) (diagonal elements are 1 & non-diagonal elements are zero)

7. Null matrix (all elements are zero)

Only matrix whose Rank is zero

8. Idempotent matrix ($A^n = A$)9. Involuntary matrix ($A^n = I$)10. Nilpotent matrix ($A^n = 0$ & $A^{n-1} \neq 0$, n is the smallest index which makes $A^n = 0$)11. Positive definite matrix (eigen values > 0 & symmetric matrix)

Cholesky decomposition:

 $AB = L \cdot LT$, any symmetric and positive definitematrix A can be decomposed into product of a unique lower triangular and its transpose.

$$I = 100$$

Cholesky decomposition:

check matrix for symmetric and positive definite matrix.

must satisfy

i) Symmetric

ii) positive definite matrix

$$A = LL^T \rightarrow \text{Transpose of lower triangular}$$

Lower triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 & a_{11} \cdot a_{12} & a_{11} \cdot a_{13} \\ a_{21} \cdot a_{11} & a_{21} \cdot a_{12} + a_{22}^2 & a_{21} \cdot a_{13} + a_{22} \cdot a_{23} \\ a_{31} \cdot a_{11} & a_{31} \cdot a_{12} + a_{32} \cdot a_{22} & a_{31} \cdot a_{13} + a_{32} \cdot a_{23} + a_{33}^2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11} \cdot a_{12} & a_{11} \cdot a_{13} \\ a_{21} \cdot a_{11} & a_{21}^2 + a_{22}^2 & a_{21} \cdot a_{13} + a_{22} \cdot a_{23} \\ a_{31} \cdot a_{11} & a_{31} \cdot a_{12} + a_{32} \cdot a_{22} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

Given a matrix,

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 11 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$$a_{11}^2 = 4 \quad a_{21} \cdot a_{11} = 2 \quad a_{31} \cdot a_{11} = 14 \quad a_{11} \cdot a_{12} = 2$$

$$a_{11} = 2$$

$$a_{21} \cdot 2 = 2$$

$$a_{21} = 1$$

$$a_{31} = 7$$

$$a_{12} = 1$$

$$a_{11} \cdot a_{13} = 14$$

$$a_{21}^2 + a_{22}^2 = 17$$

$$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = -5$$

$$g. a_{13} = 14$$

$$Q13 = 7$$

$$(1)^2 + a_{22}^2 = 17$$

$$a_{22}^2 = 17 - 1$$

$$(1) \cdot (7) + (4) a_{23} = -5$$

$$7 + 4 \cdot 0.23 = 5$$

$$a_{22}^2 = 16$$

$$a_{22} = 4$$

$$4 \alpha_{23} = -5 - 7$$

$$4a_{23} = -12$$

$$a_{23} = -\frac{1}{4}$$

$$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = -5$$

$$a_{23} = -3$$

$$(7)(1) + a_{32}(4) = -5$$

$$7 + 4 a_{32} = -5$$

$$4a_{32} = -5 - 7$$

$$4 \cdot a_{32} = -12$$

$$a_{32} = -3$$

$$a_{81}^2 + a_{32}^2 + a_{33}^2 = 83$$

$$7^2 + (-3)^2 + \alpha_{33}^2 = 83$$

$$49 + 9 + 933^2 = 83$$

$$a_{33}^2 = 83 - 49 - 9$$

$$a_{33}^2 = 83 - 58$$

$$d_{33}^2 = 85$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 0 & 33 & -31 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{pmatrix} Q(2) & Q(1) & Q(4) + 0 + 0 \\ 1(Q), & 1(1) + 4(4) & 1(7) + 4(-3) \\ Q(2) & 7(1) - 3(4) & 7(7) - 3(-3) + 5(5) \end{pmatrix}$$

$$\left[\begin{array}{ccc} 4 & 2 & 14 \\ 2 & 11 & -5 \\ 14 & -5 & 83 \end{array} \right] = \left[\begin{array}{c} 4(11 \cdot -5) + 8(11 \cdot 14) \\ 2(11 \cdot -5) + (-1) \cdot 8 - 3 \\ 14(11 \cdot -5) + 8 \cdot 83 \end{array} \right]$$

$$\therefore A = L L^T$$

$$\text{solve eqn } 25x + 15y - 5z = 35$$

$$15x + 18y + 0z = 33$$

$$-5x + 0y + 11z = 6$$

using Cholesky decomposition method

Solution:

$$AX = B$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

Take A and find whether it is symmetric & positive definite matrix.

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 18 & 0 \\ 0 & 0 & 11 \end{bmatrix} = LT$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11} \cdot a_{12} & a_{11} \cdot a_{13} \\ a_{21} \cdot a_{11} & a_{21}^2 + a_{22}^2 & a_{21} \cdot a_{13} + a_{22} \cdot a_{23} \\ a_{31} \cdot a_{11} & a_{31} \cdot a_{12} + a_{32} \cdot a_{21} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

$$a_{11}^2 = 25$$

$$a_{11} = 5$$

$$a_{11} \cdot a_{12} = 15$$

$$5 \cdot a_{12} = 15$$

$$a_{12} = 3$$

$$a_{11} \cdot a_{13} = -5$$

$$5 \cdot a_{13} = -5$$

$$a_{13} = -1$$

$$a_{21} \cdot a_{11} = 15$$

$$a_{21} \cdot 5 = 15$$

$$a_{21} = 3$$

$$a_{31} \cdot a_{11} = -5$$

$$a_{31} \cdot 5 = -5$$

$$a_{31} = -1$$

$$a_{21}^2 + a_{22}^2 = 18$$

$$3^2 + a_{22}^2 = 18$$

$$a_{22}^2 = 18 - 9$$

$$a_{22} = \sqrt{9}$$

$$a_{22} = 3$$

$$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = 0$$

$$5 \cdot (-1) + (3) a_{23} = 0$$

$$-5 + 3 a_{23} = 0$$

$$3 a_{23} = 3$$

$$a_{23} = 1$$

$$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = 0$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 11$$

$$(-1)(3) + a_{32} \cdot (3) = 0$$

$$(-1)^2 + (1)^2 + a_{33}^2 = 11$$

$$-3 + 3a_{32} = 0$$

$$a_{33}^2 = 11 - 1 - 1$$

$$3a_{32} = 3$$

$$a_{33}^2 = 9$$

$$\boxed{a_{32} = 1}$$

$$\boxed{a_{33} = 3}$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A \quad (L) \quad (L^T)$

$$S = 8K + 8$$

$$\boxed{A \mid X = B}$$

$$\boxed{I = S^T}$$

$$\boxed{L^T \cdot X = B}$$

$$\text{Let } L^T \cdot X = Y$$

$$H = SK + 8K$$

$$LY = B$$

$$H = SK + 8K$$

$$S = 8K$$

$$\boxed{I = S^T}$$

$$\text{Take } LY = B,$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

$$5y_1 = P = 8K - 8K + 10K$$

$$P = 1 - (1)8 + 10K$$

$$P = 8 + 10K$$

$$B = 10K$$

$$\boxed{I = S^T}$$

Equating right & left side,

$$5y_1 = 35 \rightarrow ①$$

$$3y_1 + 3y_2 = 33 \rightarrow ②$$

$$-y_1 - y_2 + 3y_3 = 6 \rightarrow ③$$

① \Rightarrow

$$\boxed{y_1 = 7}$$

$$③ \Rightarrow -7 + 4 + 3y_3 = 6$$

② \Rightarrow

$$3(7) + 3y_2 = 33$$

$$\boxed{y_2 = 4}$$

$$\boxed{y_3 = 3}$$

$$\text{Thus, } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$$D = \sin A \cdot \sin B + \cos A$$

$$D = (\sin A) \cdot \sin B + (\cos A) \cdot \cos B$$

$$D = \cos A \cdot \cos B$$

$$E = \sin A \cdot \cos B$$

$$F = \cos A \cdot \sin B$$

Substitute y in $L^T \cdot X = Y$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$$3x_3 = 3$$

$$x_3 = 1$$

$$3x_2 + x_3 = 4$$

$$3x_2 = 4 - 1$$

$$3x_2 = 3$$

$$x_2 = 1$$

$$5x_1 + 3x_2 - x_3 = 7$$

$$5x_1 + 3(1) - 1 = 7$$

$$5x_1 + 2 = 7$$

$$5x_1 = 5$$

$$x_1 = 1$$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

Write first 3 digit numbers

\therefore The value of x_1, x_2 and x_3 are

$$\textcircled{1} \leftarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 100$$

$$\textcircled{2} \leftarrow 100 = 100 + 100$$

$$\textcircled{3} \leftarrow 100 + 100 - 100$$

$$d = 100 + 100 + 100 \quad \text{(10)}$$

$$d = 300$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 100$$

$$100 + 100 + 100$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2) solve the following system of linear equation by using cholesky Method.

$$4x_1 + 2y + 14z = 14$$

$$2x_1 + 17y - 5z = -101$$

$$14x_1 - 5y + 83z = 155$$

Solution:

$$AX = B$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Take A & check whether it is symmetric & positive definite matrix.

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

here, $A^T = A$ & $a_{ij} = a_{ji}$. Thus "A" is symmetric.

