MA3156.

APPLIED MATHEMATICS FOR ELECTRICAL ENGINEERS

Unit-I Matrix Theory

The cholesky decomposition - Generalized Eigen vectors, canonical basis - QR factorization - Least squares method - singular value decomposition.

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Unit-II calculus of variations

Concept of variation and its properties - Euler's equation - Functionals dependent on first and higher order derivatives - Functionals dependent on functions of several independent variables variational problems with moving boundaries - Direct methods: Ritz and Kantorovich methods.

Unit-III One dimensional Random Variables.

Random variables - probability function - moments - moment generating functions and their properties - Binomial, Poisson, Geometric, uniform, Exponential, Gamma and Normal distributions - Function of a Random variable.

Unit-IV Linear programming

Formulation - Graphical solution - Simplex method - Two phase method - Transportation and Assignment Models.

Unit - V Fourier series

Fourier Trignometric series : periodic function as power signals convergence of series - Even and odd function : cosine and sine series - Non-periodic function ; Extension to other intervals - Power signals : Exponential Fourier series - Parseval's theorem and power spectrum.

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Matrix Theory

Matrix:

A system of mn numbers arranged in the form of rectangular array having m rows and n columns is called a matrix of order m×n.

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Types of matrix:

1. square matrix ($m=n$)
2. Diagonal matrix. (All off-diagonal elements are zero)
3. Scalar matrix. (diagonal matrix with all diagonal elements being equal).
4. upper triangular matrix (lower off-diagonal elements are 0)
5. lower triangular matrix (upper off-diagonal elements are 0)
6. unit matrix (or) Identity matrix (I) (diagonal elements are 1 & non-diagonal elements are zero)
7. Null matrix (all elements are zero)
   Only matrix whose Rank is zero
8. Idempotent matrix ($A^n = A$)
9. Involuntary matrix ($A^n = I$)
10. Nilpotent matrix ($A^n = 0$ & $A^{n-1} \neq 0$, n is the smallest index which makes $A^n = 0$)
11. positive definite matrix (eigen values > 0 & symmetric matrix)

Cholesky decomposition:

$A = L.L^T$, any symmetric and positive definite matrix A can be decomposed into product of a unique lower triangular and its transpose.

Cholesky decomposition:

Check matrix for symmetric & positive definite matrix.

Must satisfy

i) Symmetric

ii) positive definite matrix

$A = LL^T$ → Transpose of lower triangular

L → Lower triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{L^T}$$

$$= \begin{bmatrix} a_{11}^2 & a_{11}\cdot a_{12} & a_{11}\cdot a_{13} \\ a_{21}\cdot a_{11} & a_{21}\cdot a_{12} + a_{22}^2 & a_{21}\cdot a_{13} + a_{22}\cdot a_{23} \\ a_{31}\cdot a_{11} & a_{31}\cdot a_{12} + a_{32}\cdot a_{22} & a_{31}\cdot a_{13} + a_{32}\, a_{23} + a_{33}^2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}\cdot a_{12} & a_{11}\cdot a_{13} \\ a_{21}\cdot a_{11} & a_{21}^2 + a_{22}^2 & a_{21}\cdot a_{13} + a_{22}\cdot a_{23} \\ a_{31}\cdot a_{11} & a_{31}\cdot a_{12} + a_{32}\cdot a_{22} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

Given a matrix,

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$a_{11}^2 = 4$ $\quad$ $a_{21}\cdot a_{11} = 2$ $\quad$ $a_{31}\cdot a_{11} = 14$ $\quad$ $a_{11}\cdot a_{12} = 2$

$\boxed{a_{11} = 2}$ $\quad$ $a_{21}\cdot 2 = 2$ $\quad$ $a_{31}\cdot 2 = 14$ $\quad$ $2;\ a_{12} = 2$

$\boxed{a_{21} = 1}$ $\quad$ $\boxed{a_{31} = 7}$ $\quad$ $\boxed{a_{12} = 1}$

$a_{11} \cdot a_{13} = 14$

$2 \cdot a_{13} = 14$

$\boxed{a_{13} = 7}$

$a_{21}^2 + a_{22}^2 = 17$

$(1)^2 + a_{22}^2 = 17$

$a_{22}^2 = 17 - 1$

$a_{22}^2 = 16$

$\boxed{a_{22} = 4}$

$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = -5$

$(1) \cdot (7) + (4) a_{23} = -5$

$7 + 4 \cdot a_{23} = -5$

$4 a_{23} = -5 - 7$

$4 a_{23} = -12$

$a_{23} = -12/4$

$\boxed{a_{23} = -3}$

$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = -5$

$(7)(1) + a_{32}(4) = -5$

$7 + 4 a_{32} = -5$

$4 a_{32} = -5 - 7$

$4 a_{32} = -12$

$\boxed{a_{32} = -3}$

$a_{31}^2 + a_{32}^2 + a_{33}^2 = 83$

$7^2 + (-3)^2 + a_{33}^2 = 83$

$49 + 9 + a_{33}^2 = 83$

$a_{33}^2 = 83 - 49 - 9$

$a_{33}^2 = 83 - 58$

$a_{33}^2 = 25$

$\boxed{a_{33} = 5}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{L^T}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2) & 2(1) & 2(7)+0+0 \\ 1(2) & 1(1)+4(4) & 1(7)+4(-3) \\ 7(2) & 7(1)-3(4) & 7(7)-3(-3)+5(5) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -3 \\ 14 & -5 & 83 \end{bmatrix}$$

$\therefore \boxed{A = L \cdot L^T}$

solve eqn $25x + 15y - 5z = 35$

$15x + 18y + 0z = 33$

$-5x + 0y + 11z = 6$

using cholesky decomposition method

Solution:

$$AX = B$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

Take A and find whether it is symmetric & positive definite matrix.

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \underbrace{\begin{bmatrix} 25 & 0 & 0 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 25 & 15 & -5 \\ 0 & 18 & 0 \\ 0 & 0 & 11 \end{bmatrix}}_{L^T}$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11} \cdot a_{12} & a_{11} \cdot a_{13} \\ a_{21} \cdot a_{11} & a_{21}^2 + a_{22}^2 & a_{21} \cdot a_{13} + a_{22} \cdot a_{23} \\ a_{31} \cdot a_{11} & a_{31} \cdot a_{12} + a_{32} \cdot a_{22} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

$a_{11}^2 = 25$

$\boxed{a_{11} = 5}$

$a_{11} \cdot a_{12} = 15$

$5 \cdot a_{12} = 15$

$\boxed{a_{12} = 3}$

$a_{11} \cdot a_{13} = -5$

$5 \cdot a_{13} = -5$

$\boxed{a_{13} = -1}$

$a_{21} \cdot a_{11} = 15$

$a_{21} \cdot 5 = 15$

$\boxed{a_{21} = 3}$

$a_{31} \cdot a_{11} = -5$

$a_{31} \cdot 5 = -5$

$\boxed{a_{31} = -1}$

$a_{21}^2 + a_{22}^2 = 18$

$3^2 + a_{22}^2 = 18$

$a_{22}^2 = 18 - 9$

$a_{22} = \sqrt{9}$

$\boxed{a_{22} = 3}$

$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = 0$

$3 \cdot (-1) + (3)\ a_{23} = 0$

$-3 + 3\ a_{23} = 0$

$3\ a_{23} = 3$

$\boxed{a_{23} = 1}$

$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = 0$

$(-1)(3) + a_{32} \cdot (3) = 0$

$-3 + 3a_{32} = 0$

$3a_{32} = 3$

$\boxed{a_{32} = 1}$

$a_{31}^2 + a_{32}^2 + a_{33}^2 = 11$

$(-1)^2 + (1)^2 + a_{33}^2 = 11$

$a_{33}^2 = 11 - 1 - 1$

$a_{33}^2 = 9$

$\boxed{a_{33} = 3}$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

A (L) (LT)

