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CSA-T-TDS: A MATLAB Toolbox for Stability Analysis of Linear Time-Delay Systems

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Abstract: The objective of this paper is to present a MATLAB-based toolbox called *CSA-T-TDS* (acronym for Complete Stability Analysis Toolbox for Time-Delay Systems). By using this toolbox, one can easily find the whole stability delay-set for a linear system with commensurate delays. For a better understanding of the methodology at the origin of the software, the theoretical core of *CSA-T-TDS* (the frequency-sweeping approach together with the auxiliary characteristic function) is discussed. Then, this application of the toolbox will be demonstrated along with some numerical examples.

Keywords: Time-delay systems, complete stability problem, MATLAB-based toolbox, frequency-sweeping approach, auxiliary characteristic function.

1. INTRODUCTION

Time delays exist in dynamical systems in many fields, including engineering, physics, chemistry, biology, and economics. The dynamical systems with time delays are called time-delay systems. The stability of such systems is a classical problem and there exists an abundant literature devoted to the topic (see e.g., Gu et al. (2003), Michiels and Niculescu (2014), and the references therein).

As pointed out in the references above, understanding the effect of delay on system stability is essential. The τ-decomposition method proposed at the end of the 60s by Lee and Hsu (1969) inspired most of the relevant theoretical developments.

In the last two decades, along with the theoretical development, various powerful software toolboxes have been developed, which can be used directly or indirectly for studying the stability of time-delay systems.

For time-delay systems with known delays, some important time-domain and frequency-domain properties can be analyzed by using the Control System Toolbox (see Gumussoy et al. (2012)) in MATLAB. A variety of important analysis and control tasks, for time-delay systems with single known delay, can be studied by using the *LambertW_DDE* Toolbox (see Yi et al. (2010)), which is developed based on the Lambert W function approach. The *PID-Design-Delay* toolbox (see Li et al. (2021) and Fan et al. (2020)) is developed for designing the PID controllers for time-delay systems, from the perspective of complete stability problem, where the delay and the three controller gains are all treated as free parameters.

The TRACE-DDE (see Breda et al. (2009)), DDE-BIFTOOL (see Engelborghs et al. (2002)), and QPmR (see Vyhlidal and Zitek (2009)) are powerful software packages for estimating the distribution of characteristic roots for time-delay systems. The YALTA toolbox (see Avanessoff et al. (2013)) can be used also for fractional time-delay systems.

In this paper, we consider the generic linear time-delay systems with commensurate delays. For such systems, there is only one delay parameter τ. We focus on the *complete stability problem* w.r.t. the delay τ. It is quite fundamental and refers to as the problem of studying the stability of a time-delay system along the whole delay axis. With the solution of complete stability problem, all the stability τ-intervals can be accurately calculated.

Along the τ-decomposition idea Lee and Hsu (1969), the complete stability problem should be studied through addressing the following two subproblems:
Problem 1: Detection of critical imaginary roots (CIRs) and corresponding critical delays (CDs).
Problem 2: Analysis of the asymptotic behavior for CIRs w.r.t. the corresponding CDs.

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Both problems have received an exhaustive treatment in the open literature. A particular technical difficulty is related to the asymptotic behavior for possible multiple and/or degenerate CIRs.

For most people, it is not an easy work to learn to appropriately solve the complete stability problem of time-delay systems, due to the technicality. This motivates us to develop a specific software toolbox, by using which one can easily find the solution of a complete stability problem, without having to know very many theoretical details.

The basic framework of *CSA-T-TDS* is to combine the frequency-sweeping approach (Li et al. (2015, 2017)) and the auxiliary characteristic function (Walton and Marshall (1987)). By using this toolbox, the CIRs can be calculated and then their asymptotic behavior can be determined. As a result, the complete stability problem can be solved systematically and all the stability τ-intervals (if any) can be found.

The *CSA-T-TDS* is MATLAB based and user friendly. The process of studying the complete stability problem, referred to as the *complete stability analysis*, can be fulfilled in a semi-automatic manner (manual intervention is only needed in some specific cases). The *CSA-T-TDS* is freely available. Interested reader may directly contact us via the authors' e-mails.

The remainder of this paper is organized as follows. Some preliminaries and prerequisites are given in Section 2. In Section 3, the theoretical core of *CSA-T-TDS* is introduced. The application of the toolbox is briefly demonstrated in Section 4. Finally, discussion and conclusion are given in Section 5.

