Fourier Transforms

Introduction

The fourier series expresses any periodic function into a sum sinusoids. The fourier transform is the extension of this idea to non-periodic functions by taking the limiting form of fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant (LTI) systems etc.

Some useful results in computation of the Fourier transforms

* $\int_0^\infty e^{-ax} \sin \lambda x \, dx = \frac{\lambda}{a^2+\lambda^2}$

* $\int_0^\infty e^{-ax} \cos \lambda x \, dx = \frac{a}{a^2+\lambda^2}$

* $\int_0^\infty \frac{\sin \lambda x}{x} dx = \pi/2 , \lambda > 0$

  when $\lambda = 1$, $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$

* $\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$

* $\cos ax = \frac{e^{iax} + e^{-iax}}{2}$

* $\int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

  when $a = 1$ $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Dirichlet's Conditions

* $f(x)$ is absolutely integrable (ie) $\int_{-\infty}^{\infty} |f(x)| \, dx$ is convergent

* The function $f(x)$ has a finite number of maximuma and minima

* $f(x)$ has only finite number of discontinuities in any finite

Fourier Transform

$$f(x) \longrightarrow \boxed{\text{Fourie transform}} \longrightarrow \bar{f}(s)$$

The fourier transform of $f(x)$; $-\infty < x < \infty$ denoted by $\bar{f}(s)$ where $s \in \mathbb{N}$, is given by

$$F\{f(x)\} \equiv \bar{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

Fourier Sine Transform

$$F_s\{f(x)\} \equiv \bar{f}_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

Inverse Since Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \bar{f}_s(s) \sin sx\, ds$$

Fourier Cosine Transform

$$F_c\{f(x)\} \equiv \bar{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

Inverse Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \bar{f}_c(s) \cos sx\, ds$$

Properties of Fourier Transform

* **Linearity property**

If $x(t) \xleftrightarrow{F.T} X(\omega)$ and $y(t) \xleftrightarrow{F.T} Y(\omega)$

then $ax(t) + by(t) \xleftrightarrow{F.T} aX(\omega) + bY(\omega)$

* **Time Shifting property**

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $x(t - t_0) \xleftrightarrow{F.T} e^{-i\omega t_0} X(\omega)$

* **Frequency Shifting property**

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $e^{i\omega_0 t} \cdot x(t) \xleftrightarrow{F.T} X(\omega - \omega_0)$

* **Time Reversal property**

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $x(-t) \xleftrightarrow{F.T} X(-\omega)$

* **Change of scale property**

If $F(f(x)) = F(s)$ then $F(F(ax)) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

* $F\left[e^{iax} f(x)\right] = F(s+a)$

* $F[f(x) \cos ax) = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$

Convolution Theorem

The convolution ~~the~~ of two functions $f(x)$ and $g(x)$ is defined by $f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

The fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their fourier transform

$$F\left[f(x) * g(x)\right] = F(s) \cdot G(s)$$

Parseval's Identity

If $F(s)$ is fourier transform of $f(x)$. then

$$\int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

***

$$\boxed{\begin{aligned} \int e^{ax} \sin bx\, dx &= \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right] \\ \int e^{ax} \cos bx\, dx &= \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b \sin bx\right] \end{aligned}}$$

Pbs

Find Fourier sine transform of (i) $1/x$ (ii) $2e^{-3x} + 3e^{-2x}$

Soln By defn, we have $F_s\{f(x)\} = \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$

$$\therefore f_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx\, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\pi/2}$$

(ii) $$\therefore \bar{f}_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (2e^{-3x} + 3e^{-2x}) \sin sx\, dx$$

$\cancel{\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} 2e^{-3x} \sin sx\, dx + \sqrt{2/\pi} \int_0^{\infty} 3e^{-2x} \sin sx\, dx$$

***

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2e^{-3x}}{9+\lambda^2}\left(-3\sin\lambda x - \lambda\cos x\right)\right]_0^{\infty} + \sqrt{\frac{2}{\pi}}\left[\frac{3e^{-2x}}{4+\lambda^2}\left(-2\sin\lambda x - \lambda\cos\lambda x\right)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \frac{2\lambda}{9+\lambda^2}\right] + \sqrt{\frac{2}{\pi}}\left[0 + \frac{3\lambda}{4+\lambda^2}\right] = \sqrt{2/\pi}\left[\frac{5\lambda^{2} + 35\lambda}{(4+\lambda^2)(9+\lambda^2)}\right] \,//$$

