Random Variable

A random variable is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for any real number a, the event $[\omega : X(\omega) \leq a] \in B$.

(or)

Let E be an experiment, S be the sample space associated with the experiment. The event S is non-numeric which are associated with real number. The variable under which the real number are stored as random variable

If a coin is tossed $S = \{H, T\}$ then $X(E) = \begin{cases} 1 & H \\ 1 & T \end{cases}$

is a random variable.

There are two types of random variable * Discrete & Continuous

Discrete Random variable: The R.V is said to be discrete if it takes distinct value into it. $X = \{1, 2, 3, \dots\}$

Continous Random variable: The R.V is said to be continuous if it takes all the real values between any interval $(1 \leq x \leq 2)$

Probability Mass function: Let X be discrete random variable with $P(x)$ associated probability then $P(x)$ is said to be PMF if these two conditions are satisfied

$$(i)\ P(x) \geq 0, \quad (ii)\ \sum P(x) = 1$$

Probability density function: Let X be Continuous random variable with $P(x)$ associated probability, then $P(x)$ is said to be PDF if these two conditions are satisfied

$$(i)\ f(x) \geq 0 \quad (ii)\ \int f(x)\,dx = 1$$

Pb The random variable x has the following probability function

X :	0	1	2	3	4	5	6	7
P(x) :	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

(i) Find K

(ii) Evaluate $P(X<6)$ (iii) $P(X\geq 6)$ (IV) $P(0<X<5)$, (v) Determine the distribution function of X.

(i)

Since $\sum_{x=0}^{7} P(x) = 1 \Rightarrow K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$$\Rightarrow 10K^2 + 9K - 1 = 0$$

$$\Rightarrow (10K-1)(K+1) = 0$$

$$\Rightarrow K = 1/10 \quad K = -1$$

But since p(x) cannot be negative so $K=-1$ is rejected

Hence $K = 1/10$

(ii) $P(X<6) = P(X=0) + P(X=1) + \cdots + P(X=5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

(iii) $P(X\geq 6) = 1 - P(X<6) = 1 - \frac{81}{100} = \frac{19}{100}$

(IV) $P(0<X<5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

(v) The distribution function $F_X(x) = P(X\leq x)$ of X given in the following table

x :	0	1	2	3	4	5	6	7
$F_X(x)$: (Cummulative)	0	1/10	$3K = 3/10$	$5K = 5/10$	$8K = 8/10$	$8K+K^2 = \frac{81}{100}$	$8K+3K^2 = \frac{83}{100}$	$9K+10K^2 = 1$

Pb If $p(x) = \begin{cases} \frac{x}{15} ; & x = 1, 2, 3, 4, 5 \\ 0, & \text{elsewhere} \end{cases}$ Find (i) $P\{x = 1 \text{ or } 2\}$

(ii) $P\{\frac{1}{2} < x < \frac{5}{2} | x > 1\}$

Soln

(i) $P(x = 1 \text{ (or) } 2) = P(x=1) + P(x=2) = \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$

(ii) $P(\frac{1}{2} < x < \frac{5}{2} | x > 1) = \frac{P\{(\frac{1}{2} < x < \frac{5}{2}) \cap (x > 1)\}}{P(x > 1)}$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

$$= \frac{P\{(x = 1 \text{ (or) } 2) \cap (x > 1)\}}{P(x > 1)}$$

$$= \frac{P(x=2)}{1 - P(x=1)} = \frac{2/15}{1 - (1/15)} = \frac{\frac{2}{\cancel{15}}}{\frac{15-1}{\cancel{15}}}$$

$$= \frac{2}{14} = 1/7$$

Pb Two dice are rolled. Let X denote the random variable which counts the total number points on the upturned faces, construct a table giving the non-zero values of the probability mass function and draw the probability chart. Also find the distribution function of X

Soln If both dice are unbiased and the two rolls are independent, then each sample point of sample space S has probability $\frac{1}{36}$, then

$P(2) = P(x=2) = P\{(1,1)\} = \frac{1}{36}$

$P(3) = P(x=3) = P\{(1,2)\}\{(2,1)\} = \frac{2}{36}$

$P(4) = P(x=4) = P\{(1,3)\ (2,2), (3,1)\} = \frac{3}{36}$

$P(5) = P(x=5) = P\{(1,4)\ (2,3)\ (3,2)\ (4,1)\} = \frac{4}{36}$

④

$P(6) = P(x=6) = P\{(1,5), (2,4), (3,3), (4,2), (5,1)\} = \frac{5}{36}$

$P(7) = P(x=7) = P\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} = \frac{6}{36}$

$P(8) = P(x=8) = P\{(2,6), (3,5), (4,4), (5,3), (6,2)\} = \frac{5}{36}$

$P(9) = P(x=9) = P\{(3,6), (4,5), (5,4), (6,3)\} = \frac{4}{36}$

$P(10) = P(x=10) = P\{(4,6), (5,5), (6,4)\} = 3/36$

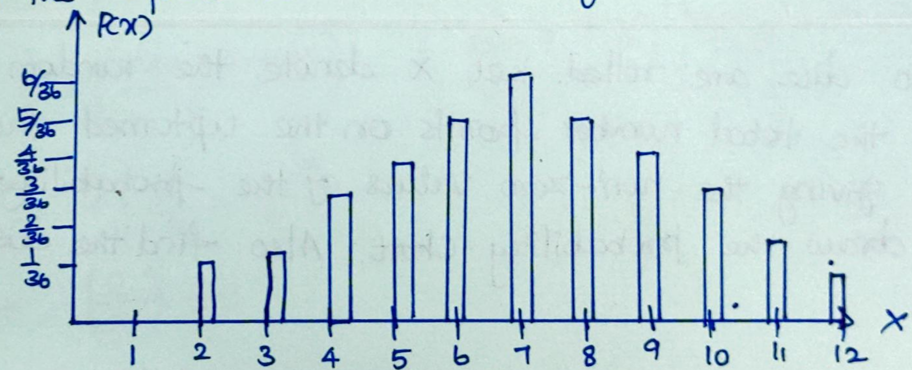
$P(11) = P(x=11) = P\{(5,6), (6,5)\} = 2/36$

$P(12) = P(x=12) = P\{(6,6)\} = 1/36$

These values are summarized in the following probability table

X :	2	3	4	5	6	7	8	9	10	11	12
P(x) :	1/36	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	5/36	4/36	3/36	$\frac{2}{36}$	$\frac{1}{36}$

