A Frequency-Sweeping Framework for Stability Analysis of Time-Delay Systems

Xu-Guang Li, Silviu-Iulian Niculescu, Arben Çela, Lu Zhang, and Xu Li

***Abstract*—For time-delay systems, the asymptotic behavior analysis of the critical imaginary roots w.r.t. the infinitely many critical delays is an open problem. In order to find a general solution, we will exploit the link between the asymptotic behavior of critical imaginary roots and the asymptotic behavior of frequency-sweeping curves, from a new *analytic curve perspective*. As a consequence, we will establish a frequency-sweeping framework with three main results: (1) A finer (regularity-singularity) classification for time-delay systems will be obtained. (2) The *general invariance property* will be proved and hence the asymptotic behavior of the critical imaginary roots w.r.t. the infinitely many critical delays can be adequately studied. (3) The complete stability problem can be fully solved. Moreover, the frequency-sweeping framework is extended to cover a broader class of time-delay systems. Finally, the geometric insights of frequency-sweeping curves are investigated. Consequently, some deeper properties on the asymptotic behavior of time-delay systems and the link to frequency-sweeping curves are found.**

***Index Terms*—Complete stability, frequency-sweeping approach, invariance property, Puiseux series, Time-delay systems.**

I. Introduction

For a linear time-delay system (TDS), we are usually interested in obtaining its exhaustive stability domain w.r.t. delays. Such a problem is referred to as the *complete stability problem*, and, according to the τ-decomposition idea [16], is divided into two problems. ***Problem 1*:** An exhaustive detection (if any) of the critical imaginary roots (CIRs). ***Problem 2*:** The asymptotic behavior analysis of the CIRs.

We will not consider Problem 1 specifically since various methods are available (see e.g., [7], [8], [25], [28]). Although Problem 2 has been largely studied (see e.g., [9], [16], [25], [28]), it is far from being fully solved. The bottleneck lies in that it is impossible to analyze a CIR's asymptotic behavior at all the (infinitely many) CDs one by one. This motivates us to study here if the *invariance property* for general TDSs (termed the *general invariance property*) holds.

The new methodology of this paper (called frequency-sweeping framework) is to further study the link between the asymptotic behavior of the CIRs and the frequency-sweeping curves (FSCs). We adopt a new *analytic curve perspective*. To be more precise, most of the reported results are based on a direct analysis of the characteristic equation $f(\lambda, \tau) = 0$[1]. Instead, we study the series expansion form $F(\Delta\lambda, \Delta\tau) = 0$ of the characteristic equation, which can be viewed as an analytic curve. On one hand, the asymptotic behavior of a CIR at a CD is determined by the Puiseux series in terms of $\Delta\tau$. On the other hand, the asymptotic behavior of the FSCs will be taken into account through the dual Puiseux series in terms of $\Delta\lambda$. As a consequence, the intrinsic spectral characteristics of a TDS will be studied by analyzing the Puiseux series, the dual Puiseux series, as well as their connection.

Then, in the commensurate delays case, we will have three results: (1) The asymptotic behavior of TDSs can be classified into regular and singular cases. The latter is much more involved and no dedicated research has been reported. (2) The *general invariance property* will be proved. (3) The complete stability problem can be solved. Furthermore, the methodology may be extended to a more general class of characteristic functions. In this way, the frequency-sweeping framework will cover a broader class of TDSs, such as distributed TDSs, fractional TDSs, as well as TDSs with incommensurate delays.

Finally, we further investigate the intrinsic properties for the asymptotic behavior of CIRs as well as FSCs. For describing all possible complex cases, we introduce a notion, *asymptotic behavior signature*, to measure how complex the asymptotic behavior of CIRs and FSCs is. Then, we will prove that the asymptotic behavior signature of a CIR is invariant w.r.t its CDs and can be reflected from the geometric shape of FSCs. As a result, more detailed information (e.g., multiplicity, number of degenerate terms, and number of conjugacy classes) may be derived from the topological structure of FSCs. This will open a new perspective to study TDSs.

This paper is organized as follows. In Section II, prerequisites are given. The new frequency-sweeping framework is presented in Section III. In Section IV, three results of the frequency-sweeping framework are proposed, in the commensurate delays case. Extension to a broader class of TDSs is discussed and illustrated in Section V. Geometric insights of the FSCs are

[1] Here, λ is the Laplace variable and τ is the delay parameter.

Manuscript received June 15, 2016; revised October 28, 2016; accepted November 21, 2016. Date of publication December 1, 2016; date of current version July 26, 2017. This work is supported by National Natural Science Foundation of China (61473065) Recommended by Associate Editor P. Pepe.

X.-G. Li, L. Zhang, and X. Li are with School of Information Science and Engineering, Northeastern University, Shenyang, Liaoning 110004, China (e-mail: masdanlee@163.com; zhanglu880226@163.com; xuli_89@163.com).

S.-I. Niculescu is with Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS-CentraleSupélec-Université Paris-Sud, Gif-Sur-Yvette 91192, France (e-mail: Silviu.Niculescu@l2s.centralesupelec.fr).

A. Çela is with UPE, ESIEE Paris, Computer Science and Telecommunication Department, Noisy-Le-Grand 93162, France (e-mail: a.cela@esiee.fr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2016.2633533

investigated in Section VI. Finally, the paper concludes in Section VII. The results of Sections II–IV were partially reported in a recent book [18], while the results of Sections V and VI are totally new.

Notations: $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of (positive) real numbers and $\mathbb{C}$ is the set of complex numbers. For $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real part and imaginary part of λ, respectively, and $\mathrm{Arg}(\lambda) \in (-\pi, \pi]$ is the principal argument of λ. $\mathbb{C}_-$, $\mathbb{C}_+$, $\mathbb{C}_L$, and $\mathbb{C}_U$ denote the sets $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$, $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$, $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) < 0\}$, and $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) > 0\}$, respectively. $\mathbb{C}_0$ is the imaginary axis and $\partial\mathbb{D}$ is the unit circle. $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$ are the sets of integers, non-negative integers, and positive integers, respectively. For a function $\phi(x, y)$, $\phi_{x^\alpha y^\beta}$ $(\alpha \in \mathbb{N}, \beta \in \mathbb{N})$ denotes the partial derivative $\frac{\partial^{\alpha+\beta}\phi(x,y)}{\partial x^\alpha \partial y^\beta}$. For a function $\varphi(x)$, $\mathrm{ord}(\varphi(x)) = \kappa$ for $x = x^*$ denotes that $\frac{d^i\varphi(x)}{dx^i} = 0$ $(i = 0, \ldots, \kappa - 1)$ and $\frac{d^\kappa\varphi(x)}{dx^\kappa} \neq 0$ for $x = x^*$. ε is a sufficiently small positive real number, mainly used to describe the infinitesimal change of λ $(\Delta\lambda = \pm\varepsilon j)$ and τ $(\Delta\tau = \pm\varepsilon)$. I is the identity matrix. For real numbers $a, b > 0$, and c, $a \bmod b = c\,(a = c(\bmod(b)))$ denotes that $a = kb + c$, where $k \in \mathbb{Z}$ and $|c| < b$. For $\gamma \in \mathbb{R}$, $\lceil\gamma\rceil$ denotes the smallest integer greater than or equal to γ. Finally, $\det(\cdot)$ denotes the determinant.

II. Prerequisites and Motivating Examples

In order to fix better the ideas, consider first a linear time-delay system (TDS) with commensurate delays:

$$\dot{x}(t) = A_0 x(t) + \sum_{\ell=1}^{m} A_\ell x(t - \ell\tau), \tag{1}$$

where $A_0, \ldots, A_m$ $(m \in \mathbb{N}_+)$ are $r \times r$ real constant matrices and $\tau \geq 0$ is the delay parameter. The characteristic function, $f(\lambda, \tau) = \det(\lambda I - \sum_{\ell=0}^{m} A_\ell e^{-\ell\tau\lambda})$, is a quasipolynomial

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}, \tag{2}$$

where $a_0(\lambda), \ldots, a_q(\lambda)$ $(q \in \mathbb{N}_+)$ are polynomials in λ with real coefficients. TDS (1) has an infinite number of characteristic roots for $\tau > 0$ and is asymptotically stable iff all the characteristic roots lie in $\mathbb{C}_-$. As in [16], [25], [26], we use $NU(\tau) \in \mathbb{N}$ to denote the number of characteristic roots in $\mathbb{C}_+$, function of τ. We preclude two trivial cases, in which the system (1) can not be asymptotically stable for any $\tau \geq 0$. Trivial case 1: $a_0(\lambda), \ldots, a_q(\lambda)$ have common zeros in $\mathbb{C}_+ \cup \mathbb{C}_0$. Trivial case 2: $\lambda = 0$ is a characteristic root.

By the root continuity argument, $NU(\tau)$ changes as τ increases only when the system has *critical imaginary roots* (CIRs) at some τ. These delays are called the *critical delays* (CDs). A pair (λ, τ), where $\tau \in \mathbb{R}_+ \cup \{0\}$ and $\lambda \in \mathbb{C}_0$, such that $f(\lambda, \tau) = 0$ is called a *critical pair*. Due to the conjugate symmetry of the spectrum, it suffices to consider only the CIRs with non-negative imaginary parts.

With the notation $z = e^{-\tau\lambda}$, $f(\lambda, \tau)$ can be rewritten as

$$p(\lambda, z) = \sum_{i=0}^{q} a_i(\lambda) z^i. \tag{3}$$

Without loss of generality, suppose that there exist u critical pairs (λ, z) $(\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D})$ for $p(\lambda, z) = 0$ denoted by $(\lambda_0 = j\omega_0, z_0), \ldots, (\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ with $\omega_0 \leq \cdots \leq \omega_{u-1}$. Once all the critical pairs $(\lambda_\alpha, z_\alpha), \alpha = 0, \ldots, u - 1$, are found, all the critical pairs (λ, τ) for $f(\lambda, \tau) = 0$ can be obtained: For each CIR λ_α, the corresponding (infinitely many) CDs are given by $\tau_{\alpha,k} \triangleq \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}, k \in \mathbb{N}, \tau_{\alpha,0} \triangleq \min\{\tau \geq 0 : e^{-\tau\lambda_\alpha} = z_\alpha\}$. The pairs $(\lambda_\alpha, \tau_{\alpha,k}), k \in \mathbb{N}$, define a set of critical pairs associated with $(\lambda_\alpha, z_\alpha)$.

Next, we need to compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ for all $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, as the main goal of Problem 2[2]. Problem 2 is actually much more involved than we used to think. We divide it into two sub-problems and they will be discussed separately in the next two subsections. ***Problem 2.1:*** Asymptotic behavior analysis of a critical pair. ***Problem 2.2:*** Asymptotic behavior analysis of a CIR w.r.t. all its infinitely many positive CDs.

A. Asymptotic Behavior of a Critical Pair

We first recall two useful indices. We denote the multiplicity of a CIR for a critical pair by $n \in \mathbb{N}_+$, which implies that

$$f_{\lambda^0} = \cdots = f_{\lambda^{n-1}} = 0, f_{\lambda^n} \neq 0. \tag{4}$$

The index $g \in \mathbb{N}_+$ for a critical pair denotes that

$$f_{\tau^0} = \cdots = f_{\tau^{g-1}} = 0, f_{\tau^g} \neq 0. \tag{5}$$

Property 1: For a critical imaginary root $\lambda_\alpha \neq 0$, the index g is a constant for all the critical delays $\tau_{\alpha,k}$.

For TDSs, the asymptotic behavior of a critical pair usually refers to how the CIR varies w.r.t. an infinitesimal variation of the CD (i.e., $\Delta\tau = \pm\varepsilon$). As recently pointed out, a simple CIR corresponds to a Taylor series (Theorem 3 of [21]) while a multiple CIR generally corresponds to some Puiseux series (Theorem 1 in [19]). We may treat the Taylor series as a specific type of Puiseux series. Through invoking the Puiseux series (a general algorithm was proposed in [19]), we may compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ for any critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with $\tau_{\alpha,k} > 0$. If the system has a critical pair $(\lambda_\alpha, \tau_{\alpha,0})$ with $\tau_{\alpha,0} = 0$, the asymptotic behavior refers to how the original CIR (when $\tau = 0$) varies as τ increases from 0 to $+\varepsilon$ and can also be analyzed from the Puiseux series. Thus, we can compute $NU(+\varepsilon)$ (the case with only simple CIRs has been investigated, see Example a1 of [26] and Example 2 of [28]).

Example 1: Consider $f(\lambda, \tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$. When $\tau = 0$, $\lambda = j$ as well as $\lambda = -j$ is a double CIR. The critical pair $(j, 0)$ corresponds to the Puiseux series

$$\Delta\lambda = (0.3536 + 0.3536j)(\Delta\tau)^{\frac{3}{2}} + o((\Delta\tau)^{\frac{3}{2}}). \tag{6}$$

Substituting $\Delta\tau = +\varepsilon$ in (6) indicates that as τ increases from 0, the double root j splits toward $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively. Thus, $NU(+\varepsilon) = +2$ by the conjugate symmetry. ■

Some fundamentals on the Puiseux series from [1], [6], [23], [24], are given in Appendix A.

Remark 1: As pointed out in Chapter 4 of [18], there are two equivalent forms to express the Puiseux series. For instance, if the Puiseux series of a CIR with index n belong to one conjugacy class, they can be expressed in two forms. The first form (as adopted in [19]): We present n individual expressions of the Puiseux series and we treat $(\Delta\tau)^{\frac{1}{n}}$ as a single-valued number. The second form, as adopted in this paper, consists in giving only one expression of the Puiseux series and taking into account all the n values of $(\Delta\tau)^{\frac{1}{n}}$.

[2] As defined in [21], $\Delta NU_\alpha(\beta)$, where (α, β) is a critical pair with $\beta > 0$, stands for the number change of unstable roots caused by the variation of the CIR $\lambda = \alpha$ as τ increases from $\beta - \varepsilon$ to $\beta + \varepsilon$.

We have the theorem for $NU(+\varepsilon)$ in the general case.

Theorem 1: If the system (1) has no critical imaginary roots at $\tau = 0$, $NU(+\varepsilon) - NU(0) = 0$. Otherwise, $NU(+\varepsilon) - NU(0)$ equals to the number of values in $\mathbb{C}_+$ of the Puiseux series for all critical imaginary roots at $\tau = 0$ with $\Delta\tau = +\varepsilon$.

Now, Problem 2.1 can be solved. The results mainly stem from the idea introduced in [19]. For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, the asymptotic behavior is determined by the equation $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau) = 0$, where $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau)$ is a convergent power series, such that $F_{(\lambda_\alpha,\tau_{\alpha,k})}(0,0) = 0$, derived from (2)[3]. We may omit the subscript "$(\lambda_\alpha, \tau_{\alpha,k})$" when no confusion occurs. Then, for any critical pair, we may obtain all the first terms of the Puiseux series by the approach in [19]. If higher-order terms are required, we continue employing the approach of [19] in an iterative manner.

Remark 2: The method in [19] only allows to acquire finitely many terms of the Puiseux series. The general expression of the Puiseux series, which is essential for the comprehensive asymptotic behavior analysis, has not been reported so far. Such a general expression will be derived in Section III and will play an important role in our study.

Since there are infinitely many CDs, solving Problem 2.1 is insufficient. We have to solve Problem 2.2, too.

