# Determine the temperature distribution in a Semi-infinite medium $x >= 0$ when the $x = 0$ is maintained at 0 temperature and the initial temperature distribution is $f(x)$. Determine the temperature distribution. ①

Sol$^n$ The given problem is described by PDE wave equation

that is $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ $0 < x < \infty$.

The boundary condition are : $u(0,t) = 0$ where $t > 0$

The above problem could be solved only through wave equation of PDE

Initial Conditions $u(x,0) = f(x)$ $0 < x < 0$

$$K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Taking Fourier Sine transform on both side we get

$$K \, F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = F_s\left[\frac{\partial u}{\partial t}\right] \quad \because K \text{ is Constant} \quad \text{———— ①}$$

By using Fourier Sine transform property

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\, \alpha u(0,t) - \alpha^2 u_s(\alpha,t) \text{ ———— } (**)$$

$$F_s\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt} u_s(\alpha,t) \text{ ———— } (***)$$

Substituting (**) and (***) in ① we get

$$K\left[\sqrt{2/\pi}\, \underbrace{\alpha(u(0,t))}_{0} - \alpha^2 u_s(\alpha,t)\right] = \frac{d}{dt} u_s(\alpha,t)$$

By boundary condition

$$-K\alpha^2 u_s(\alpha,t) = \frac{d}{dt} u_s(\alpha,t)$$

$$\frac{d}{dt} u_s(\alpha,t) + K\alpha^2 u_s(\alpha,t) = 0$$

we got an ordinary differential equation in terms of $u_s(\alpha,t)$

The auxillary equation is

$$m + K\alpha^2 = 0$$

$$m = -K\alpha^2$$

$\therefore$ Complementary function is $Ae^{-K\alpha^2 t}$

The particular integral is zero.

The general ~~equation~~ solution is

$$u_s(\alpha,t) = \text{Complementary function} + \text{particular integral}$$

$$u_s(\alpha,t) = Ae^{-K\alpha^2 t} + 0 \quad \text{———} \ (2)$$

Initial Condition : $u(x,0) = f(x)$

$$u_s(\alpha,0) = F(\alpha)$$

At $t=0$ $\quad u_s(\alpha,0) = Ae^0 = A$ from (2) and initial conditions

$$\boxed{\therefore F(\alpha) = A}$$

$$u_s(\alpha,t) = F(\alpha) \cdot e^{-K\alpha^2 t}$$

Taking inverse Fourier transform, we get

$$u_x(\alpha,t) = \sqrt{2/\pi} \int_0^{\infty} F(\alpha)\, e^{-K\alpha^2 t} \sin\alpha x \, dx$$

Hence the equation

# Solve using Fourier transform method $u_{xx} = a^{-2} u_t, \ 0 < x < \alpha, \ t > 0.$



Sol$^n$

Boundary value conditions : $u(0,t) = 0, \ t > 0$

Initial conditions : $u(x,0) = f(x), \ 0 < x < \alpha$

$$\frac{\partial^2 u}{\partial x^2} = a^{-2} \frac{\partial u}{\partial t}$$

Taking Fourier Cosine on both sides we get

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = a^{-2} F_c\left[\frac{\partial u}{\partial t}\right]$$

$$-\sqrt{2/\pi}\, \alpha \underbrace{u_x(0,t)}_{0} - \alpha^2 u_c(\alpha,t) = a^{-2}\frac{d}{dt} u_c(\alpha,t)$$