CH equation:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 2 & 14 \\ 2 & 17-\lambda & -5 \\ 14 & -5 & 83-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(17-\lambda)(83-\lambda)-25] - 2[(2(83-\lambda)+5(14))]-14[(-10-14(17-\lambda))]$$

$$(4-\lambda)[14(11+17\lambda-83\lambda+\lambda^2-25)] - 2[166-8\lambda+70]+14[-10-288+14\lambda] = 0$$

$$(4-\lambda)[1386-100\lambda+\lambda^2] - 2[96-2\lambda]+14[14\lambda-248] = 0$$

$$(4-\lambda)[1386-100\lambda+\lambda^2] - 192+4\lambda+196\lambda-3472 = 0 \quad \lambda = 1, 29$$

$$(4-\lambda)[\lambda^2(100\lambda+1386)] - 3664+200\lambda = 0 \quad \lambda = \frac{16}{11}, \frac{9}{85}, -7$$

$$4\lambda^2 - 400\lambda + 5544 - \lambda^3 + 100\lambda^2 - 1386\lambda - 3664 + 200\lambda = 0 \quad \lambda = 110$$

$$-\lambda^3 + 104\lambda^2 - 1586\lambda + 1880 = 0 \quad \lambda = 110$$

$$\lambda^3 + 104\lambda^2 - 1586\lambda + 1880 = 0$$

$$\lambda^3 - 104\lambda^2 + 1586\lambda - 1880 = 0$$

$$A_1 = \mu A_1 + \mu B_1 + \mu C_1$$

$$104 = \mu \lambda - \mu H + \mu K$$

$$221 = 288 + 88 - \lambda H$$

$$a_1 = [4] = 4 > 0$$

$$a_2 = \begin{bmatrix} 4 & 2 \\ 2 & 17 \end{bmatrix}$$

$$|a_2| = \begin{vmatrix} 4 & 2 \\ 2 & 17 \end{vmatrix} = 68 - 4 = 64 > 0$$

$$B = XA$$

$$a_3 = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} \mu I & B & C \\ B^T & \mu I & D \\ C^T & D^T & \mu I \end{bmatrix}$$

$$= 4(1411 - 25) - 2(166 + 70) + 14(-10 - 238)$$

$$= 4(1386) - 2(236) + 14(-248)$$

$$= 5544 - 472 - 3472 = 100 \quad \text{and } A = TA$$

$$|a_3| = 1600 > 0$$

$$C = [XK - A]$$

Thus positive definite matrix

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 17 & 0 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 0 & 17 & -5 \\ 0 & 0 & 83 \end{bmatrix} = \begin{bmatrix} \mu I & B & C \\ B^T & \mu I & D \\ C^T & D^T & \mu I \end{bmatrix}$$

$$\begin{aligned} A &= [(\mu I + 3\lambda K - C)^{-1}] B^T + [(\mu I + 3\lambda K - C)^{-1}] D^T \\ &= \begin{bmatrix} a_{11}^2 & a_{11}a_{12} + a_{12}a_{21} & a_{11}a_{13} + a_{13}a_{31} \\ a_{11}a_{12} + a_{12}a_{21} & a_{22}^2 + a_{22}^2 & a_{21}a_{23} + a_{23}a_{32} \\ a_{11}a_{13} + a_{13}a_{31} & a_{21}a_{23} + a_{23}a_{32} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix} (\lambda - \mu) \end{aligned}$$

$$a_{11}^2 = 4$$

$$a_{21} \cdot a_{11} = 2 \quad \text{and } a_{31} \cdot a_{11} = 14$$

$$a_{11} = \sqrt{4} = 2 \quad a_{21} = 2/2 = 1 \quad a_{31} = 14/2 = 7 \quad a_{12} = 2/2 = 1$$

$$a_{11} = 2$$

$$a_{21} = 1 \quad a_{31} = 7 \quad a_{12} = 1$$

$$a_{21}^2 + a_{22}^2 = 17$$

$$(1)^2 + a_{22}^2 = 17$$

$$a_{22}^2 = 16$$

$$\boxed{a_{22} = 4}$$

$$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = -5$$

$$(7)(1) + a_{32}(4) = -5$$

$$4a_{32} = -5 - 7$$

$$4a_{32} = -12$$

$$\boxed{a_{32} = -3}$$

$$a_{11} \cdot a_{13} = 14$$

$$(8)a_{13} = 14$$

$$\boxed{a_{13} = 7}$$

$$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = -5$$

$$(1)(7) + (4)a_{23} = -5$$

$$4a_{23} = -12$$

$$\boxed{a_{23} = -3}$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 83$$

$$(7)^2 + (-3)^2 + a_{33}^2 = 83$$

$$49 + 9 + a_{33}^2 = 83$$

$$a_{33}^2 = 83 - 49 - 9$$

$$a_{33}^2 = 25$$

$$\boxed{a_{33} = 5}$$

$$\begin{bmatrix} 4 & 8 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

A L LT

$$H = (1)E - 8K\mu$$

$$E + 15\mu = 8K\mu$$

$$H = 8K\mu$$

$$\boxed{J = 8K\mu}$$

$$AX = B$$

$$L \cdot LT \cdot X = B \quad \text{Let } LT \cdot X = Y$$

$$LY = B$$

$$\text{Take } LY = B,$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$2y_1 = 14$$

$$\boxed{y_1 = 7}$$

$$4y_2 = -108$$

$$\boxed{y_2 = -27}$$

$$J = 8K\mu$$

$$\boxed{E = 15\mu}$$

$$7y_1 - 3y_2 + 5y_3 = 155$$

$$7(7) - 3(-27) + 5y_3 = 155$$

$$49 + 81 + 5y_3 = 155$$

$$5y_3 = 155 - 49 - 81$$

$$5y_3 = 25$$

$$\boxed{y_3 = 5}$$

$$\therefore \text{If } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -27 \\ 5 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

Substitute $L^T x = y$

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -27 \\ 5 \end{bmatrix}$$

$$2x_1 + x_2 + 7x_3 = 9$$

$$5x_3 = 5$$

$$P^{-1}y - 2x_1 - x_2 = 9$$

$$x_3 = 1$$

$$2x_1 + x_2 = 8$$

$$4x_2 - 3x_3 = -27$$

$$4x_2 - 3(1) = -27$$

$$4x_2 = -27 + 3$$

$$4x_2 = -24$$

$$x_2 = -6$$

$$2x_1 + x_2 + 7x_3 = 9$$

$$2x_1 + (-6) + 7(1) = 9$$

$$2x_1 - 6 + 7 = 9$$

$$2x_1 = 6$$

$$x_1 = 3$$

\therefore The value of $x =$

$$2x_1 = 2(3) + (4)(-6) + (5)(1)$$

$$2x_1 = 6 - 24 + 5$$

$$(6 - 24 + 5) = -13$$

$$2x_1 = -13$$

$$x_1 = -6.5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \\ 10 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \\ 10 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \\ 10 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$t = 18$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underbrace{x^2 + 5x - 6}_{\text{Polynomial with deg 2}}$$

$$\underbrace{x^2 + 5x - 6 = 0}_{\text{eqn with deg 2}}$$

For quadratic eqn,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$P(x) = x - 1 \rightarrow \text{linear eqn.}$$

$$P(x) = x^2 + 5x - 1 \rightarrow \text{quadratic eqn.}$$

$$P(x) = x^3 + 5x^2 + 5x + 5 \rightarrow \text{cubic eqn}$$

To find factor for cubic eqn, there is no exact method.

factors \rightarrow equation solution $\Rightarrow x^2 + 5x + 6 = 0$

~~zeros~~

(we get two linear eqns for x)

$$(x+2)(x+3) = 0$$

\downarrow
Polynomial.

The number which makes the polynomial to zero.

$$q_1(x) = x^2 - 5x + 6$$

$$q_1(2) = 2^2 - 10 + 6 = 0$$

For cubic eqn,

first use trial method to find one root and then proceed to synthetic division.

Determinant

$$A = [a_{ij}]_{3 \times 3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Any particular row or column:

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) + (a_{11}a_{21}a_{32} - a_{11}a_{22}a_{31})$$

$$= (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{11}a_{21}a_{32} - a_{11}a_{22}a_{31})$$

$$= (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{11}a_{21}a_{32} - a_{11}a_{22}a_{31})$$

So if we take any row or column

the right answer will come

if we take any row or column

Find the eigen values and eigen vectors for the following matrix.

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

solution:

Using ch eqn,

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

Alternate method to find eigen values:

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where

s_1 = sum of the main diagonal

s_2 = sum of minors of the leading diagonal

$$s_3 = |A|$$

s_1 = sum of main diagonal elements

$$= -1 + 2 + 0 = 1$$

$$s_1 = 1$$

s_2 = sum of minors of leading diagonal

$$= \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= (+1) + (0 - 2) + (-2 - 2)$$

$$= 1 - 2 - 4$$

$$s_2 = -5$$

$$s_3 = |A| = -1(1) - 2(1) - 2(-1+2)$$

$$= -5$$

$$s_3 = -5$$

$$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

Solving $\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

$$P(\lambda) = \lambda^3 - \lambda^2 - 5\lambda + 5$$

$$\begin{aligned} P(1) &= 1^3 - 1^2 - 5(1) + 5 \\ &= 1 - 1 - 5 + 5 \end{aligned}$$

$$P(1) = 0$$

$\therefore (\lambda - 1)$ is a factor

Using synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -1 & -5 & 5 \\ & 0 & 1 & 0 & -5 \\ \hline & 1 & 0 & -5 & 0 \end{array}$$

$$\lambda^2 + \lambda - 5 = 0$$

$$\lambda^2 - 5 = 0$$

$$\lambda^2 = 5$$

$$\lambda = \pm \sqrt{5}$$

$$\therefore \lambda = 1, \sqrt{5}, -\sqrt{5}$$

To find eigen vector for $\lambda = 1$;

$$Ax = 0$$

$$\boxed{\lambda = 1}$$

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(8-8+1) + (8-0) + (1+1) =$$

when $\lambda = 1$;

$$\begin{bmatrix} -1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 1-1 &= \\ 2-1 &= \\ 1-1 &= \end{aligned}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = 8 + 6 + 8 - 8 - 1 - 8$$

$$-2x_1 + 2x_2 - 2x_3 = 0 \rightarrow ①$$

$$x_1 + x_2 + x_3 = 0 \rightarrow ②$$

$$-x_1 + 2x_2 - x_3 = 0 \rightarrow ③$$

Using cross multiplication method,

$$① \text{ & } ② \Rightarrow$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 2 & -2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ \hline \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{-2+2} = \frac{x_3}{2-2} = \frac{2}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}} = \frac{1}{1+\sqrt{-1}}$$

$$\frac{x_1}{4} = \frac{x_2}{0} = \frac{x_3}{-1} = k \quad (\sqrt{-1}) = 1+\sqrt{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k \quad \text{using no } (\sqrt{-1}) \text{ for } x_2$$

$$\therefore \text{Eigen vector for } \lambda = 1, x_1 = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{(\sqrt{5}-1)(\sqrt{5}+1)} \cdot$$

to find eigen vector for $\lambda = \sqrt{5}$; $\lambda = -\sqrt{5}$

$$AX = 0 \quad (A - \lambda I)X = 0 \quad (\sqrt{5}-1)(\sqrt{5}+1) \cdot \quad (\sqrt{5}-1)(\sqrt{5}+1)$$

$$\left[\begin{array}{ccc} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{array} \right] - (\sqrt{5}) \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] = \frac{1}{(\sqrt{5}-1)(\sqrt{5}+1)} \cdot$$

$$\left[\begin{array}{ccc} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] = \frac{1}{(\sqrt{5}-1)(\sqrt{5}+1)} \cdot$$

$$(-1-\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \rightarrow ①$$

$$x_1 + (2-\sqrt{5})x_2 + x_3 = 0 \rightarrow ②$$

$$-x_1 - x_2 - \sqrt{5}x_3 = 0 \rightarrow ③$$

using cross multiplication method,

$$② \text{ & } ③ \Rightarrow$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & 1 & 1 & 1 \\ -1 & -\sqrt{5} & -1 & -1 \\ \hline \end{array}$$

$$\frac{1}{(2-\sqrt{5})} \cdot \frac{1}{-1} = \frac{1}{-\sqrt{5}} = \frac{1}{1+\sqrt{-1}}$$

$$\frac{x_1}{-(2-\sqrt{5})\sqrt{5}+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\frac{x_1}{-2\sqrt{5}+5+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\frac{x_1}{-2\sqrt{5}+6} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\frac{x_1}{-2\sqrt{5}+6} = \frac{x_2}{-(1-\sqrt{5})} = \frac{x_3}{1-\sqrt{5}}$$