The chart of the prob distribution is gn below



$\Rightarrow F(1) = P(X \le 1) = 0$

$F(2) = P(X \le 2) = \frac{1}{36}$

$F(3) = P(X \le 3) = P(X=2) + P(X=3)$ –

$= P(2) + P(3) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

⋮

and so on

The distribution function of x is

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{36} & \text{for } 2 \le x < 3 \\ \frac{3}{36} & \text{for } 3 \le x < 4 \\ \vdots & \\ \frac{35}{36} & \text{for } 11 \le x < 12 \\ 1 & \text{for } x \ge 12 \end{cases}$$

Pb The Continuous Random Variable has the following PDF

$$f(x)=\begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Verify that it is a PDF and evaluate the following

(i) $P(x \le 1/3)$ (ii) $P(\frac{1}{3} \le x \le 1/2)$ (iii) $P(x \le 1/2 / \frac{1}{3} \le x \le 2/3)$

Sol^n Obviously, for $0 \le x \le 1$, $f(x) \ge 0$.

$$\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 3x^2 dx = \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_0^1 = 1 \quad \therefore \text{ it is PDF.}$$

(i) $$P(x \le 1/3) = \int_0^{1/3} 3x^2 dx = 3\left[\frac{x^3}{3}\right]_0^{1/3} = (1/3)^3 - 0 = \frac{1}{27}$$

(ii) $$P(\tfrac{1}{3} \le x \le 1/2) = \int_{1/3}^{1/2} 3x^2 dx = 3\left[\frac{x^3}{3}\right]_{1/3}^{1/2} = \frac{1}{8} - \frac{1}{27} = \frac{27-8}{216} = \frac{19}{216}$$

(iii) $$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \frac{P(x \le 1/2 \cap \frac{1}{3} \le x \le 1/2)}{P(1/3 \le x \le 2/3)}$$

$$= \frac{P(1/3 \le x \le 1/2)}{P(1/3 \le x \le 2/3)} = \frac{\frac{19}{216}}{P(1/3 \le x \le 2/3)}$$

$$\boxed{= \int_{1/3}^{2/3} 3x^2 dx = \frac{8}{27} - \frac{1}{27} = 7/27}$$

$$= \frac{19}{\cancel{216}_{\,8}} \times \frac{\cancel{27}}{7} = \frac{19}{56}$$

Expectation: An Average of a probability distribution is usually called expectation (or) expected value. Let x be discrete random variable with associated probability $p(x)$ then the expected value X is given by $E(x) = \sum x\,p(x)$ = mean of distribution. The variance of the distribution is given $V(x) = E(x^2) - (Ex)^2$

$$E(x^2) = \sum x^2 p(x)$$

Pb The R.v x has the probability function as follows

X :	-1	0	1
p(x) :	0.2	0.3	0.5

(i) Find mean & variance
(ii) Evaluate $E(3x+1)$

(i) $E(x) = \sum x\, p(x)$

$= -1 \times 0.2 + 0 \times 0.3 + 1 \times 0.5$

$= -0.2 + 0.5$

$\boxed{\text{mean} = 0.3}$

$E(x^2) = \sum x^2\, p(x)$

$= (-1)^2 \times 0.2 + 0 + 1^2 \times 0.5$

$= 0.2 + 0.5$

$= 0.7$

$\text{variance} = E(x^2) - (E\,x)^2$

$= 0.7 - (0.3)^2$

$= 0.61$

(ii) $E(3x+1) = \sum (3x+1)\, p(x)$

$= (3(-1)+1)(0.2) + (3(0)+1)(0.3) + (3(1)+1)(0.5)$

$= -0.4 + 0.3 + 2$

$= 1.90.$

Pb Find the mean of x for the following density function

$$f(x) = \begin{cases} kx^3, & 0 \le x \le 1 \\ k(2-x)^2, & 1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Soln

$$\int_0^1 k x^3 dx + \int_1^2 k(2-x)^2 dx = 1$$

$$\Rightarrow k\left[\frac{x^4}{4}\right]_0^1 + k\int_1^2 (4+x^2-4x)\, dx = 1$$

$$\Rightarrow k(1/4) + k\left[4x + \frac{x^3}{3} - \frac{4x^2}{2}\right]_1^2 = 1$$

$$\Rightarrow k/4 + k\left[\left(8 + \frac{8}{3} - \frac{16}{2}\right) - \left(4 + \frac{1}{3} - \frac{4}{2}\right)\right] = 1$$

$$\Rightarrow k/4 + k\left[\left(\frac{48+16-48}{6}\right) - \left(\frac{24+2-12}{6}\right)\right] = 1$$

$$\Rightarrow k/4 + k\left[\frac{16}{6} - \frac{14}{6}\right] = 1$$

$$\Rightarrow k/4 + k\left(\frac{2}{6}\right) = 1$$

$$\Rightarrow \frac{3k+4k}{12} = 1$$

$$\Rightarrow 7k = 12$$

$$\boxed{k = 12/7}$$

$$\text{Mean} = E(x) = \int x\, f(x)\, dx$$

$$= \int_0^1 x\left(\frac{12}{7}x^3\right)dx + \int_1^2 x\left(\frac{12}{7}\right)(2-x)^2 dx$$

$$= \frac{12}{7}\int_0^1 x^4 dx + \frac{12}{7}\int_1^2 (4x - 4x^2 + x^3)dx$$

$$= \frac{12}{7}\left(\frac{x^5}{5}\right)_0^1 + \frac{12}{7}\left(\frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right)_1^2$$

$$= \frac{12}{35} + \frac{12}{7}\left[\left(\frac{4\times4}{2} - \frac{4\times8}{3} + \frac{16}{4}\right) - \left(\frac{4}{2} - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{12}{35} + \frac{12}{7}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + 1/4\right)\right]$$

$$= \frac{12}{35} + \frac{12}{7}\left[\left(\frac{24-32+12}{3}\right) - \left(\frac{24-16+3}{12}\right)\right]$$

$$= \frac{12}{35} + \frac{12}{7}\left(\frac{4}{3} - \frac{11}{12}\right)$$

$$= \frac{12}{35} + \frac{\cancel{12}}{7}\left(\frac{16-11}{\cancel{12}}\right)$$

$$= \frac{12}{35} + \frac{5}{7}$$

$$= \frac{12+25}{35} = 37/35$$

$$\boxed{\therefore \text{Mean} = 37/35}$$

Moment Generating Function:

$$M_x(t) = t(e^{tx}) = \sum e^{tx} \cdot p(x), \text{ discrete}$$

$$\int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \text{ continuous}$$

$$M_x(t) = E(e^{tx}) = E\left(1 + \frac{tx}{\lfloor 1} + \frac{(tx)^2}{\lfloor 2} + \cdots\right.$$

$$= E\left(\cancel{1} + \frac{tx}{\lfloor 1} + \frac{t^2 x^2}{\lfloor 2} + \cdots\right.$$

$$= 1 + tE(x) + \frac{t^2}{\lfloor 2} E(x^2) + \cdots$$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots$$

Binomial distribution: Let X be discrete R.V with known n and probability P. Then the prob. function $P(x)$ associated with X is said to follow binomial distribution if $P(x)$

$$P(X=x) = nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2, \ldots n$$

where $p+q=1$

Remarks

* n is always finite and p, q such that p is prob of success and q is the prob of failure.
* n and p are the parameters of the binomial distribution.