by boundary conditions

$$-\alpha^2 u_c(\alpha,t) = \frac{1}{a^2}\frac{d}{dt} u_c(\alpha,t)$$

$$\frac{d}{dt} u_c(\alpha,t) + a^2\alpha^2 u_c(\alpha,t) = 0$$

$$(D + a^2\alpha^2) u_c(\alpha,t) = 0$$

The auxillary equation is $m + a^2\alpha^2 = 0$

$$m = -a^2\alpha^2$$

$$CF = Ae^{-a^2\alpha^2 t}, \quad PI = 0$$

The general sol$^n$

$$u_c(\alpha,t) = CF + PI$$

$$u_c(\alpha,t) = Ae^{-a^2\alpha^2 t} \quad \text{——— (1)}$$

Initial Condition $u(x,0) = f(x)$

$$u_c(\alpha,0) = F(\alpha)$$

At $t=0$

$$① \Rightarrow u_c(\alpha,0) = Ae^0$$

$$\boxed{F(\alpha) = A}$$

$$\therefore u_c(\alpha,t) = F(\alpha)\, e^{-a^2\alpha^2 t}$$

Taking inverse Fourier Cosine transform we get

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\alpha)\cdot e^{-\alpha^2 a^2 t} \cos\alpha x \, dx.$$

Hence the equation

# Compute the displacement $u(x,t)$ and an infinite string using the method of Fourier transform given that the string is initially at rest and initial displacement is $f(x)$; $-\infty < x < \infty$

Soln

Displacement of an infinite string is generated by PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Initial Condition: $u_t(x,0) = 0$, $-\infty < x < \infty$

$$u(x,0) = f(x)$$

Taking Fourier transform on both sides we get

$$F\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2 F\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\frac{d^2}{dt} u(\alpha,t) = c^2 \left[(-i\alpha)^2 u(\alpha,t)\right]$$

$$\frac{d^2}{dt} u(\alpha,t) = -c^2\alpha^2 u(\alpha,t)$$

$$\frac{d^2}{dt} u(\alpha,t) + c^2\alpha^2 u(\alpha,t) = 0$$

$$\left[D^2 + c^2\alpha^2\right] u(\alpha,t) = 0$$

The Auxillary equation is $m^2 + c^2\alpha^2 = 0$

$$m^2 = -c^2\alpha^2$$

$$m = \pm\sqrt{-c^2\alpha^2}$$

$$m = \pm ic\alpha$$

$$CF = e^{0t}\left[A\cos(c\alpha,t) + B\sin(c\alpha,t)\right]$$

$$CF = A\cos(c\alpha,t) + B\sin(c\alpha,t)$$

$$PI = 0$$

The general solⁿ is $u(\alpha,t) = CF + PI$

$$u(\alpha,t) = A\cos(\alpha c,t) + B\sin(\alpha ct) \quad \text{——} \; (1)$$

Initial condition: $u(x,0) = f(x)$

$$u(\alpha,0) = F(\alpha)$$

when $t=0$ (1) $\Rightarrow$ $u(\alpha,0) = A\cos 0 + B\sin 0$

$$F(\alpha) = A$$

Initial condition $u_t(x,0) \Rightarrow u_t(\alpha,0) = 0$

Diff. eq. (1) w.r.to 't'

$$u_t(\alpha,t) = -A\alpha c \sin(\alpha ct) + B\alpha c(\cos(\alpha ct)) \quad \text{--- (2)}$$

when $t=0$ (2) $\Rightarrow u_t(\alpha,0) = A\alpha c \sin 0 + B\alpha c \cos 0$

$$0 = B\alpha c$$

$$\boxed{B=0}$$

$$\therefore u(\alpha,t) = F(\alpha)\cos(\alpha ct)$$

Taking Inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\cdot\cos(\alpha ct)\cdot e^{i\alpha x}\cdot d\alpha$$

# Solve the boundary value problem (BVP) in half plane $y>0$, described by the PDE.

~~Sol~~

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0$$

$$u(x,0) = f(x); \quad -\infty < x < \infty$$

$u$ is bounded on $y \to \infty$, $u$ and $\frac{\partial u}{\partial x}$ vanish as $|x| \to \infty$

Soln

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Taking Fourier transform on both sides we get

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] + F\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$(-i\alpha)^2 u(\alpha,y) + \frac{d^2}{dy^2} u(\alpha,y) = 0$$

$$\frac{d^2}{dy^2} u(\alpha,y) - \alpha^2 u(\alpha,y) = 0 \Rightarrow (D^2 - \alpha^2)\, u(\alpha,y) = 0$$