Notations: In the sequel, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{C}_+$ ($\mathbb{C}_-$) is the right-half (left-half) plane; $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of (positive) real numbers; $\mathbb{N}$ ($\mathbb{N}_+$) denotes the set of non-negative (positive) integers; $\mathbb{C}_0$ denotes the imaginary axis and $\partial\mathbb{D}$ is the unit circle; ε denotes a sufficiently small positive real number. For $\gamma \in \mathbb{R}$, $\lceil\gamma\rceil$ is the smallest integer greater than or equal to γ.

2. PRELIMINARIES AND PREREQUISITES

Consider a linear system with commensurate delays,

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau), \tag{1}$$

where $x \in \mathbb{R}^n$ ($n \in \mathbb{N}_+$) is the system state, $A_\ell \in \mathbb{R}^{n\times n}$ ($\ell = 0, \dots, m$, $m \in \mathbb{N}_+$) are constant matrices, and $\tau \geq 0$ is the delay parameter.

The characteristic function of (1) is

$$f(\lambda, \tau) = \det(\lambda I - \sum_{\ell=0}^{m} A_\ell e^{-\ell\tau\lambda}), \tag{2}$$

which is a quasipolynomial

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}, \tag{3}$$

where $\lambda \in \mathbb{C}$ is the Laplace variable, $a_0(\lambda), \dots, a_q(\lambda)$ ($q \in \mathbb{N}_+$) are polynomials in λ with real coefficients.

It is well known that the distribution of the characteristic roots of $f(\lambda, \tau)$ determines the stability of time-delay system (1). Time-delay system (1) is asymptotically stable if and only if all the characteristic roots are located in the left half-plane $\mathbb{C}_-$.

As mentioned, in this paper we concentrate on the complete stability problem for time-delay system (1). Recently, a new frequency-sweeping approach is proposed (see Li et al. (2015, 2017)). By using it, Problems 1 and 2 mentioned in Introduction can be addressed in a unified framework and, as a result, the complete stability problem can be systematically solved.

Now, recall some useful notations. We use $NU(\tau) \in \mathbb{N}$ to denote the number of characteristic roots in the right-half plane $\mathbb{C}_+$. In order to fully solve the complete stability problem, it is necessary to obtain the $NU(\tau)$ distribution along the τ-axis. According to the definition in Li et al. (2015), $(\lambda_\alpha, \tau_{\alpha,k})$ is called a critical pair if λ_α is a CIR and $\tau_{\alpha,k}$ is a CD, and $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) \in \mathbb{N}$ stands for the change of $NU(\tau)$ due to the variation of the CIR λ_α as τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$.

3. THEORETICAL CORE OF *CSA-T-TDS*

3.1 Frequency-sweeping approach

Letting $z = e^{-\tau\lambda}$, we can rewrite the characteristic function $f(\lambda, \tau)$ (3) as

$$p(\lambda, z) = a_0(\lambda) + a_1(\lambda)z + \cdots + a_q(\lambda)z^q. \tag{4}$$

The frequency-sweeping curves (FSCs) can be generated as follows: Sweep $\omega \geq 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p(j\omega, z) = 0$ (denoted by $z_1(j\omega), \dots, z_q(j\omega)$). In this way, we obtain q FSCs $\Gamma_i(\omega) : |z_i(j\omega)|$ vs. ω, $i = 1, \dots, q$.

If $(\lambda_\alpha = j\omega_\alpha, \tau_{\alpha,k})$ is a critical pair, some FSCs intersect the line $\Im_1$ (we denote the line parallel to the abscissa with ordinate equal to 1 by $\Im_1$) at $\omega = \omega_\alpha$, which is called *critical frequency*. It is seen that *Problem 1* can be solved by means of the FSCs.

If $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$, being a set of critical pairs, there must exist some FSCs such that $z_i(j\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$ intersecting $\Im_1$ when $\omega = \omega_\alpha$. Among the FSCs, we denote the number of those above the line $\Im_1$ when $\omega = \omega_\alpha + \varepsilon$ ($\omega = \omega_\alpha - \varepsilon$) by $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$ ($NF_{z_\alpha}(\omega_\alpha - \varepsilon)$). Now introduce a notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ to describe the asymptotic behavior of FSCs. Define

$$\Delta NF_{z_\alpha}(\omega_\alpha) = NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \tag{5}$$

Theorem 1. For a critical imaginary root λ_α of time-delay system (1), it always holds that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > 0$.

