1. Find the mean of the distribution whose MGF is $\frac{0.4e^t}{1-0.6e^t}$.

Solution:

$$M_x(t) = \frac{0.4e^t}{1-0.6e^t} \iff M_x(t) = \frac{pe^t}{1-qe^t}$$

$P = 0.4$

$$E(x) = \frac{1}{p} = \frac{1}{0.4} \times \frac{10}{10} = \frac{10}{4} = \frac{5}{2}.$$

2. Find the mean of a random variable X if

$$f(x) = \begin{cases} ke^{-x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

Soln:

$$f(x) = \begin{cases} ke^{-x} & ; x>0 \\ 0 & ; \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{0}^{\infty} ke^{-x}\,dx = 1$$

$$k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -k[0-1] = 1$$

$$\boxed{k=1}$$

$$f(x) = \begin{cases} e^{-x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$= e^{-x} \; ; x>0$$

$X \sim \exp(\lambda)$

$\lambda = 1$

$E(X) = \frac{1}{\lambda} = \frac{1}{1} = 1$

3. verify that the function p(x) defined by $p(x) = \frac{3}{4}\left(\frac{1}{4}\right)^x$; $x = 0, 1, 2, \ldots$ is a PMF of a discrete random variable x.

solution:

discrete, countable

$$P(x) = \frac{3}{4}\left(\frac{1}{4}\right)^x ; \; x = 0, 1, 2, \ldots \infty$$

$$\sum p(x) = \sum_{x=0}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^x = \frac{3}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right]$$

$$= \frac{3}{4}\left[\frac{1}{1 - 1/4}\right] = \frac{3}{4}\left[\frac{4}{3}\right] = 1$$

4. If a random variable x has a Moment Generating function $M_x(t) = \frac{2}{2-t}$, determine the variance of x.

solution:

$$M_x(t) = \frac{2}{2-t}$$

$$X \sim exp(\lambda)$$

$$\lambda = 2$$

$$\text{Variance}(X) = \frac{1}{\lambda^2} = \frac{1}{4}$$

5. A test engineer discovered that the cumulative distribution function of lifetime equipment (in years) is given by $F_X(x) = \begin{cases} 0 \; ; & x < 0 \\ 1 - e^{x/5} \; ; & 0 \leq x < \infty \end{cases}$ PDF ⇒. what is the expected lifetime of the equipment.

$\underbrace{\text{expected lifetime}}_{E(X)}$

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Solution:

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{x/5} & ; x \geq 0 \end{cases} \longrightarrow \text{differentiate}$$

$\leftarrow e^{ax} = e^{ax} \cdot a.$

Expected lifetime $= E(X)$

$$f(x) = \begin{cases} 0 & ; x < 0 \\ \frac{1}{5} e^{-1/5\, x} & ; x \geq 0 \end{cases}$$

$$\boxed{f(x) = \lambda e^{-\lambda x}}$$

$X \sim exp(\lambda)$

$\lambda = \frac{1}{5}$

$E(X) = \frac{1}{\lambda} = \frac{1}{1/5} = 5.$

6. Let X be a continuous random variable with P.d.f $f(x) = cx^2$, $0 < x < 1$, then find the value of 'c'.

Solution:

continuous random variable

$f(x) = cx^2 \; ; 0 < x < 1.$

to find c.,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$c\int_{0}^{\infty} x^2 dx = 1 \Rightarrow c\left[\frac{x^3}{3}\right]_0^1 = 1 \Rightarrow \frac{c}{3}[1-0] = 1$$

$$\frac{c}{3} = 1 \Rightarrow \boxed{c = 3}$$

7. Let X be a continuous random variable with p.d.f. $f(x) = \frac{1}{3} \cdot e^{-x/3}$, $x > 0$, then find its mean.

<u>Solution:</u>

$$f(x) = \frac{1}{3} e^{-\frac{1}{3}x} \quad ; x > 0 \qquad f(x) = \lambda e^{-\lambda x}$$

$$x \sim exp(\lambda)$$

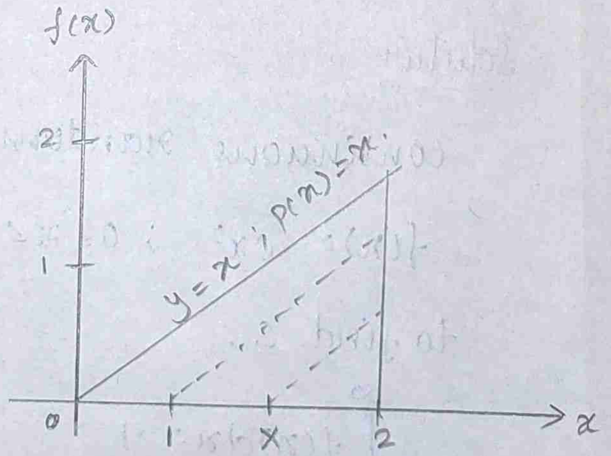
$$\lambda = \frac{1}{3}$$

$$E(X) = \frac{1}{\lambda} = \frac{1}{1/3} = 3$$

8. Let X be a continuous random variable with p.d.f

$$f(x) = \begin{cases} x & ; 0 \leq x \leq 1 \\ 2-x & ; 1 \leq x \leq 2 \end{cases}$$, then find its cumulative distribution function of X.