$AX = B$

$LL^T \cdot X = B$ Let $L^T \cdot X = Y$

$LY = B$

Take $LY = B$,

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

Equating right & left side,

$5y_1 = 35 \rightarrow (1)$

$3y_1 + 3y_2 = 33 \rightarrow (2)$

$-y_1 - y_2 + 3y_3 = 6 \rightarrow (3)$

$(1) \Rightarrow \boxed{y_1 = 7}$

$(2) \Rightarrow 3(7) + 3y_2 = 33$

$\boxed{y_2 = 4}$

$(3) \Rightarrow -7 + 4 + 3y_3 = 6$

$\boxed{y_3 = 3}$

Thus, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$

Substitute y in $L^T . X = Y$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$3x_3 = 3$

$\boxed{x_3 = 1}$

$3x_2 + x_3 = 4$

$3x_2 = 4 - 1$

$3x_2 = 3$

$\boxed{x_2 = 1}$

$5x_1 + 3x_2 - x_3 = 7$

$5x_1 + 3(1) - 1 = 7$

$5x_1 + 2 = 7$

$5x_1 = 5$

$\boxed{x_1 = 1}$

$\therefore$ The value of x_1, x_2 and x_3 are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) Solve the following system of linear equation by using cholesky Method.

$$4x_1 + 2y + 14z = 14$$

$$2x + 17y - 5z = -101$$

$$14x - 5y + 83z = 155$$

<u>Solution:</u>

$$AX = B$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Take A & check whether it is symmetric & positive definite matrix.

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

here, $A^T = A$ & $a_{ij} = a_{ji}$. Thus "A" is symmetric.

CH equation:

~~A = |A| =~~ $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 & 14 \\ 2 & 17-\lambda & -5 \\ 14 & -5 & 83-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(17-\lambda)(83-\lambda) - 25] - 2[2(83-\lambda) + 5(14)] + 14[-10 - 14(17-\lambda)] = 0$$

$$(4-\lambda)[1411 - 17\lambda - 83\lambda + \lambda^2 - 25] - 2[166 - 2\lambda + 70] + 14[-10 - 238 + 14\lambda] = 0$$

$$(4-\lambda)[1386 - 100\lambda + \lambda^2] - 2[96 - 2\lambda] + 14[14\lambda - 248] = 0$$

$$(4-\lambda)[1386 - 100\lambda + \lambda^2] - 192 + 4\lambda + 196\lambda - 3472 = 0$$

$$(4-\lambda)[\lambda^2 - 100\lambda + 1386] - 3664 + 200\lambda = 0$$

$$4\lambda^2 - 400\lambda + 5544 - \lambda^3 + 100\lambda^2 - 1386\lambda - 3664 + 200\lambda = 0$$

$$-\lambda^3 + 104\lambda^2 - 1586\lambda + 1880 = 0$$

$\lambda = 1.29$

16.9

85.7

$$-\lambda^3 + 104\lambda^2 - 1586\lambda + 1880 = 0$$

$$\lambda^3 - 104\lambda^2 + 1586\lambda - 1880 = 0$$

$$a_1 = [4] = 4 > 0$$

$$a_2 = \begin{bmatrix} 4 & 2 \\ 2 & 17 \end{bmatrix}$$

$$|a_2| = \begin{vmatrix} 4 & 2 \\ 2 & 17 \end{vmatrix} = 68 - 4$$
$$= 64 > 0$$

$$a_3 = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{vmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{vmatrix}$$

$$= 4(1411 - 25) - 2(166 + 70) + 14(-10 - 238)$$

$$= 4(1386) - 2(236) + 14(-248)$$

$$= 5544 - 472 - 3472$$

$$|a_3| = 1600 > 0$$

Thus positive definite matrix

$$\underset{A}{\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 4 & 0 & 0 \\ 2 & 17 & 0 \\ 14 & -5 & 83 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 4 & 2 & 14 \\ 0 & 17 & -5 \\ 0 & 0 & 83 \end{bmatrix}}$$

$$= \begin{bmatrix} a_{11}^2 & a_{11} \cdot a_{12} & a_{11} \cdot a_{13} \\ a_{21} \cdot a_{11} & a_{21}^2 + a_{22}^2 & a_{21} \cdot a_{13} + a_{22} \cdot a_{23} \\ a_{31} \cdot a_{11} & a_{31} \cdot a_{12} + a_{32} \cdot a_{22} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

$a_{11}^2 = 4$ $\quad a_{21} \cdot a_{11} = 2$ $\quad a_{31} \cdot a_{11} = 14$ $\quad a_{11} \cdot a_{12} = 2$

$a_{11} = \sqrt{4}$ $\quad a_{21} = 2/2$ $\quad a_{31} = 14/2$ $\quad a_{12} = 2/2$

$\boxed{a_{11} = 2}$ $\quad \boxed{a_{21} = 1}$ $\quad \boxed{a_{31} = 7}$ $\quad \boxed{a_{12} = 1}$

$$a_{21}^2 + a_{22}^2 = 17$$
$$(1)^2 + a_{22}^2 = 17$$
$$a_{22}^2 = 16$$
$$\boxed{a_{22} = 4}$$

$$a_{31} \cdot a_{12} + a_{32} \cdot a_{22} = -5$$
$$(7)(1) + a_{32}(4) = -5$$
$$4a_{32} = -5 - 7$$
$$4a_{32} = -12$$
$$\boxed{a_{32} = -3}$$

$$a_{11} \cdot a_{13} = 14$$
$$(2)\, a_{13} = 14$$
$$\boxed{a_{13} = 7}$$

$$a_{21} \cdot a_{13} + a_{22} \cdot a_{23} = -5$$
$$(1)(7) + (4)\, a_{23} = -5$$
$$4a_{23} = -12$$
$$\boxed{a_{23} = -3}$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 83$$
$$(7)^2 + (-3)^2 + a_{33}^2 = 83$$
$$49 + 9 + a_{33}^2 = 83$$
$$a_{33}^2 = 83 - 49 - 9$$
$$a_{33}^2 = 25$$
$$\boxed{a_{33} = 5}$$

$$\underset{A}{\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}}$$

$$AX = B$$
$$L \cdot L^T \cdot X = B \qquad \text{Let } L^T \cdot X = Y$$
$$LY = B$$

Take $LY = B$,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$2y_1 = 14$$
$$\boxed{y_1 = 7}$$

$$y_1 + 4y_2 = -101$$
$$4y_2 = -101 - 7$$
$$4y_2 = -108$$
$$\boxed{y_2 = -27}$$

$$7y_1 - 3y_2 + 5y_3 = 155$$
$$7(7) - 3(-27) + 5y_3 = 155$$
$$49 + 81 + 5y_3 = 155$$
$$5y_3 = 155 - 49 - 81$$
$$5y_3 = 25$$
$$\boxed{y_3 = 5}$$

$$\therefore y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

Substitute $L^T \cdot X = Y$

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

$5x_3 = 5$

$\boxed{x_3 = 1}$

$4x_2 - 3x_3 = -27$

$4x_2 - 3(1) = -27$

$4x_2 = -27 + 3$

$4x_2 = -24$

$\boxed{x_2 = -6}$

$2x_1 + x_2 + 7x_3 = 7$

$2x_1 + (-6) + 7(1) = 7$

$2x_1 - 6 + 7 = 7$

$2x_1 = 6$

$\boxed{x_1 = 3}$

$$\therefore \text{The value of } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$$

$$\underbrace{x^2+5x-6}_{\text{Polynomial with deg 2}} \qquad \underbrace{x^2+5x-6=0}_{\text{eqn with deg 2.}}$$

For quadratic eqn,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

$P(x) = x-1 \rightarrow$ linear eqn.

$P(x) = x^2+5x-1 \rightarrow$ quadratic eqn.

$P(x) = x^3+5x^2+5x+5 \rightarrow$ cubic eqn

To find factor for cubic eqn, there is no exact method.

factors $\Rightarrow$ equation · solution · $\Rightarrow$ $x^2+5x+6=0$

zeros (we get two linear eqns for) $\nearrow$ $(x+2)(x+3)=0$

$\Downarrow$ $x=-2,-3$

Polynomial.

The number which makes the polynomial to zero.

$q(x) = x^2-5x+6$

$q(2) = 2^2-10+6 = 0$

For cubic eqn,

first use trial method to find one root and then proceed to synthetic division.