B. Invariance Property for Some Specific Time-Delay Systems

A crucial property, the invariance property (for a CIR λ_α, at all the infinitely many positive CDs $\tau_{\alpha,k}$, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant), has been found and proved in some specific systems with commensurate delays. We now recall the existing results. *1) Invariance property for the case $n = 1$:* Related studies can be found in e.g., [9], [25], [26], [28]. All these results are based on computing the derivatives of λ w.r.t. τ by using the implicit function theorem and, hence, they can not be applied to the case with multiple CIRs. Such an invariance property has been systematically proved. *2) Invariance property for the case $g = 1$:* The invariance property on TDSs with multiple CIRs was first explicitly reported in [15], subject to some constraints ($n \le 2$, $g = 1$, and without the degenerate case). Later in [21], the invariance result was extended to the case with any n including the degenerate case.

In view of the above results, Problem 2.2 can be solved when $n = 1$ and/or $g = 1$. Nevertheless, extension to the general case is a nontrivial work, as discussed below.

Remark 3: The structure of the Puiseux series[4] associated with a CIR may change w.r.t. CDs. For different CDs, a CIR considered in [15] may be simple or double and a CIR for Example 1 in [21] may exhibit multiplicity 1 or 3. It will be seen (from Theorem 2) that the structure of the Puiseux series must change w.r.t. the multiplicity of the CIR. In addition, the degenerate case (the concept and the degeneracy condition will be explained in detail in Appendix B) further increases the complexity, which will be illustrated by Example 3.

Remark 4: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$ with $n = 1$, the value set of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is evidently $\{-1, 0, +1\}$. The case $g = 1$ has the same value set, by Theorem 3 of [21]. However, the general case may take more values. For instance, $\Delta NU_j(\pi) = +3$ in Example 1 of [19] (with $n = 3$ and $g = 3$).

C. Two Motivating Examples

Example 2: Consider a TDS with $f(\lambda, \tau) = (\lambda + 1)e^{-2\tau\lambda} + (\lambda + 2)e^{-\tau\lambda} + \frac{21\pi}{2}\lambda^2 + \frac{21\pi}{2} + 1$, where $\lambda = j$ is a CIR for $\tau = (2k+1)\pi$. One may tend to conclude that $\lambda = j$ is always a simple CIR, as $n = 1$ for $\tau = \pi, 3\pi, \ldots, 19\pi$. However, when $\tau = 21\pi$, $\lambda = j$ is a *double* CIR. ■

Remark 5: Example 2 shows that it is in fact difficult to exclude the possibility of multiple CIRs.

Example 3: Consider a TDS with $f(\lambda, \tau) = \sum_{i=0}^{4} a_i(\lambda)e^{-i\tau\lambda}$ with $a_0(\lambda) = \frac{15}{8}\pi^2\lambda^6 + (\frac{11}{4}\pi - \frac{15}{8}\pi^2)\lambda^4 + \frac{9}{2}\pi\lambda^3 + (1 + \frac{1}{2}\pi - \frac{75}{8}\pi^2)\lambda^2 + (3 + \frac{9}{2}\pi)\lambda + 1 - \frac{9}{4}\pi - \frac{45}{8}\pi^2$, $a_1(\lambda) = \frac{5}{4}\pi\lambda^5 + \frac{11}{2}\pi\lambda^4 + (1 + \frac{7}{2}\pi)\lambda^3 + (\pi + 7)\lambda^2 + (11 + \frac{9}{4}\pi)\lambda + 4 - \frac{9}{2}\pi$, $a_2(\lambda) = \frac{5}{4}\pi\lambda^5 + \frac{11}{4}\pi\lambda^4 + (3 - \pi)\lambda^3 + (13 + \frac{1}{2}\pi)\lambda^2 + (15 - \frac{9}{4}\pi)\lambda + 6 - \frac{9}{4}\pi$, $a_3(\lambda) = 3\lambda^3 + 9\lambda^2 + 9\lambda + 4$, and $a_4(\lambda) = \lambda^3 + 2\lambda^2 + 2\lambda + 1$.

We consider the critical pairs $(j, (2k+1)\pi)$, with $g = 2$. The multiplicity n of $\lambda = j$ is 2, 3, 4, 2, when τ is π, 3π, 5π, 7π, respectively. The Puiseux series, all degenerate, are:

$$\begin{cases} \Delta\lambda = 0.1592j\Delta\tau + (0.5371 - 0.3138j)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = 0.0796j\Delta\tau + 0.0063j(\Delta\tau)^2 + 0.0421j(\Delta\tau)^3 \\ +(0.0362 + 0.0137j)(\Delta\tau)^4 + o((\Delta\tau)^4), \end{cases}$$

$$\begin{cases} \Delta\lambda = (0.0385 + 0.0698j)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}}), \\ \Delta\lambda = 0.1592j\Delta\tau + 0.0253j(\Delta\tau)^2 + 0.6696j(\Delta\tau)^3 \\ +(1.1585 + 0.4376j)(\Delta\tau)^4 + o((\Delta\tau)^4), \end{cases}$$

$$\begin{cases} \Delta\lambda = -0.1592j\Delta\tau + (-0.5371 + 0.3644j)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = -0.0988j(\Delta\tau)^{\frac{1}{3}} + (-0.0356 + 0.0028j)(\Delta\tau)^{\frac{2}{3}} \\ +o((\Delta\tau)^{\frac{2}{3}}), \end{cases}$$

$$\begin{cases} \Delta\lambda = -0.0796j\Delta\tau + (-0.0671 + 0.0487j)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = -0.1592j\Delta\tau + 0.0253j(\Delta\tau)^2 + 0.6615j(\Delta\tau)^3 \\ +(-1.1585 - 0.4363j)(\Delta\tau)^4 + o((\Delta\tau)^4), \end{cases}$$

for $k = 0, 1, 2$, and 3, respectively. ■

Example 3 further shows the complexity for asymptotic behavior analysis. First, a CIR may be always multiple. Second, the multiplicity may change w.r.t. CDs. If so, the structure of the Puiseux series must change. Third, the Puiseux series may have multiple conjugacy classes. Finally, the first terms of the Puiseux series may be not sufficient for the stability analysis (degenerate case). For some critical pairs in Example 3, we invoke up to the fourth terms.

For a better comprehension of the asymptotic behavior for TDSs, it will be useful to introduce: (1) A finer classification of the asymptotic behavior. In the literature, all cases are typically classified into the simple CIR case and the multiple CIR case. (2) A methodology able to cover all situations. *Our idea is to exploit the "link" between the asymptotic behavior of CIRs and FSCs.* In the next section, we will introduce a new analytic curve perspective and establish a new methodology, the frequency-sweeping framework.

[3] As the characteristic function $f(\lambda, \tau)$ (2) is analytical w.r.t. both λ and τ, $f(\lambda, \tau)$ can be expanded as a convergent power series w.r.t. $\Delta\lambda$ and $\Delta\tau$.

[4] For a critical pair, the structure of the Puiseux series is determined by the number of conjugacy classes, the polydromy order, and the first exponent, of the general expression. Relevant definitions can be found in Appendix A.

Remark 6: In practical systems, multiple CIRs rarely appear[5]. For TDSs with only simple CIRs, the complete stability can be studied by the *CTCR method* [25], [26] or the *direct method* [28]. However, for the theoretical completeness and due to the difficulty in excluding the possibility of multiple CIRs (as seen in Example 2), we need to appropriately take into account the multiple CIR case. In this context, *the current paper fills the gap.*

III. Studying Frequency-Sweeping Curves From Analytic Curve Perspective

The *frequency-sweeping approach* has been largely used for TDSs, see e.g., [8], [14], [16], [27]. The idea of this paper is different. We will revisit the asymptotic behavior of CIRs as well as FSCs from an analytic curve perspective.

A. Puiseux Series and Dual Puiseux Series

Without any loss of generality, the series $F(\Delta\lambda, \Delta\tau)$, introduced in Subsection II-A, can be decomposed as:

$$F(\Delta\lambda, \Delta\tau) = U(\Delta\lambda, \Delta\tau) \prod_{l=1}^{v} F_l(\Delta\lambda, \Delta\tau), \tag{7}$$

where, in a sufficiently small neighborhood of $(0,0)$, $F_l(\Delta\lambda, \Delta\tau)$ (such that $F_l(0,0) = 0$, $l = 1, \ldots, v$) are irreducible and $U(0,0) \neq 0$ in the ring of convergent power series. For each $F_l(\Delta\lambda, \Delta\tau)$, $\operatorname{ord}(F_l(\Delta\lambda, 0))|_{\Delta\lambda=0}$ and $\operatorname{ord}(F_l(0, \Delta\tau))|_{\Delta\tau=0}$ are denoted by $n_l \in \mathbb{N}_+$ and $g_l \in \mathbb{N}_+$, respectively. It is easy to see that $\sum_{l=1}^{v} n_l = \operatorname{ord}(F(\Delta\lambda, 0))|_{\Delta\lambda=0} = n$ and $\sum_{l=1}^{v} g_l = \operatorname{ord}(F(0, \Delta\tau))|_{\Delta\tau=0} = g$. In the right-hand side of (7), neither $(\Delta\lambda)^{\alpha}$ $(\alpha \in \mathbb{N}_+)$ factor nor $(\Delta\tau)^{\beta}$ $(\beta \in \mathbb{N}_+)$ factor appears as n and g are bounded.

As an irreducible $F_l(\Delta\lambda, \Delta\tau)$ determines a conjugacy class of Puiseux series (Proposition 2.2.1 in [6]), there are v (counted with multiplicities) conjugacy classes of Puiseux series. For the sake of simplicity, in this paper, we may adopt a short expression "a (v) Puiseux series" instead of "a (v) conjugacy class(es) of Puiseux series".

We propose to consider how τ varies in $\mathbb{C}$ w.r.t. λ (viewing $\tau_{\alpha,k}$ as a g-multiple root for $f(\lambda, \tau) = 0$) and the *dual Puiseux series* will be introduced. Similarly to the Puiseux series, we have v (counted with multiplicities) dual Puiseux series, determined by $F_l(\Delta\lambda, \Delta\tau)$ respectively. Thus, for a critical pair, the Puiseux series and the dual Puiseux series have the same number of conjugacy classes v. We now present their general expressions.

Theorem 2: A critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, with indices n and g, determines v $(v \in \mathbb{N}_+)$ Puiseux series:

$$\begin{cases} \mathrm{PS}_1 : \Delta\lambda = \sum\limits_{i=g_1}^{\infty} C_{1i} (\Delta\tau)^{\frac{i}{n_1}}, \\ \vdots \\ \mathrm{PS}_v : \Delta\lambda = \sum\limits_{i=g_v}^{\infty} C_{vi} (\Delta\tau)^{\frac{i}{n_v}}, \end{cases} \tag{8}$$

where $C_{1i}, \ldots, C_{vi}$ are complex coefficients and $C_{lg_l} \neq 0$ $(l = 1, \ldots, v)$, $n_l \in \mathbb{N}_+$ and $g_l \in \mathbb{N}_+$ satisfy that $n_1 + \cdots + n_v = n$ and $g_1 + \cdots + g_v = g$. Each Puiseux series PS_l corresponds to a dual Puiseux series $\mathrm{DPS}_l : \Delta\tau = \sum_{i=n_l}^{\infty} D_{li} (\Delta\lambda)^{\frac{i}{g_l}}$, where D_{li} are complex coefficients and $D_{ln_l} \neq 0$. Thus, a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ also determines v dual Puiseux series:

$$\begin{cases} \mathrm{DPS}_1 : \Delta\tau = \sum\limits_{i=n_1}^{\infty} D_{1i} (\Delta\lambda)^{\frac{i}{g_1}}, \\ \vdots \\ \mathrm{DPS}_v : \Delta\tau = \sum\limits_{i=n_v}^{\infty} D_{vi} (\Delta\lambda)^{\frac{i}{g_v}}. \end{cases} \tag{9}$$

Proof: First, the polydromy order of PS_l is n_l, by Corollary 1.8.5 in [6]. Next, the first exponent of PS_l is $\frac{g_l}{n_l}$ (see Exercise 11.3.1 in [24]). The general expression (8) of the Puiseux series can be obtained. Analogously, we may obtain the general expression (9) of the dual Puiseux series. ■

Corollary 1: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with $v = 1$, we have the following Puiseux series

$$\Delta\lambda = \sum_{i=g}^{\infty} C_i (\Delta\tau)^{\frac{i}{n}}, \tag{10}$$

as well as the following dual Puiseux series

$$\Delta\tau = \sum_{i=n}^{\infty} D_i (\Delta\lambda)^{\frac{i}{g}}, \tag{11}$$

where C_i and D_i are complex coefficients with $C_g \neq 0$ and $D_n \neq 0$.

Remark 7: If $n = 1$ and/or $g = 1$, there is only one conjugacy class, i.e., $v = 1$. Otherwise, there may be multiple conjugacy classes and the analysis is generally more involved.

B. New Insights Into Frequency-Sweeping Curves

We will see that the dual Puiseux series are connected with the FSCs obtained by the following procedure:

Frequency-Sweeping Curves (FSCs) Sweep $\omega \geq 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p(j\omega, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q FSCs $\Gamma_i(\omega)$: $|z_i(j\omega)|$ vs. ω, $i = 1, \ldots, q$. For simplicity, we denote by $\Im_1$ the line parallel to the abscissa axis with ordinate equal to 1. If $(\lambda_\alpha, \tau_{\alpha,k})$ is a critical pair with the index g, then g FSCs intersect $\Im_1$ at $\omega = \omega_\alpha$.

The existing frequency-sweeping methods, e.g., [9], [16], [20], are only valid for asymptotic behavior analysis when $g = 1$. To address the general case, we introduce a new notation.

For a set of critical pairs $(\lambda_\alpha \neq 0, \tau_{\alpha,k}), k \in \mathbb{N}$, with the index g (have in mind that g is a constant w.r.t. different k, see Property 1), there must exist g FSCs such that $z_i(j\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$ intersecting $\Im_1$ when $\omega = \omega_\alpha$. Among such g FSCs, we denote the number of the FSCs when $\omega = \omega_\alpha + \varepsilon$ $(\omega = \omega_\alpha - \varepsilon)$ above $\Im_1$ by $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$ $(NF_{z_\alpha}(\omega_\alpha - \varepsilon))$. We introduce the notation $\Delta NF_{z_\alpha}(\omega_\alpha)$, to describe the asymptotic behavior of the FSCs, as

$$\Delta NF_{z_\alpha}(\omega_\alpha) = NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \tag{12}$$

[5] As a practical example, the inverted pendulum system controlled by delay blocks may contain multiple CIRs at the origin [3].

Remark 8: We do not define $\Delta NF_{z_\alpha}(\omega_\alpha)$ at $\omega_\alpha = 0$ since we precluded the trivial case "$\lambda = 0$ is a characteristic root".

Remark 9: The value $\Delta NF_{z_\alpha}(\omega_\alpha)$ can be easily seen from the FSCs, making our approach simple to implement.

We now equip the FSCs with the dual Puiseux series:

Property 2: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, it follows that

$$\Delta NF_{z_\alpha}(\omega_\alpha) = ND_{(\lambda_\alpha, \tau_{\alpha,k})}(+\varepsilon j) - ND_{(\lambda_\alpha, \tau_{\alpha,k})}(-\varepsilon j),$$

where $ND_{(\lambda_\alpha, \tau_{\alpha,k})}(+\varepsilon j)$ ($ND_{(\lambda_\alpha, \tau_{\alpha,k})}(-\varepsilon j)$) denotes the number of the values in $\mathbb{C}_U$ of the dual Puiseux series (9), evaluated when $\Delta\lambda = +\varepsilon j$ ($\Delta\lambda = -\varepsilon j$).

