Exponential distribution:

* Derive MGF, Mean and variance of exponential distribution.

The density function of exponential distribution is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty = \lambda \left[0 - \frac{1}{-(\lambda - t)} \right] = \frac{\lambda}{\lambda - t}$$

$$\boxed{\therefore M_x(t) = \frac{\lambda}{\lambda - t}}$$

To derive the mean and variance expand $M_x(t)$ in powers of t.

<u>Note:</u> For any cont R.v we are not differentiate but instead we are expand the powers of $M_x(t)$.

$$M_x(t) = \frac{\cancel{\lambda}}{\cancel{\lambda}(1 - t/\lambda)} = \left(1 - t/\lambda\right)^{-1}$$

$$\left[\begin{array}{l} \text{Formula} \\ (1-x)^{-1} = 1 + x + x^2 + \cdots \end{array} \right]$$

$$\Rightarrow M_x(t) = 1 + t/\lambda + t^2/\lambda^2 + \cdots$$

$$\text{Mean} = E(X) = (\text{coeff of } t) * 1! = \frac{1}{\lambda}$$

$$E(x^2) = (\text{coeff of } t^2) * 2! = \frac{1}{\lambda^2} \times 2 = 2/\lambda^2$$

Variance $= E(X^2) - (EX)^2$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2-1}{\lambda^2} = 1/\lambda^2$$

Pb The length of time a person speaks over phone follows exponential distribution with mean $\cdot 6$ mins. what is the Prob that person will talk for (i) more than 8 mins

(ii) between 4 & 8 mins

Sol^n Given that Mean = 6 mins

$$1/\lambda = 6 \Rightarrow \boxed{\lambda = 1/6}$$

$$\boxed{\int e^{ax}\,dx = \frac{e^{ax}}{a}}$$

WKT PMF $f(x) = \lambda e^{-\lambda x}$, $x > 0$

(i) P(more than 8 mins) $= \int_8^\infty f(x)\,dx = \int_8^\infty \frac{1}{6} e^{-x/6}\,dx$

$$= \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_8^\infty$$

$$= \frac{1}{6}\left[0 - \frac{e^{-8/6}}{-1/6}\right] = \cancel{\frac{1}{6}}\left[\frac{e^{-8/6}}{\cancel{1/6}}\right]$$

$$= e^{-8/6} = 0.264$$

(ii) P(between 4 & 8 mins) $= \int_4^8 f(x)\,dx = \int_4^8 \frac{1}{6} e^{-x/6}\,dx$

$$= \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_4^8 = \frac{1}{6}\left[\frac{e^{-8/6}}{-1/6} - \frac{e^{-4/6}}{-1/6}\right] = \cancel{\frac{1}{6}}\left[\frac{e^{-4/3} - e^{-2/3}}{\cancel{-1/6}}\right]$$

$$= \left[-e^{-4/3} + e^{-2/3}\right] = 0.25$$

Pb The mileage which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40000 km. Find the probabilities that one of these tyre will last (i) atleast 20000 km
(ii) atmost 30,000 km

Sol^n

X denotes the mileage

$$\text{Mean} = 1/\lambda = 40000 \Rightarrow \lambda = \frac{1}{40,000}$$

X follows exponential distribution

$$f(x) = \lambda \cdot e^{-\lambda x}, \quad x > 0$$

(i) $P(\text{atleast } 20000\text{km}) = P(x \geq 20,000)$

$$= \int_{20000}^{\infty} \lambda \cdot e^{-\lambda x}\, dx$$

$$= \int_{20000}^{\infty} \frac{1}{40,000} e^{-1/40000 \cdot x}\, dx = \frac{1}{40,000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_{20,000}^{\infty}$$

$$= \frac{1}{40,000}\left[0 - \frac{e^{-\frac{20,000}{40000}}}{-1/40,000}\right] = \frac{1}{\cancel{40,000}}\left[\frac{e^{-1/2}}{\cancel{1/40000}}\right] = e^{-1/2}$$

$$P(x \geq 20000) = 0.6065$$

(ii) $P(\text{at most } 30000\text{ km}) = P(X \leq 30000) = \int_0^{30000} f(x)\,dx$

$$= \int_0^{30000} \frac{1}{40000} e^{-x/40000}\, dx = \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_0^{30000}$$

$$= \frac{1}{40000}\left[\frac{-e^{-30000/40000} + e^{0}}{1/40000}\right] = \left[-e^{-3/4} + e^{0}\right] = \left[1 - e^{-0.75}\right]$$ (26)

$$= 0.5276$$

Gamma distribution

Find the MGF of the Random variable x whose pdf is given by $f(x) = \frac{x}{4} e^{-x/2}$, $x > 0$ hence dedue its mean and variance