Determinant

$A = [a_{ij}]_{3\times3} = |a_{ij}|_{3\times3} = a_{11}(\quad) - a_{12}(\quad) + a_{13}(\quad)$

Any particular row or column.

Find the eigen values and eigen vectors for the follow[illegible]
matrix.

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Solution:

using CH eqn,

$$|A-\lambda I|=0$$

$$\left| \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

Alternate method to find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

S_1 = sum of the main diagonal

S_2 = sum of minors of the leading diagonal

$S_3 = |A|$

S_1 = sum of main diagonal elements

$= -1+2+0 = 1$

$$\boxed{S_1 = 1}$$

S_2 = sum of minors of leading diagonal

$$= \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$$

$= (+1) + (0-2) + (-2-2)$

$= 1-2-4$

$= 1-6$

$$\boxed{S_2 = -5}$$

$S_3 = |A| = -1(1) - 2(1) - 2(-1+2)$

$= -5$

$$\boxed{S_3 = -5}$$

$$\lambda^3 - 1\lambda^2 - 5\lambda + 5 = 0$$

Solving $\lambda^3 - 1\lambda^2 - 5\lambda + 5 = 0$

$P(\lambda) = \lambda^3 - \lambda^2 - 5\lambda + 5$

$P(1) = 1^3 - 1^2 - 5(1) + 5$

$= 1 - 1 - 5 + 5$

$P(1) = 0$

$\therefore (\lambda - 1)$ is a factor

using synthetic dèvision,

$$\begin{array}{c|cccc} 1 & 1 & -1 & -5 & 5 \\ & 0 & 1 & 0 & -5 \\ \hline & 1 & 0 & -5 & \boxed{0} \end{array}$$

$\lambda^2 + 0\lambda - 5 = 0$

$\lambda^2 - 5 = 0$

$\lambda^2 = 5$

$\lambda = \pm\sqrt{5}$

$\therefore \lambda = 1, \sqrt{5}, -\sqrt{5}$.

To find eigen vector for $\lambda = 1$;

$AX = 0$

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = 1$;

$$\begin{bmatrix} -1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1+2x_2-2x_3=0 \longrightarrow ①$$

$$x_1+x_2+x_3=0 \longrightarrow ②$$

$$-x_1-x_2-x_3=0 \longrightarrow ③$$

using cross multiplication method,

① & ② ⇒

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -2 & -2 & 2 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{2+2}=\frac{x_2}{-2+2}=\frac{x_3}{-2-2}$$

$$\frac{x_1}{4}=\frac{x_2}{0}=\frac{x_3}{-4}$$

$$\frac{x_1}{1}=\frac{x_2}{0}=\frac{x_3}{-1}=K$$

∴ Eigen vector for $\lambda=1$, $X_1=K\begin{bmatrix}1\\0\\-1\end{bmatrix}$

to find eigen vector for $\lambda=\sqrt{5}$;

$$AX=0 \quad (A-\lambda I)X=0$$

$$\left(\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}-(\sqrt{5})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1-\sqrt{5} & 2 & -2\\ 1 & 2-\sqrt{5} & 1\\ -1 & -1 & -\sqrt{5}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$(-1-\sqrt{5})x_1+2x_2-2x_3=0 \longrightarrow ①$$

$$x_1+(2-\sqrt{5})x_2+x_3=0 \longrightarrow ②$$

$$-x_1-x_2-\sqrt{5}x_3=0 \longrightarrow ③$$

using cross multiplication method,

② & ③ ⇒

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ (2-\sqrt{5}) & +1 & 1 & 2-\sqrt{5} \\ -1 & -\sqrt{5} & -1 & -1 \end{array}$$

$$\frac{x_1}{-(2-\sqrt{5})\sqrt{5}+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{-1+(2-\sqrt{5})}$$

$$\frac{x_1}{-2\sqrt{5}+5+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\frac{x_1}{-2\sqrt{5}+6} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\frac{x_1}{-2\sqrt{5}+6} = \frac{x_2}{-(1-\sqrt{5})} = \frac{x_3}{1-\sqrt{5}}$$

"x" by $(1+\sqrt{5})$ on each term,

$$\frac{x_1}{(1+\sqrt{5})(6-2\sqrt{5})} = \frac{x_2}{-(1-\sqrt{5})(1+\sqrt{5})} = \frac{x_3}{(1-\sqrt{5})(1+\sqrt{5})}$$

$$\frac{x_1}{(6-2\sqrt{5})(1+\sqrt{5})} = \frac{x_2}{-(1^2-(\sqrt{5})^2)} = \frac{x_3}{1^2-(\sqrt{5})^2}$$

$$\frac{x_1}{(6-2\sqrt{5})(1+\sqrt{5})} = \frac{x_2}{-(1-5)} = \frac{x_3}{1-5}$$

$$\frac{x_1}{6+6\sqrt{5}-2\sqrt{5}-10} = \frac{x_2}{+4} = \frac{x_3}{-4}$$

$$\frac{x_1}{-4+4\sqrt{5}} = \frac{x_2}{4} = \frac{x_3}{-4}$$

$\div$ by 4

$$\frac{x_1}{-1+\sqrt{5}} = \frac{x_2}{1} = \frac{x_3}{-1} = k$$

$$\therefore X_2 = k\begin{bmatrix} -1+\sqrt{5} \\ 1 \\ -1 \end{bmatrix}$$

to find eigen values of $\lambda = -\sqrt{5}$;

$$(A-\lambda I)X = 0$$

$$\left[\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - (-\sqrt{5})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & \sqrt{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1+\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \longrightarrow ①$$

$$x_1 + (2+\sqrt{5})x_2 + x_3 = 0 \longrightarrow ②$$

$$-x_1 - x_2 + \sqrt{5}x_3 = 0 \longrightarrow ③$$

using cross-multiplication method,

② × ③ ⇒

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2+\sqrt{5} & 1 & 1 & 2+\sqrt{5} \\ -1 & \sqrt{5} & -1 & -1 \end{array}$$

$$\frac{x_1}{2\sqrt{5}+5+1} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{-1+2+\sqrt{5}}$$

$$\frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{1+\sqrt{5}}$$

$$\frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-(1+\sqrt{5})} = \frac{x_3}{1+\sqrt{5}}$$

$$\frac{x_1}{(1-\sqrt{5})(2\sqrt{5}+6)} = \frac{x_2}{-(1-\sqrt{5})(1+\sqrt{5})} = \frac{x_3}{(1+\sqrt{5})(1-\sqrt{5})}$$

$$\frac{x_1}{2\sqrt{5}+6-10-6\sqrt{5}} = \frac{x_2}{-(1^2-(\sqrt{5})^2)} = \frac{x_3}{1^2-(\sqrt{5})^2}$$

$$\frac{x_1}{-4\sqrt{5}-4} = \frac{x_2}{-(1-5)} = \frac{x_3}{1-5}$$

$$\frac{x_1}{-4(\sqrt{5}+1)} = \frac{x_2}{4} = \frac{x_3}{-4}$$

÷ by 4;

$$\frac{x_1}{-(1+\sqrt{5})} = \frac{x_2}{1} = \frac{x_3}{-1} = k$$

$$\therefore X_3 = k \begin{bmatrix} -(1+\sqrt{5}) \\ 1 \\ -1 \end{bmatrix} \quad k = \text{arbitary constant}$$

Find eigen values and Eigen vectors for the following matrix

$$\begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find the Eigen values,

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where

s_1 = sum of the main diagonal elements

s_2 = sum of minors of the leading diagonal

$s_3 = |A|$

$s_1 = 0+1+2 = 3$

$$s_2 = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}$$

$= 2+2 + 0-2 + 0-2$

$s_2 = 0$

$s_3 = 0(\quad) + 2(-2+2) - 2(1+1)$

$= -2(2)$

$s_3 = -4$

$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0$

$\lambda^3 - 3\lambda^2 + 4 = 0$

$P(\lambda) = \lambda^3 - 3\lambda^2 + 4$

$P(1) = 2 \neq 0$

$P(-1) = -1-3+4 = 0$

$\therefore \lambda+1=0$ is a factor

$$\begin{array}{c|cccc} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)(\lambda-2) = 0$