Find the Fourier transform of $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$ Hence deduce

that (i) $\int_0^\infty \frac{\sin t}{t} dt = \pi/2$ ⑤

(ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

Sol^n $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0 . e^{isx} dx + \int_{-1}^{1} 1 e^{isx} dx + \int_{1}^{\infty} 0 e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx \, dx + i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx dx + 0$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin sx}{s} \right]_0^1 = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin s}{s} - 0 \right]$$ ie $F[f(x)] = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin s}{s}\right]$

Using inverse Fourier transform, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin s}{s} \right) (\cos sx - i \sin sx) ds$$

$$= \frac{\sqrt{2}}{\sqrt{2} . \sqrt{\pi} . \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin s}{s} (\cos sx - i \sin sx) ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} (\cos sx) ds - i \frac{1}{\pi} \int_{-\infty}^{\infty} \sin sx \, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin s}{s}\cos sx\, ds \quad - 0$$

$$\therefore \int_0^\infty \left(\frac{\sin s}{s}\right)\cos sx\, ds = \frac{\pi}{2} f(x)$$

Put $x=0$ we get

$$\boxed{\begin{aligned} f(x)&=1 \\ f(0)&=1 \end{aligned}}$$

$$\int_0^\infty \frac{\sin s}{ds} = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2}(1) = \pi/2$$

$$\text{(ie)} \int_0^\infty \frac{\sin t}{t}\, dt = \pi/2$$

Using Parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}\right)^2 ds = \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1}(1)^2 dx + \int_1^\infty 0\, dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \int_{-1}^{1} dx$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = [x]_{-1}^{1} = 1-(-1) = 2$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = 2$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \cancel{2}\times\frac{\pi}{\cancel{4}\,2} \Rightarrow \int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \pi/2$$

$$\text{(ie)} \int_0^\infty \frac{\sin t}{t}\, dt = \pi/2 \quad //$$

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| \geq a \end{cases}$

Hence deduce that (i) $\displaystyle\int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \pi/4$

<u>Solⁿ</u>

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0\, e^{isx}\, dx + \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx + \int_{a}^{\infty} 0 \cdot e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx \cancel{+ i \sin sx})\, dx + i \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \sin sx\, dx$$

$$= \frac{1 \times 2}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{2a \cos as}{s^2} + \frac{2 \sin as}{s^3} - 0 \right]$$

$$\Rightarrow F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin as - a\cos as}{s^3} \right]$$

When $a = 1$ we have

$$F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

using inverse Fourier transform we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx\, ds - i\, \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \sin sx\, ds$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds - 0$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s}\right) \cos sx\, dx = \frac{\pi}{4} f(x) \quad \text{——— (1)}$$

Put $x = 0$ in equation (1) we get

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \pi/4\, f(0)$$

$$\boxed{\begin{aligned} f(x) &= a^2 - x^2 \\ f(x) &= 1 - x^2 \\ f(0) &= 1 - 0 = 1 \end{aligned}}$$

$$\therefore \int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \pi/4$$

***

$$\boxed{\text{(ii)} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{3\pi}{16}}$$ → Assignment prob (1)

Find the Fourier Transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ (9)

Hence deduce that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol^n

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^{1} (1-|x|) e^{isx} dx + \int_{1}^{\infty} 0 \cdot e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx + i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx + 0$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} 1-|x| \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{\cos s}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right]$$

(ie) $F[f(x)] = \sqrt{2/\pi} \left[\frac{1-\cos x}{s^2} \right]$

Using parseval's identity we have

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]^2\right) ds = \int_{-\infty}^{1} 0\, dx + \int_{-1}^{1}(1-|x|)^2 dx + \int_{1}^{\infty} 0 \cdot dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1}(1-|x|)^2 dx$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos 2t}{(2t)^2}\right)^2 ds = 2\int_{0}^{1}(1-x)^2 ds$$

Put $s = 2t$

$ds = 2dt$

$$\frac{8}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos 2t}{(\cancel{2}t)^2}\right)^2 dt = 2\left[0-\left\{-\frac{1}{3}\right\}\right]$$

$$\frac{\cancel{8}}{\cancel{16}_{2}\pi}\int_{0}^{\infty}\left(\frac{2\sin^2 t}{\cancel{4}\, t^2}\right)^2 dt = \frac{2}{3}$$

$$\frac{1}{2}\int_{0}^{\infty}\left(\frac{2\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$\frac{4}{2\pi}\int_{0}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\cancel{2}}{3}\times\frac{\cancel{2}\pi}{\cancel{4}}$$

(ie) $$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \pi/3$$ //