Proof: To study the FSCs, we consider (9) with $\Delta\lambda = \pm\varepsilon j$. By (3), a corresponding FSC near $\omega = \omega_\alpha$ reflects the variation of $\left|e^{-\tau\lambda}\right|$ with $\left|e^{-\tau_{\alpha,k}\lambda_\alpha}\right| = 1$. Since $\lambda_\alpha + \Delta\lambda$ is a positive imaginary number, $\left|e^{-(\tau_{\alpha,k}+\Delta\tau)(\lambda_\alpha+\Delta\lambda)}\right| - \left|e^{-\tau_{\alpha,k}\lambda_\alpha}\right| > 0$ (< 0) iff the corresponding $\mathrm{Im}(\Delta\tau) > 0$ (< 0). ■

The *new frequency-sweeping framework* is now applicable to general linear TDSs.

Remark 10: In the light of (3), $\Delta NF_{z_\alpha}(\omega_\alpha)$ is independent of CDs, representing a crucial property. Thus, if "$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ for a positive $\tau_{\alpha,k}$", the invariance property can be confirmed.

IV. Classification, Invariance Property, and Complete Stability

Using the properties developed in the last section, we will obtain some important results, as detailed below:

A. New Regularity and Singularity Classification

From an algebraic geometry viewpoint [6], the equation $F(\Delta\lambda, \Delta\tau) = 0$ defines an analytic curve in $\mathbb{C}^2$, where the point $(0, 0)$ is either a *regular point* or a *singular point* (see e.g., [13] for relevant definitions). The point $(0, 0)$ is a regular (also called non-singular) point if $\mathrm{ord}(F(\Delta\lambda, 0))|_{\Delta\lambda=0} = 1$ and/or $\mathrm{ord}(F(0, \Delta\tau))|_{\Delta\tau=0} = 1$. Otherwise, the point $(0, 0)$ is a singular point.

In this sense, a critical pair for the TDS (1) can be viewed as a *regular point* (if $n = 1$ and/or $g = 1$) or a *singular point* (if $n > 1$ and $g > 1$) of the analytic curve, and hence the complete stability problem in fact includes two possible cases: regular and singular. In particular, the regular case includes the sub-regular case $n = 1$ as well as the sub-regular case $g = 1$.

Following the above classification and the discussions in Subsection II-B, the complete stability problem in the regular case has been adequately investigated. However, *no solution covering the singular case has been reported so far*. The singular case is much more complicated than the regular case. Theorem 2 indicates some difference between two cases:

Remark 11: For a critical pair, the Puiseux series as well as the dual Puiseux series must have only one conjugacy class in the regular case. This is not necessarily true in the singular case (Examples 3 and 5). In the regular case, at least one of the first exponents of the Puiseux series and the dual Puiseux series is with numerator 1. This property may be not necessarily valid in the singular case either (Example 1).

By the aforementioned regularity and singularity classification, we are led to make use of a modern mathematical tool: *singularity analysis for analytic curves*, see e.g., [6], [24].

B. Proving General Invariance Property

First, we prove the invariance property for the case where the CIR involves only one Puiseux series (i.e., $v = 1$).

Theorem 3: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$ with $v = 1$ and any indices n and g, it follows that:

$$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha). \tag{13}$$

The proof is in Appendix B. Following Remark 10, we have:

Theorem 4: For a critical imaginary root λ_α, if $v = 1$ for all the critical pairs $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, then $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > 0$.

We next consider the general case. According to (7), a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ has v (v is allowed to be any positive integer) pairs of Puiseux series and dual Puiseux series:

$$\begin{cases} \mathrm{PS}_l : \Delta\lambda = \sum\limits_{i=g_l}^{\infty} C_{li}(\Delta\tau)^{\frac{i}{n_l}}, \\ \mathrm{DPS}_l : \Delta\tau = \sum\limits_{i=n_l}^{\infty} D_{li}(\Delta\lambda)^{\frac{i}{g_l}}, \end{cases} \quad l = 1, \ldots, v. \tag{14}$$

We call the Puiseux series PS_l together with the dual Puiseux series DPS_l expressed in (14) the *l-th dual Puiseux series pair*. In view of the decomposition form (7), we can extend Theorem 4 to the general case.

Theorem 5: For a critical imaginary root λ_α of the system (1), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > 0$.

Proof: All the v dual Puiseux series pairs for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ are determined by the convergent power series $F(\Delta\lambda, \Delta\tau)$. In the light of (7), the l-th dual Puiseux series pair is determined by $F_l(\Delta\lambda, \Delta\tau)$. We denote by $\Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k})$ the number change of the values of the l-th Puiseux series in $\mathbb{C}_+$ as τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$. Similarly, we denote by $\Delta NF_{l,z_\alpha}(\omega_\alpha)$ the number change of the values of the l-th dual Puiseux series in $\mathbb{C}_U$ as λ varies from $(\omega_\alpha - \varepsilon)j$ to $(\omega_\alpha + \varepsilon)j$. As each $F_l(\Delta\lambda, \Delta\tau)$ corresponds to one conjugacy class of Puiseux series, we can prove that $\Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{l,z_\alpha}(\omega_\alpha), l = 1, \ldots, v$, in the same spirit of Theorem 3. Since $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \sum_{l=1}^{v} \Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k})$ and $\Delta NF_{z_\alpha}(\omega_\alpha) = \sum_{l=1}^{v} \Delta NF_{l,z_\alpha}(\omega_\alpha)$, we have the result (13). Now we can complete the proof according to Remark 10. ■

Since we can *graphically determine the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$, it is not needed to explicitly invoke the dual Puiseux series,* which are mainly used in proving Theorem 3 (in the light of Property 2). Although we sometimes invoke the Puiseux series for illustrating the asymptotic behavior of a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, it is in fact not necessary either. Based on the general invariance property (Theorem 5), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ can be easily known by observing the FSCs.

C. A Unified Approach for Complete Stability Problem

In order to systematically analyze the complete stability, we need to understand the spectrum when $\tau \to +\infty$. This is the so-called *ultimate stability problem*. It is a heuristic principle that increasing τ tends to destabilize the system. This is, however, not always correct, see e.g., Example 2 in [28], the case study of [25], and Example 5.11 in [14]. One may naturally wonder the effect of τ (stabilizing or destabilizing) if it keeps increasing. We now address this ultimate stability problem via studying $\lim_{\tau\to\infty} NU(\tau)$. If this limit is not available, we will always suspect if some stability intervals distant in the τ-axis are missing.

The ultimate stability problem, has only been solved partially, see e.g., Theorem 1 in [9] for a simple form of quasipolynomials. We now develop a general conclusion.

A critical frequency ω_α is called a crossing (touching) frequency for an FSC $\Gamma_i(\omega)$, if $\Gamma_i(\omega)$ crosses (touches without crossing) $\Im_1$ as ω increases near $\omega = \omega_\alpha$. With the notions above, we have the following:

Theorem 6: If the frequency-sweeping curves have a crossing frequency, there exists a τ^* such that the system (1) is unstable for all $\tau > \tau^*$ and $\lim_{\tau\to\infty} NU(\tau) = \infty$.

One may prove the result by extending Statement (b) of Theorem 1 in [9] to the general case, based on Theorem 5.

Theorem 7: A time-delay system (1) must belong to the following three types:

Type 1: The system has a crossing frequency and $\lim_{\tau\to\infty} NU(\tau) = \infty$; Type 2: The system has neither crossing frequencies nor touching frequencies and $NU(\tau) = NU(0)$ for all $\tau > 0$; Type 3: The system has touching frequencies but no crossing frequencies and $NU(\tau)$ is a constant for all τ other than the critical delays.

Theorem 7 follows straightforwardly from Theorem 6.

First, TDSs of Type 1 are widely seen. Next, a TDS of Type 2 exhibits the well-known delay-independent stability if $NU(0) = 0$. A case of Type 2 with $NU(0) > 0$ was reported in Example 9.1 of [18]. Finally, TDSs of Type 3 can be found in Example 4 in [20] (simple CIR case) and Section 3 in [15] (involving a double CIR).

Using the new frequency-sweeping framework, we can now study the complete stability problem by the following steps.

Step 1: Generate the frequency-sweeping curves (FSCs), through which we detect all the critical imaginary roots (CIRs) and the corresponding critical delays (CDs).

Step 2: For each CIR λ_α, we may choose any positive CD $\tau_{\alpha,k}$ to compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ by invoking the Puiseux series (the value is denoted by U_{λ_α}). Alternatively, we may directly have that $U_{\lambda_\alpha} = \Delta NF_{z_\alpha}(\omega_\alpha)$ from the FSCs.

Step 3: Compute $NU(+\varepsilon)$ by using Theorem 1.

With the steps above, we obtain the explicit expression of $NU(\tau)$ for the system (1), as stated in the following theorem.

Theorem 8: For any $\tau > 0$ which is not a critical delay, $NU(\tau)$ for the system (1) can be explicitly expressed as:

$$NU(\tau) = NU(+\varepsilon) + \sum_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{15}$$

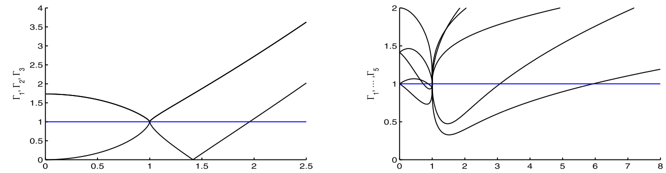
$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,0}, \end{cases} \text{if } \tau_{\alpha,0} \neq 0,$$

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,1}, \end{cases} \text{if } \tau_{\alpha,0} = 0.$$

Proof: The closed form (15) follows according to the general invariance property (Theorem 5). ■

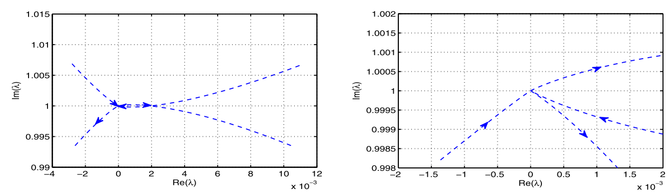
Remark 12: Similar results have been obtained for some specific TDSs, see e.g., [9], [16], [25], [28].

The TDS (1) is asymptotically stable for the range of τ with $NU(\tau) = 0$ excluding the CDs. In addition, the ultimate stability property is known by Theorem 7.



(a) (b)

Fig. 1. FSCs for Examples 4 and 5. (a) Example 4, (b) Example 5.



(a) (b)

Fig. 2. $\mathrm{Re}(\lambda)$ vs. $\mathrm{Im}(\lambda)$ for Examples 4 and 5. (a) Near $(j, 2\pi)$ for Example 4, (b) Near (j, π) for Example 5.

D. Illustrative Examples for Complete Stability Problem

For the frequency-sweeping framework, examples in the *regular* case can be found in the authors' earlier work. Here we mainly give examples in the *singular* case and verify the results by two means. We invoke the Puiseux series and use DDE-BIFTOOL [10] to numerically estimate the root loci.

Example 4 (Continuation of Example 1) The frequency-sweeping result is shown in Fig. 1(a) (there are three FSCs where two of them above $\Im_1$ coincide), from which we detect two CIRs $\lambda = j$ (with the CDs $2k\pi$) and $\lambda = 1.9566j$ (with the CDs $\frac{(2k+1)\pi}{1.9566}$).

The root $\lambda = 1.9566j$ is simple for $\tau = \frac{(2k+1)\pi}{1.9566}$. For this CIR, we may directly analyze the asymptotic behavior by the proposed frequency-sweeping approach (as this corresponds to the regular case, some existing methods can also be used). From Fig. 1(a), $\Delta NF_{-1}(1.9566) = +1$. Then, by Theorem 5, $\Delta NU_{1.9566j}(\frac{(2k+1)\pi}{1.9566}) = +1$ for all $k \in \mathbb{N}$. To verify the above analysis, we may choose a CD and invoke the Puiseux series. For instance, near $(\lambda = 1.9566j, \tau = 1.6056)$, the Puiseux series $\Delta\lambda = (0.6035 - 0.5253j)\Delta\tau + o(\Delta\tau)$ indicates that $\Delta NU_{1.9566j}(1.6056) = +1$.

The root $\lambda = j$ is double for $\tau = 2k\pi$. We have that $f_{\lambda\lambda} = -8.00$, $f_\tau = f_{\tau\tau} = 0$, $f_{\lambda\tau} = 0$, and $f_{\tau^3} = 6.00j$ for all $k \in \mathbb{N}$. The indices are $g = 3$ and $n = 2$. We have the Puiseux series (6). The root loci near $(j, 2\pi)$ are shown in Fig. 2(a). When $\tau = 0$, this system has a double CIR $\lambda = j$. We need to compute $NU(+\varepsilon)$ by using Theorem 1, which is already known from Example 1 ($NU(+\varepsilon) = +2$). We next consider the asymptotic behavior of the double CIR $\lambda = j$ for $\tau = 2k\pi$ ($k \in \mathbb{N}_+$). It is seen from the frequency-sweeping result that $\Delta NF_1(1) = 0$. Next, by Theorem 5, $\Delta NU_j(2k\pi) = 0$ for all $k \in \mathbb{N}_+$, which is consistent with the Puiseux series (6).

Now we can precisely compute $NU(\tau)$ by using Theorem 8, as shown in Fig. 3(a). In addition, $NU(\tau) \to \infty$ as $\tau \to \infty$ by Theorem 6, which is illustrated by Fig. 3(a). ■

Example 5: Consider a TDS with $f(\lambda,\tau) = \sum_{i=0}^{5} a_i(\lambda)e^{-i\tau\lambda}$, where $a_0(\lambda) = \lambda^4 + 2\lambda^3 + 5\lambda^2 + 4\lambda + 4$, $a_1(\lambda) = 5\lambda^3 + 10\lambda^2 + 15\lambda + 10$, $a_2(\lambda) = 4\lambda^3 + 14\lambda^2 +$

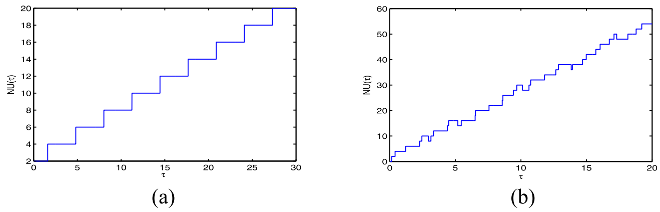


Fig. 3. $NU(\tau)$ for Examples 4 and 5. (a) Example 4, (b) Example 5.

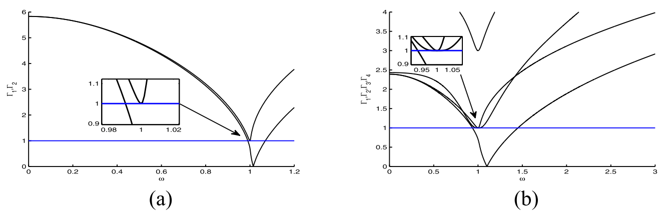


Fig. 4. FSCs for Examples 6 and 7. (a) Example 6, (b) Example 7.

$24\lambda + 14$, $a_3(\lambda) = \lambda^3 + 11\lambda^2 + 21\lambda + 11$, $a_4(\lambda) = 5\lambda^2 + 10\lambda + 5$, and $a_5(\lambda) = \lambda^2 + 2\lambda + 1$.

