Partial derivatives.

Let $z = f(x,y)$ be a function of 2 varia[bles] x and y. If we keep y as constant a[nd] vary x alone, then z becomes a funct[ion] of x only. Now, the derivative of z w[.r.t.] x, treating y as constant, is called the partial derivative of z w.r.t x and [is] denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or z_x et[c.]

Thus $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} f_x$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

Similarly,

$$\frac{\partial z}{\partial y} = f_y$$

$$= \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Here f_x & f_y are also functions of x and y and hence they can be differentiated further w.r.t x & y.

Thus

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}$$

Let (a,b) be any point ~~of~~ the domain of definition of a function $f(x,y)$.

Then

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

or

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(f_x)$$

$$f_{yy}(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y)$$

$$= \lim_{k\to 0} \frac{f_y(a, b+k) - f_y(a,b)}{k}$$

$$f_{xy}(a,b) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y)$$

$$= \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

Ex1: $f(x,y) = x^3y^2 + e^{xy^2}$.
find $\frac{\partial f}{\partial y}$ & $\left(\frac{\partial f}{\partial x}\right)_{(1,0)}$

Soln.
$$f(x,y) = x^3y^2 + e^{xy^2}.$$
$$\frac{\partial f}{\partial x} = 3x^2y^2 + e^{xy^2}.y^2$$
$$\text{or, } \left(\frac{\partial f}{\partial x}\right)_{(1,0)} = 3.1^2.0 + 0.e^0$$
$$= 0.$$

$$\frac{\partial f}{\partial y} = 2x^3y + 2xy.e^{xy^2},$$

Ex2: If $v = \log(x^2+y^2+z^2)$, prove that
$$(x^2+y^2+z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2.$$

Soln. Here $v = \log(x^2+y^2+z^2)$
Differentiating both sides w.r.t x, we get,
$$\frac{\partial v}{\partial x} = \frac{1.2x}{x^2+y^2+z^2} = \frac{2x}{x^2+y^2+z^2}$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2+z^2}\right)$$
$$= \frac{(x^2+y^2+z^2).2 - 2x.2x}{(x^2+y^2+z^2)^2}$$
$$= \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}$$

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$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2+z^2}.$$

$$\frac{\partial^2 v}{\partial y^2} = \cancel{x^2+y} \; \frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}$$

&

$$\frac{\partial^2 v}{\partial z^2} = \frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}.$$

Now,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

$$= \frac{2(y^2+z^2-\cancel{x^2}+\cancel{x^2}+\cancel{z^2}-\cancel{y^2}+x^2+\cancel{y^2}-\cancel{z^2})}{(x^2+y^2+z^2)^2}$$

$$= \frac{2\cancel{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^{\cancel{2}}}$$

$$\text{or, } (x^2+y^2+z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$$

Ex3. If $z(x+y) = x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Soln:- Given $z(x+y) = x^2+y^2$ —— ①

diff. both sides w.r.t. x, we get

$$\frac{\partial z}{\partial x}(x+y) + z(1+0) = 2x.$$

or, $(x+y)\frac{\partial z}{\partial x} = 2x - z$

or, $\frac{\partial z}{\partial x} = \frac{2x-z}{x+y}$.

diff. both sides of ① w.r.t y, we

$$\frac{\partial z}{\partial y}(x+y) + z(0+1) = 2y$$

or, $\frac{\partial z}{\partial y} = \frac{2y-z}{x+y}$.

$$L.H.S = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{2x-z}{x+y} - \frac{2y-z}{x+y}\right)^2$$

$$= \left(\frac{2x-2y}{x+y}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

$$R.H.S. = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= \frac{4}{x+y}\left(x+y-2x+z-2y+z\right)$$

$$= \frac{4}{x+y}\left(-x-y+2z\right)$$

$$= \frac{4}{x+y}\left[-(x+y) + 2\frac{(x^2+y^2)}{x+y}\right]$$

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