MGF

$$M_X(t) = \sum e^{tx}\, p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, nC_x \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x\, (pe^t)^x\, q^{n-x}$$

$$= nC_0 (pe^t)^0 q^{n-0} + nC_1 (pe^t)^1 q^{n-1} + \cdots$$

$$+ nC_r (pe^t)^r q^{n-r} + \cdots + nC_n (pe^t)^n q^{n-n}$$

$$\left[(a+b)^n = a^n + nc_1 a^{n-1} b + \cdots + nc_r a^r b^{n-r} + \cdots + b^n\right]$$

$$\boxed{M_x(t) = (q+pe^t)^n}$$

<u>Mean</u>

$$M_x'(t) = n(q+pe^t)^{n-1}(0+pe^t)$$

$$M_x'(t) = np(q+pe^t)^{n-1} \cdot e^t \quad \text{———} \quad (*)$$

$$\underset{u}{np(q+pe^t)^{n-1}} \cdot \underset{v}{e^t} \qquad d(uv) = udv + vdu$$

$$M_x''(t) = np\left[(n-1)(q+pet)^{n-2}(0+pe^t).e^t + (q+pe^t)^{n-1}.e^t\right]$$

$$\text{———} \quad (**)$$

put $t=0$ in $(*)$

$$M_x'(0) = np \cdot (q+p)^{n-1} = np(1)^{n-1} = np.$$

$\therefore$ Mean $= E(X) = np.$

put $t=0$ in $(**)$

$$M_x''(0) = np\left[(n-1)(q+p)^{n-2}.p + (q+p)^{n-1}\right]$$

$$= np\left[n-1\,(1)\,p + 1\right] = np\left[(n-1)p+1\right]$$

$$E(X^2) = np(np-p)+1 = n^2p^2 - np^2 + np$$

$$\text{Variance} = E(x^2) - (E(x))^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np - np^2$$

$$= np(1-p) = npq \qquad \because (q = 1-p)$$

$$\boxed{\text{Variance} = npq}$$

Pb The chance of running bus service according to schedule is 0.8. Calculate the prob on a day schedule with 10 drivers.
(i) exactly 1 is late (ii) atleast 1 is late

Sol^n

$n = 10 \quad q = 0.8, \quad \Rightarrow p = 1 - q = 1 - 0.8 = 0.2$

(i) exactly 1 is late

$$P(x=1) = 10C_1 (0.2)^1 (0.8)^{10-1} = 10 \cdot (0.2) \cdot (0.8)^9$$
$$= 10 \cdot (0.2) \cdot 0.1342 = 0.268.$$

(ii) atleast 1 is Late

$$P(x \geq 1) = 1 - P(x=0) = 1 - (10C_0 (0.2)^0 (0.8)^{10}$$
$$= 0.8926.$$

Pb Ten coins are thrown simultaneously. Find the prob of getting at least seven heads.

Sol^n p = prob. of getting a head = $1/2$

q = prob of not getting a head = $1/2$

$\Rightarrow$ prob of getting atleast seven heads is given by

$$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$
$$= 10C_7 (1/2)^7 (1/2)^{10-7} + 10C_8 (1/2)^8 (1/2)^{10-8} + 10C_9 (1/2)^9 (1/2)^{10-9} + 10C_{10} (1/2)^{10} (1/2)^{10-10}$$
$$= 10C_7 (1/2)^{10} + 10C_8 (1/2)^{10} + 10C_9 (1/2)^{10} + 10C_{10} (1/2)^{10}$$
$$= \left(\frac{1}{2}\right)^{10} \times (120 + 45 + 10 + 1) = \frac{176}{1024} //.$$

Pb For a binomial distribution, the parameters $n=5$ and $p=0.3$. Find the prob of getting (i) atleast 3 success (ii) atmost 3 success (iii) Exactly 3 failures.

Sol^n $n=5$ $\quad p=0.3$ $\quad X=$ no. of success, $q=1-p=1-0.3=0.7$

(i) $P(X\geq 3) = P(X=3)+P(X=4)+P(X=5)$

$= 5C_3\, p^3 q^{5-3} + 5C_4\, p^4 q^{5-4} + 5C_5 \cdot p^5 q^{5-5}$

$= 5C_3 (0.3)^3 (0.7)^2 + 5C_4 (0.3)^4 (0.7)^1 + 5C_5 (0.3)^5 (0.7)^0$

$= 10\,(0.027)(0.49) + 0.02835 + 0.00243$

$= 0.16308$

(ii) $P(X\leq 3) = 1-[P(X=4)+P(X=5)] = 1-[0.02835+0.00243] = 0.9692$

(iii) $P(X=2) = 5C_2\,(0.3)^2 (0.7)^3 = 0.3087$ // $\quad$ $\boxed{3\text{-fai} \mid 2\text{-su}}$

Pb A multiple choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 (or) 2, the second answer if he gets 3 (or) 4 and the third answers if he gets 5 (or) 6. To get a distinction, the student must secure atleast 75% correct answers. If there is no negative marking what is the prob that the student secures a distinction.

Sol^n

$p = 1/3$ so that $q = 1 - 1/3 = 2/3$

Hence the required prob of securing a distinction (ie, getting correct answers to atleast 6 out of the 8 questions) is given by

$P(6)+P(7)+P(8) = 8C_6 (1/3)^6 (2/3)^{8-6} + 8C_7 (1/3)^7 (2/3)^{8-7} + 8C_8 (1/3)^8 (2/3)^{8-8}$

$= \frac{1}{3^6}\left[28\times\frac{4}{9} + 8\times\frac{1}{3}\times\frac{2}{3} + \frac{1}{9}\right] = \frac{129}{729\times 9}$

$= 0.0197.$ //

Poisson distribution: If n is large the evaluation of binomial prob can involve considerable computation. In such a case simple approximation to binomial property that is poisson distribution can be used.