The frequency-sweeping result is shown in Fig. 1(b). Five sets of critical pairs are found: $(0.7266j, 5.1884 + \frac{2k\pi}{0.7266})$, $(0.8753j, 2.9402 + \frac{2k\pi}{0.8753})$, $(j, \pi + 2k\pi)$, $(3.0777j, 0.4083 + \frac{2k\pi}{3.0777})$, and $(5.9358j, 0.1577 + \frac{2k\pi}{5.9358})$.

We study the critical pairs $(j, \pi + 2k\pi)$ (the asymptotic behavior for the other ones is relatively simple). For all $k \in \mathbb{N}$, the indices are $n = 2$ and $g = 5$. By Theorem 5, we can directly know that $\Delta NU_j((2k+1)\pi) = +1$ for all $k \in \mathbb{N}$ from Fig. 1(b). We next verify this result. The Puiseux series for the critical pairs $(j, (2k+1)\pi)$ consists of two Taylor series:

$$\begin{cases} \Delta\lambda = (0.5 - 0.5j)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = (0.5 + 0.5j)(\Delta\tau)^3 + o((\Delta\tau)^3). \end{cases} \tag{16}$$

We also present the root loci near (j, π), see Fig. 2(b).

The ultimate stability property (by Theorem 6) is illustrated by Fig. 3(b). This system is asymptotically stable iff $\tau \in [0, 0.1577)$. ■

Remark 13: For both Examples 4 and 5, the invariance property can be *analytically proved* owing to a nice feature (for illustration) that the first terms of both the Puiseux series (6) and (16) are *invariant* w.r.t. different k (this situation rarely occurs as generally a Puiseux series varies w.r.t. k).

In the sequel, we continue the analysis for the TDSs in the motivating examples (Examples 2 and 3).

Example 6: For the system in Example 2, when $\tau = 0$, the two characteristic roots are $-0.0303 \pm 1.0585j$ ($NU(0) = 0$). We directly analyze the complete stability using the frequency-sweeping approach (the FSCs are given in Fig. 4(a)). This system has multiple stability intervals, see Fig. 5(a). ■

Example 7: For the system in Example 3, $NU(0) = 3$. The FSCs are given in Fig. 4(b). The variation of $NU(\tau)$ is shown in Fig. 5(b). The frequency-sweeping approach significantly simplifies our analysis. As seen in Example 3, the Puiseux series for this system are rather involved as: (1) They have multiple conjugacy classes. (2) They are degenerate. (3) The structure is variable. ■

Remark 14: The multiplicity information is required by most of existing methods. For instance, in [25], [26] the multiplicity

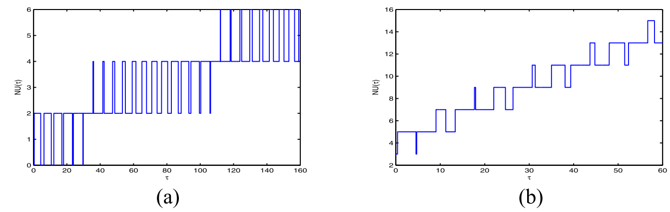


Fig. 5. $NU(\tau)$ for Examples 6 and 7. (a) Example 6, (b) Example 7.

for all CIRs is required to be 1 and the Rekasius substitution is used to detect CIRs. However, the multiplicity is not preserved by the widely-used Rekasius substitution [4]. Moreover, the multiplicity information may be directly implied from the topological structure of FSCs, see the deeper geometric analysis in Section VI of this paper.

V. A Broader Class of Time-Delay Systems

We now extend the frequency-sweeping framework to a broader class of TDSs.

A. Towards More General Characteristic Functions

Observe the characteristic function $f(\lambda, \tau)$ (2) and we call the functions $a_0(\lambda), \ldots, a_q(\lambda)$ in (2) the *coefficient functions*. In the previous sections, the coefficient functions $a_0(\lambda), \ldots, a_q(\lambda)$ *are all polynomials of* λ. To distinct with the more general characteristic function which will be considered in the sequel, such a characteristic function (i.e., $f(\lambda, \tau)$ (2) with the coefficient functions being polynomials) is called a *standard quasipolynomial*. Besides the retarded-type TDSs considered in this paper, the characteristic functions for neutral-type TDSs are standard quasipolynomials as well. Hence, the frequency-sweeping framework proposed in this paper covers the neutral-type TDSs, see Chapter 10 of [18].

We now relax the constraint on the coefficient functions such that they are no longer restricted to be polynomials: In the sequel, the coefficient functions $a_0(\lambda), \ldots, a_q(\lambda)$ of $f(\lambda, \tau)$ (2) *are only required to be analytic in* $\mathbb{C}_0 \backslash \{0\}$ (have in mind that we precluded the trivial case where $\lambda = 0$ is a characteristic root). A characteristic function (2) with the above relaxed condition is called *a general quasipolynomial*[6].

For the TDSs whose characteristic functions are general quasipolynomials, we may find some common properties related to the asymptotic behavior of CIRs. First, it is easy to see that a CIR λ_α for a general quasipolynomial corresponds to infinitely many CDs $\tau_{\alpha,k}$ ($k \in \mathbb{N}$) as well. Hence, the invariance issue still plays a crucial role in the complete stability analysis. Second, the asymptotic behavior of the CIRs w.r.t. the CDs can also be analyzed from the analytic curve perspective introduced in this paper. This is guaranteed by the analyticity in the relaxed constraint.

In the light of the above common properties, the frequency-sweeping framework presented in this paper can also be applied in the case with general quasipolynomials. First, the frequency-sweeping curves (FSCs) can be generated by the same procedure introduced in Subsection III-B. Second, the mathematical framework established in Section III is still applicable. We may

[6] The idea of studying general quasipolynomials may be traced back to [2] and [9], where a simple form $a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$ is studied.

now further generalize the invariance property stated in Theorem 5.

Theorem 9: For a critical imaginary root λ_α of the characteristic equation $f(\lambda,\tau)=0$ where $f(\lambda,\tau)$ is a general quasipolynomial, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k}>0$.

Proof: According to the relaxed constraint, for a critical pair $(\lambda_\alpha,\tau_{\alpha,k})$, the general quasipolynomial $f(\lambda,\tau)$ is analytic w.r.t. λ and τ, and the asymptotic behavior is determined by $F(\Delta\lambda,\Delta\tau)=0$, where $F(\Delta\lambda,\Delta\tau)$ is the series expansion derived from $f(\lambda,\tau)$ at $(\lambda_\alpha,\tau_{\alpha,k})$. Similar to the case with a standard quasipolynomial, the asymptotic behavior of the CIR (CD) w.r.t the CD (CIR) for a general quasipolynomial is determined by the Puiseux series in terms of $\Delta\tau$ (dual Puiseux series in terms of $\Delta\lambda$), as stated in Theorem 2.

Following the same line of Appendix B, we can prove the equivalence between a conjugacy class of Puiseux series and the corresponding conjugacy class of dual Puiseux series, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})=\Delta NF_{z_\alpha}(\omega_\alpha)$ (have in mind that $z_\alpha=e^{-\tau_{\alpha,k}\lambda_\alpha}$). In the same spirit of Theorem 5, we may prove the generalized invariance property in Theorem 9. ■

Remark 15: The frequency-sweeping framework does not depend on the expressions of characteristic functions, owing to the analytic curve perspective introduced in this paper. This significant advantage allows us to address more types of TDSs in a unified manner, as illustrated in the following subsections.

B. Distributed Time-Delay Systems

Consider the following distributed TDS

$$\dot{x}(t)=A_0x(t)+A_1\int_{-\infty}^{t}\kappa(t-\theta)x(\theta)d\theta, \tag{17}$$

where $\kappa(\theta)$ is a kernel function. Here, we consider a widely used uniform-distribution kernel:

$$\kappa(\theta)=\begin{cases}\frac{1}{d_1+d_2}, \text{ if } \tau-d_1<\theta<\tau+d_2,\\ 0, \text{ otherwise,}\end{cases} \tag{18}$$

where $\tau\geq d_1\geq 0$ and $d_2\geq 0$. The characteristic function $f(\lambda,\tau)$ of (17) subject to uniform-distribution kernel (18) is

$$\det\left(\lambda I-A_0-A_1\frac{e^{-(\tau-d_1)\lambda}-e^{-(\tau+d_2)\lambda}}{(d_1+d_2)\lambda}\right), \lambda\neq 0. \tag{19}$$

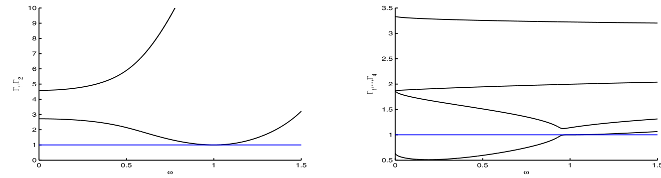
Such a characteristic function can be expressed in the form (2) where the coefficient functions $a_i(\lambda)$ are polynomials of λ and $\frac{e^{d_1\lambda}-e^{-d_2\lambda}}{\lambda}$. This characteristic function, obviously not a standard quasipolynomial, belongs to the class of general quasipolynomials.

Example 8: Consider the system (17) with

$$A_0=\begin{pmatrix}0 & 1\\ -\frac{\pi^4+3\pi^2-4}{\pi^2(\pi^2+1)} & \frac{2\pi}{\pi^2+1}\end{pmatrix}, A_1=\begin{pmatrix}\frac{1}{\pi(\pi^2+1)}\pi^2+4 & 0\\ & \frac{-1}{\pi^2+1}\end{pmatrix}.$$

Let $\kappa(\theta)$ be a uniform distribution (18) with $d_1=d_2=\frac{\pi}{2}$.

The characteristic function is a general quasipolynomial $f(\lambda,\tau)=a_0(\lambda)+a_1(\lambda)e^{-\tau\lambda}+a_2(\lambda)e^{-2\tau\lambda}$, with the coefficient functions $a_0(\lambda)=\lambda^2-\frac{4}{\pi^2}-\frac{2\pi\lambda-6}{\pi^2+1}+1$, $a_1(\lambda)=\frac{(e^{-\frac{\pi\lambda}{2}}-e^{\frac{\pi\lambda}{2}})(\pi^3\lambda-\pi^2+4)}{\pi^2\lambda(\pi^2+1)}$, and $a_2(\lambda)=-\frac{(e^{-\frac{\pi\lambda}{2}}-e^{\frac{\pi\lambda}{2}})^2}{\pi^2\lambda^2(\pi^2+1)}$.



(a) (b)

Fig. 6. FSCs for Examples 8 and 9. (a) Example 8, (b) Example 9.

At $\tau=(2k+1)\pi$, $\lambda=j$ is a CIR: $\lambda=j$ is a double CIR at $\tau=\pi$ while $\lambda=j$ is simple at all $\tau=(2k+1)\pi, k\in\mathbb{N}_+$.

According to Theorem 9, we have that $\Delta NU_j((2k+1)\pi)=\Delta NF_{-1}(1)=0$ for all $k\in\mathbb{N}$, where the value of $\Delta NF_{-1}(1)$ can be easily obtained from the FSCs shown in Fig. 6(a).

Next, we verify the above result by invoking the Puiseux series for critical pairs $(j,(2k+1)\pi)$, $k=0,1,2$:

$$\Delta\lambda=(0.2290+0.2930j)(\Delta\tau)^{\frac{1}{2}}+o((\Delta\tau)^{\frac{1}{2}}), k=0,$$

$$\Delta\lambda=-0.1592j\Delta\tau+(-0.0283+0.0324j)(\Delta\tau)^2 + o((\Delta\tau)^2), k=1,$$

$$\Delta\lambda=-0.0796j\Delta\tau+(-0.0035+0.0072j)(\Delta\tau)^2 + o((\Delta\tau)^2), k=2.$$

The analysis by frequency-sweeping approach is verified. ■

C. Fractional Time-Delay Systems

Consider the following fractional TDS

$$\frac{d^{\frac{1}{h}}x(t)}{dt^{\frac{1}{h}}}=\sum_{\ell=0}^{m}A_\ell x(t-\ell\tau), \tag{20}$$

where $h\in\mathbb{N}_+$ and $\frac{d^{\frac{1}{h}}x(t)}{dt^{\frac{1}{h}}}$ denotes the "$\frac{1}{h}$th-order derivative" of $x(t)$ (the definition of such a fractional-order derivative can be found in e.g., [12]).

We now recall some fundamentals regarding the stability from e.g., [22]. The characteristic function of (20) is $\det(\lambda^{\frac{1}{h}}I-\sum_{\ell=0}^{m}A_\ell e^{-\ell\tau\lambda})$. For a practical stability analysis, this characteristic function is treated as a single-valued function (as the stability is determined by the characteristic function on the first Riemann sheet): For a $\lambda\neq 0$ the term $\lambda^{\frac{1}{h}}$ takes the value $|\lambda|\,e^{j\frac{\mathrm{Arg}(\lambda)}{h}}$. In this context, we have the stability condition: The system (20) is *BIBO stable*[7] iff all the characteristic roots lie in $\mathbb{C}_-$.

It is easy to see that the characteristic function $f(\lambda,\tau)$ can be expressed in the form (2), where the coefficient functions $a_i(\lambda)$ are polynomials of $\lambda^{\frac{1}{h}}$. This characteristic function $f(\lambda,\tau)$ is obviously not a standard quasipolynomial, while it is included in the class of general quasipolynomials.

Example 9: Consider the system in Example 2 of [17]:

$$\frac{d^{\frac{1}{2}}x(t)}{dt^{\frac{1}{2}}}=\sum_{\ell=0}^{4}A_\ell x(t-\ell\tau),$$

[7] Unlike for a system with integral-order derivative, the link between the characteristic roots and the asymptotic stability for a fractional TDS is yet unclear, to the best of the authors' knowledge.

where $A_0 = \frac{5\sqrt{2}\pi+10\sqrt{2}}{16\pi}$, $A_1 = \frac{-3\sqrt{2}\pi+\sqrt{2}}{4\pi}$, $A_2 = \frac{-3\sqrt{2}\pi+8\sqrt{2}}{8\pi}$, $A_3 = \frac{-\sqrt{2}\pi+\sqrt{2}}{4\pi}$, and $A_4 = \frac{-3\sqrt{2}\pi+6\sqrt{2}}{16\pi}$.

The characteristic function is a general quasipolynomial $f(\lambda,\tau) = \sum_{i=0}^{4} a_i(\lambda)e^{-i\tau\lambda}$, with the coefficient functions: $a_0(\lambda) = \lambda^{\frac{1}{2}} - A_0$, $a_i(\lambda) = -A_i, i = 1,\dots,4$.

At $\tau = \frac{\pi}{2} + 2k\pi$, $\lambda = j$ is a CIR: $\lambda = j$ is a double CIR at $\tau = \frac{\pi}{2}$ while $\lambda = j$ is simple at all $\tau = \frac{\pi}{2} + 2k\pi, k \in \mathbb{N}_+$.

According to Theorem 9, $\Delta NU_j(\frac{\pi}{2} + 2k\pi) = \Delta NF_{-j}(1) = +1$ for all $k \in \mathbb{N}$, where the value of $\Delta NF_{-j}(1)$ can be easily obtained from the FSCs shown in Fig. 6(b).