$\lambda = 2, 2$

$\therefore \lambda = -1, 2, 2.$

To find Eigen vector:

for $\lambda = -1$; $(A - \lambda I)X = 0$

$$\left[\begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0+1 & -2 & -2 \\ -1 & 1+1 & 2 \\ -1 & -1 & 2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 - 2x_3 = 0$

$-x_1 + 2x_2 + 2x_3 = 0$

$-x_1 - x_2 + 3x_3 = 0$

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & \\ 2 & 2 & -1 & 2 & \\ -1 & 3 & -1 & -1 & \end{array}$$

$$\frac{x_1}{6+2} = \frac{x_2}{-2+3} = \frac{x_3}{1+2} = K$$

$$\frac{x_1}{8} = \frac{x_2}{1} = \frac{x_3}{3} = K$$

$x_1 = 8K$; $x_2 = K$; $x_3 = 3K$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$$

For $\lambda = 2$;

$$|A - \lambda I| x = 0$$

$$\begin{bmatrix} 0-\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & -2 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 - 2x_3 = 0$$

$$-x_1 - x_2 + 2x_3 = 0$$

$$-x_1 - x_2 + 0x_3 = 0$$

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & \\ -2 & -2 & -2 & -2 & \\ -1 & +2 & -1 & -1 & \end{array}$$

$$\frac{x_1}{-4-2} = \frac{x_2}{2+4} = \frac{x_3}{2-2}$$

$$\frac{x_1}{-6} = \frac{x_2}{6} = \frac{x_3}{0} = K$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{0} = K$$

$$X_2 = K \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Eigen vectors are $X_1 = K \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$, $X_2 = X_3 = K \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

[Solution of linear homogeneous eqns:

(i) If $\rho(A) = n$; $\rho(A:B) = n$

$\rho(A) = \rho(A:B) = n \Rightarrow$ consistent ; unique solution
zero solution.

(ii) $\rho(A:B) = \rho(A) < n \Rightarrow$ Infinite solution; consistent

(iii) $\rho(A) < \rho(A,B)$ $\therefore$ $\rho(A) \neq \rho(A,B) \Rightarrow$ no solution.]

Finding eigen vector for $\lambda = 2$ using row echelon form:

$$A|B = \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$2x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$\boxed{x_1 = -K_2 - K_1}$

$$\sim \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$X = \begin{bmatrix} -K_1 - K_2 \\ K_2 \\ K_1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 \\ \to x_2 \\ \to x_3 \end{array} \quad R_3 \to 3R_3 - R_2$$

$-2x_1 - 2x_2 - 2x_3 = 0$

$x_3 = 0$

$K_1 = 0$

$0(K_2) + 6K_1 = 0$

$0(x_2) = -6K_1$

(K_2)

$-2x_1 - 2K_2 - 2K_1 = 0$

$x_1 = K_1 + K_2$

③ $\Rightarrow \quad 0x_3 = 0$

$x_3 = \alpha$

② $\Rightarrow \quad 0x_2 + 0x_3 = 0$

$0(\beta) = 0$

$x_2 = \beta$

③ $-2x_1 - 2x_2 - 2x_3 = 0 \Rightarrow -2(x_1 + x_2 + x_3) = 0$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_1 = -\beta - \alpha$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(\alpha+\beta) \\ \beta \\ \alpha \end{bmatrix} \quad \begin{bmatrix} -(K_1+K_2) \\ K_2 \\ K_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (K_1) + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (K_2)$$

$$= \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} + \begin{bmatrix} -\beta \\ \beta \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -\alpha-\beta \\ \beta \\ \alpha \end{bmatrix}$$

QR decomposition 19/09/2023.

QR decomposition is a matrix factorization method that decompose the matrix A into Q, and such that $A = QR$.

The Q matrix is an orthogonal matrix, meaning that its columns are orthonormal and the R matrix is an upper triangular method.

The process of QR decomposing involves the following steps.

1) Start with a matrix A
2) Apply Gram - Schmidt process to the column of A to obtain an orthonormal set of columns. This is done by recursively subtracting the projection of each column onto the subspace spanned by the process orthonormal columns and normalizing the resulting vectors.
3) Arrange the resulting orthonormal column into a matrix Q.
4) Compute the R matrix by multiplying the transpose of Q with A. The resulting matrix will be upper triangular matrix.
5) The QR decomposition of A is the pair (Q, R).

QR decomposition with Gram - Schmidt process.

Consider the matrix $A = \{a_1, a_2, a_3, \dots a_n\}$

where $a_1, a_2, a_3, \dots a_n$ are column elements of given matrix with inner product $\langle u, v \rangle = V^T . u$.

$$\text{projection } a = \frac{\overset{\text{inner product}}{\langle u, a \rangle}}{\langle u, u \rangle} u, \quad \text{Then } u_1 = a_1 \text{ (column wise)}$$

$$u_2 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1, \quad u_3 = a_3 \frac{\langle u_1, a_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, a_3 \rangle}{\langle u_2 u_2 \rangle} u_2$$

norm $\rightarrow$ to find distance

$\|u_n\| = \sqrt{\langle u_n, u_n \rangle}$
norm

$$u = [u_1, u_2, u_3] = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right]$$

$u_1, u_2, u_3 \rightarrow$ unit vector

$$U = [u_1 \ u_2 \ u_3]$$

20/09/2023

Find QR decomposition $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$a_1 \quad a_2 \quad a_3$

soln.

orthonormal vectors.

Gram-schmidt process to find u_1, u_2, u_3:

$a_1 = (1, 1, 2)$

$a_2 = (0, 2, 2)$

$a_3 = (-1, 1, 3)$

$u_1 = (1, 1, 2)$

$$u_2 = a_2 - \frac{\langle u_1 \ a_2 \rangle}{\langle u_1 \ u_1 \rangle} u_1$$

$$= (0,2,2) - \frac{\langle (1,1,2)(0,2,2) \rangle}{\langle (1,1,2)(1,1,2) \rangle}(1,1,2)$$

$$u_2 = (0,2,2) - \frac{6}{6}(1,1,2)$$

$$= (0,2,2) - (1,1,2)$$

$$u_2 = (-1,1,0)$$

$$u_3 = a_3 - \frac{\langle u_1 \ a_3 \rangle}{\langle u_1 \ u_1 \rangle} u_1 - \frac{\langle u_2 \ a_3 \rangle}{\langle u_2 \ u_2 \rangle} u_2$$

$$= (-1,1,3) - \frac{\langle (1 \ 1 \ 2)(-1 \ 1 \ 3) \rangle}{\langle (1,1,2)(1,1,2) \rangle}(1,1,2) - \frac{\langle (-1,1,0), (-1,1,3) \rangle}{\langle (-1,1,0)(-1,1,0) \rangle}(-1,1,0)$$

$$= (-1,1,3) - \frac{-1+1+6}{6}(1,1,2) - \frac{1+1}{1+1}(-1,1,0)$$

$$u_3 = (-1,-1,1)$$

$$Q = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right]$$

Orthogonal

$A = QR \longrightarrow QQ^T = I$ (since orthogonal)

$\uparrow$ upper triangular matrix.

$\|u\|$ is used to convert orthonormal vector to orthogonal vector.