"x" by $(1+\sqrt{5})$ on each term,

$$\frac{x_1}{(1+\sqrt{5})(6-2\sqrt{5})} = \frac{x_2}{-(1-\sqrt{5})(1+\sqrt{5})} = \frac{x_3}{(1-\sqrt{5})(1+\sqrt{5})}$$

$$\frac{x_1}{(6-2\sqrt{5})(1+\sqrt{5})} = \frac{x_2}{-(1^2 - (\sqrt{5})^2)} = \frac{x_3}{1^2 - (\sqrt{5})^2}$$

$$\frac{x_1}{(6-2\sqrt{5})(1+\sqrt{5})} = \left[\begin{array}{c|cc|c} x_1 & 0 & 0 & 0 \\ \hline x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|cc|c} x_1 & 0 & 0 & 0 \\ \hline x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{array} \right]$$

$$\frac{x_1}{6+6\sqrt{5}-2\sqrt{5}-10} = \frac{x_2}{+4} = \frac{x_3}{-4}$$

$$\frac{x_1}{-4+4\sqrt{5}} = \frac{x_2}{4} = \frac{x_3}{-4}$$

÷ by 4

$$\frac{x_1}{-1+\sqrt{5}} = \frac{x_2}{1} = \frac{x_3}{-1} = k$$

$$\therefore x_2 = k \begin{bmatrix} -1+\sqrt{5} \\ 1 \\ -1 \end{bmatrix}$$

$$k = 1 + (\sqrt{5}-1)$$

$$= 1 + \sqrt{5} - 1$$

to find eigen values of $\lambda = -\sqrt{5}$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - (-\sqrt{5}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(1+\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \rightarrow ①$$

$$x_1 + (2+\sqrt{5})x_2 + x_3 = 0 \rightarrow ②$$

$$-x_1 - x_2 + \sqrt{5}x_3 = 0 \rightarrow ③$$

using cross-multiplication method,

$$② \times ③ \Rightarrow$$

$$x_1 \ x_2 \ x_3$$

$$\begin{array}{ccc|c} 2+\sqrt{5} & 1 & 1 & 2+\sqrt{5} \\ -1 & \cancel{\sqrt{5}} & \cancel{-1} & -1 \end{array}$$

$$\frac{x_1}{2\sqrt{5}+5+1} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{2-\cancel{1+2\sqrt{5}-\sqrt{5}}} = \frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-(1+\sqrt{5})} = \frac{x_3}{1+\sqrt{5}}$$

$$\frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-(1+\sqrt{5})} = \frac{x_3}{1+\sqrt{5}}$$

$$\frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-(1+\sqrt{5})} = \frac{x_3}{1+\sqrt{5}}$$

$$E = 0 + 1 + 0 = 1$$

$$\frac{x_1}{(1-\sqrt{5})(2\sqrt{5}+6)} = \frac{x_2}{-(1-\sqrt{5})(1+\sqrt{5})} = \frac{x_3}{(1+\sqrt{5})(1-\sqrt{5})}$$

$$\frac{x_1}{8\sqrt{5}+6-10-6\sqrt{5}} = \frac{x_2}{-(1^2-(\sqrt{5})^2)} = \frac{x_3}{1^2-(\sqrt{5})^2} = 0 + 8+8 = 0 = 8$$

$$\frac{x_1}{-4\sqrt{5}-4} = \frac{x_2}{-(1-5)} = \frac{x_3}{1-5}$$

$$\frac{x_1}{-4(\sqrt{5}+1)} = \frac{x_2}{4} = \frac{x_3}{-4}$$

$$0 = X(AK - A)$$

* by 4;

$$\frac{x_1}{-(1+\sqrt{5})} = \frac{x_2}{1} = \frac{x_3}{-1} = k$$

$$\therefore x_3 = k \begin{bmatrix} 0 \\ -(1+\sqrt{5}) \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] (AV) = \begin{bmatrix} s- & s & 1-\tau \\ 1 & s & 1 \\ 0 & 1-\tau & 1-\tau \end{bmatrix}$$

$k = \text{arbitrary constant}$

Find eigen values and Eigen vectors for the following matrix,

$$\textcircled{1} \leftarrow 0 = \lambda x_1 - x_2 + (\sqrt{5}+1)x_3$$

$$\textcircled{2} \leftarrow 0 = \lambda x_1 + x_2 + (\sqrt{5}+1)x_3 + x_4$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix} \leftarrow 0 = \lambda x_1 + x_2 - (\sqrt{5}-1)x_3 - x_4$$

Solution:

To find the Eigen values,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

S_1 = sum of the main diagonal elements

S_2 = sum of minors of the leading diagonal

$$S_3 = |A|$$

$$\frac{S_1}{\lambda^3} = \frac{S_2}{\lambda^2} = \frac{S_3}{\lambda} = \frac{1}{\lambda + \sqrt{5}}$$

$$S_1 = 0 + 1 + 2 = 3$$

$$S_2 = \left| \begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array} \right| + \left| \begin{array}{cc} 0 & -2 \\ -1 & 2 \end{array} \right| + \left| \begin{array}{cc} 0 & -2 \\ -1 & 1 \end{array} \right| = \frac{-x_1}{(\lambda + \sqrt{5})(\lambda - 1)} = \frac{1}{(\lambda + \sqrt{5})(\lambda - 1)}$$

$$= 2 + 2 + 0 - 2 + 0 - 2$$

$$S_2 = 0$$

$$\frac{S_3}{\lambda^3} = \frac{S_2}{\lambda^2} = \frac{S_1}{\lambda} = \frac{1}{\lambda + \sqrt{5} - 1}$$

$$s_3 = 0(1) + 2(-2+2) - 0(1+1)$$

$$= -2(0)$$

$$s_3 = -4$$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0$$

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

$$P(\lambda) = \lambda^3 - 3\lambda^2 + 4$$

$$P(1) = 2 \neq 0$$

$$P(-1) = -1 - 3 + 4 = 0$$

$\therefore \lambda + 1 = 0$ is a factor

$$\begin{array}{c} -1 \\ \hline 1 & -3 & 0 & 4 \\ 0 & -1 & 4 & -4 \\ \hline 1 & -4 & 4 & 0 \end{array}$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 2, 2$$

$$\therefore \lambda = -1, 2, 2.$$

To find Eigen vectors:

$$\text{for } \lambda = -1; (A - \lambda I)x = 0$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0+1 & -2 & -2 \\ -1 & 1+1 & 2 \\ -1 & -1 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 - 2x_3 = 0$$

$$-x_1 + 2x_2 + 2x_3 = 0$$

$$-x_1 - x_2 + 3x_3 = 0$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

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$$x_2 = -x_1 - 3x_3$$

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$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$x_1 = 2x_2 + 2x_3$$

$$x_2 = -x_1 - 3x_3$$

$$x_3 = -x_1 + x_2$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 \\ 2 & 2 & -1 & 2 \\ -1 & 3 & -1 & -1 \end{array}$$

$$0 = 8k - k + 8k + 8k - 8k$$

$$\frac{x_1}{6+2} = \frac{x_2}{-2+3} = \frac{x_3}{1+2} = k$$

$$0 = 4 + k + 8k + 8k - 8k$$

$$\frac{x_1}{8} = \frac{x_2}{1} = \frac{x_3}{3} = k$$

$$0 = 4 + 4k + 8k - 8k$$

$$x_1 = 8k ; x_2 = k ; x_3 = 3k$$

$$0 \neq 8 = 16$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$$

$$0 = 4 + 8 - 1 = 13$$

For $\lambda = 2$; $|A - \lambda I| x = 0$

$$\begin{bmatrix} 0-\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = 8 + 8k - 8k$$

$$0 = (8-\lambda)(8-\lambda)$$

$$8, 8 = \lambda$$

$$\begin{array}{ccccc} 4 & 0 & 8 & 1 & | 1 \\ 4 & 4 & 1 & 0 & \\ \hline 0 & 4 & 4 & 1 & \end{array}$$

$$\begin{bmatrix} -2 & -2 & -2 \\ -1 & -1 & 2 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$: \text{row by neg E. bring 0}$$

$$0 = x(2\lambda - A) \quad (R=1)$$

$$\begin{array}{ccccc} 8 & 8 & 8 & 0 & | 1 \\ 8 & 1 & 1 & 1 & \\ 8 & 1 & 1 & 1 & \end{array}$$

$$-x_1 - x_2 + 0x_3 = 0$$

$$\begin{bmatrix} x_1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2 \\ +2 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{array}{ccccc} 8 & 8 & 8 & 1 & | 0 \\ 8 & 1 & 1 & 1 & \\ 1+8 & 1 & 1 & 1 & \end{array}$$

$$\frac{x_1}{-4-2} = \frac{x_2}{2+4} = \frac{x_3}{-2-2}$$

$$0 = 8k - 8k - 0$$

$$0 = 8k + 8k + 0 =$$

$$\frac{x_1}{-b} = \frac{x_2}{b} = \frac{x_3}{0} = K$$

$$x_1 = -Kb$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{0} = K$$

$$x_1 = -K, x_2 = K, x_3 = 0$$

$$x_2 = K \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \text{Eigen vectors are } x_1 = K \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, x_2 = x_3 = K \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

[The process of decomposing $A = gI + qK + rE$]

Solution of linear homogeneous eqns:

(i) If $\rho(A) = n$; $\rho(A:B) = n$

$\rho(A) = \rho(A:B) = n \Rightarrow$ consistent; unique solution
zero solution.

(ii) $\rho(A:B) = \rho(A) < n \Rightarrow$ infinite solution; consistent

(iii) $\rho(A) < \rho(A:B) \therefore \rho(A) \neq \rho(A:B) \Rightarrow$ no solution.