Theorem 1 is rather significant. First, the invariance property allows to overcome the peculiarity that a CIR has *infinitely many* CDs (it is impossible to analyze a CIR's asymptotic behavior at all the CDs one by one). Second, the value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ can be computed in a geometric manner.

Without loss of generality, suppose that there are u critical pairs (λ, z) ($\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$) for $p(\lambda, z) = 0$: $(\lambda_0 = j\omega_0, z_0), \dots, (\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ where $\omega_0 > \cdots > \omega_{u-1} > 0$. For each CIR λ_α, we can obtain the

CDs by $\tau_{\alpha,k} = \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}, k \in \mathbb{N}$, where $\tau_{\alpha,0} = \min\{\tau \geq 0 : e^{-\tau\lambda_\alpha} = z_\alpha\}$. The value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ for any critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ can be denoted by U_{λ_α}. We also can directly obtain $U_{\lambda_\alpha} = \Delta NF_{z_\alpha}(\omega_\alpha)$ from the FSCs.

Theorem 2. Consider the characteristic function $f(\lambda, \tau)$ (3). For any $\tau > 0$ which is not a critical delay, we can obtain the explicit expression of $NU(\tau)$:

$$NU(\tau) = NU(+\varepsilon) + \sum\nolimits_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{6}$$

where

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,0}, \end{cases} \text{if } \tau_{\alpha,0} \neq 0,$$

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,1}, \end{cases} \text{if } \tau_{\alpha,0} = 0.$$

The quantity $NU(+\varepsilon)$ can be calculated in light of Theorem 5.1 (Li et al. (2015)) or Theorem 1 (Li et al. (2017)). Based on Theorem 2, we can obtain the solution of complete stability problem for time-delay system (1).

3.2 Auxiliary characteristic function

The frequency-sweeping approach recalled in the last subsection normally needs to be performed manually. In order to solve the complete stability problem in a more automatic manner, we integrate the auxiliary characteristic function into the frequency-sweeping approach.

For the characteristic function $f(\lambda, \tau)$ (3), let

$$a_k^{(1)}(\lambda) = a_0(-\lambda)a_k(\lambda) - a_q(\lambda)a_{q-k}(\lambda), \tag{7}$$

where $k = 0, \ldots, q-1$. And define

$$a_k^{(r+1)}(\lambda) = a_0^{(r)}(-\lambda)a_k^{(r)}(\lambda) - a_{q-r}^{(r)}(\lambda)a_{q-r-k}^{(r)}(-\lambda), \tag{8}$$

where $r = 1, \ldots, q-1$, $k = 0, \ldots, q-r-1$.

The auxiliary characteristic function $F(W)$ is given by

$$F(W) = a_0^{(q)}(j\omega), \tag{9}$$

with $W = \omega^2$. As the right-hand side of (9) is a polynomial of ω with only even orders, it can be denoted by a function of W.

Obviously, we are only interested to consider the positive real W roots of $F(W) = 0$. If $\lambda = j\omega \neq 0$ is a CIR for time-delay system (1), then $W = \omega^2$ must be a positive real W root of $F(W) = 0$. Such a W root is called an *effective W root.*

Conversely, if a $W \in \mathbb{R}_+$ satisfies that $F(W) = 0$, $j\sqrt{W}$ is not necessarily a CIR for time-delay system (1). Such a W root is called a *fake W root.*

3.3 Basic framework

In this subsection, we explain how the frequency-sweeping approach and the auxiliary characteristic function work together in the framework of *CSA-T-TDS*.

For the auxiliary characteristic function $F(W)$ (9), we can easily find the set containing all the positive real W roots such that $F(W) = 0$. For simplicity, we denote this set by S_F, then $W_\alpha = \omega_\alpha^2$ must belong to the set S_F [1]. However, an element in S_F does not necessarily correspond to a CIR, i.e., a positive real W root with $F(W) = 0$ does not mean that $j\sqrt{W}$ must be a CIR. As mentioned, such positive real W roots are fake W roots.

According to the FSCs, the number of effective W roots can be determined and hence the fake W roots (if any) can be filtered out. Then, all the CIRs and the associated CDs can be computed.