$u_1 = (1,1,2)$

$u_2 = (-1,1,0)$

$u_3 = (-1,-1,1)$

$\|u\| = \sqrt{\langle u,u \rangle}$

$\|u_1\| = \sqrt{\langle (1,1,2) \cdot (1,1,2) \rangle} = \sqrt{1+1+4} = \sqrt{6}$

$\|u_2\| = \sqrt{\langle (-1,1,0)\,(-1,1,0) \rangle} = \sqrt{1+1+0} = \sqrt{2}$

$\|u_3\| = \sqrt{\langle (-1,-1,1)\,(-1,-1,1) \rangle} = \sqrt{1+1+1} = \sqrt{3}$

$$Q = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right]$$ $\Rightarrow$ To change orthonormal to orthogonal matrix

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\Rightarrow A = QR$

Pre multiply by Q^T

$Q^TQ \cdot R = Q^TA$

$I \cdot R = Q^TA$ ($\because Q^TQ = I$ orthogonal matrix)

$R = Q^TA$

$$R = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} & 0 + \frac{2}{\sqrt{6}} + \frac{4}{\sqrt{6}} & \frac{-1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{6}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & 0 + \frac{2}{\sqrt{2}} + 0 & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \\ \frac{-1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} & 0 - \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{6}} & \frac{6}{\sqrt{6}} & \frac{6}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & 0 & \frac{3}{\sqrt{3}} \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{2}} + 0 & \frac{\sqrt{6}}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{3}} \\ \frac{\sqrt{6}}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{2}} + 0 & \frac{\sqrt{6}}{\sqrt{6}} + \frac{2}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{3}} \\ \frac{2\sqrt{6}}{\sqrt{6}} & \frac{2\sqrt{6}}{\sqrt{6}} + 0 + 0 & \frac{2\sqrt{6}}{\sqrt{6}} + 0 + \frac{\sqrt{3}}{\sqrt{3}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\therefore A = QR.$$

LEAST SQUARES METHOD

1) $x_1 + 0x_2 + 2x_3 = 3$

$0x_1 + 2x_2 + 2x_3 = -3$

$-x_1 - x_2 - x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = -3.$

Solve the above equations by least square method.

Solution:

Since it is a non-homogeneous equation, we can write

$$AX = B$$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -2 & 0 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}}$$

Premultiply both sides by A^T,

$$A^T . A . X = A^T B$$

$$A^T A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1+1 & 0+0+1+2 & 2+0+1+0 \\ 0+0+1+2 & 0+4+1+4 & 0+4+1+0 \\ 2+0+1+0 & 0+4+1+0 & 4+4+1+0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix}$$

$$A^T.B = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0+3 \\ 0-6+0+6 \\ 6-6+0+0 \end{bmatrix}$$

$$A^TB = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T.A.X = A^TB.$$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Augumented matrix:

$$[A^TA|A^TB] \sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 6 \\ 3 & 9 & 5 & 0 \\ 3 & 5 & 9 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 3 & 3 & 3 & 6 \\ 0 & 6 & 2 & -6 \\ 0 & 2 & 6 & -6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 6 \\ 0 & 6 & 2 & -6 \\ 0 & 2 & 6 & -6 \end{array}\right] \quad R_3 \to 3R_3 - R_2$$

$-18-(-6)$

$-18+6$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 6 \\ 0 & 6 & 2 & -6 \\ 0 & 0 & 16 & -12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -3 \\ 0 & 0 & 4 & -3 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/3 \\ R_2 \to R_2/2 \\ R_3 \to R_3/4 \end{array}$$

$4x_3 = -3$

$\boxed{x_3 = -3/4}$

$3x_2 + x_3 = -3$

$3x_2 = -3 - x_3$

$3x_2 = -3 - x_3$

$x_2 = \dfrac{-3 - x_3}{3}$

$x_2 = \dfrac{-3 + \frac{3}{4}}{3} \Rightarrow \dfrac{\frac{-12+3}{4}}{3}$

$x_2 = \dfrac{-9}{4 \times 3} = \dfrac{-9}{12} = \dfrac{-3}{4}$

$\boxed{x_2 = \dfrac{-3}{4}}$

$x_1 + x_2 + x_3 = 2$

$x_1 = 2 - x_2 - x_3$

$x_1 = 2 + \dfrac{3}{4} + \dfrac{3}{4}$

$x_1 = 2 + \dfrac{6}{4}$

$x_1 = 2 + \dfrac{3}{2} \Rightarrow \dfrac{4+3}{2} = 7/2$

$x_1 = 7/2$

$x_2 = -3/4$

$x_3 = -3/4$

2) $x_1 + 0x_2 + 2x_3 = 3$

$0x_1 + 2x_2 + 2x_3 = -3$ Solve by least square method

$-x_1 - x_2 - x_3 = 0$

$-x_1 + 2x_2 + 0x_3 = -3$

Solution:

Since it is a non-homogeneous eqn, A

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

Premultiply both sides by A^T

$$A^T A \cdot X = A^T B$$

$$A^T \cdot A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1+0+1+1 & 0+0+1-2 & 2+0+1+0 \\ 0+0+1-2 & 0+4+1+4 & 0+4+1+0 \\ 2+0+1+0 & 0+4+1+0 & 4+4+1+0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix}$$

$$A^T \cdot B = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0+3 \\ 0-6+0-6 \\ 6-6+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix}$$

$A^T \cdot A \cdot X = A^T B$

$$\begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & 5 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix}$$

$A^T A \qquad X \qquad B$

Augumented Matrin:

$$[A^T A / B] = \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ -1 & 9 & 5 & -12 \\ 3 & 5 & 9 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 6 & 6 & -6 \end{array}\right] \quad \begin{array}{l} R_2 \rightleftharpoons 3R_2 + R_1 \\ R_3 \rightleftharpoons R_3 - R_1 \end{array}$$

$27+1$

$15+3$

$3(-12)+6$

$-36+6$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 1 & 1 & -1 \end{array}\right] \quad R_3 \rightleftharpoons \frac{R_3}{6}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 3 & 6 \\ 0 & 26 & 18 & -30 \\ 0 & 0 & 8 & 4 \end{array}\right] \quad R_3 \rightleftharpoons 26R_3 - R_2$$

$26-18$

$-26+30$

(1) $\Rightarrow$

$8x_3 = 4$

$x_3 = 4/8 = 1/2 = 0.5$

$$\boxed{x_3 = 0.5}$$

$x_3 = 1/2$

② $\Rightarrow$ $26x_2 + 18x_3 = -30$

" $\div$ " by 2

$$13x_2 + 9x_3 = -15$$

Put $x_3 = 0.5$;

$$13x_2 = -15 - 9(0.5)$$

$$= -15 - 4.5$$

$$13x_2 = -19.5$$

$$x_2 = \frac{-19.5}{13}$$

$$\boxed{x_2 = -1.5} \quad x_2 = \frac{-39}{26}$$

③ $\Rightarrow$ $3x_1 - x_2 + 3x_3 = 6$

$$3x_1 = 6 + x_2 - 3x_3$$

$$3x_1 = 6 - 1.5 - 3(0.5)$$

$$3x_1 = 3$$

$$\boxed{x_1 = 1}$$

$\therefore$ $x_1 = 1$

$x_2 = -1.5$

$x_3 = 0.5$

Find the singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$(A-\lambda I)X=0$

<u>Solution:</u>

$$AA^T = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+2(2)+2(2) & 2(2)+2(2)+2(-2) & 2(2)+2(-2)+2(6) \\ 4+4-4 & 4+4+4 & 4-4-12 \\ 4-4+12 & 4-4-12 & 4+4+36 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 12 & 4 & 12 \\ 84 & 12 & -12 \\ 12 & -12 & 44 \end{bmatrix}$$

$$\begin{bmatrix} 4+4+4 & 4+4+4 & -4-4-12 \\ 4+4+4 & 4+4+4 & -4-4-12 \\ -4-4-12 & -4-4-12 & 4+4+36 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$$

$$AA^T = 4\begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{bmatrix}$$

$$AA^T = 4B$$

$$B = \begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{bmatrix}$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 3+3+11 = 17$$

$$S_2 = \begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} + \begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix}$$

$$= (33-25) + (33-25) + 0$$

$$S_2 = 8+8 = 16$$

$$S_3 = 3(33-25) - 3(33-25) - 5(0)$$

$$= 3(8) - 3(8).$$

$$S_3 = 0$$

substitute S_1, S_2, S_3

$$\lambda^3 - 17\lambda^2 + 16\lambda - 0 = 0$$

$$\lambda^3 - 17\lambda^2 + 16\lambda = 0$$

$$\lambda\,[\lambda^2 - 17\lambda + 16] = 0$$

$\therefore \lambda = 0$ is a factor

$$\lambda^2 - 17\lambda + 16 = 0$$

$$\lambda^2 - \lambda - 16\lambda + 16 = 0$$

$$\lambda(\lambda-1) - 16(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-16) = 0$$