To verify the above result, we give the Puiseux series for critical pairs $(j,(2k+\frac{1}{2})\pi)$, $k = 0,1,2$:

$$\Delta\lambda = 0.3757(\Delta\tau)^{\frac{1}{2}} + (0.1584 - 0.6734j)\Delta\tau + o(\Delta\tau), k = 0,$$

$$\begin{aligned}&\Delta\lambda = -0.1592j\Delta\tau - 0.0032j(\Delta\tau)^2\\&+(0.0102 - 0.0395j)(\Delta\tau)^3 + o((\Delta\tau)^3)\end{aligned}, k = 1,$$

$$\begin{aligned}&\Delta\lambda = -0.0796j\Delta\tau + 0.0028j(\Delta\tau)^2\\&+(0.0006 - 0.0024j)(\Delta\tau)^3 + o((\Delta\tau)^3)\end{aligned}, k = 2.$$

The frequency-sweeping approach considerably simplifies the analysis as the above Puiseux series are all degenerate. ■

D. Multiple Time-Delay Systems

Consider the following multiple TDS

$$\dot{x}(t) = A_0x(t) + \sum_{\ell=1}^{m} A_\ell x(t-\tau_\ell), \tag{21}$$

where $\tau_\ell \geq 0$ $(\ell = 1,\dots,m)$ are *independent* delays. Note that the delays are commensurate in system (1), while *incommensurate* here. The characteristic function for (21) is $\det(\lambda I - A_0 - \sum_{\ell=1}^{m} A_\ell e^{-\tau_\ell\lambda})$, see e.g., [11].

We study the invariance property of system (21) w.r.t. one delay parameter. Without loss of generality, suppose $\tau_2 = \tau_2^\sharp,\dots,\tau_m = \tau_m^\sharp$ are fixed and τ_1 is the free parameter denoted by τ.

The corresponding characteristic function can be formulated in the form $f(\lambda,\tau)$ (2) where the coefficient functions $a_i(\lambda)$ are polynomials of $e^{-\tau_2^\sharp j},\dots,e^{-\tau_m^\sharp j}$, and λ. Such a characteristic function $f(\lambda,\tau)$, apparently not a standard quasipolynomial, falls in the class of general quasipolynomials.

Example 10: Consider a system involving two delays τ_1 and τ_2, with the characteristic function $\frac{1}{\pi^2} + 2 + (1-\frac{3}{\pi})\lambda + \lambda^2 + ((\frac{2}{\pi^2}+3) + (3-\frac{4}{\pi})\lambda)e^{-\tau_1\lambda} + ((\frac{1}{\pi^2}+3) + (3-\frac{1}{\pi})\lambda)e^{-\tau_2\lambda} + (1+\lambda)e^{-(\tau_1+\tau_2)\lambda}$. Suppose $\tau_2 = 2\pi$ is fixed and τ_1 is the free delay parameter denoted by τ.

The characteristic function can be expressed by a general quasipolynomial $f(\lambda,\tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$ with the coefficient functions $a_0(\lambda) = \frac{1}{\pi^2} + 2 + (1-\frac{3}{\pi})\lambda + \lambda^2 + ((\frac{1}{\pi^2}+3) + (3-\frac{1}{\pi})\lambda)e^{-2\pi\lambda}$ and $a_1(\lambda) = ((\frac{2}{\pi^2}+3) + (3-\frac{4}{\pi})\lambda) + (1+\lambda)e^{-2\pi\lambda}$.

At $\tau = (2k+1)\pi$, $\lambda = j$ is a CIR and, in particular, $\lambda = j$ is triple at $\tau = \pi$ (it is simple at all $\tau = (2k+1)\pi, k \in \mathbb{N}_+$). According to Theorem 9, $\Delta NU_j((2k+1)\pi) = \Delta NF_{-1}(1) = +1$ for all $k \in \mathbb{N}$, where the value of $\Delta NF_{-1}(1)$ is easily obtained from the FSC (Fig. 7(a)).

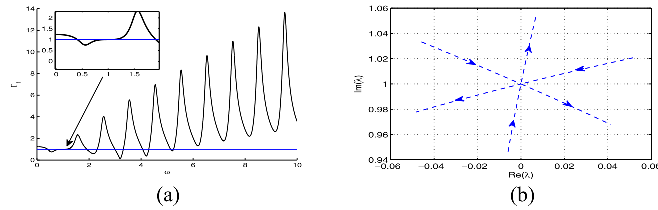


Fig. 7. FSC and $\mathrm{Re}(\lambda)$ vs. $\mathrm{Im}(\lambda)$ for Example 10. (a) FSC, (b) $\mathrm{Re}(\lambda)$ vs. $\mathrm{Im}(\lambda)$ near (j,π).

To verify the above result, we give the Puiseux series for critical pairs $(j,(2k+1)\pi)$, $k = 0,1,2$:

$$\Delta\lambda = (0.3801 - 0.2846j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}}), k = 0,$$

$$\begin{aligned}&\Delta\lambda = -0.1592j\Delta\tau + 0.0253j(\Delta\tau)^2\\&+(0.0021 - 0.0096j)(\Delta\tau)^3 + o((\Delta\tau)^3), k = 1,\end{aligned}$$

$$\begin{aligned}&\Delta\lambda = -0.0796j\Delta\tau + 0.0063j(\Delta\tau)^2\\&+(0.0001 - 0.0009j)(\Delta\tau)^3 + o((\Delta\tau)^3), k = 2.\end{aligned}$$

To further illustrate the asymptotic behavior of the triple CIR, we give the root loci near critical pair (j,π) in Fig. 7(b)[8]. ■

VI. Geometric Insights Into Frequency-Sweeping Curves

As seen, the CIRs may exhibit some complex asymptotic behavior, due to e.g., multiple CIRs, degenerate cases, and multiple conjugacy classes. In this section, we will further investigate the connection between CIRs and FSCs such that all involved cases may be understood more thoroughly. From the FSCs, we may not only analyze the stability as we did in the earlier sections, but also know the multiplicity, the number of degenerate terms, and the number of conjugacy classes.

A. Geometry in Case $g = 1$

To the best of the authors' knowledge, there does not exist a proper index to measure the complexity concerning the asymptotic behavior of TDSs. For instance, one may tend to think that a multiple CIR is "more complex" than a simple one. In fact, a simple CIR's asymptotic behavior may be complex as well (even more complex) if degenerate terms occur. This motivates us to first introduce an appropriate index, before proceeding further.

We start with a specific case, as the motivation for the complexity analysis (the general notion will be proposed later in Subsection VI-B). In this subsection, let $(\lambda_\alpha, \tau_{\alpha,k})$ be a set of critical pairs with $g = 1$. That is, at the critical frequency ω_α, one FSC intersects the line $\Im_1$.

Recall that the FSCs are generated according to solutions of z for $p(j\omega, z) = 0$ as we sweep ω. When no confusion occurs, we may use $|z|$ to describe the FSCs at the critical frequency under consideration.

Without loss of generality, suppose

$$\frac{d\,|z|}{d\omega} = \cdots = \frac{d^{\kappa-1}\,|z|}{d\omega^{\kappa-1}} = 0, \frac{d^\kappa\,|z|}{d\omega^\kappa} \neq 0. \tag{22}$$

[8] For this example, one may also analyze the stability by observing the stability crossing set, obtained by the CTCR method, in the τ_1-τ_2 plane [11].

TABLE I
LINK BETWEEN ASYMPTOTIC BEHAVIOR OF CIRS AND FSC IN CASE $g = 1$

	Puiseux series	κ
Examples 1-3 of [20] with $n = 1$	$\Delta\lambda = C_1(\Delta\tau) + o(\Delta\tau)$	1
Example 4 of [20] with $n = 1$	$\Delta\lambda = C_1(\Delta\tau) + C_2(\Delta\tau)^2 + o((\Delta\tau)^2)$, C_1 is degenerate	2
Example 5 of [20] with $n = 1$	$\Delta\lambda = \sum_{i=1}^{3} C_i(\Delta\tau)^i + o((\Delta\tau)^3)$, C_1 and C_2 are degenerate	3
Example in Section 3 of [15] with $n = 2$	$\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$	2
Example 2 of [21] with $n = 2$	$\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{2}} + C_2(\Delta\tau) + o(\Delta\tau)$, C_1 is degenerate	3
Example 1 of [21] with $n = 3$	$\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$	3

It is known that if κ is odd (even) the FSC crosses (touches without crossing) $\Im_1$ and that the FSC is tangent to $\Im_1$ if $\kappa > 1$. Furthermore, we adopt the geometric description from [5]: The FSC with $|z| = 1$ and condition (22) is said to have κ-point contact with $\Im_1$ (at the critical frequency). As κ increases (higher contact), the FSC becomes "flatter".

Using the index κ, we observe the connection between asymptotic behavior of the CIRs and the FSCs. The details for some representative examples are listed in Table I, where we see that κ is always equal to the multiplicity n plus the number of degenerate terms for the Puiseux series. This is a clue of possible deeper link between CIRs and FSCs in the general case.

B. Geometry in General Case

We now address the geometric properties of FSCs in the general case, where the index κ defined in (22) is no longer valid. The following theorem gives some general mathematical expression of the variation of FSCs, $\Delta|z|$.

Theorem 10: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, the variation of the frequency-sweeping curves, $\Delta|z|$, is determined by

$$\begin{aligned}\Delta|z| &= \mathrm{Im}(\Delta\tau(\Delta\omega j))(\omega_\alpha + \Delta\omega)\\ &+\frac{1}{2}(\mathrm{Im}(\Delta\tau(\Delta\omega j))(\omega_\alpha + \Delta\omega))^2 + \cdots,\end{aligned} \tag{23}$$

where $\Delta\tau(\Delta\omega j)$ denotes the $\Delta\tau$ obtained by substituting $\Delta\lambda = \Delta\omega j$ in the dual Puiseux series (9).

Proof: The local variation of FSCs is caused by a perturbation $\Delta\omega j$ in λ. Substituting $\Delta\lambda = \Delta\omega j$ into the dual Puiseux series (9) yields the expression of $\Delta\tau$. As $z = e^{-(\tau_{\alpha,k}+\Delta\tau)(\lambda_\alpha+\Delta\omega j)} = e^{-\tau_{\alpha,k}(\lambda_\alpha+\Delta\omega j)}e^{-\Delta\tau(\lambda_\alpha+\Delta\omega j)}$ with $\left|e^{-\tau_{\alpha,k}(\lambda_\alpha+\Delta\omega j)}\right| = 1$, $|z| = \left|e^{-\Delta\tau(\lambda_\alpha+\Delta\omega j)}\right| = e^{\mathrm{Im}(\Delta\tau)(\omega_\alpha+\Delta\omega)}$. Furthermore, $|z|$ can be expanded as

$$1 + \mathrm{Im}(\Delta\tau)(\omega_\alpha + \Delta\omega) + \frac{1}{2}(\mathrm{Im}(\Delta\tau)(\omega_\alpha + \Delta\omega))^2 + \cdots.$$

The proof is complete. ■

Remark 16: According to Theorem 10, the variation of FSCs $\Delta|z|$ is actually in the form of Puiseux series of $\Delta\omega j$. Furthermore, the topological structure is determined by $\omega_\alpha \mathrm{Im}(\Delta\tau(\Delta\omega j))$ (the higher order terms can be neglected), i.e., fully by the dual Puiseux series. In other words, the complexity index of FSCs may be defined according to the dual Puiseux series.

Remark 17: Since the variation of a CIR $\Delta\lambda$ is in the form of Puiseux series of $\Delta\tau$, similar to the discussions in Remark 16, the complexity index of a CIR may be defined according to the Puiseux series.

Following Remarks 16 and 17, we introduce the notion *asymptotic behavior signature* for the general case:

Definition 1: For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, we define the *asymptotic behavior signature of the CIR* λ_α as $n + \sum_{\ell=1}^{v}(\mathcal{N}_\ell^C - 1)$ ($C_{\ell\mathcal{N}_\ell^C}$ denotes the first non-degenerate coefficient for the Puiseux series PS_ℓ in (8)). We define the *asymptotic behavior signature of the FSCs* (at the critical frequency ω_α) as $g + \sum_{\ell=1}^{v}(\mathcal{N}_\ell^D - 1)$ ($D_{\ell\mathcal{N}_\ell^D}$ denotes the first non-degenerate coefficient for the dual Puiseux series DPS_ℓ in (9)).

The meaning of above notions is easy to understand: When analyzing the CIR, n number of root loci are involved and we have to calculate up to the term $C_{\ell\mathcal{N}_\ell^C}(\Delta\tau)^{\frac{\mathcal{N}_\ell^C}{n_l}}$ for each Puiseux series PS_ℓ (i.e., totally $\sum_{\ell=1}^{v}(\mathcal{N}_\ell^C - 1)$ degenerate terms). When analyzing the FSCs, g number of FSCs are involved and we have to calculate up to the term $D_{\ell\mathcal{N}_\ell^D}(\Delta\lambda)^{\frac{\mathcal{N}_\ell^D}{g_l}}$ for each dual Puiseux series DPS_ℓ (i.e., totally $\sum_{\ell=1}^{v}(\mathcal{N}_\ell^D - 1)$ degenerate terms). Obviously, the case $g = 1$ discussed in Subsection VI-A is covered.

C. Equivalence and Invariance for Asymptotic Behavior Signature

In the sequel, we first point out the equivalence between the asymptotic behavior signature for the CIRs and the corresponding FSCs.

Theorem 11: For a critical imaginary root λ_α, the asymptotic behavior signature of λ_α at a positive critical delay $\tau_{\alpha,k}$ equals to the asymptotic behavior signature of the frequency-sweeping curves.

Proof: We only address the case $v = 1$ (i.e., when there is only one conjugacy class), as the result may be extended to the case $v > 1$ following the line of proof for Theorem 5.

Observe the Puiseux series (10) and the dual Puiseux series (11). Suppose C_{g+M} ($M \in \mathbb{N}$) is the first non-degenerate coefficient of (10). The asymptotic behavior signature for the CIR is $n + g + M - 1$. In the light of the proof of Theorem 4, a degenerate term in the Puiseux series is always concurrent with a degenerate term in the dual Puiseux series and vice versa. Thus, D_{n+M} is the first non-degenerate coefficient for (11). The asymptotic behavior signature for the FSCs is $g + n + M - 1$. ■

Since the FSCs do not depend on τ, we have the following invariance on the asymptotic behavior signature of CIRs:

Theorem 12: The asymptotic behavior signature of a critical imaginary root λ_α is invariant at all its positive critical delays $\tau_{\alpha,k}$.

Example 11: Consider the system in Example 3. We illustrate Theorems 11 and 12 by analyzing critical pairs $(j, (2k + 1)\pi)$. The Puiseux series and FSCs were given in Example 3 and Fig. 4(b). The details are listed in Table II. ■

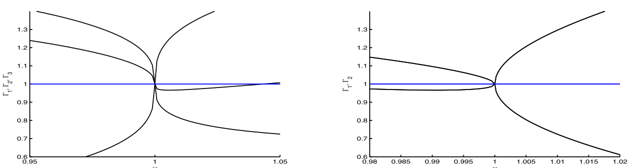
In view of Theorems 11 and 12, for a set of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, we may denote the asymptotic behavior signature (for the CIR as well as FSCs) by *one* notation $\mathcal{S}(\lambda_\alpha, \tau_{\alpha,k})$ or simply $\mathcal{S}$ when no confusion occurs.

D. Classification According to Topological Structure

Based on the results above, all complex cases associated with the asymptotic behavior of CIRs can be implied from the geometry of FSCs, without invoking the Puiseux series or the dual

TABLE II
ASYMPTOTIC BEHAVIOR SIGNATURE FOR EXAMPLE 11

	Asymptotic behavior signature of CIR	Asymptotic behavior signature of FSCs
$k = 0$	$2 + 4 = 6$	$2 + 4 = 6$
$k = 1$	$3 + 3 = 6$	$2 + 4 = 6$
$k = 2$	$4 + 2 = 6$	$2 + 4 = 6$
$k = 3$	$2 + 4 = 6$	$2 + 4 = 6$



(a) (b)

Fig. 8. Some typical FSCs when $g > 1$. (a) $g = 3$ and $\mathcal{S} = 3$, (b) $g = 2$ and $\mathcal{S} = 2$.