Furthermore, the value of U_{λ_α} for each CIR $\lambda_\alpha = j\omega_\alpha$ can be determined in the light of FSCs' asymptotic behavior, which can be described by the notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ (as introduced in Subsection 3.1).

Along the above idea, a complete stability problem can be largely solved by *CSA-T-TDS* [2].

4. APPLICATION OF *CSA-T-TDS*

In this section, we will introduce how to apply *CSA-T-TDS*. As mentioned, the complete stability analysis is performed in a semi-automatic manner. In general, the implementation is simple. The user mainly uses "Previous" button and "Next" button.

In each dialog box of *CSA-T-TDS*: Click "Previous" button to return to the previous step and the user may check or correct the previous operations. When finishing the needed operations in the current dialog box, click "Next" button to move on to the next step of complete stability analysis.

In the following two subsections, the case $q = 1$ and the case $q > 1$ will be separately addressed by using *CSA-T-TDS.*

4.1 Application in case $q = 1$

When $q = 1$, the characteristic function (3) is reduced to

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}, \tag{10}$$

where

$$a_0(\lambda) = b_{\mathrm{m}}\lambda^{\mathrm{m}} + b_{\mathrm{m}-1}\lambda^{\mathrm{m}-1} + \cdots + b_0,$$
$$a_1(\lambda) = c_{\mathrm{n}}\lambda^{\mathrm{n}} + c_{\mathrm{n}-1}\lambda^{\mathrm{n}-1} + \cdots + c_0.$$

When the toolbox is started, the user needs to enter the information concerning characteristic function (10), following the prompts.

The inputs to *CSA-T-TDS* in the command window are as follows.

Input "1" after "$q =$".
Input the coefficient vector "$[b_{\mathrm{m}}, b_{\mathrm{m}-1}, \ldots, b_0]$" of $a_0(\lambda)$ after "a0=".
Input the coefficient vector "$[c_{\mathrm{n}}, c_{\mathrm{n}-1}, \ldots, c_0]$" of $a_1(\lambda)$ after "a1=".

Then, the type of time-delay system (retarded, neutral, or advanced type) is known and the 'Settings' dialog box will pop up for the user to provide some necessary information (refer to Fig. 1).

[1] Since $F(W)$ is a real-coefficient polynomial, the set S_F can be (sufficiently) accurately calculated by computer.
[2] In most cases, extra calculation by the user is not needed.

The inputs for edit boxes in 'Settings' dialog box are as follows.

- "tau_max": Enter a positive value τ_{max}. The delay τ is treated as a free parameter with $\tau \in [0, \tau_{max}]$.
- "omega_max": Enter a positive value for ω_{max}. The FSCs will be generated for $\omega \in [0, \omega_{max}]$.
- "number of grids for FSCs": Enter a positive value N_ω. The step length of ω for generating the FSCs is $\frac{\omega_{max}}{\mathrm{N}_\omega}$.
- "number of grids for NU(tau)": Enter a positive value N_τ. The step length of τ for plotting $NU(\tau)$ is $\frac{\tau_{max}}{\mathrm{N}_\tau}$.

Some explanation regarding the above inputs is as follows. ω_{max} and N_ω (τ_{max} and N_τ) are, respectively, the range and number of grids for plotting the FSCs ($NU(\tau)$). In practice, we need to set ω_{max}, N_ω, τ_{max}, and N_τ sufficiently large. With the aid of outputs of *CSA-T-TDS*, it is easy to appropriately set them.

Then, the 'Frequency-Sweeping Curve' and 'W roots' dialog boxes (refer to Fig. 2) will pop up. The user can tick the effective W roots.

For auxiliary characteristic function (9), if there is no effective W root for $F(W) = 0$, click the check box at the bottom of the 'W roots' dialog box. The result of complete stability analysis will be straightforwardly obtained: Such system is delay-independent stable or unstable.

After the effective W roots are checked, the dialog boxes for determining the number of associated FSCs to each critical frequency[3] and the corresponding z satisfying $|z| = 1$ will pop up in sequence (refer to Fig. 3(a) and Fig. 3(b)).

Then, the 'Delta NU' dialog box (refer to Fig. 4(a)) will pop up, where the values of associated $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ are calculated by *CSA-T-TDS*. The user can correct such values, if necessary, in the light of FSCs.