$\lambda = 16, \lambda = 1$ are factors

$$\therefore \lambda = 0, 1, 16$$

$$\lambda = 16, 1, 0$$

$$4[B] = 64, 4, 0$$

$$\lambda = 64, 4, 0$$

$\lambda = 64;$

$$\begin{bmatrix} 12-64 & 12 & -20 \\ 12 & 12-64 & -20 \\ -20 & -20 & 44-64 \end{bmatrix} = \begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ -20 & -20 & -20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ -20 & -20 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-52x_1 + 12x_2 - 20x_3 = 0 \Rightarrow -13x_1 + 3x_2 - 5x_3 = 0$

$12x_1 - 52x_2 - 20x_3 = 0 \Rightarrow 3x_1 - 13x_2 - 5x_2 = 0.$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & -5 & -13 & 3 \\ -13 & -5 & 3 & -13 \end{array}$$

$$\frac{x_1}{-15-65} = \frac{x_2}{-15-65} = \frac{x_3}{169-9} = K$$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{160} = K$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2} = K$$

$x_1 = K; \quad x_2 = K; \quad x_3 = -2K$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ K \\ -2K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\|X_1\| = \sqrt{\langle X_1, X_1 \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{1^2+1^2+(-2)^2}$$

$$= \sqrt{1+1+4} = \sqrt{6} = \sqrt{6}$$

$$x_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$$ normalized eigen vectors

$\lambda = 4$;

$$\begin{bmatrix} 12-4 & 12 & -20 \\ 12 & 12-4 & -20 \\ -20 & -20 & 44-4 \end{bmatrix} = \begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 40 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 + 12x_2 - 20x_3 = 0$$
$$12x_1 + 8x_2 - 20x_3 = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 12 & -20 & 8 & 12 \\ 8 & -20 & 12 & 8 \end{array}$$

$$\frac{x_1}{12(-20) - 8(-20)} = \frac{x_2}{12(-20) - 8(-20)} = \frac{x_3}{64 - 12(12)}$$

$$\frac{x_1}{-240 + 160} = \frac{x_2}{-240 + 160} = \frac{x_3}{64 - 144}$$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{-80}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = K$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$||X_2|| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$X_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$ normalized Eigen vector

When $\lambda = 0$;

$$\Rightarrow \begin{bmatrix} 12-0 & 12 & -20 \\ 12 & 12-0 & -20 \\ -20 & -20 & 44-0 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① & ③ $\Rightarrow$

$$12x_1 + 12x_2 - 20x_3 = 0$$

$$-20x_1 - 20x_2 + 44x_3 = 0$$

"$\div$" by 2;

$$6x_1 + 6x_1 - 10x_3 = 0$$

$$-10x_1 - 10x_2 + 22x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 6 & -10 & 6 & 6 \\ -10 & 22 & -10 & -10 \end{array}$$

$$\frac{x_1}{132-100} = \frac{x_2}{100-132} = \frac{x_3}{-60+60}$$

$$\frac{x_1}{82} = \frac{x_2}{-32} = \frac{x}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$||X_3|| = \sqrt{(1)^2 + (-1)^2 + 0^2} = \sqrt{1+1} = \sqrt{2}$$

$$X_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{2}} \end{bmatrix}$$

$$u = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

$\cancel{V = [x_1]}$

$$V = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

$$A = [u] \underset{\text{Sqrt}(\lambda)}{\overset{E}{[\text{Diagonal matrix}]}} [V = u^T]$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{+1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{+1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{-16}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{+1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{-16}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} \\ \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{8}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} \\ \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{-16}{\sqrt{36}} + \frac{2}{\sqrt{9}} & \frac{32}{\sqrt{36}} + \frac{2}{\sqrt{9}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{8}{6} + \frac{2}{3} & \frac{8}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} \\ \frac{8}{6} + \frac{2}{3} & \frac{8}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} \\ \frac{-16}{6} + \frac{2}{3} & \frac{-16}{6} + \frac{2}{3} & \frac{32}{6} + \frac{2}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{8+4}{6} & \frac{8+4}{6} & \frac{-16+4}{6} \\ \frac{8+4}{6} & \frac{8+4}{6} & \frac{-16+4}{6} \\ \frac{-16+4}{6} & \frac{-16+4}{6} & \frac{32+4}{6} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{12}{6} & \frac{12}{6} & \frac{-12}{6} \\ \frac{12}{6} & \frac{12}{6} & \frac{-12}{6} \\ \frac{-12}{6} & \frac{-12}{6} & \frac{36}{6} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} = A$$

$$\therefore \; A = U\Sigma V^T$$

Problem

Find the singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

To prove

$A = U\Sigma V$

Solution:

$$\cancel{AA^T} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

A^TA

Since A is not a symmetric matrix

$$= \begin{bmatrix} 0+1 & 0+1 & 0+0 \\ 0+1 & 1+1 & 1+0 \\ 0+0 & 1+0 & 1+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

To find Eigen values:

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0 \longrightarrow (1)$$

where

s_1 = sum of the main diagonal

s_2 = sum of the minors of the leading diagonal

$s_3 = |A|$

s_1 = sum of main diagonal elements

$= 1+2+1$

$= 4$

s_2 = sum of minors of leading diagonal

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= (2-1) + (1-0) + (2-1)$

$= 1+1+1$

$= 3$

$$S_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) - 1(1-0) + 0(1-0)$$

$$= 1(1) - 1(1)$$

$$S_3 = 0$$

Substitute S_1, S_2, S_3 in eqn ①,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0; \quad \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 0; \quad (\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda = 0, 1, 3}$$

$\therefore$ Eigen values are $0, 1, 3$
$3, 1, 0$

When $\lambda = 0$;

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1-0 & 1 & 0 \\ 1 & 2-0 & 1 \\ 0 & 1 & 1-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① → $x_1 + x_2 + 0x_3 = 0$

② → $x_1 + 2x_2 + x_3 = 0$

③ → $0x_1 + x_2 + x_3 = 0$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 0 & & 1 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$||X_1|| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$ normalized eigen vector.

when $\lambda = 1$; $|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 $\rightarrow$ $0x_1 + x_2 + 0x_3 = 0$

2 $\rightarrow$ $x_1 + x_2 + x_3 = 0$

3 $\rightarrow$ $0x_1 + x_2 + 0x_3 = 0$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 0 & & 0 & & 1 \\ 1 & & 1 & & 1 & & 1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\|X_2\| = \sqrt{(1)^2 + 0^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

when $\lambda = 3$;

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 $\rightarrow -2x_1 + x_2 + 0x_3 = 0$

2 $\rightarrow x_1 - x_2 + x_3 = 0$

3 $\rightarrow 0x_1 + x_2 - 2x_3 = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -2 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0+2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\|x_3\| = \sqrt{(1)^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$$

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$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\cancel{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & 2/\sqrt{6} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & 1/\sqrt{6} \end{bmatrix}$$

$$U = \begin{array}{c} \begin{matrix} \lambda=3 & \lambda=1 & \lambda=0 \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \end{array}$$

$A = [U]$ [Diagonal matrix] $[V = U^T]$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \Bigg[$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}}+0+0 & 0+\frac{1}{\sqrt{2}}+0 & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}} & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{6}+\frac{1}{2} & \frac{2\sqrt{3}}{6} & \frac{\sqrt{3}}{6}-\frac{1}{2} \\ \frac{2\sqrt{3}}{6} & \frac{4\sqrt{3}}{6} & \frac{2\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6}-\frac{1}{2} & \frac{2\sqrt{3}}{6} & \frac{\sqrt{3}}{6}+\frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A = [U] \, [\underset{E}{\text{diagonal matrix}}] \, [V]$$

$$\Rightarrow \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}}+0+0 & 0+\frac{1}{\sqrt{2}}+0 & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}}+0+0 & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}}+0+0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2\sqrt{3}}{\sqrt{6}} & 0 & 0 \\ \frac{\sqrt{3}}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{6}+\frac{1}{2}+0 & \frac{2\sqrt{3}}{6}+0+0 & \frac{\sqrt{3}}{6}-\frac{1}{2}+0 \\ \frac{2\sqrt{3}}{6}+0+0 & \frac{4\sqrt{3}}{6}+0+0 & \frac{2\sqrt{3}}{6}+0+0 \\ \frac{\sqrt{3}}{6}-\frac{1}{2}+0 & \frac{2\sqrt{3}}{6}+0+0 & \frac{\sqrt{3}}{6}+\frac{1}{2}+0 \end{bmatrix}$$