<u>solution:</u>



$$f(x) = \begin{cases} x ; & 0 \leq x \leq 1 \\ 2-x ; & 1 \leq x \leq 2 \end{cases}$$

$\rightarrow$ cummulative

$F(x) = p(X \leq n)$

for $x < 0$

$$F(x) = 0$$

for $0 \leq x \leq 1$

$$F(x) = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^x x\, dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

for $1 \leq x \leq 2$,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_0^1 x\, dx + \int_1^x (2-x)\, dx.$$

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$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$F(x) = 2x - \frac{x^2}{2} - 1$$

9. Let x be a discrete random variable with p.m.f $P(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$, where $x = 1, 2, 3, \ldots$ then find its mean, variance and $P(x>4/x>2)$.

(same as sum 3)

Solution:

Let x be a discrete random variable.

$$P(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1} \quad ; x = 1, 2, 3, \ldots$$

To find: $E(X) = ?$

$var(x) = ?$

$P(X>4/x>2) = ?$

$$\longrightarrow = P(X>2)$$

$x \sim geo(P)$

$$= P(X \geq 3)$$

$$= \sum_{x=3}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\right]$$

$$= \frac{3}{4}\left(\frac{1}{4}\right)^2\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots\right]$$

$$= \frac{3}{4} \times \frac{1}{16}\left[\frac{1}{1 - 1/4}\right] = \frac{3}{64}\left[\frac{4}{3}\right] = \frac{1}{16}.$$

10) The probability density function of continuous random variable x is given by

$$f(x) = \begin{cases} 2e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

. Find (i) E(X) and var(x)

(ii) $P(X \leq 3)$

(i) $E(X) = \frac{1}{\lambda} = \frac{1}{2}$

$E(X^2) = \frac{1}{\lambda^2} = \frac{1}{2^2} = \frac{1}{4}$

(ii) $P(X \leq 3)$

$$F(3) = \int_{-\infty}^{3} f(x)\,dx$$

$$= \int_{0}^{3} 2e^{-2x}\,dx = 2.\left[\frac{e^{-2x}}{-2}\right]_0^3 = -1\left[e^{-6} - 1\right]$$

$$= 1 - e^{-6} \Rightarrow \frac{e^6 - 1}{e^6}.$$

11) Assume that the length of a phone call in minutes is an exponential random variable x with parameter $\lambda = \frac{1}{10}$. If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait a) less than 5 minutes, and (ii) between 5 and 10 minutes.

solution:

$X \sim exp(\lambda)$

$f(x) = \lambda e^{-\lambda x} \; ; x > 0$

$\lambda = \frac{1}{10}.$

(i) $P(X < 5)$

(ii) $P(5 < x < 10)$

(i) $P(x < 5)$

$$= \int_0^5 \frac{1}{10} e^{-\frac{1}{10} \cdot x} \cdot dx$$

$$= \frac{1}{10} \left[\frac{e^{-\frac{1}{10} \cdot x}}{-\frac{1}{10}} \right]_0^5 \Rightarrow -[e^{-0.5} - 1]$$

$$= 1 - e^{-0.5}$$

(ii) $P(5 < x < 10)$

$$= \int_5^{10} \frac{1}{10} \cdot e^{-\frac{1}{10} \cdot x} \cdot dx$$

$$= \frac{1}{10} \left[\frac{e^{-\frac{1}{10} x}}{-\frac{1}{10}} \right]_5^{10} \Rightarrow -[e^{-1} - e^{-0.5}] = e^{-0.5} - e^{-1}.$$

12) you are taking a multiple choice quiz that consists of five questions. Each question has four possible answers, only one of which is correct. To complete the quiz, you randomly guess the answer to each question. Find the probability of guessing (a) exactly three answers correctly, (b) atleast three answers correctly and (c) less than three answers correctly.

Solution:

$n=5$ (no. of. trial)

$^nC_n = 1$

Probability of success $= \frac{1}{4}$

$^nC_{n-1} = n$

Probability of failure $= \frac{3}{4}$

$X \rightarrow$ number of correct answers

$X \sim$ Binomial (n, p)

$$P(x) = {}^nC_x p^x q^{n-x} \; ; \; x = 0, 1, 2, 3, \dots n$$

(i) $P(X=3)$

$$= 5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 = \frac{5 \times 4}{1 \times 2} \times \frac{1}{64} \times \frac{9}{16}$$

$$= \frac{5 \times 9}{2 \times 4 \times 64} = \frac{45}{512}$$

(ii) $P(X \geq 3)$

$$= P(X=3) + P(X=4) + P(X=5)$$

$$= 5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 + 5C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^1 + 5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$$

$$= 5C_3 \times \frac{1}{64} \times \frac{9}{16} + 5C_4 \times \frac{1}{256} \times \frac{3}{4} + 5C_5 \times \frac{1}{1024} \times 1$$

$$= \frac{5 \times 4}{1 \times 2} \times \frac{1}{64} \times \frac{9}{16} + \frac{5 \times 3}{256 \times 4} + 1 \times \frac{1}{1024}$$

$$= \frac{45}{512} + \frac{15}{1024} + \frac{1}{1024} = \frac{1 + 15 + 90}{1024} = \frac{106}{1024}$$

(iii) $P(X < 3) = 1 - P(X \geq 3)$

$$= 1 - \frac{106}{1024} = \frac{918}{1024}$$

13) A survey indicates that for trip to the supermarket, a shopper spends an average of 45 minutes with a standard deviation of 12 minutes in the store. The lengths of time spent in the store are normally distributed and are represented by the variable x. A shopper enters the store. a) Find the probability that the shopper will be in the store for each interval given below.
b) Interprete your answer if 200 shoppers enter the store. How many shoppers would you expect to be in the store for each interval of time listed below?