Finding eigen vector for $\lambda = 2$ using row echelon form:

$$A|B = \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 + \text{R}_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 + \text{R}_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right] \xrightarrow{\text{R}_2 \rightarrow 2\text{R}_2 - \text{R}_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right] \xrightarrow{\text{R}_3 \rightarrow 2\text{R}_3 - \text{R}_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right]$$

$$X = \begin{bmatrix} K_1 - K_2 \\ K_2 \\ K_1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 \rightarrow -\frac{1}{2}\text{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_2 \rightarrow \frac{1}{6}\text{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{R}_1 \rightarrow \text{R}_1 - \text{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_3 &\neq 0 \\ K_1 &\neq 0 \\ K_2 &\neq 0 \end{aligned} \quad \begin{aligned} 0(K_2 + 6K_1) &= 0 \\ 0(K_2) &= -6K_1 \\ (K_2) & \end{aligned} \quad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 0 &= 0 \\ x_1 + x_2 &= 0 \\ -2x_1 - 2x_2 - 2x_3 &= 0 \end{aligned}$$

$$\begin{aligned} -2x_1 - 2K_2 - K_1 &= 0 \\ -2(K_1 + K_2) - K_1 &= 0 \\ -3K_1 - 2K_2 &= 0 \end{aligned}$$

$$-3K_1 - 2K_2 = 0$$

$$\textcircled{3} \Rightarrow 0x_3 = 0 \quad x = \frac{\alpha x}{0} = \frac{\alpha x}{1} = \frac{\alpha x}{1}$$

$$x_3 = \alpha$$

$$\textcircled{2} \Rightarrow 0x_2 + 0x_3 = 0$$

$$x = \frac{\alpha x}{0} = \frac{\alpha x}{1} = \frac{\alpha x}{1}$$

$$0(B) = 0$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} x = \alpha x$$

$$\textcircled{3} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -2 & 0 \end{bmatrix} x_2 = \beta \quad , \quad \begin{bmatrix} 8 \\ 1 \\ 8 \end{bmatrix} x = \alpha x \quad \text{Eigenvector bei } \alpha = 0$$

$$-2x_1 - 2x_2 - 2x_3 = 0 \Rightarrow -2(x_1 + x_2 + x_3) = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3 \quad \text{wegen } x_1 \text{ frei}$$

$$x_1 = -\beta - \alpha$$

$$\alpha = (B:A) \beta \quad (\alpha = (A)\beta \quad \text{I})$$

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(\alpha + \beta) \\ -(\alpha + \beta) \\ -(\alpha + \beta) \end{bmatrix} \leftarrow \alpha = (A)\beta \quad \text{II} \\ &\in (B:A)\beta + \alpha(A)\beta \quad \leftarrow \alpha = (A)\beta \quad \text{III} \end{aligned}$$

$$= \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{oder } \alpha = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{IV}$$

$$= \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} + \begin{bmatrix} -\beta \\ \beta \\ 0 \end{bmatrix} \quad \text{V}$$

$$\alpha x = \begin{bmatrix} -\alpha - \beta \\ \beta \\ \alpha \end{bmatrix} \quad \text{VI}$$

$$\alpha x = \beta \quad \text{VII}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

QR decomposition

19/09/2023.

QR decomposition is a matrix factorization method that decompose the matrix A into Q , and such that $A = QR$.

The Q matrix is an orthogonal matrix, meaning that its columns are orthonormal and the R matrix is an upper triangular matrix.

The process of QR decomposing involves the following steps.

- 1) Start with a matrix A
- 2) Apply Gram-Schmidt process to the column of A to obtain an orthonormal set of columns. This is done by recursively subtracting the projection of each column onto the subspace spanned by the previous orthogonal columns and normalizing the resulting vectors.
- 3) Arrange the resulting orthonormal column into a matrix Q .
- 4) Compute the R matrix by multiplying the transpose of Q with A . The resulting matrix will be upper triangular matrix.
- 5) The QR decomposition of A is the pair (Q, R) .

QR decomposition with Gram-Schmidt process:

Consider the matrix $A = \{a_1, a_2, a_3, \dots, a_n\}$

where $a_1, a_2, a_3, \dots, a_n$ are column elements of given matrix, with inner product $\langle u, v \rangle = v^T u$.

projection $u = \frac{\langle u, a \rangle}{\langle u, u \rangle} u$, Then $u_i = a_i$ (column wise)

$$u_2 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1, \quad u_3 = a_3 - \frac{\langle u_1, a_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, a_3 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$\|u\| = \sqrt{\langle u, u \rangle}$$

norm \rightarrow to find distance

$$\text{norm} \quad \text{norm of vector is 2, norm of } u_1, u_2, u_3 \text{ is 1}$$

$$u = [u_1, u_2, u_3] = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right]$$

$u_1, u_2, u_3 \rightarrow$ unit vector : $1 - 810 = A$ fast line

unit length $u = [u_1, u_2, u_3]$ norm is 1

all basis transformations are equivalent iff fast function
homework requirement regular no 3 20/09/2023

Find QR decomposition

introduction of basis functions basis for basis

cols:

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Gram-Schmidt process to find u_1, u_2, u_3

$a_1 = (1, 1, 2)$ 1st of vectors forming matrix

$$u_1 = (1, 1, 2)$$

$a_2 = (0, 2, 2)$ 2nd of vectors forming matrix

$a_3 = (-1, 1, 3)$ 3rd of vectors forming matrix

$$\text{ext projection} = (0, 2, 2) - \frac{\langle (1, 1, 2), (0, 2, 2) \rangle}{\langle (1, 1, 2), (1, 1, 2) \rangle} (1, 1, 2)$$

$$\text{ext projection} = (0, 2, 2) - \frac{6}{6} (1, 1, 2) = (0, 2, 2) - (1, 1, 2)$$

$$\text{ext projection} = (0, 2, 2) - (1, 1, 2)$$

then calculate projections at $u_2 = (-1, 1, 0)$

$$u_2 = a_3 - \frac{\langle u_1, a_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, a_3 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$= (-1, 1, 3) - \frac{\langle (-1, 1, 0), (-1, 1, 3) \rangle}{\langle (-1, 1, 2), (-1, 1, 2) \rangle} (-1, 1, 2)$$

$$= (-1, 1, 3) - \frac{\langle (-1, 1, 0), (-1, 1, 3) \rangle}{\langle (-1, 1, 0), (-1, 1, 0) \rangle} (-1, 1, 0)$$

$$= (-1, 1, 3) - \frac{6}{6} (-1, 1, 2) + \frac{1}{1} (-1, 1, 0)$$

$$= \frac{6}{6} (-1, 1, 1) + \frac{1}{1} (-1, 1, 0)$$

$$Q = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Orthogonal

$$A = Q R \quad Q Q^T = I \quad (\text{since orthogonal})$$

Upper triangular matrix.

$\|u\|$ is used to convert orthonormal vector to

$$\begin{bmatrix} \frac{\partial}{\partial V} & \frac{\partial}{\partial V} & \frac{\partial}{\partial V} \\ \frac{\partial}{\partial V} & \frac{\partial}{\partial V} & \frac{\partial}{\partial V} \\ \frac{\partial}{\partial V} & \frac{\partial}{\partial V} & \frac{\partial}{\partial V} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$u_1 = (1, 1, 2)$
 $u_2 = (-1, 1, 0)$
 $u_3 = (-1, -1, 1)$

$$\|u\| = \sqrt{\langle u, u \rangle}$$

$$\|u_1\| = \sqrt{\langle (1, 1, 2), (1, 1, 2) \rangle} = \sqrt{\frac{1}{2}(1+1+4)} = \sqrt{6}$$

$$\|u_2\| = \sqrt{\langle (-1, 1, 0), (-1, 1, 0) \rangle} = \sqrt{\frac{1}{2}(1+1+0)} = \sqrt{2}$$

$$\|u_3\| = \sqrt{\langle (-1, -1, 1), (-1, -1, 1) \rangle} = \sqrt{1+1+1} = \sqrt{3}$$

$$Q = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \rightarrow \text{change orthonormal to orthogonal matrix}$$

$$\Rightarrow A = Q R$$

Pre multiply by Q^T

$$Q^T Q \cdot R = Q^T A$$

$$I \cdot R = Q^T A \quad (\because Q^T Q = I \text{ orthogonal matrix})$$

$$R = Q^T A = \begin{bmatrix} \frac{\partial}{\partial V} & \frac{\partial}{\partial V} & \frac{\partial}{\partial V} \\ 0 & \frac{\partial}{\partial V} & \frac{\partial}{\partial V} \\ 0 & 0 & \frac{\partial}{\partial V} \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} & 0 + \frac{2}{\sqrt{6}} + \frac{4}{\sqrt{6}} & \frac{-1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{6}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & 0 + \frac{2}{\sqrt{2}} + 0 & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \\ -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} & 0 - \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} \end{bmatrix}$$

$\langle u, v \rangle = 0 \Rightarrow u \perp v$

$$= \begin{bmatrix} \frac{6}{\sqrt{6}} = \frac{6}{\sqrt{6+1+6}} = \frac{6}{\sqrt{13}} & \langle (0,1,1), (\sqrt{6}, 1, \sqrt{6}) \rangle = \sqrt{6} = \|u\| \\ 0 = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} & \Rightarrow \langle (0,1,1), (0,1,\sqrt{2}) \rangle = \sqrt{2} = \|v\| \\ 0 = \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{1+1+1}} = \frac{3}{\sqrt{3}} & \langle (1,1,-1), (0,1,-1) \rangle = \sqrt{3} = \|w\| \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = Q$$

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{13}} & \frac{\sqrt{6}}{\sqrt{13}} & \frac{\sqrt{6}}{\sqrt{13}} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = P$$

$$= \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{6}} + 0 & \frac{\sqrt{6}}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{6}} - \frac{\sqrt{3}}{\sqrt{6}} \\ \frac{\sqrt{6}}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{6}} + 0 & \frac{\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{6}} - \frac{\sqrt{3}}{\sqrt{6}} \\ \frac{2\sqrt{6}}{\sqrt{6}} & 2\frac{\sqrt{6}}{\sqrt{6}} + 0 + 0 & \frac{2\sqrt{6}}{\sqrt{6}} + 0 + \frac{\sqrt{3}}{\sqrt{6}} \end{bmatrix}$$

$A^T P = Q \cdot I$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\therefore A = QR.$$

$$\begin{bmatrix} 2 & 3 & 8 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = A^T b$$

$$1) \quad x_1 + 0x_2 + 2x_3 = 3$$

$$0x_1 + 2x_2 + 2x_3 = -3$$

$$-x_1 - x_2 - x_3 = 0$$

$$-x_1 - 2x_2 + 0x_3 = -3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8.7$$

Solve the above equations by least square method.