Soln

The probability density function of Gamma distribution is $f(x) = \frac{\lambda^{k}}{\Gamma k} x^{k-1} e^{-\lambda x}$, $x > 0$ | when $\lambda = 1/2$, $k = 2$

MGF

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \frac{x}{4} e^{-x/2}\,dx$$

$$= \frac{1}{4}\int_0^\infty x e^{-(1/2 - t)x}\,dx$$

$\int u\,dv = uv - u'v_1 + \cdots$

$u = x$ $\quad dv = e^{-(1/2 - t)x}dx$

$u' = 1$ $\quad v = \frac{e^{-(1/2-t)x}}{-(1/2 - t)}$

$u'' = 0$ $\quad v_1 = \frac{e^{-(1/2-t)x}}{\left(\frac{1}{2} - t\right)^2}$

$$= \frac{1}{4}\left[\frac{x e^{-(1/2 - t)x}}{-(1/2 - t)} - \frac{e^{-(1/2-t)x}}{(1/2 + t)^2}\right]_0^\infty$$

$$= \frac{1}{4}\left[(0-0) - \left(0 - \frac{1}{(1/2 - t)^2}\right)\right] = \frac{1}{4}\,\frac{1}{\left(\frac{1}{2} - t\right)^2}$$

Expanding MGF in powers of t

$$= \frac{1}{4}\left[\left(\frac{1}{2}-t\right)^{-2}\right] = \frac{1}{4}\left(\frac{1}{2}\right)^{-2}\left(1-2t\right)^{-2}$$

$$(1-x)^{-2} = 1+2x+3x^2+\cdots$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{2}}{1}\right)^{2}\left(1-2t\right)^{-2}$$

$$= \left[1+2(2t)+3(2t)^2+\cdots\right]$$

$$\text{Mean} = E(X) = (\text{coeff of } t) * \underline{|1} = 4$$

$$E(X^2) = (\text{coeff of } t^2) \times 2! = (3\times 4)2! = 12\times 2 = 24.$$

$$\text{variance} = E(X^2) - E(X)^2$$

$$= 24 - 4^2 = 24 - 16 = 8$$

<u>Gamma distribution</u> (MGF, Mean & variance)

$$\text{MGF} \quad M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma\lambda} e^{-x} x^{\lambda-1}\,dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-t)x} \cdot x^{\lambda-1}\,dx$$

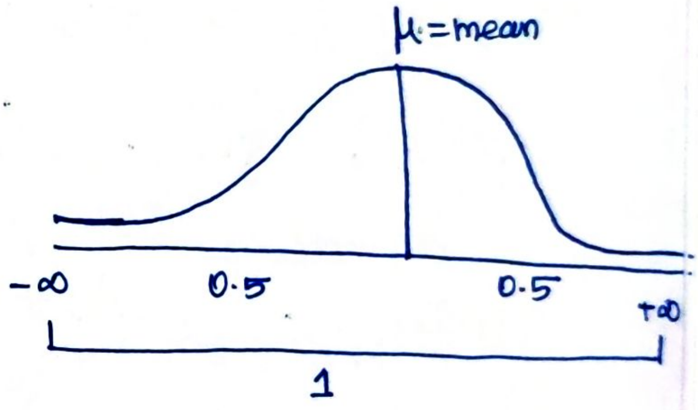
$$\left[\int_0^\infty e^{-ax} x^{\lambda-1}\,dx = \frac{\Gamma(\lambda)}{a^\lambda}\right] \text{ formula}$$

$$= \frac{1}{\cancel{\Gamma\lambda}} \frac{\cancel{\Gamma(\lambda)}}{(1-t)^\lambda} = \frac{1}{(1-t)^\lambda} = M_x(t) = (1-t)^{-\lambda}$$

$$\boxed{\text{Mean} = \text{variance} = \lambda}$$

Normal distribution

Let x be Continuous random variable with f(x) associated probability then f(x) is said to follow normal distribution if $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



Proporties of Normal distribution

* Normal distribution is a symmetrical distribution
* The curve has single peak point that is unimodel.
* The mean of normal distribution is lies at the centre and also mean = median = mode is coinside
* The tails of Normal distribution extends indefenitely and never touch the x-axis
* Area property : In a Normal distribution 68% lies b/w $P(\mu \pm \sigma)$ about 95% of the observation will lies b/w $P(\mu + 2\sigma)$ and about 99.7% of the observation will lies b/w $P(\mu \pm 3\sigma)$.

Standard Normal distribution

If X is normally distributed random variable with μ and σ as its mean and standard deviation respectively, then

$Z = \frac{X-\mu}{\sigma}$ is called Standard Normal Variate such that

$\mu = 0$ and $\sigma^2 = 1$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2)}, \quad -\infty < z < \infty.$$

MGF Mean and variance of Normal distribution

The probability density function of standard normal variate is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$

where $z = \frac{x-\mu}{\sigma}$.