(i) Between 24 and 54 minutes

(ii) More than 39 minutes.

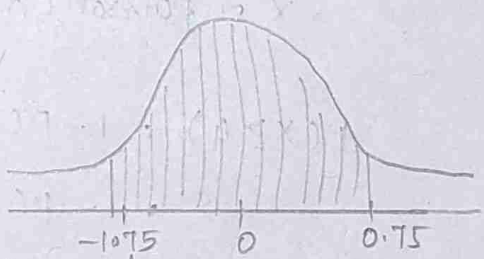
soln:

$\underbrace{\text{45 minutes}}_{\text{time spent in shop.}}$ average with standard of 12 minutes.

so, $\mu = 45$ min

$\sigma = 12$ min

$X \sim N(\mu, \sigma^2)$

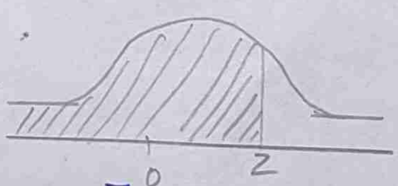


convert to standard normal

(i) $P(24 < x < 54) = P\left(\frac{24-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{54-\mu}{\sigma}\right)$

$= P\left(\frac{24-45}{12} < Z < \frac{54-45}{12}\right)$

$= P(-1.75 < Z < 0.75)$



$= P(Z < 0.75) - [1 - P(Z < 1.75)]$

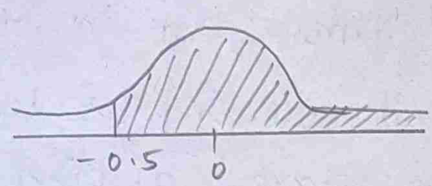
In table $= 0.7734 - [1 - 0.9599] = 0.7333.$

(ii) $P(X>39) = P\left(\frac{X-\mu}{\sigma} > \frac{39-\mu}{\sigma}\right)$

$= P\left(Z > \frac{39-45}{12}\right) = P(Z>-0.5)$

$= P(Z<0.5)$

$= 0.6915$



14) A book of 2021 pages contains 2021 mistakes. Find the probability that there are atleast 4 mistakes in randomly selected pages.

Solution:

$\lambda = \frac{2021}{2021} = 1$

$X \sim \text{poisson}(\lambda)$, $P(X) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$, $x = 0, 1, \ldots$

$P(X \geq 4) = 1 - P(X<4)$

$= 1 - P(X<4)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[\frac{e^{-1}(1)^{0}}{0!} + \frac{e^{-1}(1)^{1}}{1!} + \frac{e^{-1}(1)^{2}}{2!} + \frac{e^{-1}(1)^{3}}{3!}\right]$

$= 1 - e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right]$

$= 1 - e^{-1}!$

15) Suppose the cumulative distribution function of random variable X is

$$F(x) = \begin{cases} 0 & ; x < -2 \\ 0.25x + 0.5 & ; -2 \leq x \leq 2 \\ 1 & ; x \geq 2 \end{cases}$$

Determine $P(X<1.8)$, $P(X>-1.5)$, $P(X<-2)$.

Solution:

$$F(x) = \begin{cases} 0 \ ; & x < -2 \\ 0.25x + 0.5 \ ; & -2 \le x \le 2 \\ 1 \ ; & x \ge 2 \end{cases}$$

Another way

cd.f

$f(x) = \frac{1}{4} \ ; -2 \le x \le 2$

(or)

$F(x) = P(X \le x)$

i) $P(X<1.8) = P(X \le 1.8) = F(1.8) = 0.25(1.8)+0.5$
$= 0.95$

ii) $P(X>-1.5) = 1 - P(X \le -1.5)$
$= 1 - F(-1.5)$
$= 1 - [0.25(-1.5)+0.5]$
$= 0.875$

iii) $P(X<-2)$
$= P(X \le -2) = F(-2) = 0$.

16) A surgical technique is performed on seven patients. The results say that there is a 70 %. chance of success. Find the probability that the surgery is successful for exactly five patients, atleast five patients and less than five patients.