The A.E is

$$m^2 = +\alpha^2$$

$$m^2 = \alpha^2$$

$$m = \pm \alpha$$

$$CF = Ae^{\alpha y} + Be^{-\alpha y}, \quad PI = 0$$

The general solution is

$$u(\alpha, y) = Ae^{\alpha y} + Be^{-\alpha y}$$

$|x| = \{ x, \; x \geq 0$

$$= (A+B)\, e^{-|\alpha| y}$$

where $(A+B)$ is constant

$$= e^{-|\alpha| y} \quad \text{——— (1)}$$

Bandary condition $u(x,0) = f(x)$, $u(\alpha,0) = F(\alpha)$

when $y=0$ (1) $\Rightarrow$ $u(\alpha, y) = \text{constant} \cdot e^{-|\alpha| 0}$

$$F(\alpha) = \text{constant}$$

$$\therefore u(\alpha, y) = F(\alpha) \cdot e^{-|\alpha| y}$$

Apply inverse Farrier transformation, we get

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-|\alpha| y}$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{i\xi\alpha}\, d\xi$$

$$\therefore u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{i\xi\alpha}\, d\xi\; e^{-|\alpha| y} \cdot \cancel{dx}\, e^{-i\alpha x}\, d\alpha$$

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} e^{i\xi x} \cdot e^{-|\alpha| y} \cdot e^{-i\alpha x}\, d\alpha$$

Consider $\int_{-\infty}^{\infty} e^{-|\alpha| y} e^{i\xi\alpha} e^{-i\alpha x} \, d\alpha = \int_{-\infty}^{0} e^{+\alpha y} \cdot e^{i\xi\cdot\alpha} \cdot e^{-i\alpha x} \, d\alpha$

$$+ \int_{0}^{\infty} e^{-\alpha y} e^{i\xi\alpha} e^{-i\alpha x} \, d\alpha$$

$$= \int_{0}^{\infty} e^{-\alpha (y + i(x-\xi))} d\alpha + \int_{-\infty}^{0} e^{\alpha[y + i(\xi - x))} d\alpha$$

$$= \left[ \frac{e^{-\alpha [y + i(x-\xi)]}}{-[y + i(x-\xi)]} \right]_{0}^{\infty} + \left[ \frac{e^{\alpha [y + i(\xi + x)]}}{y + i(\xi - x)} \right]_{-\infty}^{0}$$

$$= \left[ 0 + \frac{1}{y + i(x-\xi)} \right] + \left[ \frac{1}{y + i(\xi - x)} - 0 \right]$$

$$= \frac{1}{y + i(\xi - x)} + \frac{1}{y + i(\xi - x)} = \frac{2y}{y^2 + (\xi - x)^2}$$

$$\therefore u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \cdot \frac{2y}{y^2 + (\xi - x)^2} \, d\xi.$$

# Using Fourier transform solve the PDE

$$u_{xx} + u_{yy} = 0, \quad -\alpha < x < \alpha, \quad y > 0$$ (9)

BC: $u_y(x,0) = F(x); \; -\alpha < x < \alpha$

$u$ is bounded on $y \to \alpha$ $\quad u$ and $\frac{\partial u}{\partial x}$ both vanish as $|x| \to \alpha$

**Sol$^n$**

Let us define a function

$$\phi(x,y) = u_y(x,y) = \frac{\partial}{\partial y} u(x,y)$$

$$\phi_{xx} + \phi_{yy} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial y} u(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\partial}{\partial y}[u(x,y)\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial}{\partial y}(0)$$

$$= \phi_{xx} + \phi_{yy} = 0$$

$\therefore$ PDE: $\nabla^2 \phi = 0$ (or) $\phi_{xx} + \phi_{yy} = 0$

BC: $\phi(x,0) = F(x) \; ; \; -\alpha < x < \alpha$

By using the result

$$\phi(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{1}{y^2 + (\xi - x)^2} \, d\xi$$

$$u_y(x,y) = \phi(x,y)$$

Then $$u(x,y) = \int \left[ \frac{y}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{1}{y^2 + (\xi - x)^2} \, d\xi \right] dy$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int \frac{y\,dy}{y^2+(\xi-x)^2}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \cdot \frac{1}{2}\,\frac{2y\,dy}{y^2+(\xi-x)^2}$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \left[\ln[y^2+(\xi-x)^2]\right] + c$$