The 'characteristic roots when tau=0' dialog box will appear (refer to Fig. 4(b)). If there are CIRs at $\tau = 0$, the user needs to click the check box at the bottom of the dialog box and then manually enter the value of $NU(+\varepsilon)$ in the pop-up dialog box.

Finally, the full solution for complete stability problem will be given in command window and the $NU(\tau)$ plot will be displayed in 'NU Distribution' dialog box (refer to Fig. 5).

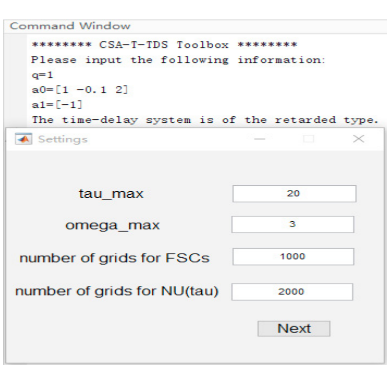


Fig. 1. Inputs in 'Command Window' and 'Settings' dialog box for Example 1

Example 1. Consider a time-delay system with the characteristic function

$$f(\lambda, \tau) = -e^{-\tau\lambda} + \lambda^2 - 0.1\lambda + 2. \quad (11)$$

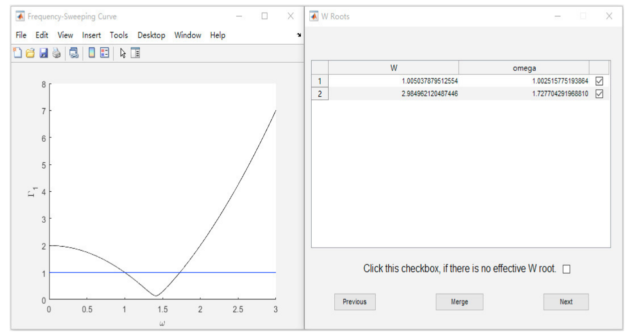
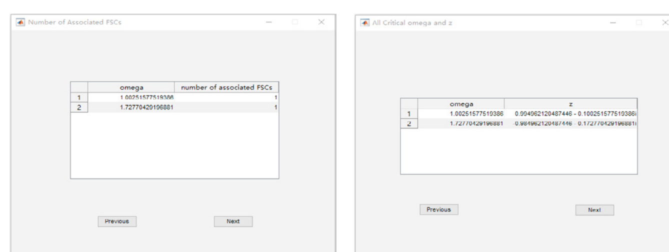


Fig. 2. 'Frequency-Sweeping Curve' and 'W Roots' dialog boxes for Example 1

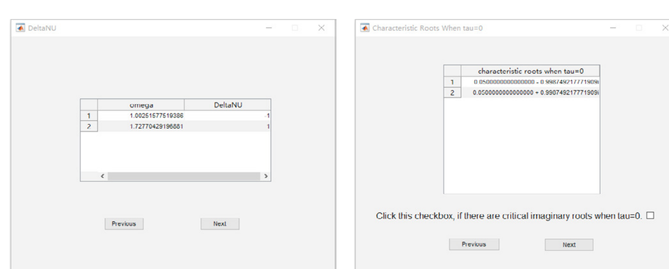
We first enter "1", "[1 -0.1 2]", and "[-1]" after "q=", "a0=", and "a1=" in the command window, respectively. Next, we fill "tau_max" edit box with "20", "omega_max" edit box with "3", "number of grids for FSCs" edit box with "1000" and "number of grids for NU(tau)" edit box with "2000" in 'Settings' dialog box. The above inputs are as shown in Fig. 1.



(a) 'Number of Associated FSCs' (b) 'All Critical omega and z'

Fig. 3. 'Number of Associated FSCs' and 'All Critical omega and z' dialog boxes for Example 1

Check the effective W roots in 'W roots' dialog box (Fig. 2). The critical frequencies and the number of associated FSCs are shown (Fig. 3(a)) and then the corresponding z values with $|z| = 1$ are listed (Fig. 3(b)). Furthermore, the values of ΔNU for all CIRs are given (Fig. 4(a)).



(a) 'DeltaNU' (b) 'Characteristic Roots When tau=0'

Fig. 4. 'DeltaNU' and 'Characteristic Roots When tau=0' dialog boxes for Example 1

[3] Apparently, in the case $q = 1$, the number of FSCs intersecting $\Im_1$ at a critical frequency is necessarily 1.