Solution:-

$X \sim B(n, p)$

$n = 7, \ p = 0.7, \ q = 0.3$

$P(x) = nC_x p^x q^{n-x}; \ x = 0, 1, 2, \ldots n$

$P(X=5) = 7C_5 (0.7)^5 (0.3)^2 = \frac{7 \times 6}{1 \times 2} (0.7)^5 (0.3)^2 = 0.3176$

$P(X \ge 5) = P(X=5) + P(X=6) + P(X=7)$
$= 0.3176 + 7C_6 (0.7)^6 (0.3)^1 + 7C_7 (0.7)^7 (0.3)^0$
$= 0.3176 + 7(0.7)^6 (0.3)^1 + 1(0.7)^7 (0.3)^0$

$= 0.3176 + 7(0.7)^6(0.3)^1 + (0.7)^7(0.3)^0$

$= 0.3176 + 7(0.3)(0.1176) + (0.0823)$

$= 0.64686.$

$P(X<5) = 1 - P(X \geq 5)$

$= 1 - 0.64686$

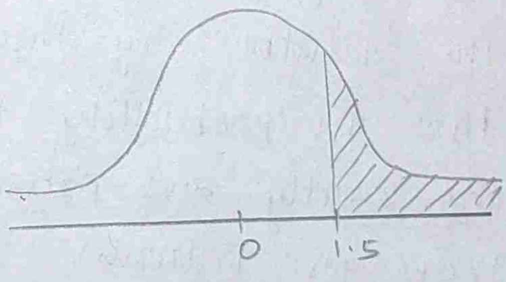
$= 0.35314$

17) The amounts a soft drink machine is designed to dispense for each drink are normally distributed, with mean of 12 fluid ounces and a standard deviation of 0.2 fluid ounces. A drink is randomly selected. Find the probability that the drink is less than 11.9 fluid ounces, between 11.8 and 11.9 fluid ounces, and more than 12.3 fluid ounces.

Solution:

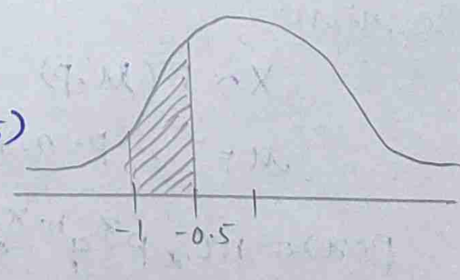
$\mu = 12, \ \sigma = 0.2$

$X \sim N(\mu, \sigma^2)$



$P(X<11.9) = P\left(\frac{X-\mu}{\sigma} < \frac{11.9-\mu}{\sigma}\right)$

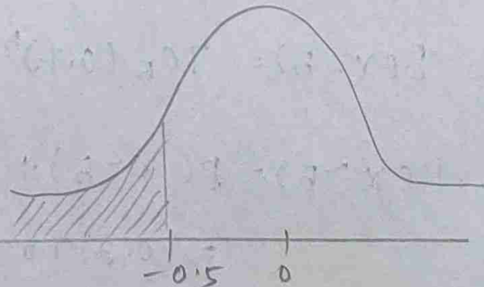
$= P\left(z < \frac{11.9-12}{0.2}\right)$



$= P(z<-0.5) = 1 - P(z<0.5)$

$= 1 - 0.6915$

$= 0.3085$



$$P(11.8 < X < 11.9) = P\left(\frac{11.8-12}{0.2} < z < \frac{11.9-12}{0.2}\right)$$

$$= P(-1 < z < -0.5) = P(z<1) - P(z<0.5)$$

$$= 0.8413 - 0.6915$$

$$= 0.1498$$

$$P(X > 12.3) = P\left(z > \frac{12.3-12}{0.2}\right) = P(z > 1.5)$$

$$= 1 - P(z < 1.5) = 1 - 0.9332 = 0.0668$$

18) Suppose that the life of an industrial lamp (in thousands of hours) is exponentially distributed with mean life of 3000 hours. Find the probability that

a) the lamp will last more than mean life

b) the lamp will last between 2000 and 3000 hours

c) the lamp will last another 1000 hours given that it has already lasted for 2500 hours.

Solution:

$X \sim exp(\lambda)$

Mean = 3000 hours ; $\frac{1}{\lambda} = \frac{1}{3}$

$p(x) = \lambda e^{-\lambda x}$

(i) $P(X>3) = \int_3^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_3^\infty = -[0 - e^{-1}] = e^{-1}$.

(ii) $P(2 \le X \le 3) = \int_2^3 \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}x}\right]_2^3 = -[e^{-1} - e^{-0.667}]$

$= e^{-0.667} - e^{-1}$.

(iii) $P(X > 3.5 / X > 2.5)$

$$= P(X>1) = \int_1^{\infty} \frac{1}{3} e^{-\frac{1}{3}x} \, dx$$

$P(X>s+t / x>s)$ with >1000 above t and $\downarrow$ 2500 below s

$$= \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_1^{\infty} = -\left[0 - e^{-0.333}\right]$$

$$= e^{-0.333}$$

19) The cumulative distribution function of a continuous random variable X is given by $F(x) = \begin{cases} 1 - e^{-2x} ; & x \geq 0 \\ 0 ; & x < 0 \end{cases}$.
Find the p.d.f of X, $P(X>2)$ and variance of X.

