Geometric distribution

Drive MGF, Mean and variance of Geometric distribution.

Soln

The probability Mass function of Geometric distribution is

$$P[X=x] = q^{x-1}.p, \quad x = 1, 2, \dots \infty$$

where $p+q=1$

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} . q^{x-1} . p$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x} . q^{-1} p = \sum_{x=1}^{\infty} (qe^t)^x \frac{p}{q}$$

$$= \frac{p}{q}\left[(qe^t)^1 + (qe^t)^2 + (qe^t)^3 + \dots\right]$$

$$= \frac{p}{\cancel{q}}(\cancel{q}e^t)\left[1+(qe^t)^1+(qe^t)^2+\dots\right]$$

$$= pe^t\left[(1-qe^t)^{-1}\right]$$

formula

$$\left[1+x+x^2+\dots = (1-x)^{-1} = \frac{1}{(1-x)}\right]$$

$$\boxed{M_x(t) = \frac{pe^t}{1-qe^t}}$$

Mean & variance

Differentiate MGF with respect to t

$$\left[d\left(\frac{u}{v}\right) = \frac{vdu-udv}{v^2}\right]$$

$$M'_x(t) = \frac{(1-qe^t)\,pe^t - pe^t(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - \cancel{pq}\cancel{e^{2t}} + \cancel{pq}e^{2t}}{(1-qe^t)^2}$$

$$M'_x(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{———— } *$$

$$M''_x(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} \quad \text{———— } **$$

Put $t=0$ in $*$

$$M'_x(0) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = \text{mean}$$

Put $t=0$ in $**$

$$M''_x(0) = \frac{(1-qe^0)^2 pe^0 - pe^0 \cdot 2(1-qe^0)(-qe^0)}{(1-qe^0)^4}$$

$$= \frac{(1-q)^2 p - p \cdot 2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{(1-q)^2 p - p \cdot 2(p)(-q)}{(1-q)^4}$$

$$= \frac{p^2 \cdot p + p \cdot 2 \cdot pq}{p^4} = \frac{p^3}{p^4} + \frac{2p^2q}{p^4} = \frac{1}{p} + \frac{2q}{p^2}$$

$$\text{Variance} = E(x^2) - (Ex)^2 \Rightarrow \frac{1}{p} + \frac{2q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2} = \frac{p+q+q-1}{p^2} = \frac{\cancel{1}+q-\cancel{1}}{p^2} = \frac{q}{p^2}$$

Pb Suppose a R.v x has a geometric distribution $P(x=x) = (1/3)(2/3)^{x-1}$, $x = 1, 2, \cdots \infty$. Determine (i) $P(x \le 2)$ (ii) $P(x>4 \mid x>2)$

Soln

If x is a Geometric random variable then

$$P(x=x) = p q^{x-1}, \quad x = 1, 2, 3, \dots \infty$$

Given $P(x) = (1/3)(2/3)^{x-1}$

$$p = 1/3 \quad q = 2/3$$

(i) $P(x \le 2) = P(x=1) + P(x=2) = \left(\frac{1}{3}\right)(2/3)^0 + (1/3)(2/3)^1$

$= \frac{1}{3} + (1/3)(2/3)$

$= 1/3\,(1 + 2/3) = \frac{1}{3} \times \frac{5}{3} = 5/9$

(ii) $P(x>4 \mid x>2) = P(x > 2+2 \mid x > 2)$

$= P(x>2)$

$= 1 - P(x=1) + P(x=2)$

$= 1 - 5/9 = \frac{4}{9}$

Memory less property

$P(x > m+n \mid x > m)$
$= P(x > n)$

Pb Suppose the R.v x has a geometric distribution $P(x=x)$

$$= \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, 3, \dots \infty \\ 0 & \text{otherwise} \end{cases}$$