Then, the characteristic roots when $\tau = 0$ are listed in the dialog box shown in Fig. 4(b). We can see that there is no CIR when $\tau = 0$.

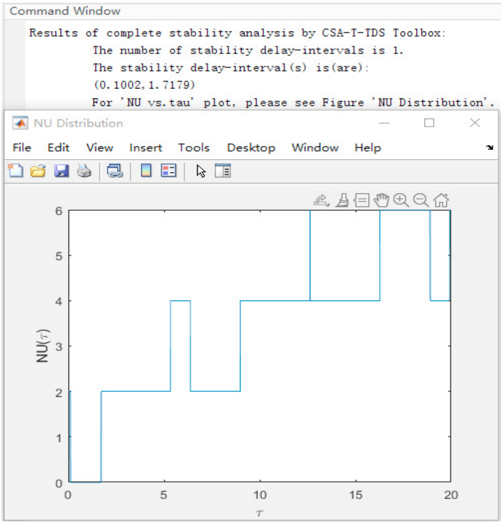


Fig. 5. Final results given in 'Command Window' and 'NU Distribution' dialog box for Example 1

Finally, the complete stability problem can be solved by *CSA-T-TDS*, and the result is presented as shown in Fig. 5. □

4.2 Application in case $q > 1$

Compared to the case $q = 1$, there may be some additional technical issues in the case $q > 1$. In the sequel, we give an example where a fake W root appears and illustrate how to handle it when using *CSA-T-TDS*. More additional technical issues will be discussed in the full version of paper.

Example 2. Consider a time-delay system with the characteristic function

$$f(\lambda, \tau) = a_3(\lambda)e^{-3\tau\lambda} + a_2(\lambda)e^{-2\tau\lambda} + a_1(\lambda)e^{-\tau\lambda} + a_0(\lambda), \tag{12}$$

where $a_3(\lambda) = \lambda - 20.3$, $a_2(\lambda) = \lambda^2 + 8.7\lambda + 7.4$, $a_1(\lambda) = 0.8\lambda^2 + 2\lambda + 5.8$, and $a_0(\lambda) = \lambda^3 + 5\lambda^2 + 7.3\lambda + 10.4$.

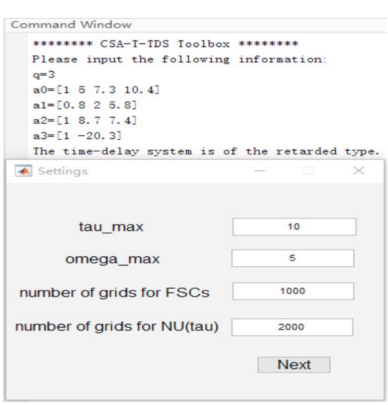


Fig. 6. Inputs in 'Command Window' and 'Settings' dialog box for Example 2

After inputting the required information as shown in Fig. 6, the 'Frequency-Sweeping Curves' and 'W roots' dialog boxes will pop up as shown in Fig. 7.

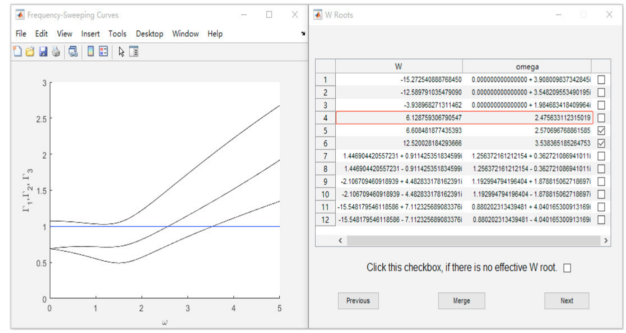
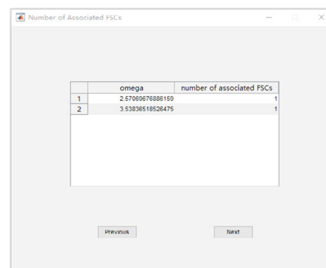
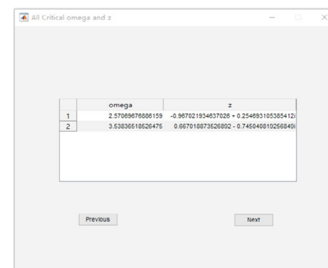