<u>solution</u>:

$$F(x) = \begin{cases} 1 - e^{-2x} ; & x \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$\longrightarrow \lambda e^{-\lambda x}$$

$$f(x) = \begin{cases} 2e^{-2x} ; & x \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$X \sim exp(\lambda), \lambda = 2$$

$$P(X>2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - [1 - e^{-4}] = e^{-4}$$

$$Var(X) = \frac{1}{\lambda^2} = \frac{1}{4}$$

20) Let X be a continuous random variable whose pdf is $f(x) = \begin{cases} \frac{x^3}{4}, & 0 < x < c \\ 0, & \text{otherwise} \end{cases}$. What is the value of 'c' that makes $f(x)$ a valid probability density function?

Solution:

$$f(x) = \begin{cases} \frac{x^3}{4} ; & 0 < x < c \\ 0 ; & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\frac{1}{4}\int_0^c x^3 dx = 1 \Rightarrow \frac{1}{4}\left[\frac{x^4}{4}\right]_0^c = 1 \Rightarrow \frac{1}{16}\left[c^4 - 0\right] = 1$$

$$c^4 = 16 = 2^4.$$

$$\boxed{c = 2}$$

21) If x is uniformly distributed in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Find the pdf of $y = \tan x$.

Solution:-

$$x \sim \text{unif}\left(-\frac{\pi}{2}, \pi/2\right)$$

$$f(x) = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2} - (-\pi/2)} = \frac{1}{\pi}.$$

22) Assume that x is continuous random variable with the following pdf

$$f(x) = \begin{cases} 0 ; & x < 0.5 \\ ke^{-2(x-0.5)} ; & x \geq 0.5 \end{cases}$$

Find the value of k and cdf of x. Also $P(X \leq 1.5)$, $P(1.2 < X < 2.4)$.

Solution:

$$f(x) = \begin{cases} 0 ; & x < 0.5 \\ ke^{-2(x-0.5)} ; & x \geq 0.5 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$K\int_{0.5}^{\infty} e^{-2x} \cdot e\, dx = 1 \Rightarrow Ke\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = 1$$

$$\frac{Ke}{-2}\left[0 - e^{-1}\right] = 1 \Rightarrow \frac{K}{2} = 1 \Rightarrow \boxed{K=2}$$

For $x < 0.5$,

$$F(x) = 0.$$

For $x \geq 0.5$,

$$F(x) = \int_{0.5}^{x} 2e^{-2x} \cdot e\, dx = 2e\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{x}$$

$$= -e\left[e^{-2x} - e^{-1}\right]$$

$$= 1 - e^{-2x+1}$$

(i) $P(X \leq 1.5) = F(1.5) = 1 - e^{-2(1.5)+1} = 1 - e^{-2}$

(ii) $P(1.2 < x < 2.4) = F(2.4) - F(1.2) = \left[1 - e^{-2(2.4)+1}\right] - \left[1 - e^{-2(1.2)+1}\right]$

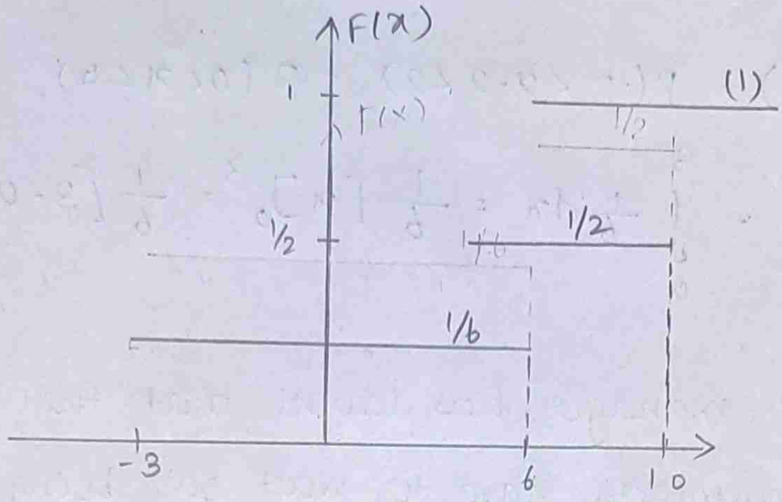
$$= e^{-1.4} - e^{-3.8}$$

23) Let X be a discrete random variable whose cdf is given by

$$F(x) = \begin{cases} 0, & \text{if } x < -3 \\ 1/6, & \text{if } -3 \leq x < 6 \\ 1/2, & \text{if } 6 \leq x < 10 \\ 1, & \text{if } x \geq 10 \end{cases}$$

Find $P(X \leq 4)$, $P(-5 < X \leq 4)$ and probability mass function of X.

solution:



$\frac{3}{3} \times \frac{1}{2} - \frac{1}{6} = \frac{2}{6}$

$P(X \leq 4) = F(4) = \frac{1}{6}$

$P(-5 < x \leq 4) = F(4) - F(-5) = \frac{1}{6} - 0 = \frac{1}{6}$

x	-3	6	10
$P(x)$	$1/6$	$1/3$	$1/2$

24) A random variable $X \sim \text{unif}(-3,3)$. compute

(i) $P(|X| < 2)$, (ii) $E(X)$ (iii) $P(|X-2| < 2)$

solution:

$X \sim \text{unif}(-3,3)$.