Def: A random variable x is said to follow poisson distribution if it assumes only non-negative values and its prob Mass function is given by

$$P(X=x) = P(x,\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{\underline{|x}}, & x=0,1,2,\dots \quad \lambda>0 \\ 0, & \text{elsewhere} \end{cases}$$

MGF

$$M_x(t) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{\underline{|x}} e^{tx}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{\underline{|x}}$$

$$= e^{-\lambda}\left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \dots \infty\right]$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$M_x(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)} \dots$$

Mean

$$M_x'(t) = e^{\lambda(e^t-1)} \cdot \lambda(e^t-0) = \underset{(u)}{e^{\lambda(e^t-1)}} \cdot \underset{(v)}{\lambda e^t} \quad \text{—— } (*)$$

$d(uv) = udv + vdu$

$$M_x''(t) = \lambda\left[e^{\lambda(e^t-1)} \cdot e^t + e^t e^{\lambda(e^t-1)} \lambda(e^t-0)\right] \quad \text{—— } (**)$$

Put $t=0$ in $(*)$

$$M_x'(0) = e^{\lambda(1-1)} \cdot \lambda e^0 = e^0 \cdot \lambda e^0 = \lambda$$

$$\text{Mean} = E(X) = \lambda$$

Put $t=0$ in (**)

$$M''_x(0) = \lambda\left[e^{\lambda(1-1)}(1) + e^0 e^{\lambda(1-1)} \cdot (1)\lambda\right]$$

$$E(x^2) = \lambda[1+\lambda] = \lambda+\lambda^2 \quad \text{(Second order) Moment)}$$

$$\text{variance} = E(x^2) - (Ex)^2$$
$$= \lambda+\lambda^2 - (\lambda)^2$$
$$= \cancel{\lambda} + \lambda^2 - \cancel{\lambda^2} = \lambda \quad \Rightarrow \text{variance} = \lambda$$

$$\boxed{\therefore \text{Mean} = \text{variance} = \lambda}$$

#Pb Find the prob that atmost five defection fuses will be found in a box of two hundred fuses, if experience shows that 2% of fuses are defective.

Sol^n Given $n=200$ $p=0.02$ $\lambda = np = 200\times 0.02 = 4$

$$\Rightarrow P(x\le 5) = \frac{e^{-4}4^0}{\lfloor 0} + \frac{e^{-4}4^1}{\lfloor 1} + \frac{e^{-4}4^2}{\lfloor 2} + \frac{e^{-4}4^3}{\lfloor 3} + \frac{e^{-4}4^4}{\lfloor 4} + \frac{e^{-4}4^5}{\lfloor 5}$$

$$= e^{-4}\left[\frac{4^0}{\lfloor 0} + \frac{4^1}{\lfloor 1} + \frac{4^2}{\lfloor 2} + \frac{4^3}{\lfloor 3} + \frac{4^4}{\lfloor 4} + \frac{4^5}{\lfloor 5}\right]$$

$$= e^{-4}\left[\frac{1}{1} + \frac{4}{1} + \frac{16}{2} + \frac{64}{6} + \frac{256}{24} + \frac{1024}{120}\right]$$

$$= e^{-4}\left[1+4+8+\frac{32}{3}+\frac{32}{3}+8.53\right]$$

$$= e^{-4}\left[1+4+8+10.66+10.66+8.53\right]$$

$$= e^{-4}[42.863] = 0.018315\times 42.863 = 0.785$$

If 3% of electric bulbs manufactured by a Company are defective. Find the prob that in a sample of 100 bulbs exactly 5 are defective.

Sol^n $n=100$ $p=0.03$ $\lambda=np=100\times 0.03=3$

$$P(x=5)=\frac{e^{-3}3^5}{\lfloor 5}=\frac{e^{-3}(243)}{120}=\frac{0.049\times 243}{120}=\frac{12.09}{120}$$

$$=0.1008$$

An insurance company insures 4000 people against loss of both eyes in a car accident based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00,000 will have car accident each year that result in this type of injury. what is the probability that more than 3 of the insured will collect on their policy in a given year?

Sol^n Gn $n=4000$ and $p=$ prob of loss of both eyes in a car accident $=\frac{10}{100,000}=0.0001$

Since p is very small and n is large we can use poisson distribution

$$\lambda=np=4000\times 0.0001=0.4$$

$$P(x>3)=1-\left[P(x=0)+P(x=1)+P(x=2)+P(x=3)\right]$$

$$=1-\left[\frac{e^{-0.4}(0.4)^0}{\lfloor 0}+\frac{e^{-0.4}(0.4)^1}{\lfloor 1}+\frac{e^{-0.4}(0.4)^2}{\lfloor 2}+\frac{e^{-0.4}(0.4)^3}{\lfloor 3}\right]$$

$$=1-e^{-0.4}\left((0.4)^0+(0.4)^1+\frac{(0.4)^2}{2}+\frac{(0.4)^3}{6}\right]$$

$$=1-0.6703(1+0.4+0.08+0.0107]=1-0.6703\times 1.4907$$

$$=0.0008$$

 # Pb A Manufacturer, who produces medicine bottles find that 0.1% of the bottles are defective. The bottles are packed in boxes; each containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottle. Using poisson distribution, find how many boxes will contain.

(i) no defective

(ii) at most 3 defective

(iii) at least two defective

$\boxed{e^{-0.5} = 0.6065}$

Sol^n $N=100, \quad n=500 \quad p = 0.001$ and $\lambda = np = 500 \times 0.001 = 0.5$

$$\Rightarrow P(X=x) = \frac{e^{-0.5}(0.5)^x}{\lfloor x}, \quad x = 0, 1, 2, \ldots$$

Hence in consignment of 100 boxes, the frequency number of boxes containing x defective bottle is

$$f(x) = N \times P(X=x) = \frac{100 \times e^{-0.5}(0.5)^x}{\lfloor x} \quad x = 0, 1, 2 \ldots 500$$

(i) No defective $\Rightarrow 100 \times P(X=0) = 100 \times 0.6065 = 60.65 \simeq 61$ bottles

(ii) atmost 3 defective $= 100 \times P[X \le 3] = 100 \times [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 100 \times 0.6065 [1 + 0.5 + 0.125 + 0.0208]$

$= 100 \times 0.6065 \times 1.6458 = 100 \times 0.9981 \approx 100$ bottles

(iii) atleast 2 defective $= 100 \times [P(X \ge 2)] = 100 (1 - [P(X=0) + P(X=1)])$

$= 100 [1 - 0.6065 - 0.6065 \times 0.5]$

$= 100 \times 0.09025 \approx 9$ bottles

Geometric distribution

Drive MGF, Mean and variance of Geometric distribution.