Fig. 7. 'Frequency-Sweeping Curves' and 'W Roots' dialog boxes for Example 2



(a) 'Number of Associated FSCs'

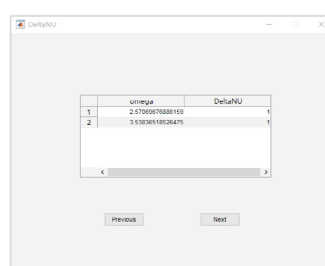


(b) 'All Critical omega and z'

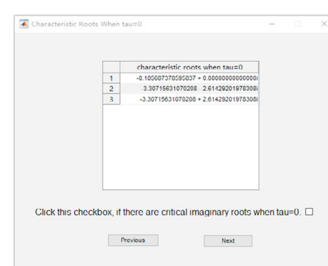
Fig. 8. 'Number of Associated FSCs' and 'All Critical omega and z' dialog boxes for Example 2

From the FSCs shown in Fig. 7, it can be found that there are two CIRs. However, three positive real W roots are obtained (see the 'W roots' dialog box). Thus, there exists a fake W root, which corresponds to the 4th row of table in the 'W Roots' dialog box shown in Fig. 7.

Click the corresponding two check boxes in 'W roots' dialog box to select the two effective W roots. Consequently, the fake W root is filtered out in this way.



(a) 'DeltaNU'



(b) 'Characteristic Roots When tau=0'

Fig. 9. 'DeltaNU' and 'Characteristic Roots When tau=0' dialog boxes for Example 2

The remaining steps are similar to those for Example 1 and as shown in Fig. 8(a) - Fig. 9(b). The solution of complete stability problem is given as shown in Fig. 10. □

Recall the following properties: In the case $q = 1$, 'W^* is an effective W root' is a *necessary and sufficient condition* for '$\lambda^* = j\sqrt{W^*}$ is CIR'. While, in the case $q > 1$, the former is only a *necessary condition* for the latter.

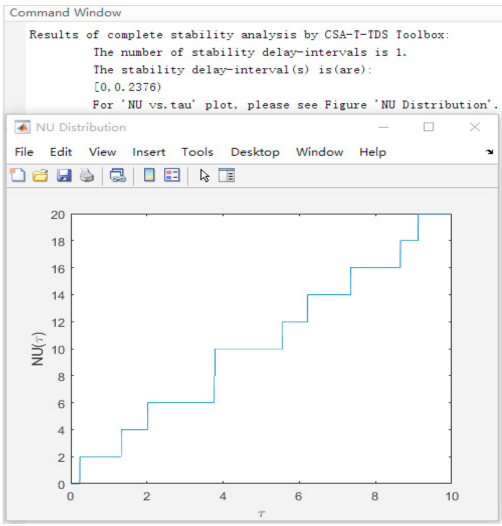


Fig. 10. Final results given in 'Command Window' and 'NU Distribution' dialog box for Example 2.

Thus, in the case $q > 1$, some fake W roots may appear. As illustrated in Example 2, one can easily distinguish the fake W roots in the light of FSCs.

5. DISCUSSION AND CONCLUSION

The complete stability problem is one the most fundamental theoretical problems for time-delay systems. It is a nontrivial work to find the solution, especially for those who are not sufficiently familiar with the stability study of time-delay systems. In this paper, we introduced a MATLAB-based toolbox *CSA-T-TDS* (initials for Complete Stability Analysis Toolbox for Time-Delay Systems).

By using this toolbox, one can easily solve the complete stability problem for a linear time-delay system with commensurate delays. The whole process of complete stability analysis can be performed in a semi-automatic manner, and all the stability delay intervals (if any) can be precisely calculated. The *CSA-T-TDS* is user friendly and the user does not have to understand the stability of time-delay systems very deeply.

The objective of this paper is to briefly demonstrate the application of *CSA-T-TDS*, along with two representative examples. Related theoretical issues will be specifically discussed and reported in another paper. For instance, an important technical point is as follows.

In some cases, Problem 1 and Problem 2 mentioned in Introduction may be entangled and it is difficult to determine the structural classifications of CIRs. Such cases will be studied with the aid of some polynomial algebra tools (see e.g., Xia and Yang (2016) and Yang (1999)). In this way, the "frequency-sweeping approach plus auxiliary characteristic function" methodology, used in this paper, will be complete qualitatively as well as quantitatively.

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