$f(x) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$; $-3 < x < 3$

$P(|X| < 2) = P(-2 < x < 2) = \int_{-2}^{2} \frac{1}{6}\, dx = \frac{1}{6}[x]_{-2}^{2}$

$= \frac{1}{6}[2-(-2)] = 2/3$.

$E(X) = \frac{a+b}{2} = \frac{-3+3}{2} = 0$

$$P(|X-2|<2) = P(-2<x-2<2) = P(0<x<4)$$

$$= \int_0^3 \frac{1}{6}\,dx = \frac{1}{6}[x]_0^3 = \frac{1}{6}[3-0] = \frac{3}{6} = \frac{1}{2}$$

25) A bank manager has learnt that the length of time the customers have to wait for being attended by the teller is normally distributed with mean time of 5 minutes (μ) and standard deviation of 0.8 mins (σ). Find the probability that a customer has to wait

(i) for more than 3.5 minutes

(ii) between 3.4 minutes and 6.2 minutes.

<u>solution:</u>

$$X \sim N(\mu, \sigma^2)$$

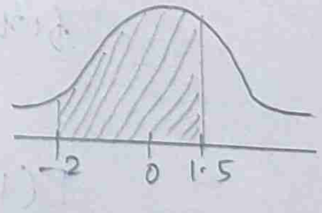
$$\mu = 5 \quad , \sigma = 0.8$$

$$P(X>3.5) = P\left(Z> \frac{3.5-5}{0.8}\right) = P(Z>-1.87)$$

$$= P(Z<1.87)$$

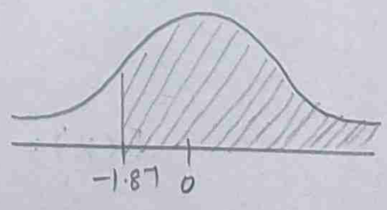
$$= 0.9693$$

$$P(3.4<x<6.2) = P\left(\frac{3.4-5}{0.8} < 5 < \frac{6.2-5}{0.8}\right)$$



$$= P(-2<Z<1.5)$$

$$= P(Z<1.5) - [1-P(Z<2)]$$



$$= 0.9332 - 1 + 0.9772$$

$$= 0.9104$$

26) Find the M.G.F of the random variable 'x' whose Pdf is $f(x) = \begin{cases} \frac{x}{4} e^{-x/2} & ; \text{ if } x > 0 \\ 0 & ; \text{ otherwise} \end{cases}$

Solution:

$$f(x) = \begin{cases} \frac{x}{4} e^{-x/2} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$\int u\,dv = uv - \int v\,du$

$= uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \frac{x}{4} e^{-x/2}\, dx$$

$$= \frac{1}{4} \int_0^\infty x e^{-(1/2 - t)x}\, dx \Rightarrow \frac{1}{4}\left[x\left(\frac{e^{-(1/2-t)x}}{-(\frac{1}{2}-t)}\right) - 1\left(\frac{e^{-(1/2-t)x}}{(1/2-t)^2}\right)\right]_0^\infty$$

$$= \frac{1}{4}\left[0 - \left\{0 - \frac{1}{(\frac{1}{2}-t)^2}\right\}\right]$$

$$= \frac{1}{4}\left[\frac{4}{(1-2t)^2}\right] = \frac{1}{(1-2t)^2}$$

$$M_x(t) = \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

$$(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$M_x(t) = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + 5(2t)^4 + \cdots$$

$$= 1 + 4t + 12t^2 + 32t^3 + 80t^4 + \cdots$$

$$= 1 + \underbrace{\frac{4t^1}{1!}}_{E(X)} + \underbrace{\frac{12t^2}{2!} \times 2!}_{E(X2)} + \underbrace{\frac{32t^3}{3!} \times 3!}_{E(X^3)} + \frac{80t^4}{4!} \times 4!$$

$E(X) = 4$

$E(X^2) = 24$

$E(X^3) = 192$

$E(X^4) = 1920$

27) In a certain town, 20% samples of the population are literate. Assume that 200 investigator each take samples of 10 individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literate in the sample?

Solution:

$n = 10$; $p = 0.2$; $q = 0.8$;

$p(x) = nC_x p^x q^{n-x}$; $x = 0,1,2,\ldots$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 10C_0 (0.2)^0 (0.8)^{10} + 10C_1 (0.2)^1 (0.8)^9 + 10C_2 (0.2)^2 (0.8)^8$

$+ 10C_3 (0.2)^3 (0.8)^7$

$= (0.8)^{10} + 10(0.2)(0.8)^9 + 45(0.2)(0.8)^8 +$

$120 (0.2)^3 (0.8)^7$

$= 2.087 \times 200$

$= 417.4$

28) The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$. what is the probability that the repair time exceeds 2 hours? what is the conditional probability that the repair time takes atleast 10 hours given that its duration exceeds 9 hours?

Solution:

$X \sim exp(\lambda)$

$\lambda = \frac{1}{2}$ $\quad f(x) = \lambda e^{-\lambda x}$; $x > 0$

$$P(X>2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_2^\infty = -[0 - e^{-1}] = e^{-1}.$$

$$P(X>10 \mid X>9) = P(X>1)$$

$$= \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx \Rightarrow \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_1^\infty$$

$$= -[0 - e^{-0.5}] = e^{-0.5}$$

29) A candidate applying for driving licence has the probability of 0.8 in passing the road test in a given trial. what is the probability that he will pass the test (i) on the fourth trial, (ii) in less than four trials.