Puiseux series. That is, via observing the FSCs, we may not only study the complete stability but also acquire deeper information of TDSs. Furthermore, we can classify the FSCs, e.g., according to the indices g and $\mathcal{S}$, as each type of FSCs may share a same topological structure. Some representative geometric shapes of the FSCs are as follows.

Case $g = 1$: The FSC with $\mathcal{S} = 1$ (the most common case) crosses the line $\Im_1$ without tangency; the FSC with $\mathcal{S} = 2$ touches without crossing $\Im_1$ (e.g., Figs. 4(a) and 6(a)); the FSC with $\mathcal{S} = 3$ crosses $\Im_1$ in a tangent way (e.g., Fig. 7(a)); the FSC with $\mathcal{S} = 4$ touches without crossing $\Im_1$ (e.g., Fig. 4(b)). A comparison between an FSC with $\mathcal{S} = 2$ and an FSC with $\mathcal{S} = 4$ can be made in Fig. 4(b). At $\omega = 1$ both FSCs touch without crossing $\Im_1$. The one with $\mathcal{S} = 4$ is in a flatter shape.

Case $g > 1$: A type of FSCs with $g = 3$ and $\mathcal{S} = 4$ can be found in Fig. 1(a). For comparison, we borrow Example 3.2 of [18] where the FSCs are with $g = 3$ and $\mathcal{S} = 3$ (Fig. 8(a)). Another typical case can be found in Example 7.3 of [18] with $g = 2$ and $\mathcal{S} = 2$, see the FSCs in Fig. 8(b).

Remark 18: We compare the topological structure between the FSCs in Fig. 1(a) and the FSCs in Fig. 8(a). In Fig. 1(a), the FSCs when $\Delta\omega = -\varepsilon$ and when $\Delta\omega = +\varepsilon$ are nearly symmetric (the first term of the dual Puiseux series $D_2(\Delta\omega j)^{\frac{2}{3}}$ has the same values when $\Delta\omega = -\varepsilon$ and $+\varepsilon$). While, in Fig. 8(a), the FSCs when $\Delta\omega = -\varepsilon$ and $\Delta\omega = +\varepsilon$ are nearly skew-symmetric (the first term of the dual Puiseux series $D_1(\Delta\omega j)^{\frac{1}{3}}$ has opposite values when $\Delta\omega = -\varepsilon$ and $+\varepsilon$). In addition, the variation of FSCs in Fig. 1(a) is of order $(\Delta\omega j)^{\frac{2}{3}}$, while the variation of FSCs in Fig. 8(a) is of order $(\Delta\omega j)^{\frac{1}{3}}$. Thus, the local geometric shape of the FSCs in Fig. 1(a) is flatter than the FSCs in 8(a).

In our opinion, the analysis of topological structure in this section will open an interesting perspective. First, we may collect typical topological shapes of FSCs and construct some appropriate *library*. Everybody around the world may put new typical types into this open access library, such that more and more (involved) types of asymptotic behavior of TDSs will be included. As a consequence, the asymptotic behavior of a TDS may be directly implied if its type falls in the library. Some more detailed explanation is given below.

TABLE III
ESTIMATING n ACCORDING TO THEOREM 13

Critical pairs for FSCs at $\omega = 1$	g	$\mathcal{S}$	Possible n by Theorem 13	Actual n
Fig. 1(a)	3	4	1, 2	2
Fig. 7(a)	1	3	1, 2, 3	1, 3
Fig. 4(a)	1	2	1, 2	1, 2
Fig. 8(b)	2	2	1	1
Fig. 8(a)	3	3	1	1

As already seen, for a set of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$ the multiplicity of the CIR λ_α may vary w.r.t. k and it is not a trivial work to check the possible multiplicities (Examples 2 and 3, and Remark 14). Now, we can determine all the possible values of n (i.e., multiplicities) of a CIR from the FSCs:

Theorem 13: Suppose a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ involves one conjugacy class. It follows that

$$n \leq \mathcal{S} + 1 - g. \tag{24}$$

Proof: As the first coefficient for the Puiseux series is C_g (Corollary 1), the minimum asymptotic behavior signature for the CIR may be reached as $n + g - 1$ if C_g is non-degenerate. We obtain the relation (24) as the CIR and the FSCs share an identical asymptotic behavior signature. ■

Remark 19: Since g and $\mathcal{S}$ may be graphically observed, we may directly know the possible multiplicities of the CIR.

A straightforward corollary from Theorem 13 is as follows:

Corollary 2: Suppose a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ involves one conjugacy class. If $\mathcal{S} = g$, the critical imaginary root is simple.

Remark 20: Corollary 2 can be used quite often, as the types of FSCs with $\mathcal{S} = g$ are frequently encountered and can be easily identified. The case $\mathcal{S} = g = 1$ represents the most common situation where a simple CIR crosses $\mathbb{C}_0$ without tangency. The FSCs given in Figs. 8(a) and 8(b) correspond to the cases $\mathcal{S} = g = 3$ and $\mathcal{S} = g = 2$, respectively.

Example 12: We verify Theorem 13 through some representative cases. The details are listed in Table III, where the last two cases can also be analyzed by using Corollary 2. ■

The above results may be used in the case with multiple conjugacy classes, as discussed below (*decoupling* idea).

Remark 21: The number of conjugacy classes can be identified graphically. More precisely, if the FSCs are found to be a mixture of v typical types of FSCs corresponding to one conjugacy class, the number of conjugacy classes is v. For instance, the FSCs at $\omega = 1$ in Figs. 1(b) and 4(b) both involve 2 types of typical FSC shapes for one conjugacy class (i.e., $v = 2$ for both cases). Clearly, such a *decoupling* idea is insightful for the asymptotic behavior analysis.

Finally, for a better understanding of the results in this section, we give the correspondence between the topological structure of FSC and the root loci in the case $g = 1$. For the case $g > 1$, we may construct the library in the same way.

Due to the limited space, we only list the cases when $\mathcal{S} = 1, 2,$ and 3, in Fig. 9. As mentioned, we may not only analyze the complete stability from the FSCs, but also obtain more detailed information on the asymptotic behavior from the topological structure of FSCs.

Case $\mathcal{S} = 1$: The CIR is simple and it crosses the imaginary axis $\mathbb{C}_0$ without tangency. Case $\mathcal{S} = 2$: The CIR may be simple or double. If simple, it touches without crossing $\mathbb{C}_0$. If double,

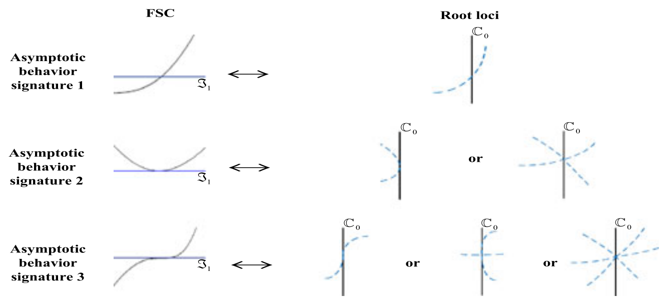


Fig. 9. Topological correspondence between FSC and root loci when $g = 1$.

it splits and none of the branches is tangent to $\mathbb{C}_0$. Case $\mathcal{S} = 3$: The CIR may be simple, double, or triple. If simple, it crosses $\mathbb{C}_0$ in a tangent way. If double, it splits and two of the branches, corresponding to $\Delta\tau = +\varepsilon$ or $-\varepsilon$, are tangent to $\mathbb{C}_0$ (such a case can be found in Example 2 of [21]). If triple, it splits and none of the branches is tangent to $\mathbb{C}_0$.

With the topological analysis, the complete stability study for TDSs becomes deeper and more intuitive.

VII. Concluding Remarks

The complete stability of time-delay systems has remained an open problem, due to the difficulty in characterizing the asymptotic behavior of the critical imaginary roots w.r.t. the infinitely many critical delays. As proved in this paper, such an asymptotic behavior may be much more intricate than expected. We established a new frequency-sweeping framework from an analytic curve perspective. Then, the asymptotic behavior of the critical imaginary roots and the frequency-sweeping curves as well as the relation between the two types of asymptotic behavior can be adequately studied.

Within this new frequency-sweeping framework, we obtained three important results. First, the asymptotic behavior concerning time-delay systems can be classified into the regular and the singular cases. Such a new classification leads us to a powerful mathematical tool, singularity analysis for analytic curves. Second, we proved the general invariance property for linear systems with commensurate delays. Finally, we fully solve the complete stability problem and the frequency-sweeping approach is simple to implement. Furthermore, the frequency-sweeping framework was applied to a more general class of characteristic functions. As a consequence, for a broad class of time-delay systems, the asymptotic behavior can be covered and the invariance property was confirmed. Finally, deeper connection between the asymptotic behavior of critical imaginary roots and the topological structure of frequency-sweeping curves was studied. This opens a new perspective for the asymptotic behavior analysis of time-delay systems.

Appendix A
Basics on Puiseux Series

A series of the form

$$y = \sum_{i=1}^{\infty} \alpha_i x^{\frac{i}{\mathrm{N}}} \tag{25}$$

(where $x \in \mathbb{C}$, $y \in \mathbb{C}$, α_i are complex coefficients, and $\mathrm{N} \in \mathbb{N}_+$) is called a Puiseux series. N is called the *polydromy order* for this Puiseux series, if N and all i with $a_i \neq 0$ have no common factor greater than 1. The term $\alpha_i x^{\frac{i}{\mathrm{N}}}$, with the minimal i such that $\alpha_i \neq 0$, is called the *first term* of the Puiseux series (25), and the corresponding α_i and $\frac{i}{\mathrm{N}}$ are called the *first coefficient* and the *first exponent*, respectively. Without loss of generality, for a Puiseux series given in the form (25), N is assumed to be the polydromy order.

In this paper we address the Puiseux series as follows: For each $x \in \mathbb{C}$, the Puiseux series (25) has N values for $y \in \mathbb{C}$ since $x^{\frac{1}{\mathrm{N}}}$ has N values in $\mathbb{C}$. As these N values of y exhibit some conjugacy feature (see Section 1.2 of [6]), the Puiseux series (25) is called *a conjugacy class of* Puiseux series.

If a value of the Puiseux series (25) is a solution for $\Phi(y, x) = 0$, i.e., $\Phi(\sum_{i=1}^{\infty} \alpha_i x^{\frac{i}{\mathrm{N}}}, x) = 0$ for a value of $x^{\frac{1}{\mathrm{N}}}$, in a small neighborhood of $(0, 0)$ ($\Phi(y, x)$ is a convergent power series of y and x), then all the N values of the Puiseux series (25) (corresponding to all the N values of $x^{\frac{1}{\mathrm{N}}}$) are solutions. Note that the solutions for $\Phi(y, x) = 0$ may correspond to more than one conjugacy class of Puiseux series.

Appendix B
Proof of Theorem 3

When a CIR has only one Puiseux series (i.e., $v = 1$), the dual Puiseux series pair is given by (10) and (11). For the first coefficients, it is true that $D_n = (\frac{1}{C_g})^{\frac{n}{g}}$. For higher-order coefficients, we have the following relation.

Property 3: For any integer $h \geq 1$, it follows that $D_{n+h} = (\sum_{i_1+2i_2+\cdots+hi_h=h} \alpha_{i_1,\ldots,i_h} \prod_{\mathrm{w}=1}^{h} \gamma_{h,\mathrm{w}}^{i_\mathrm{w}})(\frac{1}{C_g})^{\frac{n+h}{g}}$, where $\alpha_{i_1,\ldots,i_h}$ ($i_1, \ldots, i_h$ are non-negative integers) are real coefficients, $\gamma_{h,\mathrm{w}} = 1$ if $C_{g+\mathrm{w}} = 0$ and $i_\mathrm{w} = 0$, $\gamma_{h,\mathrm{w}} = \frac{C_{g+\mathrm{w}}}{C_g}$ otherwise.

Proof: Substituting (10) in (11), we have that

$$\Delta\tau = \sum_{h=0}^{\infty} D_{n+h} C_g^{\frac{n+h}{g}} (\Delta\tau)^{\frac{n+h}{n}} \left(\sum_{i=0}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}} \right)^{\frac{n+h}{g}} \tag{26}$$

By the Binomial Theorem (pp. 90 in [1]), $(\sum_{i=0}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}})^{\frac{n+h}{g}} = 1 + \sum_{k=1}^{\infty} (\frac{(\frac{n+h}{g})\cdots(\frac{n+h}{g}-k+1)}{k!} (\sum_{i=1}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}})^k)$. Thus, the right-hand side of (26) is of the form $\sum_{h=0}^{\infty} \chi_h (\Delta\tau)^{\frac{n+h}{n}}$, where χ_h are complex coefficients such that $\chi_0 = 1$ and $\chi_h = 0$ for all $h \geq 1$. We have that for any $h \geq 1$, χ_h is of the form $\sum_{\mathrm{v}=0}^{h-1} D_{n+\mathrm{v}} C_g^{\frac{n+\mathrm{v}}{g}} (\sum_{i_1+\cdots+(h-\mathrm{v})i_{h-\mathrm{v}}=h-\mathrm{v}} \beta^{(\mathrm{v})}_{i_1,\ldots,i_{h-\mathrm{v}}} \prod_{\mathrm{w}=1}^{h-\mathrm{v}} \gamma_{h-\mathrm{v},\mathrm{w}}^{i_\mathrm{w}}) + D_{n+h} C_g^{\frac{n+h}{g}}$ ($\beta^{(\mathrm{v})}_{i_1,\ldots,i_{h-\mathrm{v}}}$ are real coefficients). First, D_n is directly known from $\chi_0 = 1$. It can be seen that after determining $D_n, \ldots, D_{n+h-1}$, we can proceed to determine D_{n+h} from $\chi_h = 0$, which can be viewed as a linear equation with the variable D_{n+h} ($D_n, \ldots, D_{n+h-1}$ are already known). In this way, as stated by the property, we obtain the general forms of D_{n+h} by induction. ∎

For $\Delta\tau = +\varepsilon$ ($-\varepsilon$), the n principal arguments of $(\Delta\tau)^{\frac{1}{n}}$ are denoted by a set $\Theta^{(+)}$ ($\Theta^{(-)}$) with elements $\theta_i^{(+)}$ ($\theta_i^{(-)}$), $i = 1, \ldots, n$. For $\Delta\lambda = +\varepsilon j$ ($-\varepsilon j$), the g principal arguments of $(\frac{\Delta\lambda}{C_g})^{\frac{1}{g}}$ are denoted by a set $\Psi^{(+)}$ ($\Psi^{(-)}$) with elements $\psi_i^{(+)}$ ($\psi_i^{(-)}$), $i = 1, \ldots, g$.

Without any loss of generality, suppose that the greatest common factor of n and g is $\eta \in \mathbb{N}_+$, i.e., $\frac{g}{n} = \frac{\eta\widetilde{g}}{\eta\widetilde{n}} = \frac{\widetilde{g}}{\widetilde{n}}$, where $\widetilde{n} \in \mathbb{N}_+$ and $\widetilde{g} \in \mathbb{N}_+$ are co-prime. Hence, there are three possible cases (Case 1: $\widetilde{n}$ is odd and $\widetilde{g}$ is odd, Case 2: $\widetilde{n}$ is odd and $\widetilde{g}$ is even, and Case 3: $\widetilde{n}$ is even and $\widetilde{g}$ is odd).