Solution:

Since it is a non-homogeneous equation, we can write

$$Ax = B$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

$$B^T A = X \cdot A^T A$$

Premultiply both sides by A^T ,

$$A^T \cdot A \cdot X = A^T B$$

$$A^T A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1+1 & 0+0+1+2 & 2+0+1+0 \\ 0+0+1+2 & 0+4+1+4 & 0+4+1+0 \\ 2+0+1+0 & 0+4+1+0 & 4+4+1+0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix} \quad . AB = A \therefore$$

$$A^T B = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

$\Sigma = 3 \cdot 1 + 0 \cdot 0 + (-1) \cdot (-1) + (-1) \cdot (-1)$
 $\Sigma = 3 \cdot 0 + 2 \cdot 2 + (-1) \cdot (-2) + (-2) \cdot (-2)$
 $1 \cdot 2 + 2 \cdot 2 - 1 \cdot (-1) + 0 \cdot (-2)$
 $\Sigma = 3 \cdot 0 + 2 \cdot 2 - 1 \cdot (-1) + 0 \cdot (-2)$

$$= \begin{bmatrix} 3+0+0+3 \\ 0+6+0+6 \\ 6-6+0+0 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Solution
 Ansatz

$$A^T A \cdot X = A^T B$$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Augmented matrix: A^T und $A^T B$ nach Gauß-Jordan Elimination

$$\begin{bmatrix} 3 & 3 & 3 & | & 6 \\ 3 & 9 & 5 & | & 0 \\ 3 & 5 & 9 & | & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{R_1 \rightarrow R_1 - R_3}$

$$\sim \begin{bmatrix} 3 & 3 & 3 & | & 6 \\ 0 & 6 & 2 & | & -6 \\ 0 & 2 & 6 & | & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{R_2 \rightarrow R_2 - R_1}$

$$\sim \begin{bmatrix} 3 & 3 & 3 & | & 6 \\ 0 & 3 & -1 & | & -12 \\ 0 & 2 & 6 & | & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{R_3 \rightarrow R_3 - R_2}$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 6 \\ 0 & 6 & 2 & -6 \\ 0 & 2 & 6 & -6 \end{array} \right] \quad R_3 \rightarrow 3R_3 - R_2$$

$$-18 - (-6)$$

$$-18 + 6$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 6 \\ 0 & 6 & 2 & -6 \\ 0 & 0 & 16 & -12 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -3 \\ 0 & 0 & 4 & -3 \end{array} \right] \quad R_1 \rightarrow R_1/3$$

$$R_2 \rightarrow R_2/2$$

$$R_3 \rightarrow R_3/4$$

$$4x_3 = -3$$

$$x_3 = -\frac{3}{4}$$

$$3x_2 + x_3 = -3$$

$$3x_2 = -3 - x_3$$

$$3x_2 = -3 - x_3$$

$$x_2 = \frac{-3 - x_3}{3}$$

$$x_2 = \frac{-3 + \frac{3}{4}}{3} \Rightarrow \frac{\frac{-12 + 3}{4}}{3}$$

$$x_2 = \frac{-9}{4 \times 3} = \frac{-9}{12} = -\frac{3}{4}$$

$$x_2 = -\frac{3}{4}$$

$$x_1 + x_2 + x_3 = 2$$

$$x_1 = 2 - x_2 - x_3$$

$$x_1 = 2 + \frac{3}{4} + \frac{3}{4}$$

$$x_1 = 2 + \frac{6}{4}$$

$$x_1 = 2 + \frac{3}{2} \Rightarrow \frac{4+3}{2} = \frac{7}{2}$$

$$x_1 = \frac{7}{2}$$

$$x_2 = -\frac{3}{4}$$

$$x_3 = -\frac{3}{4}$$

$$a) \quad x_1 + 0x_2 + 2x_3 = 3$$

$$0x_1 + 2x_2 + 0x_3 = -3 \quad \text{Solve by Least square}$$

$$-x_1 - x_2 - x_3 = 0$$

$$-x_1 + 2x_2 + 0x_3 = -3$$

Solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix} \quad \text{method}$$

Since it is a non-homogeneous eqn.

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix} \quad \text{non homogeneous}$$

Premultiply both sides by A^T

$$A^T A \cdot X = A^T B$$

$$A^T A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1+0+1+1 & 0+0+1-2 & 2+0+1+0 \\ 0+0+1-2 & 0+4+1+4 & 0+4+1+0 \\ 2+0+1+0 & 0+4+1+0 & 4+4+1+0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0+3 \\ 0-6+0-6 \\ 6-6+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix}$$

$$A^T A x = A^T B$$

$$x_1 = 2x_2 + 3x_3 \quad x_1 = 2x_2 + 3x_3 - x_3$$

$$\begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix}$$

$A^T A$ ist invertierbar B - nach \Rightarrow es gibt eine

$$x = xB$$

Augmented Matrix:

$$[A^T A / B] = \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ -1 & 9 & 5 & -12 \\ 3 & 5 & 9 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 6 & 6 & -6 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow 3R_2 + R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 1 & 1 & -1 \end{array} \right] \quad \begin{array}{l} R_3 \rightarrow R_3 - \frac{1}{2}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 0 & 8 & 4 \end{array} \right] \quad \begin{array}{l} R_3 \rightarrow 26R_3 - R_2 \\ R_3 \rightarrow R_3 - \frac{1}{2}R_1 \end{array}$$

$$\textcircled{1} \Rightarrow 8R_3 = 4$$

$$R_3 = 4/8 = 1/2 = 0.5$$

$$\boxed{x_3 = 0.5} \rightarrow \begin{bmatrix} 8+0+0+8 \\ 0+0+0-0 \\ 0+0+0-0 \end{bmatrix}$$

$$x_3 = 1/2$$

$$② \Rightarrow 26x_2 + 18x_3 = -30$$

÷ by 2

$$13x_2 + 9x_3 = -15$$

Put $x_3 = 0.5$;

$$13x_2 = -15 - 9(0.5)$$

$$= -15 - 4.5$$

$$13x_2 = -19.5$$

$$x_2 = \frac{-19.5}{13}$$

$$\boxed{x_2 = -1.5}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} = A$$

$$x_2 = \frac{-39}{26}$$

$$③ \Rightarrow 3x_1 - x_2 + 3x_3 = 6$$

$$3x_1 = 6 + x_2 - 3x_3$$

$$3x_1 = 6 - 1.5 - 3(0.5)$$

$$3x_1 = 3$$

$$\boxed{x_1 = 1}$$

$$\begin{bmatrix} 3 & -1 & 3 \\ 3 & -1 & 3 \\ 3 & -1 & 3 \end{bmatrix} = TAA$$

$$\therefore x_1 = 1 \quad S1 - H - H \quad H + H + H \quad H + H + H$$

$$x_2 = -1.5 \quad H - H - \quad H + H + H \quad H + H + H$$

$$x_3 = 0.5 \quad H + H \quad S1 - H - H \quad S1 - H - H$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = TAA$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = H = TAA$$

$$H = TAA$$

Find the singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$\lambda_1 = 2\sqrt{10} + 2i\sqrt{6}$
 $(2\sqrt{10})^2 = 40 + 4i^2 = 40 - 4 = 36$
 $(2\sqrt{6})^2 = 24 + 24i^2 = 24 - 24 = 0$
 $(2\sqrt{10})(2\sqrt{6}) = 4\sqrt{60} = 4\sqrt{12 \cdot 5} = 8\sqrt{15}$

Solution:

$$AA^T = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2) + 2(2) + -2(-2) & 2(2) + 2(-2) + 2(-2) & 2(2) + 2(-2) + 2(6) \\ 4+4-4 & 4+4+4 & 4-4-12 \\ 4-4+12 & 4-4-12 & 4+4+36 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 12 & 4 & 12 \\ 84 & 12 & -12 \\ 12 & -12 & 44 \end{bmatrix}$$

$\epsilon = 108$
 $\boxed{1 = 108}$

$$\begin{bmatrix} 4+4+4 & 4+4+4 & -4-4-12 \\ 4+4+4 & 4+4+4 & -4-4-12 \\ -4-4-12 & -4-4-12 & 4+4+36 \end{bmatrix}$$

$\boxed{1 = 108}$
 $\boxed{2 = 0}$
 $\boxed{3 = 80}$

$$AA^T = \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$$

$$AA^T = 4 \begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{bmatrix}$$

$$AA^T = 4B$$

$$B = \begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{bmatrix}$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 3+3+11=17$$

$$S_2 = \begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} + \begin{vmatrix} 3 & -5 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix}$$

$$= (33-25) + (33-25) + 0$$

$$S_2 = 8+8=16$$

$$S_3 = 3(33-25) - 3(33-25) - 5(0)$$

$$= 3(8) - 3(8)$$

$$S_3 = 0$$

Substitute S_1, S_2, S_3

$$\lambda^3 - 17\lambda^2 + 16\lambda - 0 = 0$$

$$\lambda^3 - 17\lambda^2 + 16\lambda = 0$$

$$\lambda [\lambda^2 - 17\lambda + 16] = 0$$

$\therefore \lambda = 0$ is a factor

$$\lambda^2 - 17\lambda + 16 = 0$$

$$\lambda^2 - \lambda - 16\lambda + 16 = 0$$

$$\lambda(\lambda-1) - 16(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-16) = 0$$