<u>Solution:</u>

$X \sim$ Geometric (P)

$$p(x) = pq^{x-1} \quad ; \quad x = 1, 2, \ldots$$

$$P = 0.8 \quad ; \quad q = 0.2$$

(i) $P(X=4) = (0.8)(0.2)^3 = 0.0064$

(ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$$= 0.8 + (0.2)(0.8) + (0.2)^2(0.8)$$

$$= 0.8[1 + 0.2 + 0.04]$$

$$= 0.8[1.24]$$

$$= 0.992$$

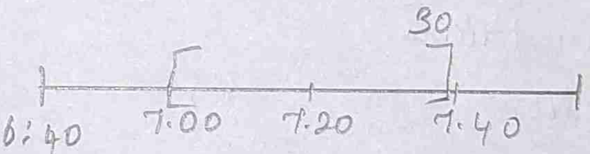
30) A bus arrives every 20 minutes, at a specified stop, beggining at 6.40 am. and contineing until 8.40 a.m. A passenger arrives randomly between 7.00 am to 7.30 a.m. what is the probability that the passenger has to wait for more than 5 mins for a bus?

Solution:

$X \sim$ uniform ; $0 \le x \le 30$ (7.00 to 7.30)

$$f(x) = \frac{1}{b-a} = \frac{1}{30-0} = \frac{1}{30}$$

P(a passenger has to wait for more than 5 mins) $= P(0 < x < 15) + P(20 < x < 30)$



$$= \int_0^{15} \frac{1}{30}\,dx + \int_{20}^{30} \frac{1}{30}\,dx \Rightarrow \frac{1}{3}[x]_0^{15} + \frac{1}{3}[x]_{20}^{30}$$

$$= \frac{1}{2} + \frac{1}{3} = 5/6.$$

31) A system has a component whose time to failure is exponentially distributed with parameter $\lambda = 1/6$. If 6 such components are installed in different systems, what is the probability that at least 2 are still working at the end of 9 year?

Solution:

$x \sim exp(\lambda)$

$\lambda = 1/6 \qquad f(x) = \lambda e^{-\lambda x} \; ; \; x > 0 \; ; \; P(X \geq 9)$

$n = 6$

$X \sim B(n, p), \; P = P(X \geq 9)$

$$P = P(X \geq 9) = \int_9^\infty \frac{1}{6} e^{-1/6 x} dx$$

$$= \frac{1}{6}\left[\frac{e^{-1/6 x}}{-1/6}\right]_9^\infty \Rightarrow -[0 - e^{-1.5}]$$

$$= e^{-1.5} \Rightarrow 0.2231$$

so, $q = 0.7769$

$$P(Y \geq 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [6C_0 (0.2231)^0 (0.7769)^6 + 6C_1 (0.2231)^1 (0.7769)^5]$$

$$= 1 - [(1)(0.7769)^6 + 6(0.2231)(0.7769)^5]$$

$$= 1 - [0.21988 + 0.378857]$$

$$= 1 - [0.5987]$$

$$= 0.40126$$

32) The daily consumption of milk in excess of 20,000 litres is approximately distributed as Gamma variable with parameter $K = 2$, $\lambda = \frac{1}{10000}$. If the city has a daily stock of 30,000 litres on a given day, find the probability that the stock is insufficient.

solution:

$K=2;\ \lambda = \frac{1}{10,000}$

$Y = X - 20,000$

$Y \sim Gamma(K,\lambda)$; $f(x) = \frac{\lambda^{K} x^{K-1} e^{-\lambda x}}{\Gamma(K)}$, $x>0$

P (stock is insufficient) = P (X > 30,000)

= P (X − 20,000 > 30,000 − 20,000)

= P(Y > 10,000)

use formula,

$= \frac{\left(\frac{1}{10,000}\right)^2 x \cdot e^{-\frac{1}{10,000}x}}{1}$

$= \int_{10,000}^{\infty} \left(\frac{1}{10,000}\right)^2 \underset{u}{x}\ \underset{dv}{e^{-\frac{1}{10,000}x}}\, dx$

$= \left(\frac{1}{10,000}\right)^2 \left[uv - u'v_1 + u''v_2 - \cdots\right]$

$= \left(\frac{1}{10,000}\right)^2 \left[x\left(\frac{e^{-\frac{1}{10,000}x}}{-\frac{1}{10,000}}\right) - 1\left(\frac{e^{-\frac{1}{10,000}x}}{\left(\frac{1}{10,000}\right)^2}\right)\right]^{x}$

$= \left(\frac{1}{10,000}\right)^2 \left[0 - \{-(10,000)^2 e^{-1} - (10,000)^2 e^{-1}\right]$

$= \frac{1}{(10,000)^2}\left[2e^{-1}(10,000)^2\right] = 2e^{-1}$.

