Canonical Basis problems

1. $\begin{bmatrix} 7 & 1 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix} = A.$ $\lambda = 7, 7, 7$ $m = 3$ times repeated.

n = rank of matrix = 3

min - possible rank = $n - m = 3 - 3 = 0$.

$\boxed{\text{rank}\left((A - \lambda I)^k\right) = 0.}$

$A - \lambda I = A - 7I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2.

$(A - \lambda I)^2 = (A - 7I)^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 1.

$(A - \lambda I)^3 = (A - 7I)^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 0

$\boxed{\text{rank}\left((A - \lambda I)^k\right) = 0}$ where $\boxed{k = 3.}$

Rank number: $N_k = \text{rank}(A - \lambda I)^{k-1} - \text{rank}(A - \lambda I)^k$

$N_3 = \text{rank}(A - \lambda I)^2 - \text{rank}(A - \lambda I)^3 = 1 - 0 = 1$

$N_2 = \text{rank}(A - \lambda I)^1 - \text{rank}(A - \lambda I)^2 = 2 - 1 = 1$

$N_1 = \text{rank}(A - \lambda I)^0 - \text{rank}(A - \lambda I)^1 = 3 - 2 = 1.$

Therefore there are,

$N_3 = 1$ generalised eigen vector of rank 3

$N_2 = 1$ generalised eigen vector of rank 2

$N_1 = 1$ generalised eigen vector of rank 1

For generalised eigen vector of rank 3,

$\underbrace{(A - \lambda I)^3 X_3 = 0}_{\text{Satisfied.}}$ and $(A - \lambda I)^2 \neq 0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$(A-\lambda I)^2 X_3 \neq 0 \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equating we get $x_3 \neq 0$.

So the possible eigen vector can be written as

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ Generalised eigen vector of rank 3.

To get generalised eigen vector of rank 2, rank 1 we use chain method.

$$X_2 = (A-\lambda I) X_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$X_1 = (A-\lambda I) X_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

* So we get one eigen vector generalised each on rank 3, 2, 1.

$$\{ X_3, X_2, X_1 \}.$$

Complete canonical basis for the 3x3 eigen vector is set of 3 vectors

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(2)

$$A = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\lambda = 3, 3, 3, 3, 1.$

3 repeated 4 times. $m = 4$

$n = \text{rank}(A) = 5$

min. possible rank $= 5 - 4 = n - m = 1$

$$\boxed{\text{rank}\left((A-\lambda I)^k\right) = 1}$$

$$A - \lambda I = A - 3I = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ -1 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{rank} = 3$$

$$(A - \lambda I)^2 = (A - 3I)^2 = \begin{bmatrix} 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & -2 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \quad \text{rank} = 2$$

$$(A - \lambda I)^3 = (A - 3I)^3 = \begin{bmatrix} 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 9 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \quad \text{rank} = 1$$

$\boxed{\text{rank}((A - \lambda I)^k) = 1}$ where $\boxed{k = 3}$

Rank number: $N_k = \text{rank}(A - \lambda I)^{k-1} - \text{rank}(A - \lambda I)^k$

$N_3 = \text{rank}(A - \lambda I)^2 - \text{rank}(A - \lambda I)^3 = 2 - 1 = 1$

$N_2 = \text{rank}(A - \lambda I)^1 - \text{rank}(A - \lambda I)^2 = 3 - 2 = 1$

$N_1 = \text{rank}(A - \lambda I)^0 - \text{rank}(A - \lambda I)^1 = 5 - 3 = 2$

Therefore there are,

$N_3 = 1$ generalised eigen vector of rank 3.

$N_2 = 1$ generalised eigen vector of rank 2

$N_1 = 2$ generalised eigen vectors of rank 1.

Find generalised eigen vector of rank 3.

$$(A - \lambda I)^3 X_3 = 0 \quad \text{and} \quad (A - \lambda I)^2 X_3 \neq 0.$$

$$(A - \lambda I)^3 X_3 = \begin{bmatrix} 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 9 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-3x_4 + 3x_5 = 0$ (or) $-9x_4 + 9x_5 = 0$ Every equation are similar

$$\boxed{x_5 - x_4 = 0}$$

$$\boxed{x_5 = x_4} \quad (1)$$

$$(A-\lambda I)^2 X_3 \neq 0 \Rightarrow \begin{bmatrix} 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & -2 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_3 + 7x_4 \neq 0$

$2x_3 \neq 7x_4$

$$\boxed{2x_3 + 5x_4 + 2x_5 \neq 0} \quad (2)$$

$$\boxed{-2x_3 - 8x_4 + x_5 \neq 0} \quad (3)$$

Therefore, $\boxed{X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$

The possible way to satisfy 1, 2, 3 by minimum values.

We can ~~used~~ use chain method for generalized eigen vectors for rank 2, rank 1

$$\boxed{X_2 = (A - \lambda I) X_1 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ -1 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{X_1 = (A - \lambda I) X_2 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ -1 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

We used to obtain,

1 generalized eigen vector of rank 3
1 generalised eigen vector of rank 2
1 generalized eigen vector of rank 1

But there should be 2 $\rightarrow$ generalised eigen vectors, we get only one.

So, we can obtain other generalised eigen vector for rank 1 by $(A-\lambda I)X_1' = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ -1 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1' = \begin{bmatrix} 0 \\ -1 \\ -7 \\ 2 \\ 2 \end{bmatrix}.$$ is one of the possible solution eigen vector for rank 1. (Criss-cross method)

So, x_3, x_2, x_1, x_1' are the generalised eigen vectors for $\lambda = 3$

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3. $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\lambda = 2, 2, 2$ $m = 3$. $n = 3$.

min. possible rank $= n - m = 3 - 3 = \underline{\underline{0}}$

$\boxed{\text{rank}\left((A - \lambda I)^k\right) = 0}$

$A - \lambda I = A - 2I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rank $= 1$.

$(A - 2I)^2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rank $= 0$ ✓

$\boxed{\text{rank}((A-\lambda I)^k) = 0}$ where $\boxed{k = 2}$

Rank numbers: $N_k = \text{rank}(A-\lambda I)^{k-1} - \text{rank}(A-\lambda I)^k$

$N_2 = \text{rank}(A-\lambda I)^1 - \text{rank}(A-\lambda I)^2 = 1 - 0 = 1.$

$N_1 = \text{rank}(A-\lambda I)^0 - \text{rank}(A-\lambda I)^1 = 3 - 1 = 2.$

There are, 1 generalised eigen vector for rank 2.
2 generalised eigen vectors for rank 1.

Find generalised eigen vector for rank = 2..

$\underbrace{(A-\lambda I)^2 X_2 = 0}_{\text{satisfies.}}$ and $(A-\lambda I)X_2 \neq 0.$

$(A-\lambda I)X_2 \neq 0 \rightarrow \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_2 + x_3 \neq 0$

we can use the minimum possible ~~matrix~~ vector with $x_2 = 1$, $x_3 = 1$

$\therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

By chain rule, find generalised eigen vector for rank 1.

$X_1 = (A-\lambda I)X_2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$ are now the 2 eigen vectors found.

But we have to find 2 generalised eigen vectors for rank 1.

One is X_1 and the another gen. eigen vector is found ~~from~~ the rank 1 is.

$$(A - \lambda I) X_1' = 0$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$2x_2 + x_3 = 0 \qquad \frac{x_2}{-1} = \frac{x_3}{2}$$

where $x_1 = 0$.

$$X_1' = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Therefore for $\lambda = 2$

generalised eigen vectors are $\{X_2, X_1, X_1'\}$.