(a) Find eigen value, eigen vector and QR factorization of the matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

Given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

S_1 = sum of main diagonal elements

$= 8+7+3$

$= 18$

S_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= (21-16) + (24-4) + (56-36)$

$= 5 + 20 + 20$

$= 45$

$S_3 = |A|$

$= 8(5) + 6(-18+8) + 2(24-14)$

$= 40 + 6(-10) + 2(10)$

$= 40 - 60 + 20$

$= 65 - 60$

$= 0$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

45

-3 -15

-18

$$\lambda = 0;$$

$$\lambda^2 - 18\lambda + 45 = 0$$

$$(\lambda - 3) = 0$$

$$(\lambda - 15) = 0$$

$$\lambda = 15, 3, 0.$$

Eigen vectors:

When $\lambda = 0$;

$$(A - \lambda I)X = 0$$

$$\left(\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

① $\rightarrow$ $8x_1 - 6x_2 + 2x_3 = 0$

② $\rightarrow$ $-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

solve ① & ②;

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24 - 14} = \frac{x_2}{-12 + 32} = \frac{x_3}{56 - 36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

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$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

To find eigen vector for $\lambda = 3$;

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0 \rightarrow (1)$

$-6x_1 + 4x_2 - 4x_3 = 0 \rightarrow (2)$

$2x_1 - 4x_2 + 0x_3 = 0 \rightarrow (3)$

solve (1) & (2), we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

To find eigen vector for $\lambda = 15$;

$$(A - 15I)X = 0$$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \rightarrow (1)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \rightarrow (2)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \rightarrow (3)$$

solve (1) & (2), we get

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -6 & 2 & -7 & -6 \\ -8 & -4 & -6 & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$A_1 = \begin{array}{ccc} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}$$

$a_1 = (1,2,2)$

$a_2 = (2,1,-2)$

$a_3 = (2,-2,1)$

$u_1 = (1,2,2)$

$$u_2 = a_2 - \frac{\langle u_1, a_2 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$= (2,1,-2) - \frac{\langle (1,2,2)(2,1,-2) \rangle}{\langle (1,2,2)(1,2,2) \rangle} (1,2,2)$$

$$= (2,1,-2) - \frac{2+2-4}{1+4+4} (1,2,2)$$

$$u_2 = (2,1,-2)$$

$$u_3 = a_3 - \frac{\langle u_1\ a_3 \rangle}{\langle u_1\ u_1 \rangle} u_1 - \left[- \frac{\langle u_2\ a_3 \rangle}{\langle u_2\ u_2 \rangle} u_2 \right]$$

$$= (2,-2,1) - \frac{\langle (1,2,2)(2,-2,1) \rangle}{\langle (1,2,2)(1,2,2) \rangle} (1,2,2) - \left[(2,-2,1) - \frac{\langle u_2\ a_3 \rangle}{\langle u_2\ u_2 \rangle} u_2 \right]$$

$$= \left[(2,-2,1) - \frac{2-4+2}{1+4+4} (1,2,2) \right] - \left[- \frac{\langle (2,1,-2)(2,-2,1) \rangle}{\langle (2,1,-2)(2,1,-2) \rangle} (2,1,-2) \right]$$

$$= (2,-2,1) - \left[- \frac{4-2-2}{4+1-4} (2,1,-2) \right]$$

$$u_3 = (2,-2,1)$$

$$\|u_1\| = \sqrt{1^2+2^2+2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\|u_2\| = \sqrt{2^2+1^2+(-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\|u_3\| = \sqrt{2^2+(-2)^2+1^2} = \sqrt{9} = 3.$$

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}+\frac{4}{3}+\frac{4}{3} & \frac{2}{3}+\frac{2}{3}-\frac{4}{3} & \frac{2}{3}-\frac{4}{3}+\frac{2}{3} \\ \frac{2}{3}+\frac{2}{3}-\frac{4}{3} & \frac{4}{3}+\frac{1}{3}+\frac{4}{3} & \frac{4}{3}-\frac{2}{3}-\frac{2}{3} \\ \frac{2}{3}-\frac{4}{3}+\frac{2}{3} & \frac{4}{3}-\frac{2}{3}-\frac{2}{3} & \frac{4}{3}+\frac{2}{3}+\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{3} & 0 & 0 \\ 0 & \frac{9}{3} & 0 \\ 0 & 0 & 9/3 \end{bmatrix}$$

$$R = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$A = QR$.

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

(4) Least singular method.

$x - 2y + z = 1$

$2x - 3y - z = 0$

$-x + y + 2z = 1$

$3x - 5y + 0z = 0$

solutions: $AX = B$

$A^T . A . X = A^T B$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

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$$A^T.A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1+9 & -2-6-1-15 & 1-2-2 \\ -2-6-1-15 & 4+9+1+25 & -2+3+2 \\ 1-2-2 & -2+3+2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 15 & -24 & -3 \\ -8 & 13 & 1 \\ -3 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^TB = \underset{3\times4}{\begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 1+0-1+0 \\ -2+0+1+0 \\ 1+0+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$A^T.A.X = A^TB$$

$$\begin{bmatrix} 5 & -8 & -1 \\ -8 & 13 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 5 & -8 & -1 & 0 \\ -8 & 13 & 1 & -1 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -3 & 9 & 15 \end{bmatrix} \quad \begin{array}{l} R_2 \to 5R_2 + 8R_1 \\ R_3 \to 5R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 3 & 5 \end{bmatrix} \quad R_3 \to R_3/3$$

$$\sim \left[\begin{array}{ccc|c} 5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$0x_3 = 0$

$\boxed{x_3 = \alpha}$

$x_2 - 3x_3 = -5$

$x_2 = -5 + 3x_3$

$\boxed{x_2 = -5 + 3\alpha}$

$5x_1 - 8x_2 - x_3 = 0$

$5x_1 = 8x_2 + x_3$

$x_1 = \dfrac{8x_2 + x_3}{5}$

$x_1 = \dfrac{8(-5+3\alpha) + \alpha}{5}$

$= \dfrac{-45 + 24\alpha + \alpha}{5}$

$= \dfrac{-45 + 25\alpha}{5}$

$\boxed{x_1 = -9 + 5\alpha}$

$\therefore\ x_1 = -9 + 5\alpha$

$x_2 = -5 + 3\alpha$

$x_3 = \alpha$

3) Singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Solution:

Premultiply by A^T since A is not a symmetric matrix.

$$A^TA = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 2+20 & 3+24 \\ 2+20 & 4+25 & 6+30 \\ 3+24 & 6+30 & 9+36 \end{bmatrix} \Rightarrow \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$$

To find eigen values of A^TA:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of the main diagonal elements

$= 17 + 29 + 45$

$= 91$

$$S_2 = \begin{vmatrix} 29 & 36 \\ 36 & 45 \end{vmatrix} + \begin{vmatrix} 17 & 27 \\ 27 & 45 \end{vmatrix} + \begin{vmatrix} 17 & 22 \\ 22 & 29 \end{vmatrix}$$

$= [1305 - 1296] + [765 - 729] + [493 - 484]$

$= 9 + 36 + 9$

$= 54$

$S_3 = |A| = 17[(29 \times 45) - (36 \times 36)] - 22[(22 \times 45) - (36 \times 27)]$
$+ 27[(22 \times 36) - (27 \times 29)]$

$= 153 - 396 + 243$

$= 0$

$$\lambda^3 - 91\lambda^2 + 54\lambda = 0$$

$$\lambda(\lambda^2 - 91\lambda + 54) = 0$$

$\lambda = 0$;