33) In a newly constructed township, 2000 electrical lamps are installed with an average the life of 1000 burning hours & standard deviation of 200 hours. The normal approximation is a close approximation to this case. find (i) the number of lamps expected to fail during the first 700 hours, (ii) in what period of burning hours 10 % of the lamps fail.

Soln

$X \sim N(\mu, \sigma^2)$

$\mu = 1000$, $\sigma = 200$

10% of 2000 = 200

i) $P(X < 700)$ ii) $P(X < K)$

i) $P(X < 700)$

$$= P\left(Z < \frac{700-1000}{200}\right) = P(Z < 1.5)$$

$$= P(Z > 1.5) = 1 - P(Z < 1.5)$$

$$= 1 - 0.9332$$

$$= 0.0668$$

The number of lamps fails in first 700 hours

$$= 0.0668 \times 2000$$

$$\simeq 1.33.$$

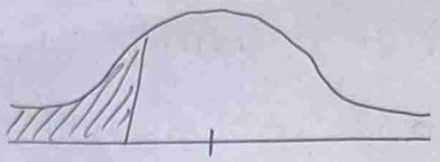
ii) $P(X < K) = \frac{10}{100}$

$$P\left(\frac{X-\mu}{\sigma} < \frac{K-\mu}{\sigma}\right) = 0.1$$

$$P\left(Z < \frac{K-1000}{200}\right) = 0.1$$

$$P\left(Z > \frac{1000-K}{200}\right) = 0.1$$

$1 - P\left(z < \frac{1000-k}{200}\right) = 0.1$



$0.9 = P\left(z < \frac{1000-k}{200}\right)$

$\Rightarrow \frac{1000-k}{200} = 1.28$

$1000 - k = 1.28 \times 200$

$1000 - 256 = k$

$\boxed{k = 744}$

34) Given the random variable 'x' with the probability distribution

$f(x) = \begin{cases} 2x & ; 0 < x < 1 \\ 0 & ; \text{otherwise.} \end{cases}$ Find the probability distribution of $y = 8x^3$.

'x' $f(x) = \begin{cases} 2x & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$

$y = 8x^3$

$f(y) = f(x) \left| \frac{dx}{dy} \right|$

$y = 8x^3$

$x^3 = y/8$

$\boxed{x = \frac{1}{2} y^{1/3}}$

$\frac{dx}{dy} = \frac{1}{2} \cdot \frac{1}{3} y^{\frac{1}{3}-1}$

$= \frac{1}{6} y^{\frac{1}{3}-1}$

$= \frac{1}{6} y^{-2/3}$

Range of y

$0 < x < 1$

$0 < \frac{1}{2} y^{1/3} < 1$

$0 < y^{1/3} < 2$

$0 < y < 8$

$$f(y) = 2x \cdot \frac{1}{6} y^{-2/3}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot y^{1/2} \cdot y^{-2/3}$$

$$= \frac{1}{6} y^{-1/3} \Rightarrow \frac{1}{6y^{1/3}}$$

35) If X is uniformly distributed in $\left(-\frac{\pi}{2}, \pi/2\right)$. Find the probability density function of $y = \tan x$.

Solution:

$$X \sim \text{unif}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$f(x) = \frac{1}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{1}{\pi} \quad ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$y = \tan x$$

$$x = \tan^{-1} y$$

$$\frac{dx}{dy} = \frac{1}{1+y^2}$$

$$f(y) = f(x) \left|\frac{dx}{dy}\right| \Rightarrow \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

Range of y,

$$-\frac{\pi}{2} < x < \pi/2$$

$$\tan(-\pi/2) < \tan x < \tan \pi/2$$

$$-\infty < y < \infty.$$

36) A random variable x has p.m.f

x	1	2	3	4	5	6	7
$P(x)$	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find the value of K, cdf of X and if $P(X \le K) > \frac{1}{2}$, then find the minimum value of K.

Solution:

$$\sum P(X)=1$$

$$\sum_{x=1}^{7} 10K^2+9K=1$$

$$10K^2+9K=1$$

$$10K^2+10K-K-1=0$$

$$10K(K+1)-1(K+1)=0$$

$$(10K-1)(K+1)=0$$

$$K=\frac{1}{10}, -1$$

-1 not possible

$$K=1/10$$

c.d.f of x is

$F(x)=P(X \le x)$

$F(1)=P(X \le 1)=1/10$

$F(2)=P(X \le 2)=K+2K=3K=3/10$

$F(3)=P(X \le 3)=5K=1/2$

$F(4)=8K=8/10.$

$F(5)=8K+K^2=8/10+1/100=81/100$

$F(6)=8K+3K^2=8/10+3/100=83/100$

$F(7)=1$

$P(X \le K) > 1/2$

$P(X \le 1) = \frac{1}{10} < \frac{1}{2}$

$P(X \le 2) = \frac{3}{10} \ngtr 1/2$

$P(X \le 3) = \frac{5}{10} \ngtr \frac{1}{2}$

$P(X \le 4) = \frac{8}{10} > \frac{1}{2}$

$P(X \le 5) = \frac{81}{100} > 1/2$

$P(X \le 6) = \frac{83}{100} \ngtr 1/2$

$P(X \le 7) = 1 > 1/2$

$\therefore$ The min value of $K = 4$.