Obtain (i) $P(x \le 2)$ (ii) $P(x>4 \mid x>2)$

Sol^n

(i) $P(x \leq 2) = P(x=1) + P(x=2)$

$= \left(\frac{1}{2}\right) + (1/2)^2$

$= 1/2 + 1/4 = \frac{3}{4}$

(ii) $P(x > 4 | x > 2) = P(x > 2+2 | x > 2)$

$= P(x > 2)$

$= 1 - P(x \leq 2)$

$= 1 - 3/4 = 1/4.$

Pb If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test

(i) on the 4th trial (ii) in fewer than 4 trials

Sol^n x = No. of trials required to achieve the 1st success

$p = 0.8 \quad q = 1-p = 1-0.8 = 0.2$

(i) P(on the 4th trial) $= P(X=4) = 0.8 \times (0.2)^{4-1}$

$= 0.8 \times (0.2)^3 = 0.8 \times 0.008$

$= 0.0064$

(ii) P(in fewer than 4 trials) $= P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.8)(0.2)^{1-1} + (0.8)(0.2)^{2-1} + (0.8)(0.2)^{3-1}$

$= (0.8).(0.2)^0 + (0.8)(0.2)^1 + (0.8)(0.2)^2$

$= 0.8[1 + 0.2 + 0.04] = 0.992$ //.

Uniform distribution

Derive MGF, Mean and variance of uniform distribution

Sol^n

The probability density function of uniform random variable is $f(x) = \dfrac{1}{b-a}, \quad a \le x \le b$

MGF

Moment generating function is defined by $M_x(t) = \int e^{tx} f(x)\,dx$

$$M_x(t) = \int_a^b e^{tx} \frac{1}{b-a}\, dt$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{b-a}\left[e^{tb} - e^{at}\right] \times \frac{1}{t}$$

Expand $M_x(t)$ in powers of 't'

$$= \frac{1}{(b-a)t}\left[\left(1 + \frac{bt}{\lfloor 1} + \frac{b^2t^2}{\lfloor 2} + \frac{b^3t^3}{\lfloor 3} + \cdots\right) - \left(1 + \frac{at}{\lfloor 1} + \frac{a^2t^2}{\lfloor 2} + \frac{a^3t^3}{\lfloor 3} + \cdots\right)\right]$$

$$= \frac{1}{(b-a)t}\left[\frac{bt - at}{\lfloor 1} + \frac{b^2t^2 - a^2t^2}{\lfloor 2} + \frac{b^3t^3 - a^3t^3}{\lfloor 3} + \cdots\right]$$

$$= \frac{1}{(b-a)t}\left[\frac{(b-a)t}{\lfloor 1} + \frac{(b^2-a^2)t^2}{\lfloor 2} + \frac{(b^3-a^3)t^3}{\lfloor 3} + \cdots\right]$$

$$= \frac{1}{\cancel{(b-a)}t}\,\cancel{(b-a)}t\left[1 + \frac{(b+a)t}{\lfloor 2} + \frac{(b^2+ab+a^2)t^2}{\lfloor 3} + \cdots\right]$$

$$= 1 + (b+a)\frac{t}{\lfloor 2} + (b^2+ab+a^2)\frac{t^2}{\lfloor 3} + \cdots$$

$$\text{Mean} = E(X) = \left(\text{Coeff of } t \text{ in } M_x(t)\right) \underline{|1}$$

$$= \frac{b+a}{\underline{|2}} = \frac{b+a}{2}$$

$$\text{Variance} = E(X^2) - (EX)^2$$

$$E(X^2) = \left(\text{Coeff of } t^2 \text{ in } M_x(t)\right) 2!$$

$$= \frac{b^2+ab+a^2}{\underline{|3}} \times \underline{|2} = \frac{b^2+ab+a^2 \times \cancel{\underline{|2}}}{3 \times \cancel{\underline{|2}}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{(2)^2}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{\begin{aligned} \therefore \text{Mean} &= \frac{b+a}{2} \\ \text{Variance} &= \frac{(b-a)^2}{12} \end{aligned}}$$