If the first term $C_g(\Delta\tau)^{\frac{g}{n}}$ when $\Delta\tau = \pm\varepsilon$ of the Puiseux series (10) contains purely imaginary numbers, we need to further consider higher-order terms until we can conclude on the value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. This corresponds to the so-called *degenerate case*. The higher-order terms $C_{g+1}(\Delta\tau)^{\frac{g+1}{n}}, \ldots$ are also called degenerate if the corresponding branches of these terms still contain purely imaginary numbers plus 0. Similarly, if the first term $D_n(\Delta\lambda)^{\frac{n}{g}}$ when $\Delta\lambda = \pm\varepsilon j$ of the dual Puiseux series (11) contains purely real numbers, we need to further consider the higher-order terms until we can conclude on the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$ according to Property 2. This also corresponds to the so-called *degenerate case*. The higher-order terms $D_{n+1}(\Delta\lambda)^{\frac{n+1}{g}}, \ldots$ are also called degenerate if the corresponding branches of these terms still contain purely real numbers.

It is necessary to know the condition causing a degenerate case (i.e., the degeneracy condition). We will see (by Lemmas 2, 3, 5, 6, 8, and 9 given later) that a degenerate Puiseux series (10) must be concurrent with a degenerate dual Puiseux series (11) under the degeneracy condition: When $\widetilde{n}$ is odd (even), the degenerate case occurs iff $\mathrm{Re}(C_g^{\widetilde{n}}) = 0$ ($\mathrm{Im}(C_g^{\widetilde{n}}) = 0$).

We first consider the non-degenerate case, for which $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is as follows. Case 1: (1) When $\widetilde{n} \bmod 4 = 1$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \eta$ ($-\eta$); (2) When $\widetilde{n} \bmod 4 = 3$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = -\eta$ (η). Case 2 and Case 3: $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$.

Regarding the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$, we can precisely calculate it via studying the dual Puiseux series (11), according to Property 2. As $D_n(\Delta\lambda)^{\frac{n}{g}} = (\frac{\Delta\lambda}{C_g})^{\frac{n}{g}} = (\frac{\Delta\lambda}{C_g})^{\widetilde{\frac{n}{g}}}$ and we are interested in the case with $\Delta\lambda = \pm\varepsilon j$, we have that

$$\left(\frac{\Delta\lambda}{C_g}\right)^{\widetilde{\frac{n}{g}}} = \left(\frac{(\pm\varepsilon)^{\widetilde{n}} j^{\widetilde{n}} (\mathrm{Re}(C_g^{\widetilde{n}}) - \mathrm{Im}(C_g^{\widetilde{n}})j)}{|C_g^{\widetilde{n}}|^2}\right)^{\widetilde{\frac{1}{g}}}. \quad (27)$$

Following the idea of [16], we have the results for the non-degenerate case. Case 1: (1) When $\widetilde{n} \bmod 4 = 1$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NF_{z_\alpha}(\omega_\alpha) = \eta$ ($-\eta$); (2) When $\widetilde{n} \bmod 4 = 3$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NF_{z_\alpha}(\omega_\alpha) = -\eta$ (η). Case 2 and Case 3: $\Delta NF_{z_\alpha}(\omega_\alpha) = 0$.

In conclusion, we have the following lemma.

Lemma 1: Theorem 3 holds for the non-degenerate case.

In the forthcoming subsections, we suppose that the degenerate cases occur and we will explicitly address Cases 1, 2, and 3. The following notations will be adopted. For the elements in $\Theta^{(+)}$ ($\Theta^{(-)}$) causing a degenerate case, we denote them by a set $\Theta_{\mathbb{D}}^{(+)}$ ($\Theta_{\mathbb{D}}^{(-)}$). For the elements in $\Psi^{(+)}$ ($\Psi^{(-)}$) causing a degenerate case, we denote them by a set $\Psi_{\mathbb{D}}^{(+)}$ ($\Psi_{\mathbb{D}}^{(-)}$).

A. Case 1:

According to the degeneracy condition, $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ or $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$, $k_1 \in \mathbb{Z}$. In this subsection, we let $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ (the proof when $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ can be completed by similarity). For an odd $\widetilde{n}$, it has two possibilities: $\widetilde{n} \bmod 4 = 1$ and $\widetilde{n} \bmod 4 = 3$. In this subsection, we let $\widetilde{n} \bmod 4 = 1$ (the proof when $\widetilde{n} \bmod 4 = 3$ can be completed in the same spirit). To avoid confusion, we emphasize that the results to be developed below are for the case where $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ and $\widetilde{n} \bmod 4 = 1$. For the other possibilities, we may prove the result analogously.

We have that $\Theta_{\mathbb{D}}^{(+)}$ has η elements satisfying the condition

$$g\theta_i^{(+)} + \mathrm{Arg}(C_g) = \frac{\pi}{2}(\mathrm{mod}(2\pi)), \quad (28)$$

and that $\Theta_{\mathbb{D}}^{(-)}$ has η elements satisfying the condition

$$g\theta_i^{(-)} + \mathrm{Arg}(C_g) = -\frac{\pi}{2}(\mathrm{mod}(2\pi)). \quad (29)$$

In the light of (27), $\Psi_{\mathbb{D}}^{(+)}$ contains η elements such that

$$n\psi_i^{(+)} = 0(\mathrm{mod}(2\pi)), \quad (30)$$

and $\Psi_{\mathbb{D}}^{(-)}$ contains η elements such that

$$n\psi_i^{(-)} = \pi(\mathrm{mod}(2\pi)). \quad (31)$$

With proper relabeling, we let $\Theta_{\mathbb{D}}^{(+)} : \{\theta_1^{(+)}, \ldots, \theta_\eta^{(+)}\}$, $\Theta_{\mathbb{D}}^{(-)} : \{\theta_1^{(-)}, \ldots, \theta_\eta^{(-)}\}$, $\Psi_{\mathbb{D}}^{(+)} : \{\psi_1^{(+)}, \ldots, \psi_\eta^{(+)}\}$, and $\Psi_{\mathbb{D}}^{(-)} : \{\psi_1^{(-)}, \ldots, \psi_\eta^{(-)}\}$.

Property 4: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}, 1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}}^{(+)}, 1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. For different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.

Proof: From (28) and (30), $\eta\theta_1^{(+)}, \ldots, \eta\theta_\eta^{(+)}$ correspond to a same principal argument (denoted by $\overline{\theta}^{(+)} \in (-\pi, \pi]$) and $\eta\psi_1^{(+)}, \ldots, \eta\psi_\eta^{(+)}$ correspond to a same principal argument (denoted by $\overline{\psi}^{(+)} \in (-\pi, \pi]$). The proof will be complete if $\overline{\psi}^{(+)} = \overline{\theta}^{(+)}$.

Since $\overline{\psi}^{(+)}$ corresponds to a principal argument of $(\frac{\pm\varepsilon j}{C_g})^{\frac{1}{g}}$, it follows that

$$\overline{\psi}^{(+)} = \frac{1}{\widetilde{g}}\left(\frac{\pi}{2} + 2s_1\pi - \mathrm{Arg}(C_g)\right), \quad (32)$$

where $s_1 \in \mathbb{Z}$. From (28), we have that $\mathrm{Arg}(C_g) = -\widetilde{g}\overline{\theta}^{(+)} + \frac{\pi}{2} + 2s_2\pi$, $s_2 \in \mathbb{Z}$. Then, from (32),

$$\overline{\psi}^{(+)} = \overline{\theta}^{(+)} + \frac{(s_1 - s_2)2\pi}{\widetilde{g}}. \quad (33)$$

Note that $\overline{\theta}^{(+)} = \frac{2s_3\pi}{\widetilde{n}}, s_3 \in \mathbb{Z}$ (corresponding to a principal argument of $(+\varepsilon)^{\frac{1}{\widetilde{n}}}$) and that $\overline{\psi}^{(+)} = \frac{2s_4\pi}{\widetilde{n}}, s_4 \in \mathbb{Z}$ (by (30)). According to (33), it must be true that $s_1 - s_2 = 0$, as $\widetilde{n}$ and $\widetilde{g}$ are co-prime. Thus, $\overline{\psi}^{(+)} = \overline{\theta}^{(+)}$. ■

Similarly, from (29) and (31), we have:

Property 5: For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}, 1 \le b \le \eta$, there exists a unique $\psi_{d(b)}^{(-)} \in \Psi_{\mathbb{D}}^{(-)}, 1 \le d(b) \le \eta$, such that $\psi_{d(b)}^{(-)} = \theta_b^{(-)}$. For different $\theta_b^{(-)}$, the corresponding $d(b)$ are different.

Lemma 2: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}, 1 \le a \le \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \le c(a) \le \eta$, satisfying the following properties:

(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.

(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Proof: We start by proving property (1) for $h = 1, \ldots, M(a)-1$ (the degeneracy of $D_n(+\varepsilon j)^{\frac{n}{g}}$ is apparent from (27)). First, assume that $C_{g+h} \ne 0$ for $h = 1, \ldots, M(a)-1$. According to the conditions of the lemma, $((g+h)\theta_a^{(+)} + \mathrm{Arg}(C_{g+h})) \bmod 2\pi = \pm\frac{\pi}{2}$, $h = 0, \ldots, M(a)-1$. It follows that for $h = 1, \ldots, M(a)-1$,

$$(\mathrm{Arg}(C_{g+h}) - \mathrm{Arg}(C_g) + h\theta_a^{(+)}) \bmod 2\pi = 0 \text{ or } \pi. \quad (34)$$

In the light of Property 3, for $h = 1, \ldots, M(a)-1$, $D_{n+h}(+\varepsilon j)^{\frac{n+h}{g}}$ is the sum of a finite number of terms subject to the form $\alpha_{i_1,\ldots,i_h} \prod_{\mathrm{w}=1}^{h} \gamma_{h,\mathrm{w}}^{i_\mathrm{w}} (\frac{+\varepsilon j}{C_g})^{\frac{n+h}{g}}$. For each such term, the argument corresponding to $\psi_{c(a)}^{(+)}$ is calculated by (34) as $(n+h)\psi_{c(a)}^{(+)} - h\theta_a^{(+)} + \kappa\pi, \kappa \in \mathbb{Z}$, having in mind that $i_1 + \cdots + hi_h = h$ and $\alpha_{i_1,\ldots,i_h} \in \mathbb{R}$. Then, according to Property 4 and (30), for $h = 1, \ldots, M(a)-1$, the value of $D_{n+h}(+\varepsilon j)^{\frac{n+h}{g}}$ associated with $\psi_{c(a)}^{(+)}$ is purely real. We may also prove property (1) in the same spirit if some $C_{g+h} = 0$ ($1 \le h \le M(a)-1$) by noting that 0 corresponds to a degenerate term for both the Puiseux series and the dual Puiseux series.

We next consider property (2). Clearly, $C_{g+M(a)} \ne 0$ according to the conditions of the lemma. It follows from Property 3 that $D_{n+M(a)} = (D' + D'')(\frac{1}{C_g})^{\frac{n+M(a)}{g}}$ where $D' = \sum_{i_1+\cdots+M(a)i_{M(a)}=M(a)} \alpha_{i_1,\ldots,i_{M(a)}} \prod_{\mathrm{w}=1}^{M(a)} \gamma_{M(a),\mathrm{w}}^{i_\mathrm{w}}$ with $i_{M(a)} = 0$ and $D'' = \alpha_{0,\ldots,0,1} \frac{C_{g+M(a)}}{C_g}$ with $\alpha_{0,\ldots,0,1} = -\frac{n}{g}$. By the same idea of the proof for property (1), we have that the value of $D'(\frac{+\varepsilon j}{C_g})^{\frac{n+M(a)}{g}}$ associated with $\psi_{c(a)}^{(+)}$ is purely real. Consequently, property (2) can be proved if the following condition holds

$$\begin{aligned} &\mathrm{Arg}\left(D''(\frac{+\varepsilon j}{C_g})^{\frac{n+M(a)}{g}}\right) \\ &-\mathrm{Arg}\left(C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}\right) = \frac{\pi}{2}. \end{aligned} \quad (35)$$

Taking into account (28), (30), and Property 4, we see that (35) is satisfied and hence the proof is complete. ■

Lemma 3: For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}, 1 \le b \le \eta$, satisfying that the corresponding branches of $C_g(-\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(b)-1}(-\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $1 \le d(b) \le \eta$, satisfying the following properties:

(1) For $\psi_{d(b)}^{(-)}$, the corresponding branches of $D_n(-\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(b)-1}(-\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.

(2) For $\theta_b^{(-)}$ and $\psi_{d(b)}^{(-)}$, the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(b)}(-\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

The proof is in the same spirit of that of Lemma 2.

Combining the above discussions, we have:

Lemma 4: Theorem 3 holds if $\widetilde{n}$ is odd and $\widetilde{g}$ is odd.

Proof: For the elements in $\Theta^{(+)} - \Theta_{\mathbb{D}}^{(+)}$ ($\Theta^{(-)} - \Theta_{\mathbb{D}}^{(-)}$), the number of the corresponding values of the Puiseux series (10) in $\mathbb{C}_+$ is denoted by $NU_{\Theta^{(+)}-\Theta_{\mathbb{D}}^{(+)}}(+\varepsilon)$ ($NU_{\Theta^{(-)}-\Theta_{\mathbb{D}}^{(-)}}(-\varepsilon)$). Similarly, for the elements in $\Psi^{(+)} - \Psi_{\mathbb{D}}^{(+)}$ ($\Psi^{(-)} - \Psi_{\mathbb{D}}^{(-)}$), the number of the corresponding values of the dual Puiseux series (11) in $\mathbb{C}_U$ is denoted by $ND_{\Psi^{(+)}-\Psi_{\mathbb{D}}^{(+)}}(+\varepsilon j)$ ($ND_{\Psi^{(-)}-\Psi_{\mathbb{D}}^{(-)}}(-\varepsilon j)$). It follows that $NU_{\Theta^{(+)}-\Theta_{\mathbb{D}}^{(+)}}(+\varepsilon) = NU_{\Theta^{(-)}-\Theta_{\mathbb{D}}^{(-)}}(-\varepsilon)$ and $ND_{\Psi^{(+)}-\Psi_{\mathbb{D}}^{(+)}}(+\varepsilon j) = ND_{\Psi^{(-)}-\Psi_{\mathbb{D}}^{(-)}}(-\varepsilon j)$ for Case 1. Thus, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is determined by the effect of the elements in $\Theta_{\mathbb{D}}^{(+)}$ and $\Theta_{\mathbb{D}}^{(-)}$ on the Puiseux series (10) and $\Delta NF_{z_\alpha}(\omega_\alpha)$ is determined by the effect of the elements in $\Psi_{\mathbb{D}}^{(+)}$ and $\Psi_{\mathbb{D}}^{(-)}$ on the dual Puiseux series (11). From Lemmas 2 and 3, the above two effects are equivalent. ■

B. Case 2: According to the degeneracy condition, $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$ or $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$, $k_2 \in \mathbb{Z}$. For an odd $\widetilde{n}$, there are two possibilities: $\widetilde{n} \bmod 4 = 1$ and $\widetilde{n} \bmod 4 = 3$. In this subsection, we let $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$ and $\widetilde{n} \bmod 4 = 1$.