$a = 1$; $b = -91$; $c = 54$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{-(-91) \pm \sqrt{(-91)^2 - 4(1)(54)}}{2(1)}$$

$$\Rightarrow \frac{91 \pm \sqrt{8281 - 216}}{2}$$

$\Rightarrow 90.4 ; 0.59$

$\therefore \lambda_1 = 90.4 \quad \lambda_2 = 0.6 \quad \lambda_3 = 0$

For $\lambda = 90.4$;

$$\begin{bmatrix} 17-90.4 & 22 & 27 \\ 22 & 29-90.4 & 36 \\ 27 & 36 & 45-90.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -73.4 & 22 & 27 \\ 22 & -61.4 & 36 \\ 27 & 36 & -45.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-73.4x_1 + 22x_2 + 27x_3 = 0 \rightarrow (1)$

$22x_1 - 61.4x_2 + 36x_3 = 0 \rightarrow (2)$

$27x_1 + 36x_2 - 45.4x_3 = 0 \rightarrow (3)$

(1) & (2)

$x_1 \qquad x_2 \qquad x_3$

$22 \qquad 27 \qquad -73.4 \qquad 22$

$-61.4 \qquad 36 \qquad 22 \qquad -61.4$

$$\frac{x_1}{792+1657.8} = \frac{x_2}{594+2642.4} = \frac{x_3}{4506.76-484} = K$$

$$\frac{x_1}{2449.8} = \frac{x_2}{3236.4} = \frac{x_3}{4022.76} = K$$

$$\frac{x_1}{1.2249} = \frac{x_2}{1.6182} = \frac{x_3}{2.01138} = K$$

$$\frac{x_1}{1.2} = \frac{x_2}{1.6} = \frac{x_3}{2.0} = K$$

$$\|x_1\| = \sqrt{(1.2)^2 + (1.6)^2 + (2.0)^2} = \sqrt{1.44+2.56+4}$$

$$\|x_1\| = \sqrt{8}$$

$$X_1 = \begin{bmatrix} \frac{1.2}{\sqrt{8}} \\ \frac{1.6}{\sqrt{8}} \\ \frac{2}{\sqrt{8}} \end{bmatrix}$$

For $\lambda = 0.6$;

$$\begin{bmatrix} 17-0.6 & 22 & 27 \\ 22 & 29-0.6 & 36 \\ 27 & 36 & 45-0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16.4 & 22 & 27 \\ 22 & 28.4 & 36 \\ 27 & 36 & 44.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$16.4x_1 + 22x_2 + 27x_3 = 0 \rightarrow ①$

$22x_1 + 28.4x_2 + 36x_3 = 0 \rightarrow ②$

$27x_1 + 36x_2 + 44.4x_3 = 0 \rightarrow ③$

x_1	x_2	x_3	
22	27	16.4	22
28.4	36	22	28.4

$$\frac{x_1}{792 - 766.8} = \frac{x_2}{594 - 590.4} = \frac{x_3}{465.76 - 484}$$

$$\frac{x_1}{25.2} = \frac{x_2}{3.6} = \frac{x_3}{-18.24}$$

$$\frac{x_1}{7} = \frac{x_2}{1} = \frac{x_3}{-5.06}$$

$$\|X_2\| = \sqrt{7^2 + 1^2 + (-5.06)^2} = \sqrt{49 + 1 + 25.6036}$$

$$= \sqrt{75.6036}$$

$$= 8.69$$

$$= 8.7$$

$$X_2 = \begin{bmatrix} 7/8.7 \\ 1/8.7 \\ \frac{-5.06}{8.7} \end{bmatrix}$$

For $\lambda = 0$;

$$\begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$17x_1 + 22x_2 + 27x_3 = 0 \rightarrow ①$$

$$22x_1 + 29x_2 + 36x_3 = 0 \rightarrow ②$$

$$27x_1 + 36x_2 + 45x_3 = 0 \rightarrow ③$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 22 & 27 & 17 & 22 \\ 29 & 36 & 22 & 29 \end{array}$$

$$\frac{x_1}{792-783} = \frac{x_2}{594-612} = \frac{x_3}{493-484}$$

$$\frac{x_1}{9} = \frac{x_2}{-18} = \frac{x_3}{9}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\|x_3\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = \sqrt{5}$$

$$X_3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1.2}{\sqrt{8}} & 7/8.7 & 1/\sqrt{5} \\ \frac{1.6}{\sqrt{8}} & 1/8.7 & -2/\sqrt{5} \\ 2/\sqrt{8} & \frac{-5.06}{8.7} & 1/\sqrt{5} \end{bmatrix} \quad E = \begin{bmatrix} \sqrt{90.4} & 0 & 0 \\ 0 & \sqrt{0.6} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = U^T = \begin{bmatrix} \frac{1.2}{\sqrt{8}} & \frac{1.6}{\sqrt{8}} & 2/\sqrt{8} \\ 7/8.7 & 1/8.7 & \frac{-5.06}{8.7} \\ 1/\sqrt{5} & -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$A = [U][E][V]^T$$

$$= \begin{bmatrix} \frac{1.2}{\sqrt8} & \frac{7}{8.7} & \frac{1}{\sqrt5} \\ \frac{1.6}{\sqrt8} & \frac{1}{8.7} & -2/\sqrt5 \\ 2/\sqrt8 & \frac{-5.06}{8.7} & \frac{1}{\sqrt5} \end{bmatrix} \begin{bmatrix} 9.5 & 0 & 0 \\ 0 & 0.77 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1.2}{\sqrt8} & \frac{1.6}{\sqrt8} & 2/\sqrt8 \\ \frac{7}{8.7} & \frac{1}{8.7} & -5.06/8.7 \\ \frac{1}{\sqrt5} & \frac{-2}{\sqrt5} & 1/\sqrt5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11.4}{\sqrt8} & \frac{5.39}{8.7} & 0 \\ \frac{15.2}{\sqrt8} & \frac{0.77}{8.7} & 0 \\ \frac{19}{\sqrt8} & \frac{-3.9}{8.7} & 0 \end{bmatrix} \begin{bmatrix} \frac{1.2}{\sqrt8} & \frac{1.6}{\sqrt8} & \frac{2}{\sqrt8} \\ \frac{7}{8.7} & \frac{1}{8.7} & \frac{-5.06}{8.7} \\ \frac{1}{\sqrt5} & \frac{-2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13.68}{8}+\frac{37.73}{75.69} & \frac{18.24}{8}+\frac{5.39}{75.69} & \frac{22.8}{8}-\frac{27.2734}{75.69} \\ \frac{18.24}{8}+\frac{5.39}{75.69} & \frac{24.32}{8}+\frac{0.77}{75.69} & \frac{30.4}{8}-\frac{3.8962}{75.69} \\ \frac{22.8}{8}-\frac{27.3}{75.69} & \frac{11.4}{8}-\frac{3.9}{75.69} & \frac{38}{8}+\frac{19.734}{75.69} \end{bmatrix}$$

$$= \begin{bmatrix} 1.71+0.4984 & 2.28+0.07121 & 2.85-0.3603 \\ 2.28+0.07121 & 3.04+0.01017 & 3.8-0.05147 \\ 2.85-0.36068 & 1.425-0.0515 & 4.75+0.2607 \end{bmatrix}$$

$$A = \begin{bmatrix} 2.2084 & 2.35121 & 2.4897 \\ 2.35121 & 3.05017 & 3.75853 \\ 2.48932 & 1.3735 & 5.0107 \end{bmatrix}$$

Canonical Basis

Generalized Eigen vector:

→ can be find only for particular rank.

condition (i) $(A-\lambda I)^{r} X_r = 0$
and
(ii) $(A-\lambda I)^{r-1} X_r \neq 0$

Chain $X_r \to$ Rank r

$X_{r-1} = (A-\lambda I) X_r$

$X_{r-2} = (A-\lambda I) X_{r-1}$

$X_{r-3} = (A-\lambda I) X_{r-2}$

we are finding different generalized eigenvectors for the same matrix.

Q.
$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Step 1: Find the eigen values of the matrix.

$\lambda = 3, 3, 3, 3$

Step 2: Rank - multiplicity $= 4 - 4 = 0$.
N ↳ multip..

Rank $((A-\lambda I)^{r}) = 0$

Step 3: $A-\lambda I = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2.

$(A-\lambda I)^{2} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 0