MGF is defined as $M_z(t) = E(e^{tz})$

$$= \int_{-\infty}^{\infty} e^{tz} f(z)\, dz = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{z^2}{2} - tz\right]} dz$$

The power $\left(\frac{z^2}{2} - tz\right)$ of e looks like $A^2 - 2AB$. We make it this as $A^2 - 2AB + B^2$ To write in the form of $A^2 - 2AB + B^2$

Consider $A^2 = z^2/2$, $2AB = tz$

$A = z/\sqrt{2}$ $\quad 2\left(\frac{z}{\sqrt{2}}\right)B = tz$

$$\sqrt{2} \times \cancel{\sqrt{2}} \times \frac{z}{\cancel{\sqrt{2}}} \quad B = tz \Rightarrow B = \frac{tz}{\sqrt{2}\cancel{z}} = t/\sqrt{2}$$

$$\boxed{B^2 = \frac{t^2}{2}}$$

Let us add and subtract B^2 so that the value is

$$\frac{z^2}{2} - tz + \frac{t^2}{2} - \frac{t^2}{2} = \left(\frac{z-t}{\sqrt{2}}\right)^2 - \frac{t^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-t}{\sqrt{2}}\right)^2} \cdot e^{t^2/2} \, dz$$

sub $u = \frac{z-t}{\sqrt{2}}$ $\quad \frac{du}{dz} = \frac{1}{\sqrt{2}}$ $\quad \sqrt{2}\, du = dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} \cdot \sqrt{2} \, du$$

$$= \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \cdot \cancel{\sqrt{2}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} du$$

$$= e^{t^2/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = e^{t^2/2} \frac{1}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$\boxed{M_z(t) = e^{t^2/2}}$$

$$M_z(t) = E(e^{tz}) = e^{t^2/2} \Rightarrow E\left(e^{t \cdot \left(\frac{x-\mu}{\sigma}\right)}\right) = e^{t^2/2}$$

$$E\left(e^{xt/\sigma} \cdot e^{-t\mu/\sigma}\right) = e^{t^2/2} \Rightarrow E\left[e^{x\,t/\sigma}\right] = e^{t^2/2} \cdot e^{t\mu/\sigma}$$

$$\therefore M_x(t) = E\left(e^{xt}\right) = e^{\sigma^2 t^2/2} \cdot e^{t\mu}$$

$$= \left[1 + \frac{\frac{\sigma^2 t^2}{2}}{\lfloor 1} + \frac{\frac{\sigma^4 t^4}{4}}{\lfloor 2} + \cdots\right]\left[1 + \frac{t\mu}{\lfloor 1} + \frac{t^2\mu^2}{\lfloor 2} + \cdots\right]$$

$$= 1 + \frac{t\mu}{\lfloor 1} + \frac{t^2\mu^2}{2!} + \frac{\sigma^2 t^2}{\lfloor 2} + \cdots$$

$$E(x) = \text{mean} = (\text{coeff. of } t) * \lfloor 1 = \mu$$

$$E(x^2) = (\text{Coeff. of } t^2) * \lfloor 2 = \left(\frac{\mu^2+\sigma^2}{2}\right) 2 = \mu^2 + \sigma^2$$

$$\text{variance} = E(x^2) - E(x)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$$\boxed{\text{Mean} = \mu \qquad \text{variance} = \sigma^2}$$

#Pb

Students of Class were given Mechanical aptitude test. The grade were found to be normally distributed with mean 60 and standard deviation 5. what percent of student score (i) more than 60 grades

(ii) Less than 56 grades

(iii) Between 45 and 65 grades

Show that with a sketch for all)

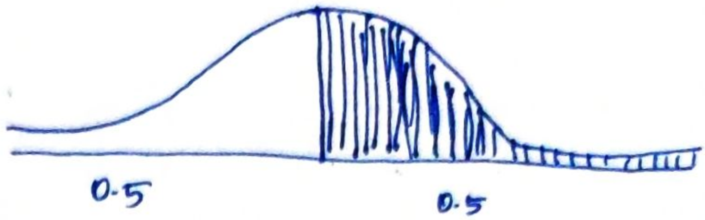
Sol^n

Given Mean $\mu = 60$ and Standard deviation $\sigma = 5$.