We have that $\Theta_{\mathbb{D}}^{(+)}$ has η elements satisfying (28) and that $\Theta_{\mathbb{D}}^{(-)}$ has η elements satisfying the condition

$$g\theta_i^{(-)} + \mathrm{Arg}(C_g) = \frac{\pi}{2}(\bmod\ (2\pi)). \quad (36)$$

By (27), $\Psi_{\mathbb{D}}^{(+)}$ contains two subsets ($\Psi_{\mathbb{D}_1}^{(+)}$ and $\Psi_{\mathbb{D}_2}^{(+)}$) and $\Psi_{\mathbb{D}}^{(-)}$ is empty. More precisely, $\Psi_{\mathbb{D}_1}^{(+)}$ has η elements satisfying (30) and $\Psi_{\mathbb{D}_2}^{(+)}$ has η elements satisfying the condition

$$n\psi_i^{(+)} = \pi(\bmod(2\pi)). \quad (37)$$

After proper relabeling, we let $\Theta_{\mathbb{D}}^{(+)} : \{\theta_1^{(+)}, \ldots, \theta_\eta^{(+)}\}$, $\Theta_{\mathbb{D}}^{(-)} : \{\theta_1^{(-)}, \ldots, \theta_\eta^{(-)}\}$, and $\Psi_{\mathbb{D}}^{(+)} : \{\psi_1^{(+)}, \ldots, \psi_{2\eta}^{(+)}\} = \Psi_{\mathbb{D}_1}^{(+)} \cup \Psi_{\mathbb{D}_2}^{(+)}$ where $\Psi_{\mathbb{D}_1}^{(+)} : \{\psi_1^{(+)}, \ldots, \psi_\eta^{(+)}\}$ and $\Psi_{\mathbb{D}_2}^{(+)} : \{\psi_{\eta+1}^{(+)}, \ldots, \psi_{2\eta}^{(+)}\}$.

Property 6: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}, 1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}_1}^{(+)}, 1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. For different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.

Property 7: For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}, 1 \le b \le \eta$, there exists a unique $\psi_{d(b)}^{(+)} \in \Psi_{\mathbb{D}_2}^{(+)}, \eta+1 \le d(b) \le 2\eta$, such that $\psi_{d(b)}^{(+)} = \theta_b^{(-)}$. For different $\theta_b^{(-)}$, the corresponding $d(b)$ are different.

Lemma 5: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}, 1 \le a \le \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \dots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \le c(a) \le \eta$, satisfying the following properties:

(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \dots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.

(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Lemma 6: For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}, 1 \le b \le \eta$, satisfying that the corresponding branches of $C_g(-\varepsilon)^{\frac{g}{n}}, \dots, C_{g+M(b)-1}(-\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $\eta+1 \le d(b) \le 2\eta$, satisfying the following properties:

(1) For $\psi_{d(b)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \dots, D_{n+M(b)-1}(+\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.

(2) For $\theta_b^{(-)}$ and $\psi_{d(b)}^{(+)}$, the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(b)}(+\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_L$ ($\mathbb{C}_U$).

Combining the above properties and lemmas, we have:

Lemma 7: Theorem 3 holds if $\widetilde{n}$ is odd and $\widetilde{g}$ is even.

C. Case 3: According to the degeneracy condition, $\operatorname{Arg}(C_g) = \frac{2k_3\pi}{\widetilde{n}}$ or $\operatorname{Arg}(C_g) = \frac{\pi}{\widetilde{n}} + \frac{2k_3\pi}{\widetilde{n}}, k_3 \in \mathbb{Z}$. For an even $\widetilde{n}$, there are two possibilities: $\widetilde{n} \bmod 4 = 0$ and $\widetilde{n} \bmod 4 = 2$. In this subsection, we let $\operatorname{Arg}(C_g) = \frac{2k_3\pi}{\widetilde{n}}$ and $\widetilde{n} \bmod 4 = 0$.

$\Theta_{\mathbb{D}}^{(+)}$ has two subsets ($\Theta_{\mathbb{D}_1}^{(+)}$ and $\Theta_{\mathbb{D}_2}^{(+)}$), while $\Theta_{\mathbb{D}}^{(-)}$ is empty. $\Theta_{\mathbb{D}_1}^{(+)}$ contains η elements satisfying (28), and $\Theta_{\mathbb{D}_2}^{(+)}$ contains η elements satisfying that

$$g\theta_i^{(+)} + \operatorname{Arg}(C_g) = -\frac{\pi}{2}(\operatorname{mod}(2\pi)). \tag{38}$$

$\Psi_{\mathbb{D}}^{(+)}$ has η elements satisfying (30), and $\Psi_{\mathbb{D}}^{(-)}$ has η elements satisfying that

$$n\psi_i^{(-)} = 0(\operatorname{mod}(2\pi)). \tag{39}$$

With suitable relabeling, we define $\Theta_{\mathbb{D}}^{(+)}: \{\theta_1^{(+)}, \dots, \theta_{2\eta}^{(+)}\} = \Theta_{\mathbb{D}_1}^{(+)} \cup \Theta_{\mathbb{D}_2}^{(+)}$ (where $\Theta_{\mathbb{D}_1}^{(+)}: \{\theta_1^{(+)}, \dots, \theta_\eta^{(+)}\}$ and $\Theta_{\mathbb{D}_2}^{(+)}: \{\theta_{\eta+1}^{(+)}, \dots, \theta_{2\eta}^{(+)}\}$), $\Psi_{\mathbb{D}}^{(+)}: \{\psi_1^{(+)}, \dots, \psi_\eta^{(+)}\}$, and $\Psi_{\mathbb{D}}^{(-)}: \{\psi_1^{(-)}, \dots, \psi_\eta^{(-)}\}$.

Property 8: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}_1}^{(+)}, 1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}}^{(+)}, 1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. For different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.

Property 9: For each $\theta_b^{(+)} \in \Theta_{\mathbb{D}_2}^{(+)}, \eta+1 \le b \le 2\eta$, there exists a unique $\psi_{d(b)}^{(-)} \in \Psi_{\mathbb{D}}^{(-)}, 1 \le d(b) \le \eta$, such that $\psi_{d(b)}^{(-)} = \theta_b^{(+)}$. For different $\theta_b^{(+)}$, the corresponding $d(b)$ are different.

Lemma 8: For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}_1}^{(+)}, 1 \le a \le \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \dots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \le c(a) \le \eta$, satisfying the following properties:

(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \dots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.

(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Lemma 9: For each $\theta_b^{(+)} \in \Theta_{\mathbb{D}_2}^{(+)}, \eta+1 \le b \le 2\eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \dots, C_{g+M(b)-1}(+\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(+\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $1 \le d(b) \le \eta$, satisfying the following properties:

(1) For $\psi_{d(b)}^{(-)}$, the corresponding branches of $D_n(-\varepsilon j)^{\frac{n}{g}}, \dots, D_{n+M(b)-1}(-\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.

(2) For $\theta_b^{(+)}$ and $\psi_{d(b)}^{(-)}$, the corresponding branch of $C_{g+M(b)}(+\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) iff the corresponding branch of $D_{n+M(b)}(-\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_L$ ($\mathbb{C}_U$).

We have the following result for Case 3:

Lemma 10: Theorem 3 holds if $\widetilde{n}$ is even and $\widetilde{g}$ is odd.

The proofs of the properties and lemmas for Cases 2 and 3 are omitted as they follow the same idea as for Case 1.

The proof of Theorem 3 is now complete according to Lemmas 1, 4, 7, and 10.

Acknowledgment

The authors wish to thank P. Pepe, R. Tempo, and M. Krstic, and the anonymous reviewers for their help and constructive comments.

References

[1] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers.* Providence, RI, American Mathematical Society, 1990.

[2] F. G. Boese, "Stability with respect to the delay: On a paper of K. L. Cooke and P. van den Driessche," *J. Math. Anal. Appl.*, vol. 228, pp. 293–321, 1998.

[3] I. Boussaada, I.-C. Morărescu, and S.-I. Niculescu, "Inverted pendulum stabilization: Characterization of codimension-three triple zero bifurcation via multiple delayed proportional gains," *Syst. Control Lett.*, vol. 82, pp. 1–9, 2015.

[4] I. Boussaada and S.-I. Niculescu, "Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials: A Vandermonde-based approach," *IEEE Trans. Autom. Control*, vol. 61, pp. 1601–1606, 2016.

[5] J. W. Bruce and P. J. Giblin, *Curves and Singularities.* Cambridge, U.K., Cambridge Univeristy Press, 1984.

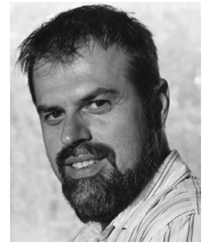
[6] E. Casas-Alvero, *Singularities of Plane Curves.* Cambridge, U.K., Cambridge University Press, 2000.

[7] J. Chen, P. Fu, S.-I. Niculescu, and Z. Guan, "An eigenvalue perturbation approach to stability analysis, Part II: When will zeros of time-delay systems cross imaginary axis?" *SIAM J. Contr. Optim.*, vol. 48, pp. 5583–5605, 2010.

[8] J. Chen and H. A. Latchman, "Frequency sweeping tests for stability independent of delay," *IEEE Trans. Autom. Control*, vol. 40, pp. 1640–1645, 1995.
[9] K. L. Cooke and P. van den Driessche, "On zeros of some transcendental equations," *Funkcialaj Ekvacioj*, vol. 29, pp. 77–90, 1986.
[10] K. Engelborghs, T. Luzyanina, and D. Roose, "Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL," *ACM Trans. Math. Software*, vol. 28, pp. 1–21, 2002.
[11] H. Fazelinia, R. Sipahi, and N. Olgac, "Stability robustness analysis of multiple time-delayed systems using "Building Block" concept," *IEEE Trans. Autom. Control*, vol. 52, pp. 799–810, 2007.
[12] A. R. Fioravanti, *$\mathcal{H}_\infty$ Analysis and Control of Time-Delay Systems by Methods in Frequency-Domain*. Ph.D. Thesis. Université Paris-Sud XI, France, 2011.
[13] W. Fulton, *Algebraic Curves: An Introduction to Algebraic Geometry.* New York, Benjamin, 1969.
[14] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems.* Boston, MA, Birkhäuser, 2003.
[15] E. Jarlebring and W. Michiels, "Invariance properties in the root sensitivity of time-delay systems with double imaginary roots," *Automatica*, vol. 46, pp. 1112–1115, 2010.
[16] M. S. Lee and C. S. Hsu, "On the τ-decomposition method of stability analysis for retarded dynamical systems," *SIAM J. Control*, vol. 7, pp. 242–259, 1969.
[17] X. Li and L. Zhang, "Comments on "Stability of linear time invariant fractional delay systems of retarded type in the space of delay parameters [Automatica 49 (2013) 1287-1294]," *Automatica*, vol. 72, pp. 253–254, 2016.
[18] X.-G. Li, S.-I. Niculescu, and A. Çela, *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays.* London, U.K., Springer, 2015.
[19] X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, and T.-Y. Cai, "On computing Puiseux series for multiple imaginary characteristic roots of LTI systems with commensurate delays," *IEEE Trans. Autom. Control*, vol. 58, pp. 1338–1343, 2013.
[20] X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, and T.-Y. Cai, "On τ-decomposition frequency-sweeping test for a class of time-delay systems. Part I: Simple imaginary roots case," *Proc. 10th IFAC Workshop Time-Delay Syst.*, Boston, MA USA, 2012.
[21] X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, and T.-Y. Cai, "Invariance properties for a class of quasipolynomials," *Automatica*, vol. 50, pp. 890–895, 2014.
[22] A. Mesbahi and M. Haeri, "Stability of linear time invariant fractional delay systems of retarded type in the space of delay parameters," *Automatica*, vol. 49, pp. 1287–1294, 2013.
[23] W. Michiels and S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach.* Philadelphia, PA, SIAM, 2007.
[24] L. I. Nicolaescu, *Notes on the Topology of Complex Singularities*, 2013.
[25] N. Olgac and R. Sipahi, "An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 793–797, 2002.
[26] R. Sipahi and N. Olgac, "Degenerate cases in using the direct method," *ASME J. Dyn. Syst. Measur. Control*, vol. 125, pp. 194–201, 2003.
[27] Ya. Z. Tsypkin, "The systems with delayed feedback," *Avtomathika i Telemech.*, vol. 7, pp. 107–129, 1946.
[28] K. Walton and J. E. Marshall, "Direct method for TDS stability analysis," *Proc. IEE, Part D*, vol. 134, pp. 101–107, 1987.

Xu-Guang Li was born in Shenyang, Liaoning, China, in 1980. He received the Ph.D. degree in automatic control from Shanghai Jiaotong University, China, in 2008.

He was a postdoc at ESIEE, Paris, France, and CNRS-Supélec, France, in 2008 and 2009. He joined Northeastern University, China, in 2010. His research interests include time-delay systems, networked control systems, and application of analytic curve perspective to control systems.



Silviu-Iulian Niculescu received the B.S. degree from the Polytechnical Institute of Bucharest, Romania, in 1992, the M.Sc., and Ph.D. degrees from the Institut National Polytechnique de Grenoble, France, in 1993 and 1996, respectively, and the French Habilitation (HDR) from Université de Technologie de Compiègne, in 2003, all in automatic control.

He is currently Research Director at the CNRS(French National Center for Scientific Research) and the head of L2S (Laboratory of Signals and Systems), a joint research unit of CNRS with CentraleSupélec and Université Paris-Sud located at Gif-sur-Yvette. He is author/coauthor of 9 books and of more than 450 scientific papers. His research interests include delay systems, robust control, operator theory, and numerical methods in optimization, and their applications to the design of engineering systems. He is the responsible of the IFAC Research Group on "Time-delay systems" since its creation in October 2007.

Dr. Niculescu received the CNRS Silver and Bronze Medals for scientific research and the Ph.D. Thesis Award from INPG, Grenoble (France) in 2011, 2001, and 1996, respectively. He served as Associate Editor for several journals in Control area, including the IEEE Transactions on Automatic Control (2003–2005).

Arben Cela received the engineering diploma from the Polytechnic University of Tirana, Albania, in 1984, and the Ph.D. degree and HDR diploma from Université Paris-Sud, in 1993 and 2011, respectively.

From 1984 to 1986, he was with the Department of Power Generation of Faculty of Engineering of Polytechnic University of Tirana, Albania. Currently he is with Computer Science and Telecommunication Department of ESIEE Paris, UPE, Noisy-Le-Grand 93162, France. He is author of more than 85 conference and journal papers in the area of optimization, control of distributed real time systems and delayed systems. His research activities are mainly concerned with stability analysis and optimization of distributed real time systems with particular emphasis on the delayed system modeling and analysis tools.

Lu Zhang was born in Fuxin, Liaoning, China, in 1988. She received the B.S. degree in mathematics science from Inner Mongolia Normal University, China, in 2011 and the M.S. degree in mathematics science from Northeastern University, China, in 2013 where she is currently pursuing the Ph.D. degree in automatic control.

Her research interests include time-delay systems and application of analytic curve perspective to control systems.

Xu Li was born in Shenyang, Liaoning, China, in 1989. He received the B.S. degree in mathematics and applied mathematics from China University of Mining and Technology, in 2012 and the M.S. degree in automatic control from Northeastern University, China, in 2014 where he is currently pursuing the Ph.D. degree in automatic control.

His research interests include time-delay systems, multi-agent systems, and application of analytic curve perspective to control systems.