Sol^n

The probability Mass function of Geometric distribution is

$$P[X=x] = q^{x-1} \cdot p, \quad x = 1, 2, \cdots \infty$$

where $p+q=1$

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} \, P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \, q^{x} \cdot q^{-1} p = \sum_{x=1}^{\infty} (qe^t)^x \frac{p}{q}$$

$$= \frac{p}{q}\left[(qe^t)^1 + (qe^t)^2 + (qe^t)^3 + \cdots\right]$$

$$= \frac{p}{\cancel{q}} (\cancel{q}e^t)\left[1 + (qe^t)^1 + (qe^t)^2 + \cdots\right]$$

$$= pe^t \left[(1-qe^t)^{-1}\right]$$

formula

$$\left[1 + x + x^2 + \cdots = (1-x)^{-1} = \frac{1}{(1-x)}\right]$$

$$\boxed{M_x(t) = \frac{pe^t}{1-qe^t}}$$

Mean & variance

Differentiate MGF with respect to t

$$\left[d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}\right]$$

$$M_x'(t) = \frac{(1-qe^t)\,pe^t - pe^t\,(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2}$$

$$M'_x(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{———} \; (*)$$

$$M''_x(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} \quad \text{———} \; (**)$$

Put $t=0$ in $*$

$$M'_x(0) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = \text{mean}$$

Put $t=0$ in $(**)$

$$M''_x(0) = \frac{(1-qe^0)^2 pe^0 - pe^0 \cdot 2(1-qe^0)(-qe^0)}{(1-qe^0)^4}$$

$$= \frac{(1-q)^2 p - p \cdot 2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{(1-q)^2 p - p \cdot 2(p)(-q)}{(1-q)^4}$$

$$= \frac{p^2 \cdot p + p \cdot 2 \cdot pq}{p^4} = \frac{p^3}{p^4} + \frac{2p^2q}{p^4} = \frac{1}{p} + \frac{2q}{p^2}$$

$$\text{Variance} = E(x^2) - (Ex)^2 \Rightarrow \frac{1}{p} + \frac{2q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2} = \frac{p+q+q-1}{p^2} = \frac{\cancel{1}+q-\cancel{1}}{p^2} = \frac{q}{p^2}$$

Pb Suppose a R.v x has a geometric distribution $P(X=x)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{x-1}$, $x=1,2,\cdots\infty$. Determine (i) $P(X\leq 2)$ (ii) $P(X>4 \mid x>2)$

Sol^n

If x is a Geometric random variable then

$$P(X=x)=pq^{x-1},\ x=1,2,3,\dots\infty$$

Given $p(x)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{x-1}$

$$p=\tfrac{1}{3} \quad q=\tfrac{2}{3}$$

(i) $P(X\leq 2)=P(x=1)+P(x=2)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{0}+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{1}$

$$=\frac{1}{3}+\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)$$

$$=\tfrac{1}{3}\left(1+\tfrac{2}{3}\right)=\frac{1}{3}\times\frac{5}{3}=\tfrac{5}{9}$$

(ii) $P(X>4 \mid x>2)=P(X>2+2 \mid x>2)$

$$=P(X>2)$$

$$=1-P(x=1)+P(x=2)$$

$$=1-\tfrac{5}{9}=\frac{4}{9}$$

Memory ~~Memorymory~~less property

$$P(X>m+n \mid x>m)$$
$$=P(x>n)$$

Pb Suppose the R.v x has a geometric distribution $P(X=x)$

$$=\begin{cases}\left(\frac{1}{2}\right)^{x}, & x=1,2,3,\dots\infty\\ 0 & \text{otherwise}\end{cases}$$

Obtain (i) $P(X\leq 2)$ (ii) $P(x>4 \mid x>2)$

Sol^n

(i) $P(x \leq 2) = P(x=1) + P(x=2)$

$= \left(\frac{1}{2}\right) + (1/2)^2$

$= 1/2 + 1/4 = \frac{3}{4}$

(ii) $P(x>4 \mid x>2) = P(x > 2+2 \mid x > 2)$

$= P(x>2)$

$= 1 - P(X \leq 2)$

$= 1 - 3/4 = 1/4.$

Pb If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test

(i) on the 4th trial (ii) in fewer than 4 trials

Sol^n X = No. of trials required to achieve the 1st success

$p = 0.8 \quad q = 1-p = 1-0.8 = 0.2$

(i) $P(\text{on the } 4^{th} \text{ trial}) = P(X=4) = 0.8 \times (0.2)^{4-1}$

$= 0.8 \times (0.2)^3 = 0.8 \times 0.008$

$= 0.0064$

(ii) $P(\text{in fewer than 4 trials}) = P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.8)(0.2)^{1-1} + (0.8)(0.2)^{2-1} + (0.8)(0.2)^{3-1}$

$= (0.8).(0.2)^0 + (0.8)(0.2)^1 + (0.8)(0.2)^2$

$= 0.8\left[1 + 0.2 + 0.04\right] = 0.992$ //.

Uniform distribution

Derive MGF, Mean and variance of uniform distribution

Sol^n

The probability density function of uniform random variable is $f(x) = \frac{1}{b-a}$, $a \le x \le b$

MGF

Moment generating function is defined by $M_x(t) = \int e^{tx} f(x)dx$

$$M_x(t) = \int_a^b e^{tx} \frac{1}{b-a} dt$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{b-a}\left[e^{tb} - e^{at}\right] \times \frac{1}{t}$$