# Using Fourier transform Solve the PDE

$$u_{xx} + u_{yy} = 0\ , \quad -\alpha < x < \alpha\ , \ y > 0$$

BC: $u(x,0) = F(x)\ ; \ -\alpha < x < \alpha$

$u \to 0$ as $p \to \alpha$ where $p = \sqrt{x^2+y^2}$

$$f(x) = \begin{cases} T_0\ , & |x| < b \\ 0\ , & |x| > b \end{cases}$$

Sol$^n$ By the result

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} f(\xi)\frac{1}{y^2+(\xi-x)^2}\,d\xi$$

$$\therefore f(\xi) = \begin{cases} T_0, & |\xi| < b \Rightarrow -b < \xi < b \\ 0, & |\xi| > b \end{cases}$$

$$\therefore u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{y^2+(\xi-x)^2}\,d\xi.$$

$$= \frac{y}{\pi}\int_{-b}^{b} \frac{T_0}{y^2+(\xi-x)^2}\,d\xi \Rightarrow \frac{yT_0}{\pi}\int_{-b}^{b}\frac{d\xi}{y^2+(\xi-x)^2}$$

Let $t = \xi - x$, $dt = d\xi$ | when $\xi = -b$, $t = -b-x = -(b+x)$; $\xi = b$, $t = b-x$

$$u(x,y) = \frac{yT_0}{\pi}\int_{-(b+x)}^{b-x} \frac{dt}{y^2+t^2} = \frac{yT_0}{\pi}\left[\frac{1}{y}\tan^{-1}\left(\frac{t}{y}\right)\right]_{-(b+x)}^{b-x}$$

$$= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{b-x}{y}\right) - \tan^{-1}\left(\frac{-(b+x)}{y}\right)\right]$$

$$= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{b-x}{y}\right) + \tan^{-1}\left(\frac{b+x}{y}\right)\right]$$

$$= \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{\frac{b-x}{y}+\frac{b+x}{y}}{1-\frac{(b-x)}{y}\frac{(b+x)}{y}}\right)\right)$$

$$= \frac{T_0}{\pi}\tan^{-1}\left(\frac{2b/y}{y^2-(b^2-x^2)}\right) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2by}{y^2+x^2-b^2}\right)$$

## Assignment problems

① Show that using fourier transform solve the PDE

$$u_{xx} + u_{yy} = -xe^{-x^2}, \quad -\alpha < x < \alpha, \; y > 0$$

BC: $u(x,0) = 0 \quad -\alpha < x < \alpha$

$u(x,y) \to 0$ as $y \to \alpha$

Show that $u(\alpha,y) = \left(1 - e^{-|\alpha| y}\right)\frac{i}{2\sqrt{2}\cdot\alpha}\, e^{-\alpha^2/4}$

(2) solve using Fourier transform $a^2 u_{xx} = u_t \quad 0 < x < \infty$

BC : $u(0,t) = f(t)$ $\quad$ $\left.\begin{array}{l} u(x,t) \to 0 \\ u_x(x,t) \to 0 \end{array}\right\}$ as $x \to \infty$

IC : $u(x,0) = 0, \quad 0 < x < \infty$

(3) $u_{tt} = c^2 u_{xx} \; ; \quad 0 < x < \infty$

IC : $u(x,0) = F(x) \quad , u_t(x,0) = g(x).$

Solve the problem by using Fourier transform.