Lexicographic Breadth First Search – A Survey

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Abstract. Lexicographic Breadth First Search, introduced by Rose, Tarjan and Lueker for the recognition of chordal graphs is currently the most popular graph algorithmic search paradigm, with applications in recognition of restricted graph families, diameter approximation for restricted families and determining a dominating pair in an AT-free graph. This paper surveys this area and provides new directions for further research in the area of graph searching.

1 Introduction

Graph searching is a fundamental paradigm that pervades graph algorithms. A search of a graph visits all vertices and edges of the graph and will visit a new vertex only if it is adjacent to some previously visited vertex. Such a generic search does not, however, indicate the rules to be followed in choosing the next vertex to be visited. The two fundamental search strategies are *Breadth First Search* (BFS) and *Depth First Search* (DFS). As the names indicate, BFS visits all previously unvisited neighbours of the currently visited vertex before visiting the previously unvisited non-neighbours, whereas DFS follows unvisited edges (if possible) from the most recently visited vertex. Both searches seem to have been "discovered" in the 19th century (and probably earlier) as algorithms for maze traversal. DFS, as popularized by Tarjan [41], has been used for such diverse applications as connectivity, planarity, topological ordering and strongly connected components of digraphs. BFS has been applied to shortest path problems, network flows and the recognition of various graph classes.

In the mid 1970s, Rose, Tarjan and Lueker [42] introduced a variant of BFS called *Lexicographic Breadth First Search* (LBFS). Their application of LBFS was to the recognition of chordal graphs. This algorithm is one of the classic graph algorithms and, given the current interest in LBFS, it is somewhat surprising that little work was done on LBFS until the mid 1990s.

In this paper, we survey many of the applications of LBFS (in Section 4). Before doing so, we provide the graph theoretical background for the paper as well as a description of LBFS and its two most common variants (Section 2) and, in Section 3, present some LBFS structural results. Concluding remarks are made in the final section.

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2 Background

Before presenting LBFS and its various variants, we give some relevant definitions. We start with standard graph theoretical definitions and then define various graph families and indicate some characterizations that will be used in the relevant LBFS algorithms. Further information regarding the definitions and families can be found in [6].

2.1 Definitions and Notation

All graphs will be assumed to be undirected and finite. For a graph $G(V,E)$, we use n to denote $|V|$ and m to denote $|E|$. K_n, C_n and P_n denote the Clique, Cycle and Path respectively on n vertices. A *House, Hole* and *Domino* are respectively: a C_4 sharing an edge with a K_3; an induced $C_k, k > 4$; a pair of C_4s sharing an edge. A subset of vertices M is a *module* if for all vertices $x, y \in M$ and $z \in V \setminus M, xz \in E$ if and only if $yz \in E$. Module M is *trivial* if $M = V, M = \emptyset$ or $|M| = 1$. A *maximal clique module* is a module that is a clique and is maximal with respect to both properties. Subset S of V is a *separator* if the graph induced on $V \setminus S$ is disconnected. A *moplex* is a maximal clique module whose neighbourhood is a minimal separator.

The *distance* between two vertices u and v is the length of a shortest path between u and v and is denoted $d(u,v)$. For vertex $v, ecc(v)$, the *eccentricity* of v is the length of a longest shortest path with v as an endpoint. The *diameter* ($diam(G)$) is the maximum eccentricity of all vertices in G. A vertex is *simplicial* if its neighbourhood is a clique. An ordering $v_1, v_2, \cdots, v_n$ of V is a *perfect elimination ordering* (PEO) if for all $i, 1 < i \leq n, v_i$ is simplicial in the graph induced on $v_1, \cdots, v_i$. A vertex v is *semisimplicial* if v is not the midpoint of any induced P_4. An ordering $v_1, v_2, \cdots, v_n$ of V is a *semiperfect elimination ordering* if for all $i, 1 < i \leq n, v_i$ is semisimplicial in the graph induced on $v_1, \cdots, v_i$. A vertex v is *2-simplicial* if there is no induced P_4 in the graph induced on $\{u : d(u,v) \leq 2\}$. An ordering $v_1, v_2, \cdots, v_n$ of V is a *2-simplicial elimination ordering* if for all $i, 1 < i \leq n, v_i$ is 2-simplicial in the graph induced on $v_1, \cdots, v_i$.

We say that path P *misses* vertex v if $P \cap N(v) = \emptyset$ (i.e., no vertex of P is adjacent to v). A path P is a *dominating path* if no vertex of G is missed by P. A pair of vertices x, y is a *dominating pair* if every path between x and y is a dominating path. Two vertices x, y are *unrelated with respect to vertex* v if there are paths P between x and v and Q between y and v such that P misses y and Q misses x. An independent triple of vertices x, y, z is an *Asteroidal Triple (AT)*, if between every pair of vertices, there is a path that misses the third. A vertex v is *admissible* if there are no unrelated vertices with respect to v. An ordering $v_1, v_2, \cdots, v_n$ of V is an *admissible elimination ordering* (AEO) if for all $i, 1 < i \leq n, v_i$ is admissible in the graph induced on $v_1, \cdots, v_i$.

For $t \geq 1$, an ordering $v_1, v_2, \cdots, v_n$ of V is a *strong t-cocomparability ordering* (strong t-CCPO) if for all $i, j, 1 \leq i < j < k \leq n, d(v_i, v_k) \leq t$ implies $d(v_i, v_j) \leq t$ or $d(v_j, v_k) = 1$. Note that a graph is a *cocomparability* graph (there is a