Expand $M_x(t)$ in powers of 't'

$$= \frac{1}{(b-a)t}\left[\left(1 + \frac{bt}{\lfloor 1} + \frac{b^2t^2}{\lfloor 2} + \frac{b^3t^3}{\lfloor 3} + \cdots\right) - \left(1 + \frac{at}{\lfloor 1} + \frac{a^2t^2}{\lfloor 2} + \frac{a^3t^3}{\lfloor 3} + \cdots\right)\right]$$

$$= \frac{1}{(b-a)t}\left[\frac{bt - at}{\lfloor 1} + \frac{b^2t^2 - a^2t^2}{\lfloor 2} + \frac{b^3t^3 - a^3t^3}{\lfloor 3} + \cdots\right]$$

$$= \frac{1}{(b-a)t}\left[\frac{(b-a)t}{\lfloor 1} + \frac{(b^2-a^2)t^2}{\lfloor 2} + \frac{(b^3-a^3)t^3}{\lfloor 3} + \cdots\right]$$

$$= \frac{1}{\cancel{(b-a)}t}\cancel{(b-a)}t\left[1 + \frac{(b+a)t}{\lfloor 2} + \frac{(b^2+ab+a^2)t^2}{\lfloor 3} + \cdots\right]$$

$$= 1 + (b+a)\frac{t}{\lfloor 2} + (b^2+ab+a^2)\frac{t^2}{\lfloor 3} + \cdots$$

$$\text{Mean} = E(x) = \left(\text{Coeff of } t \text{ in } M_x(t)\right) 1!$$

$$= \frac{b+a}{\lfloor 2} = \frac{b+a}{2}$$

$$\text{Variance} = E(x^2) - (Ex)^2$$

$$E(x^2) = \left(\text{Coeff of } t^2 \text{ in } M_x(t)\right) 2!$$

$$= \frac{b^2+ab+a^2}{\lfloor 3} \times \lfloor 2 = \frac{b^2+ab+a^2 \times \cancel{\lfloor 2}}{3 \times \cancel{\lfloor 2}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{(2)^2}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{\begin{aligned} \therefore\ \text{Mean} &= \frac{b+a}{2} \\ \text{Variance} &= \frac{(b-a)^2}{12} \end{aligned}}$$

Exponential distribution:

* Derive MGF, Mean and Variance of exponential distribution.

The density function of exponential distribution is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right]_0^\infty = \lambda \left[0 - \frac{1}{-(\lambda - t)} \right] = \frac{\lambda}{\lambda - t}$$

$$\boxed{\therefore M_x(t) = \frac{\lambda}{\lambda - t}}$$

To derive the mean and variance expand $M_x(t)$ in powers of t.

Note: For any cont R.v we are not differentiate but instead we are expand the powers of $M_x(t)$.

$$M_x(t) = \frac{\cancel{\lambda}}{\cancel{\lambda}(1 - t/\lambda)} = \left(1 - t/\lambda\right)^{-1}$$

$$\left[\begin{array}{l}\text{Formula} \\ (1-x)^{-1} = 1 + x + x^2 + \cdots\end{array}\right]$$

$$\Rightarrow M_x(t) = 1 + t/\lambda + t^2/\lambda^2 + \cdots$$

$$\text{Mean} = E(X) = (\text{coeff of } t) * \underline{|1} = \frac{1}{\lambda}$$

$$E(X^2) = (\text{coeff of } t^2) * \underline{|2} = \frac{1}{\lambda^2} \times 2 = 2/\lambda^2$$

$$\text{Variance} = E(x^2) - (Ex)^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2-1}{\lambda^2} = 1/\lambda^2$$

Pb The length of time a person speaks over phone follows exponential distribution with mean ·6 mins. What is the prob that person will talk for (i) more than 8 mins
(ii) between 4 & 8 mins

Soln Given that Mean = 6 mins

$$1/\lambda = 6 \Rightarrow \boxed{\lambda = 1/6}$$

$$\boxed{\int e^{ax} dx = \frac{e^{ax}}{a}}$$

WKT PMF $f(x) = \lambda e^{-\lambda x}, \; x > 0$

(i) $P(\text{more than 8 mins}) = \int_8^\infty f(x)\,dx = \int_8^\infty \frac{1}{6} e^{-x/6} dx$

$$= \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_8^\infty$$

$$= \frac{1}{6}\left[0 - \frac{e^{-8/6}}{-1/6}\right] = \frac{1}{6}\left[\frac{e^{-8/6}}{1/6}\right]$$

$$= e^{-8/6} = 0.264$$

(ii) $P(\text{between 4 \& 8 mins}) = \int_4^8 f(x)dx = \int_4^8 \frac{1}{6} e^{-x/6} dx$

$$= \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_4^8 = \frac{1}{6}\left[\frac{e^{-8/6}}{-1/6} - \frac{e^{-4/6}}{-1/6}\right] = 1/6\left[\frac{e^{-4/3} - e^{-2/3}}{-1/6}\right]$$

$$= \left[-e^{-4/3} + e^{-2/3}\right] = 0.25$$

Pb The mileage which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40000 Km. Find the probabilities of that one of these tire will last (i) atleast 20000 km

(ii) atmost 30,000 km

Sol^n

X denotes the mileage

$$\text{Mean} = \frac{1}{\lambda} = 40000 \Rightarrow \lambda = \frac{1}{40,000}$$

X follows exponential distribution

$$f(x) = \lambda \cdot e^{-\lambda x}, \quad x > 0$$

(i) P(atleast 20000km) $= P(x \geq 20,000)$

$$= \int_{20000}^{\infty} \lambda \cdot e^{-\lambda x}\, dx$$

$$= \int_{20000}^{\infty} \frac{1}{40,000} e^{-\frac{1}{40000}\cdot x}\, dx = \frac{1}{40,000}\left[\frac{e^{-x/40,000}}{-1/40,000}\right]_{20,000}^{\infty}$$

$$= \frac{1}{40,000}\left[0 - \frac{e^{-\frac{20,000}{40000}}}{-1/40,000}\right] = \frac{1}{\cancel{40,000}}\left[\frac{e^{-1/2}}{\cancel{1/40,000}}\right] = e^{-1/2}$$

$$P(x \geq 20000) = 0.6065$$

(ii) P(at most 30000 km) $= P(X \leq 30000) = \int_{0}^{30000} f(x)\, dx$

$$= \int_{0}^{30000} \frac{1}{40000} e^{-x/40000}\, dx = \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_{0}^{30000}$$

$$= \frac{1}{40000}\left[\frac{-e^{-30000/40000} + e^{0}}{1/40000}\right] = \left[-e^{-3/4} + e^{0}\right] = \left[1 - e^{-0.75}\right]$$

$$= 0.5276$$

Gamma distribution

Find the MGF of the Random variable x whose pdf is given by $f(x) = \frac{x}{4} e^{-x/2}$, $x > 0$ hence deduce its mean and variance

Sol^n

The probability density function of Gamma distribution is $f(x) = \frac{\lambda^{K}}{\Gamma K} x^{K-1} e^{-\lambda x}$, $x > 0$ | when $\lambda = 1/2$, $K = 2$