Standard normal variate $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-60}{5}$

(i) $P(\text{more than } 60) = P(x>60) = P\left(z > \frac{60-60}{5}\right) = P(z>0)$

$= P(0<z<\infty) = 0.500$

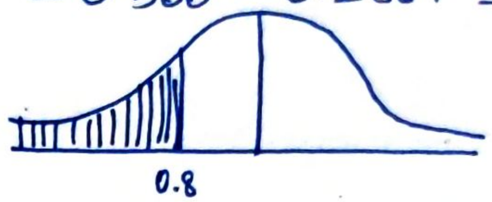


Students scored more than 60 marks $= 0.5000 \times 100 = 50\%$

(ii) $P(\text{less than } 56) = P(x<56) = P\left(z < \frac{56-60}{5}\right) = P(z<-0.8)$

$= P(-\infty<z<0) - P(-0.8<z<0) = 0.500 - 0.2881 = 0.2119$

$\boxed{\text{From table } +0.8 = 0.2881}$



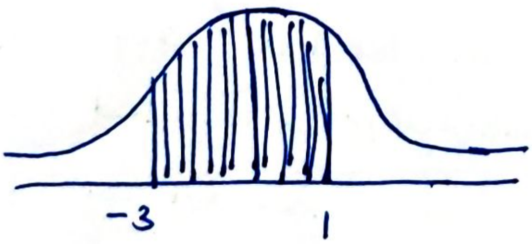
Number of students scored less than 56 marks $= 0.2119 \times 100 = 21.19\%$

(iii) $P(\text{between 45 and 65 marks}) = P(45<x<65)$

$= P\left(\frac{45-60}{5} < z < \frac{65-60}{5}\right) = P(-3<z<1)$

$= P(0<z<3) + P(0<z<1) = 0.4987 + 0.3413 = 0.8399$

$\therefore$ No. of students scored b/w 45 & 60 $= 0.8399 \times 100 = 83.99\%$.

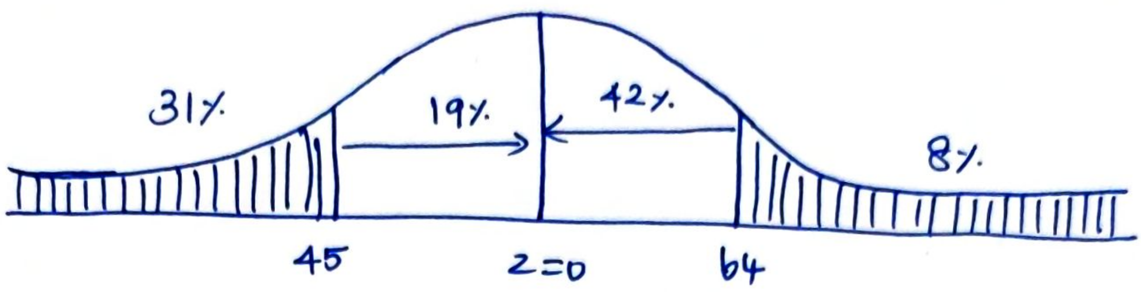


#Pb In a normal distribution 31% of the item are under 45 and 8% are over 64. Find mean and SD of the distribution

Soln $z = \frac{x-\mu}{\sigma}$. Sub $x = 45$ Sub $x = 64$

$z_1 = \frac{45-\mu}{\sigma}$ —(1) $z_2 = \frac{64-\mu}{\sigma}$ —(2)

Z_1 (under 50%) $\qquad$ Z_2 (over 50%)



$$P(\bar{z}_1 \le z \le 0) = 0.19, \quad P(0 \le z \le z_2) = 0.42$$

$$z_1 = -0.49 \qquad z_2 = 1.4$$

Sub value of z_1 & z_2 value in (1) and (2)

$$-0.49 = \frac{45-\mu}{\sigma} \qquad 1.4 = \frac{64-\mu}{\sigma}$$

$$-0.49\sigma = 45-\mu \qquad 1.4\sigma = 64$$

$$\mu - 0.49\sigma = 45 \quad \text{—(3)} \qquad \mu + 1.4\sigma = 64 \quad \text{—(4)}$$

Solve (3) and (4)

$$\cancel{\mu} \cancel{-} 0.49\sigma = 45$$
$$\cancel{\mu} + 1.4\sigma = 64$$
$$(-) \quad (-) \quad (-)$$

$$\cancel{-}1.89\sigma = \cancel{-}19$$

$$\sigma = +\frac{19}{1.89}$$

$$\boxed{\sigma = 10}$$

Sub $\sigma = 10$ in (4)

$$\mu + 1.4 \times 10 = 64$$

$$\mu = 64 - 14$$

$$\boxed{\mu = 50}$$

$\therefore$ Mean = 50 Standard deviation = 10

Formula

* $\Gamma(1/2) = \sqrt{\pi}$

* $\Gamma(n+1) = n\Gamma n = n!$

* $\Gamma 1 = 1$

* $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$

* $\int_0^\infty x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n}$

* $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$

* $(1-x)^{-1} = 1 + x + x^2 + \cdots$

* Memoryless property

$$P(X > s+t \mid X > s) = P(X > t)$$

* $e^x = 1 + x + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$