$\lambda = 16, \lambda = 1$ are factors

$$\therefore \lambda = 0, 1, 16$$

$$\lambda = 16, 1, 0$$

$$4[B] = 64, 4, 0$$

$$\lambda = 64, 4, 0$$

$$\lambda = 64;$$

$$\begin{bmatrix} 12-64 & 12 & -20 \\ 12 & 12-64 & -20 \\ -20 & -20 & 44-64 \end{bmatrix} = \begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ 0 & -20 & -20+20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ -20 & -20 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-52x_1 + 12x_2 - 20x_3 = 0 \Rightarrow -13x_1 + 3x_2 - 5x_3 = 0$$

$$12x_1 - 52x_2 - 20x_3 = 0 \Rightarrow 3x_1 - 13x_2 - 5x_3 = 0.$$

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & & \\ \hline 3 & -5 & -13 & 3 & \\ & & & & \\ -13 & -5 & 3 & -13 & \end{array} \quad \begin{array}{l} 3(28-88) \\ -5(28-88) \\ -13(28-88) \\ \hline 0 = 820+820 \end{array}$$

$$\frac{x_1}{-15-65} = \frac{x_2}{-15-65} = \frac{x_3}{160-9} = k$$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{160} = k$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2} = k$$

$$x_1 = k; x_2 = k; x_3 = -2k$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ -2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\|x_1\| = \sqrt{\langle x_1, x_1 \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{1^2 + 1^2 + (-2)^2}$$

$$= \sqrt{1+1+4} = \sqrt{6} = \sqrt{6}$$

$$\sqrt{1+1+4} = \sqrt{6}$$

$$\sqrt{1+1+4} = \sqrt{6}$$

$$x_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}$$

normalized eigen vectors

$$\lambda = 4;$$

$$\begin{bmatrix} 12-4 & 12 & -20 \\ 12 & 12-4 & -20 \\ -20 & -20 & 44-4 \end{bmatrix} = \begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 40 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 + 12x_2 - 20x_3 = 0$$

$$12x_1 + 8x_2 - 20x_3 = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 \\ 12 & -20 & 8 & 12 \end{array}$$

$$\begin{array}{cccc} 8 & -20 & 12 & 8 \end{array}$$

$$\frac{x_1}{12(-20) - 8(-20)} = \frac{x_2}{12(-20) - 8(-20)} = \frac{x_3}{64 - 16(12)}$$

$$\frac{x_1}{-240 + 160} = \frac{x_2}{-240 + 160} = \frac{x_3}{64 - 144}$$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{-80}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|x_2\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$x_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \text{ normalized Eigen vector}$$

When $\lambda=0$:

$$\Rightarrow \begin{bmatrix} 12-0 & 12 & -20 \\ 12 & 12-0 & -20 \\ -20 & -20 & 44-0 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① & ③ \Rightarrow

$$12x_1 + 12x_2 - 20x_3 = 0$$

$$-20x_1 - 20x_2 + 44x_3 = 0$$

" \div " by 2;

$$6x_1 + 6x_2 - 10x_3 = 0$$

$$-10x_1 - 10x_2 + 22x_3 = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 \\ 6 & -10 & 6 & 6 \\ -10 & 22 & -10 & -10 \end{array}$$

$$\frac{x_1}{132-100} = \frac{x_2}{100-132} = \frac{x_3}{-60+60}$$

$$\frac{x_1}{82} = \frac{x_2}{-32} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$||x_3|| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{1+1} = \sqrt{2}$$

$$x_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{0}{\sqrt{2}} \end{bmatrix}$$

$$u = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

$$A = [u] \begin{bmatrix} \text{Diagonal matrix} \\ S_{\text{rot}}(\lambda) \end{bmatrix} [v^T = u]$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{-16}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ -\frac{16}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} \\ \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} \\ \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} \\ \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} \\ \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{32}{\sqrt{36}} + \frac{2}{\sqrt{9}} \end{bmatrix} = U$$

$$\Rightarrow \begin{bmatrix} \frac{8}{6} + \frac{2}{3} & \frac{8}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} \\ \frac{8}{6} + \frac{2}{3} & \frac{8}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} \\ \frac{-16}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} & \frac{32}{6} + \frac{2}{3} \end{bmatrix} = U$$

$$\Rightarrow \begin{bmatrix} \frac{8+4}{6} & \frac{8+4}{6} & \frac{-16+4}{6} \\ \frac{8+4}{6} & \frac{8+4}{6} & \frac{-16+4}{6} \\ \frac{-16+4}{6} & \frac{-16+4}{6} & \frac{32+4}{6} \end{bmatrix} = U$$

$$\Rightarrow \begin{bmatrix} \frac{12}{6} & \frac{12}{6} & \frac{-12}{6} \\ \frac{12}{6} & \frac{12}{6} & \frac{-12}{6} \\ \frac{-12}{6} & \frac{-12}{6} & \frac{36}{6} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} = A$$

$$\therefore A = UEV$$

Problem

Find the singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

↓

TO PROVE

$$(0-1)\delta + (0-1)A = (UEV)$$

$$(0-1)(0-1) =$$

$$\delta = \sqrt{2}$$

Solution:

$$AA^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$A^T A$
Since A is
not a symmetric
matrix

$$= \begin{bmatrix} 0+1 & 0+1 & 0+0 \\ 0+1 & 1+1 & 1+0 \\ 0+0 & 1+0 & 1+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\delta = \sqrt{1+2+1} = \sqrt{4} = 2$$

TO find Eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \rightarrow ①$$

where

S_1 = sum of the main diagonal

S_2 = sum of the minors of leading diagonal

$$S_3 = |A|$$

S_1 = sum of main diagonal elements

$$= 1+2+1$$

$$= 4$$

S_2 = sum of minors of leading diagonal

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= (2-1) + (1-0) + (0-1)$$

$$= 1+1+1$$

$$= 3$$

$$\delta = \sqrt{0+1+1} = \sqrt{2} \quad ①$$

$$\delta = \sqrt{0+1+1} = \sqrt{2} \quad ②$$

$$\delta = \sqrt{0+1+1} = \sqrt{2} \quad ③$$

$$S_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) - 1(1-0) + 0(1-0)$$

$$= 1(1) - 1(1)$$

$$S_3 = 0$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = A$$

Substitute S_1, S_2, S_3 in eqn ①,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0; \quad \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 0; \quad (\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda = 0, 1, 3}$$

Eigen values are $0, 1, 3$

When $\lambda = 0$:

$$AAT = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1-0 & 0 & 0 \\ 1 & 2-0 & 1 \\ 0 & 1 & 1-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\textcircled{1} \rightarrow x_1 + x_2 + 0x_3 = 0$$

$$\textcircled{2} \rightarrow x_1 + 2x_2 + x_3 = 0$$

$$\textcircled{3} \rightarrow 0x_1 + x_2 + x_3 = 0$$

$$x_1 \quad x_2 \quad x_3$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\|x\| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \text{ normalized eigen vector.}$$

$$\text{when } \lambda=1; \quad |A-\lambda I| = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 = g(x_1) + g(x_2) + g(x_3) \leftarrow \textcircled{1}$
 $0 = g(x_1) + g(x_2) - g(x_3) \leftarrow \textcircled{2}$
 $0 = g(x_1) - g(x_2) + g(x_3) \leftarrow \textcircled{3}$

$$\textcircled{1} \rightarrow 0x_1 + x_2 + 0x_3 = 0$$

$$\textcircled{2} \rightarrow x_1 + x_2 + x_3 = 0$$

$$\textcircled{3} \rightarrow 0x_1 + x_2 + 0x_3 = 0$$

$$\begin{array}{ccc|c|c|c|c} & x_1 & x_2 & x_3 & g(x_1) & g(x_2) & g(x_3) \\ \hline 1 & 0 & 0 & 1 & 1+0 & 0+0 & 0-1 \end{array}$$

$$\begin{array}{cccc|c|c|c} & 1 & 1 & 1 & 1 & g(x_1) & g(x_2) \\ \hline & 1 & 1 & 1 & 1 & 1+1+1 & 1+1-1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1} = \frac{x_1+x_2+x_3}{1+1+1} = \frac{x_1+x_2-x_3}{1+1-1} = \frac{x_1-x_2+x_3}{1+1+1} \quad \|x\| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\|x_2\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

when $\lambda = 3$;

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textcircled{1} \rightarrow -2x_1 + x_2 + 0x_3 = 0$$

$$\textcircled{2} \rightarrow x_1 - x_2 + x_3 = 0$$

$$\textcircled{3} \rightarrow 0x_1 + x_2 - 2x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 0 & -2 \\ -1 & 1 & 1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0+2} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\|x_3\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A = [U] [Diagonal\ matrix] [V = U^T]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} + 0 + 0 & 0 + \frac{1}{\sqrt{2}} + 0 & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}} & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{6} + \frac{1}{2} & \frac{2\sqrt{3}}{6} & \frac{\sqrt{3}}{6} - \frac{1}{2} \\ \frac{2\sqrt{3}}{6} & \frac{4\sqrt{3}}{6} & \frac{2\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} - \frac{1}{2} & \frac{2\sqrt{3}}{6} & \frac{\sqrt{3}}{6} + \frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A = [U] [\text{diagonal matrix}] [V]$$

$$\Rightarrow \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} + 0 + 0 & 0 + \frac{1}{\sqrt{2}} + 0 & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}} + 0 + 0 & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}} + 0 + 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}} & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{6} + \frac{1}{2} + 0 & \frac{2\sqrt{3}}{6} + 0 + 0 & \frac{\sqrt{3}}{6} - \frac{1}{2} + 0 \\ \frac{2\sqrt{3}}{6} + 0 + 0 & \frac{4\sqrt{3}}{6} + 0 + 0 & \frac{2\sqrt{3}}{6} + 0 + 0 \\ \frac{\sqrt{3}}{6} - \frac{1}{2} + 0 & \frac{2\sqrt{3}}{6} + 0 + 0 & \frac{\sqrt{3}}{6} + \frac{1}{2} + 0 \end{bmatrix}$$

(a) Find eigen value, eigen vector and QR factorization of the matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

Given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

$$S_1 = \text{sum of main diagonal elements}$$

$$\begin{aligned} &= 8 + 7 + 3 \\ &= 18 \end{aligned}$$

$$S_2 = \text{sum of the minors of the main diagonal elements.}$$

$$\begin{aligned} &= \left| \begin{array}{cc} 7 & -4 \\ -4 & 3 \end{array} \right| + \left| \begin{array}{cc} 8 & 2 \\ 2 & 3 \end{array} \right| + \left| \begin{array}{cc} 8 & -6 \\ -6 & 7 \end{array} \right| \\ &= (21 - 16) + (24 - 4) + (56 - 36) \\ &= 5 + 20 + 20 \\ &= 45 \end{aligned}$$

$$S_3 = |A|$$

$$\begin{aligned} &= 8(5) + 6(-18 + 8) + 2(24 - 14) \\ &= 40 + 6(-10) + 2(10) \\ &= 40 - 60 + 20 \\ &= 65 - 60 \\ &= 5 \end{aligned}$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0;$$

$$\lambda^2 - 18\lambda + 45 = 0$$

$$(\lambda - 3) = 0$$

$$(\lambda - 15) = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda$$

$$\begin{bmatrix} 45 \\ -3 \\ -15 \end{bmatrix}$$

$\therefore \lambda = 0$ is a root of $\lambda^2 - 18\lambda + 45 = 0$

$$0 = \lambda(\lambda^2 - 18)$$

$$\lambda = 15, 3, 0$$

Eigen vectors:

$$\text{When } \lambda = 0;$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