MGF

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \frac{x}{4} e^{-x/2}\,dx$$

$$= \frac{1}{4}\int_0^\infty x e^{-(1/2 - t)x}\,dx$$

$\int u\,dv = uv - u'v_1 + \cdots$

$u = x$ $\quad dv = e^{-(1/2 - t)x}dx$

$u' = 1$ $\quad v = \frac{e^{-(1/2 - t)x}}{-(1/2 - t)}$

$u'' = 0$ $\quad v_1 = \frac{e^{-(1/2 - t)x}}{(\frac{1}{2} - t)^2}$

$$= \frac{1}{4}\left[\frac{x e^{-(1/2 - t)x}}{-(1/2 - t)} - \frac{e^{-(1/2 - t)x}}{(1/2 + t)^2}\right]_0^\infty$$

$$= \frac{1}{4}\left[(0 - 0) - \left(0 - \frac{1}{(1/2 - t)^2}\right)\right] = \frac{1}{4}\,\frac{1}{(\frac{1}{2} - t)^2}$$

Expanding MGF in powers of t

$$= \frac{1}{4}\left[\left(\frac{1}{2}-t\right)^{-2}\right] = \frac{1}{4}\left(\tfrac{1}{2}\right)^{-2}(1-2t)^{-2}$$

$$(1-x)^{-2} = 1+2x+3x^2+\cdots$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{2}}{1}\right)^2 (1-2t)^{-2}$$

$$= \left[1+2(2t)+3(2t)^2+\cdots\right]$$

$$\text{Mean} = E(x) = (\text{coeff of } t) * \lfloor 1 = 4$$

$$E(x^2) = (\text{coeff of } t^2)\times 2! = (3\times 4)2! = 12\times 2 = 24.$$

$$\text{variance} = E(x^2) - E(x)^2$$

$$= 24 - 4^2 = 24 - 16 = 8$$

<u>Gamma distribution</u> (MGF, Mean & variance)

MGF $\quad M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx$

$$= \int_0^\infty e^{tx}\cdot \frac{1}{\Gamma\lambda} e^{-x} x^{\lambda-1}\,dx = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-t)x}\cdot x^{\lambda-1}\,dx$$

$$\left[\int_0^\infty e^{-ax} x^{\lambda-1}\,dx = \frac{\Gamma(\lambda)}{a^\lambda}\right] \text{formula}$$

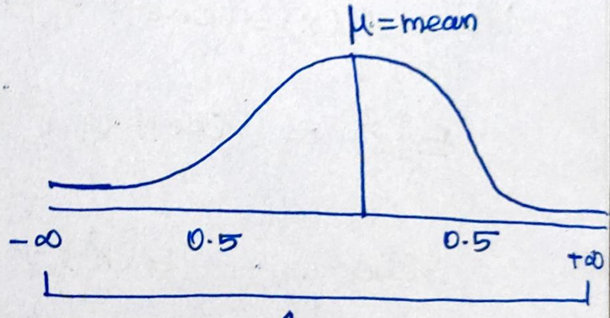
$$= \frac{1}{\cancel{\Gamma\lambda}} \frac{\cancel{\Gamma(\lambda)}}{(1-t)^{\lambda}} = \frac{1}{(1-t)^{\lambda}} = M_x(t) = (1-t)^{-\lambda}$$

$$\boxed{\text{Mean} = \text{variance} = \lambda}$$

Normal distribution

Let x be Continuous random variable with f(x) associated probability then f(x) is said to follow normal distribution if

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Properties of Normal distribution

* Normal distribution is a symmetrical distribution
* The curve has single peak point that is unimodel.
* The mean of normal distribution is lies at the centre and also mean = median = mode is coinside
* The tails of Normal distribution extends indefenitely and never touch the x-axis
* Area property : In a Normal distribution 68% lies b/w $P(\mu \pm \sigma)$ about 95% of the observation will lies b/w $P(\mu + 2\sigma)$ and about 99.7% of the observation will lies b/w $P(\mu \pm 3\sigma)$.

Standard Normal distribution

If X is normally distributed random variable with μ and σ as its mean and standard deviation respectively, then

$Z = \frac{X-\mu}{\sigma}$ is called Standard Normal Variate such that

$\mu = 0$ and $\sigma^2 = 1$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2)}, -\infty < z < \infty.$$

MGF Mean and variance of Normal distribution

The probability density function of standard normal variate is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$

where $z = \frac{x-\mu}{\sigma}$

MGF is defined as $M_z(t) = E(e^{tz})$

$$= \int_{-\infty}^{\infty} e^{tz} f(z)\, dz = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{z^2}{2} - tz\right]} dz$$

The power $\left(\frac{z^2}{2} - tz\right)$ of e looks like $A^2 - 2AB$. We make it this as $A^2 - 2AB + B^2$ To write in the form of $A^2 - 2AB + B^2$

Consider $A^2 = z^2/2$, $2AB = tz$

$A = z/\sqrt{2}$ $\quad 2\left(\frac{z}{\sqrt{2}}\right)B = tz$

$$\sqrt{2} \times \cancel{\sqrt{2}} \times \frac{z}{\cancel{\sqrt{2}}} B = tz \Rightarrow B = \frac{t\cancel{z}}{\sqrt{2}\cancel{z}} = t/\sqrt{2}$$

$$\boxed{B^2 = \frac{t^2}{2}}$$

Let us add and subtract B^2 so that the value is

$$\frac{z^2}{2} - tz + \frac{t^2}{2} - \frac{t^2}{2} = \left(\frac{z-t}{\sqrt{2}}\right)^2 - \frac{t^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-t}{\sqrt{2}}\right)^2} \cdot e^{t^2/2} \, dz$$

Sub $u = \frac{z-t}{\sqrt{2}}$ $\quad \frac{du}{dz} = \frac{1}{\sqrt{2}}$ $\quad \sqrt{2}\, du = dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} \cdot \sqrt{2}\, du$$

$$= \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \cdot \cancel{\sqrt{2}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{t^2/2} du$$

$$= e^{t^2/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = e^{t^2/2} \frac{1}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$\boxed{M_z(t) = e^{t^2/2}}$$

$$M_z(t) = E(e^{tz}) = e^{t^2/2} \Rightarrow E\left(e^{t \cdot \left(\frac{x-\mu}{\sigma}\right)}\right) = e^{t^2/2}$$

$$E\left(e^{xt/\sigma} \cdot e^{-t\mu/\sigma}\right) = e^{t^2/2} \Rightarrow E\left[e^{x t/\sigma}\right] = e^{t^2/2} \cdot e^{t\mu/\sigma}$$

$$\therefore M_x(t) = E\left(e^{xt}\right) = e^{\sigma^2 t^2/2} \cdot e^{t\mu}$$

$\sigma^2 = 1$

$$= \left[1 + \frac{\sigma^2 t^2 / 2}{\lfloor 1} + \frac{\sigma^4 t^4 / 4}{\lfloor 2} + \cdots\right]\left[1 + \frac{t\mu}{\lfloor 1} + \frac{t^2\mu^2}{\lfloor 2} + \cdots\right]$$

$$= 1 + \frac{t\mu}{\lfloor 1} + \frac{t^2\mu^2}{2!} + \frac{\sigma^2 t^2}{\lfloor 2} + \cdots$$

$$E(x) = \text{mean} = (\text{coeff. of } t) * \lfloor 1 = \mu$$

$$E(x^2) = (\text{Coeff. of } t^2) * \lfloor 2 = \left(\frac{\mu^2 + \sigma^2}{2}\right) 2 = \mu^2 + \sigma^2$$

$$\text{variance} = E(x^2) - E(x)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$$\boxed{\text{Mean} = \mu \quad \text{variance} = \sigma^2}$$

#Pb

Students of class were given Mechanical aptitude test. The grade were found to be normally distributed with mean 60 and standard deviation 5. What percent of student score (i) more than 60 grades

(ii) Less than 56 grades

(iii) Between 45 and 65 grades

Show that with a sketch for all)

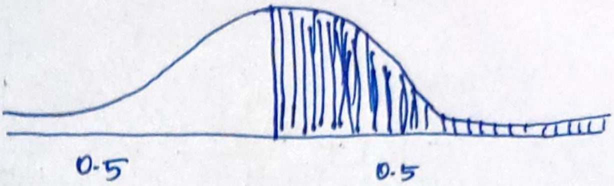
Solⁿ

Given Mean $\mu = 60$ and Standard deviation $\sigma = 5$

Standard normal variate $z = \dfrac{x - \mu}{\sigma} = \dfrac{x - 60}{5}$

(i) P(more than 60) $= P(x > 60) = P\left(z > \frac{60-60}{5}\right) = P(z > 0)$

$= P(0 < z < \infty) = 0.500$

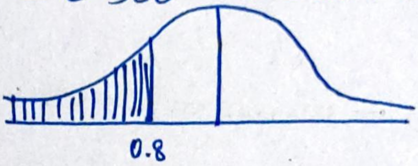


Students scored more than 60 marks $= 0.5000 \times 100 = 50\%$

(ii) P(less than 56) $= P(x < 56) = P\left(z < \frac{56-60}{5}\right) = P(z < -0.8)$

$= P(-\infty < z < 0) - P(-0.8 < z < 0) = 0.500 - 0.2881 = 0.2119$

$\boxed{\text{From table } +0.8 = 0.2881}$



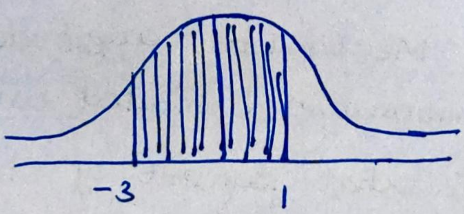
Number of students scored less than 56 marks $= 0.2119 \times 100 = 21.19\%$

(iii) P(between 45 and 65 marks) $= P(45 < x < 65)$

$= P\left(\frac{45-60}{5} < z < \frac{65-60}{5}\right) = P(-3 < z < 1)$

$= P(0 < z < 3) + P(0 < z < 1) = 0.4987 + 0.3413 = 0.8399$

$\therefore$ No. of students scored b/w 45 & 60 $= 0.8399 \times 100 = 83.99\%$



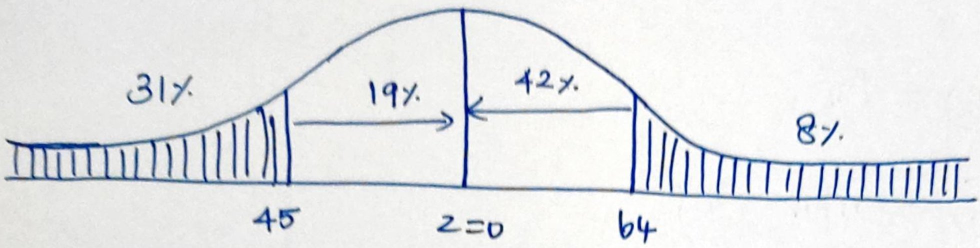
#Pb In a normal distribution 31% of the item are under 45 and 8% are over 64. Find mean and SD of the distribution

Soln

$z = \frac{x-\mu}{\sigma}$. Sub $x = 45$ Sub $x = 64$

$z_1 = \frac{45-\mu}{\sigma}$ — ① $z_2 = \frac{64-\mu}{\sigma}$ — ②

Z_1 (under 50%) Z_2 (over 50%)



$P(\bar{z}_1 \le z \le 0) = 0.19$, $P(0 \le z \le z_2) = 0.42$

$z_1 = -0.49$ $z_2 = 1.4$

Sub value of z_1 & z_2 value in ① and ②

$$-0.49 = \frac{45-\mu}{\sigma}$$

$$-0.49\sigma = 45-\mu$$

$$\mu - 0.49\sigma = 45 \quad —③$$

$$1.4 = \frac{64-\mu}{\sigma}$$

$$1.4\sigma = 64$$

$$\mu + 1.4\sigma = 64 \quad —④$$

Solve ③ and ④

$$\cancel{\mu} - 0.49\sigma = 45$$
$$\cancel{\mu} + 1.4\sigma = 64$$
$$(-) \quad (-) \quad (-)$$

$$\cancel{+}1.89\sigma = \cancel{+}19$$

$$\sigma = +\frac{19}{1.89}$$

$$\boxed{\sigma = 10}$$

Sub $\sigma = 10$ in ④

$$\mu + 1.4 \times 10 = 64$$

$$\mu = 64 - 14$$

$$\boxed{\mu = 50}$$

$\therefore$ Mean = 50 Standard deviation = 10

Formula

* $\Gamma(1/2) = \sqrt{\pi}$

* $\Gamma(n+1) = n\Gamma n = n!$

* $\Gamma 1 = 1$

* $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$

* $\int_0^\infty x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n}$

* $\int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}$

* $(1-x)^{-1} = 1 + x + x^2 + \cdots$

* Memoryless property

$$P(X > s+t \mid x > s) = P(X > t)$$

* $e^x = 1 + x + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$