① $\rightarrow 0 = 8x_1 - 6x_2 + 2x_3$
 ② $\rightarrow 0 = -6x_1 + 7x_2 - 4x_3$
 ③ $\rightarrow 0 = 2x_1 - 4x_2 + 3x_3$

$$\textcircled{1} \rightarrow 8x_1 - 6x_2 + 2x_3 = 0$$

$$\textcircled{2} \rightarrow -6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

解得 3x1, ③ ② - ① 3x1

$$\begin{array}{cccc|c} 8 & -6 & 2 & 0 \\ -6 & 7 & -4 & 0 \\ 2 & -4 & 3 & 0 \end{array}$$

$$\begin{array}{cccc|c} 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{5}{2} & 0 \end{array}$$

solve ① & ③;

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & & \\ \hline -6 & 8 & 2 & -6 & 0 \\ 7 & -4 & -6 & 7 & 0 \end{array} \xrightarrow{\text{消去法}} \begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \end{array}$$

$$\frac{x_1}{-4-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} \xrightarrow{\text{消去法}} \frac{x_1}{-18} = \frac{x_2}{16} = \frac{x_3}{20}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

To find eigenvector for $\lambda = 3$;

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 0 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \rightarrow ①$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \rightarrow ②$$

$$2x_1 - 4x_2 + 0x_3 = 0 \rightarrow ③$$

Solve ① & ②, we get

$$\begin{array}{cccc} x_1 & x_2 & x_3 \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\text{Given matrix } A = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

To find eigenvector for $\lambda = 15$:

$$(A - 15I)x = 0$$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(8, 2, 1) = 0$
 $(-1, 1, 8) = 0$
 $(1, 8, -8) = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\text{Row reduction method}$
 $(-1, 1, 8) = 0$
 $(1, 8, -8) = 0$

$$-7x_1 - 6x_2 + 2x_3 = 0 \rightarrow ① \quad (8, 2, 1) = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \rightarrow ② \quad \frac{\text{L.H.S. N.D.}}{\text{R.H.S. N.D.}} = 0 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0 \rightarrow ③ \quad (8, 2, 1) = 0$$

Solve ① & ②, we get $(8, 2, 1), (0, 0, 1)$

$$\begin{array}{r} x_1 \quad x_2 \quad x_3 \\ -6 \quad (8, 2, 1) \quad -7 \quad -6 \\ \hline p+q+r \end{array} \quad (1, 1, 8) = 0$$

$$\begin{array}{r} x_1 \quad x_2 \quad x_3 \\ -8 \quad -4 \quad -6 \quad -8 \\ \hline (8, 2, 1) \quad \frac{2}{p} \end{array} \quad (1, 8, -8) = 0$$

$$\frac{x_1}{24+16} = \left(\frac{x_2}{-40} \right) = \left(\frac{x_3}{56-32} \right) \Rightarrow \left(\frac{81}{p} + \frac{2}{p} + \frac{21}{p} \right) = 0$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \quad \left(\frac{81}{p} + \frac{2}{p} + \frac{21}{p} \right) = 0$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow -\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad (8, 2, 1) = 0 \quad (1, 8, -8) = 0$$

$$A_1 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{array}{l} \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}$$

$$a_1 = (1, 2, 2)$$

$$a_2 = (2, 1, -2)$$

$$a_3 = (2, -2, 1)$$

$$u_1 = (1, 2, 2)$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \mathbf{I}$$

$$u_2 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$A^T D = \mathbf{I}$$

$$= (2, 1, -2) - \frac{\langle (1, 2, 2), (2, 1, -2) \rangle}{\langle (1, 2, 2), (1, 2, 2) \rangle} (1, 2, 2)$$

$$= (2, 1, -2) - \frac{2+2-4}{1+4+4} (1, 2, 2)$$

$$u_2 = (2, 1, -2)$$

$$u_3 = a_3 - \frac{\langle u_1, a_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \left[\cancel{(2, 1, -2)} - \frac{\langle u_2, a_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \right]$$

$$= (2, -2, 1) - \frac{\langle (1, 2, 2), (2, -2, 1) \rangle}{\langle (1, 2, 2), (1, 2, 2) \rangle} (1, 2, 2) - \left[\cancel{(2, -2, 1)} - \frac{\langle u_2, a_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \right]$$

$$= \left[(2, -2, 1) - \frac{2-4+2}{1+4+4} (1, 2, 2) \right] - \left[\cancel{(2, -2, 1)} - \frac{\langle (2, 1, -2), (2, -2, 1) \rangle}{\langle (2, 1, -2), (2, 1, -2) \rangle} (2, 1, -2) \right]$$

$$= (2, -2, 1) - \left[\cancel{(2, -2, 1)} - \frac{4-2-2}{4+1-4} (2, 1, -2) \right]$$

$$u_3 = (2, -2, 1)$$

$$\|u_1\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\|U_2\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\|u_3\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$$

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} \frac{1}{3}(8+12) & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3}(8+12) \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{4(26+8)}{3(8+12)} & 1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} \frac{(8+12)}{3(8+12)} & 1 \\ 2 & -2 \end{bmatrix}$$

$$\begin{aligned}
 &= \left[\frac{1}{3} + \frac{4}{3} + \frac{4}{3} \right] - \left[\frac{2}{3} + \frac{2}{3} - \frac{4}{3} \right] \\
 &= \left[\frac{2}{3} + \frac{2}{3} - \frac{4}{3} \right] - \left[\frac{4}{3} + \frac{1}{3} - \frac{4}{3} \right] + \left[\frac{4}{3} - \frac{2}{3} - \frac{2}{3} \right] \\
 &= \left[\frac{2}{3} - \frac{4}{3} + \frac{2}{3} \right] - \left[\frac{4}{3} - \frac{2}{3} - \frac{2}{3} \right] + \left[\frac{4}{3} + \frac{2}{3} + \frac{1}{3} \right]
 \end{aligned}$$

$$\frac{(x_1 - 3)(x_2 - 1)}{(x_1 + 2)(x_2 + 2)} \begin{bmatrix} \frac{9}{3} & -1 \\ 0 & \frac{9}{3} \end{bmatrix} = \begin{bmatrix} 0 & \frac{9}{3} \\ 0 & 0 \end{bmatrix} \cdot \frac{x_1 + 2}{x_1 + 2} = (1, \frac{9}{3})$$

(1, 5, 9) 1

$$R = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A^T R$$

$$A = QR.$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^T A$$

(4) Least singular method

$$x - 2y + z = 1$$

$$2x - 3y - z = 0$$

$$-x + y + 2z = 1$$

$$3x - 5y + 0z = 0$$

solutions: $Ax = B$

$$A^T A x = A^T B$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$$

$$A^T A = X \cdot A^T A$$

$$\begin{bmatrix} 37 & 18 & 1 & 21 \\ 18 & 38 & 1 & 21 \\ 1 & 1 & 8 & 1 \\ 21 & 21 & 1 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -3 & 1 & -5 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 8 & 1 \\ 2 & -3 & -1 & 1 \\ -1 & 1 & 2 & -5 \\ 3 & -5 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1+9 & -2-6-1-15 & 1-2-2 \\ -2-6-1-15 & 4+9+1+25 & -2+3+2 \\ 1-2-2 & -2+3+2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 15 & -24 & -3 \\ -8 & 13 & 1 \\ -3 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 58 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3x4 4x1

$$= \begin{bmatrix} 1+0-1+0 \\ -2+0+1+0 \\ 1+0+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$S = S - 8E - 10G$$

$$I = 5S + 8G + 10E$$

$$G = 5O + 8D - 10E$$

$$S = X_A \quad ; \text{durchsetzen}$$

$$S^T B = X \cdot A^T A$$

$$A^T \cdot A \cdot X = A^T B$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ -8 & 13 & 1 & -1 \\ -1 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -3 & 9 & 15 \end{bmatrix} \quad R_2 \rightarrow 5R_2 + 8R_1$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 3 & 5 \end{bmatrix} \quad R_3 \rightarrow R_3 / 3$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

$$0x_3 = 0$$

$$\boxed{x_3 = \alpha}$$

$$x_2 - 3x_3 = -5$$

$$x_2 = -5 + 3x_3$$

$$\boxed{x_2 = -5 + 3\alpha}$$

$$5x_1 - 8x_2 - x_3 = 0$$

$$5x_1 = 8x_2 + x_3$$

$$x_1 = \frac{8x_2 + x_3}{5}$$

$$x_1 = \frac{8(-5 + 3\alpha) + \alpha}{5}$$

$$= \frac{-40 + 24\alpha + \alpha}{5}$$

$$= \frac{-40 + 25\alpha}{5}$$

$$\boxed{x_1 = -8 + 5\alpha}$$

$$\therefore x_1 = -8 + 5\alpha$$

$$x_2 = -5 + 3\alpha$$

$$x_3 = \alpha$$

3) Singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Solution:

premultiply by A^T since A is not a symmetric matrix.

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 17 & 2+30 & 3+24 \\ 2+30 & 4+25 & 6+30 \\ 3+24 & 6+30 & 9+36 \end{bmatrix} \Rightarrow \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix} \end{aligned}$$

To find eigen values of $A^T A$:

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$s_1 = \text{sum of the main diagonal elements}$

$$= 17 + 29 + 45 = 91$$

$$\begin{aligned} s_2 &= \begin{vmatrix} 29 & 36 \\ 36 & 45 \end{vmatrix} + \begin{vmatrix} 17 & 27 \\ 27 & 45 \end{vmatrix} = 17 \cdot 27 - 22 \cdot 36 + 493 - 484 = 54 \\ &= [1305 - 1296] + [765 - 729] + [493 - 484] = 9 + 36 + 9 = 54 \end{aligned}$$

$$\begin{aligned} s_3 &= |A| = 17 [(29 \times 45) - (36 \times 36)] - 22 [(29 \times 45) - (36 \times 27)] \\ &\quad + 27 [(29 \times 36) - (27 \times 29)] = 153 - 396 + 243 = 60 \end{aligned}$$

$$\lambda^3 - 91\lambda^2 + 54\lambda = 0$$

$$\lambda(\lambda^2 - 91\lambda + 54) = 0$$

$$\lambda = 0$$

$$a=1; b=-91; c=54$$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{-(-91) \pm \sqrt{(-91)^2 - 4(1)(54)}}{2(1)}$$

$$\Rightarrow \frac{91 \pm \sqrt{8281 - 816}}{2}$$

$$\Rightarrow 90.4, 0.59 \quad \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

$$\therefore \lambda_1 = 90.4 \quad \lambda_2 = 0.6 \quad \lambda_3 = 0$$

For $\lambda = 90.4$;

$$\left[\begin{array}{ccc} 17-90.4 & 22 & 27 \\ 22 & 29-90.4 & 36 \\ 27 & 36 & 45-90.4 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

$$\left[\begin{array}{ccc} -73.4 & 22 & 27 \\ 22 & -61.4 & 36 \\ 27 & 36 & 45.4 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] + \left[\begin{array}{c} d_8 \\ d_8 \\ d_8 \end{array} \right]$$

$$-73.4x_1 + 22x_2 + 27x_3 = 0 \rightarrow ① \quad d_8 =$$

$$22x_1 - 61.4x_2 + 36x_3 = 0 \rightarrow ② \quad d_8 =$$

$$27x_1 + 36x_2 - 45.4x_3 = 0 \rightarrow ③ \quad d_8 =$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \downarrow & \downarrow & \downarrow & \\ 22 & 27 & 22 & -73.4 \\ & 36 & 22 & 22 \\ -61.4 & 36 & -61.4 & 22 \end{array}$$

$$0 = k_1d_8 + k_2d_8 + k_3d_8$$

$$\begin{aligned} & \text{①} \leftarrow 0 = 8K + 12 + 18P + 12 \\ & \frac{x_1}{192 + 1657.8} = \frac{x_2}{594 + 2642.4} = \frac{x_3}{4506.76 - 484} = K \\ & \text{②} \leftarrow 0 = 8K + 12 + 18P + 12 \end{aligned}$$

$$\frac{x_1}{2449.8} = \frac{x_2}{3236.4} = \frac{x_3}{4022.76} = K$$

$$\frac{x_1}{1.2249} = \frac{x_2}{1.6182} = \frac{x_3}{2.01138} = K$$

$$\frac{x_1}{1.2} = \frac{x_2}{1.6} = \frac{x_3}{2.0} = K$$

$$\|x_1\| = \sqrt{(1.2)^2 + (1.6)^2 + (2.0)^2} = \sqrt{1.44 + 2.56 + 4} = \sqrt{8}$$

$$\|x_1\| = \sqrt{8}$$

$$x_1 = \begin{bmatrix} 1.2 \\ 1.6 \\ 2.0 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \\ 2.0 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \\ 2.0 \end{bmatrix}$$

For $\lambda = 0.6$;

$$\begin{bmatrix} 17 - 0.6 & 22 & 27 \\ 22 & 89 - 0.6 & 36 \\ 27 & 36 & 45 - 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16.4 & 22 & 27 \\ 22 & 88.4 & 36 \\ 27 & 36 & 44.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$16.4x_1 + 22x_2 + 27x_3 = 0 \rightarrow ①$$

$$22x_1 + 28.4x_2 + 36x_3 = 0 \rightarrow ②$$

$$27x_1 + 36x_2 + 44.4x_3 = 0 \rightarrow ③$$

$$\begin{matrix} x_1 & \frac{16.4}{22} & \frac{22}{28.4} \\ x_1 & \frac{88.2}{22} & \frac{28.4}{28.4} \end{matrix} = \begin{matrix} x_2 & \frac{16.4}{36} & \frac{22}{36} \\ x_2 & \frac{88.2}{36} & \frac{28.4}{36} \end{matrix} = \begin{matrix} x_3 & \frac{27}{44.4} & \frac{36}{44.4} \\ x_3 & \frac{18.9}{44.4} & \frac{36}{44.4} \end{matrix}$$

$$\frac{x_1}{-192 - 766.8} = \frac{x_2}{594 - 590.4} = \frac{x_3}{465.76 - 484} = \frac{1}{8.1}$$

$$\frac{x_1}{25.2} = \frac{x_2}{3.6} = \frac{x_3}{-18.24} = \frac{1}{11.04} \quad 8.1 = 11.04 \times$$

$$\|x_2\| = \sqrt{7^2 + 1^2 + (-5.06)^2} = \sqrt{49 + 1 + 25.6036} = \sqrt{75.6036} = 8.69$$

$$x_2 = \begin{bmatrix} 7 \\ 1 \\ -5.06 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} 1 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$17x_1 + 22x_2 + 27x_3 = 0 \rightarrow \textcircled{1} \quad [17, 22, 27] = 0$$

$$22x_1 + 29x_2 + 36x_3 = 0 \rightarrow \textcircled{2} \quad [22, 29, 36] = 0$$

$$27x_1 + 36x_2 + 45x_3 = 0 \rightarrow \textcircled{3} \quad [27, 36, 45] = 0$$

$$\begin{array}{ccc|c} & x_1 & x_2 & x_3 \\ \hline 17 & 17 & 22 & 27 \\ 22 & 22 & 29 & 36 \\ 27 & 27 & 36 & 45 \end{array}$$

$$\begin{array}{ccc|c} 29 & 36 & 22 & 17.29 \\ 8 & 8 & 8 & 8 \\ \hline x_1 & x_2 & x_3 & 0 \end{array}$$

$$\frac{x_1}{792 - 783} = \frac{x_2}{594 - 612} = \frac{x_3}{493 - 484}$$

$$\frac{x_1}{9} = \frac{x_2}{-18} = \frac{x_3}{9}$$

$$\frac{x_1}{9} = \frac{x_2}{-2} = \frac{x_3}{8}$$

$$||x_3|| = \sqrt{1^2 + (-2)^2 + 8^2} = \sqrt{1+4+64} = \sqrt{73}$$

$$x_3 = \frac{1}{\sqrt{73}} \begin{bmatrix} 8.8 \\ -1.1 \\ 8.8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 8 \\ -2 & 1 & -8 \\ 8 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8.8 \\ -1.1 \\ 8.8 \end{bmatrix} = \begin{bmatrix} 18.8 & 0 & 17.7 \\ 0 & 17.7 & 0 \\ 0 & 0 & 17.7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8.8 \\ -1.1 \\ 8.8 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{7}{8.7} & \frac{1}{\sqrt{5}} \\ \frac{1.6}{\sqrt{8}} & \frac{1}{8.7} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{8}} & \frac{-5.06}{8.7} & \frac{1}{\sqrt{5}} \end{bmatrix} \quad E = \begin{bmatrix} \sqrt{90.4} & 0 & 0 \\ 0 & \sqrt{10.6303} & 0 \\ 0 & 0 & \sqrt{28.13} \end{bmatrix}$$

$$V = U^T = \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{1.6}{\sqrt{8}} & \frac{2}{\sqrt{8}} \\ \frac{7}{8.7} & \frac{1}{8.7} & \frac{-5.06}{8.7} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$A = [U][E][V] \leftarrow O = \Sigma U E F_{\text{load}} V \text{, now}$$

$$= \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{7}{8.7} & \frac{1}{\sqrt{5}} \\ \frac{1.6}{\sqrt{8}} & \frac{1}{8.7} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{8}} & -\frac{5.06}{8.7} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 9.5 & 0 & 0 \\ 0 & 0.77 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{1.6}{\sqrt{8}} & \frac{1}{\sqrt{5}} \\ \frac{7}{8.7} & \frac{1}{8.7} & -\frac{2}{\sqrt{5}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11.4}{\sqrt{8}} & \frac{5.39}{8.7} & 0 \\ \frac{15.2}{\sqrt{8}} & \frac{0.77}{8.7} & 0 \\ \frac{19}{\sqrt{8}} & \frac{-3.9}{8.7} & 0 \end{bmatrix} \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{1.6}{\sqrt{8}} & \frac{1}{\sqrt{5}} \\ \frac{7}{8.7} & \frac{1}{8.7} & -\frac{2}{\sqrt{5}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13.68}{8} + \frac{37.73}{75.69} & \frac{18.24}{8} + \frac{5.39}{75.69} & \frac{22.8}{8} - \frac{27.2734}{75.69} \\ \frac{18.24}{8} + \frac{5.39}{75.69} & \frac{24.32}{8} + \frac{0.77}{75.69} & \frac{30.4}{8} - \frac{3.8962}{75.69} \\ \frac{22.8}{8} - \frac{27.2734}{75.69} & \frac{11.4}{8} - \frac{3.9}{75.69} & \frac{38}{8} + \frac{19.734}{75.69} \end{bmatrix}$$

$$= \begin{bmatrix} 1.91 + 0.4984 & 2.28 + 0.07121 & 2.85 - 0.3603 \\ 2.28 + 0.07121 & 3.04 + 0.01017 & 3.8 - 0.05147 \\ 2.85 - 0.36068 & 10.425 - 0.0515 & 4.75 + 0.2607 \end{bmatrix}$$

$$A = \begin{bmatrix} 2.2084 & 2.35121 & 2.4897 \\ 2.35121 & 3.05017 & 3.75853 \\ 2.4897 & 1.3735 & 5.0107 \end{bmatrix} \begin{bmatrix} 2.2084 & 2.35121 & 2.4897 \\ 2.35121 & 3.05017 & 3.75853 \\ 2.4897 & 1.3735 & 5.0107 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \frac{1}{8V} & \frac{1}{8V} & \frac{1}{8V} \\ \frac{1}{8V} & \frac{1}{8V} & \frac{1}{8V} \\ \frac{1}{8V} & \frac{1}{8V} & \frac{1}{8V} \end{bmatrix} = T_U = V$$

Canonical Basis

Generalized Eigen Vector: → can be find only for particular condition

$$(i) (A - \lambda I)^n X_r = 0 \quad \text{rank } n$$

$$(ii) (A - \lambda I)^{n-1} X_r \neq 0$$

$\Rightarrow r = n$

Chain

$$((I\lambda - A)^r)_{\text{Rank}} = ((I\lambda - A))_{\text{Rank}} = n$$

$$X_{r-1} = (A - \lambda I) X_r$$

$$X_{r-2} = (A - \lambda I) X_{r-1}$$

$$X_{r-3} = (A - \lambda I) X_{r-2}$$

we are finding different generalized eigenvectors for the same matrix.

$$\textcircled{1} \quad \begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$0 = s X^1 (I\lambda - A) \quad \textcircled{2}$$

$$0 \neq s X^{1+1} (I\lambda - A)$$

$$i.e. s = \text{Rank } (I\lambda - A)$$

$$0 = s X^2 (I\lambda - A)$$

Step 1: Find the eigen values of the matrix A

$$\lambda = 3, 3, 3, 3$$

Step 2: Rank- multiplicity = $4 - 4 = 0$. $0 = s X^3 (I\lambda - A)$
 $N \downarrow \text{multip.}$

$$0 \neq s X^3 (I\lambda - A)$$

$$\text{Rank } ((A - \lambda I)^3) = 0$$

$$\text{Step 3: } A - \lambda I = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank } = 2$$

$$(A - \lambda I)^2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank } = 0$$