Jacobians

If $u(x,y)$ and $v(x,y)$ are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v w.r.t x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Similarly, the Jacobian of u, v, w w.r.t. x, y, z is $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J\left(\frac{u,v,w}{x,y,z}\right)$.

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Ex. If $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Soln. $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Now, $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

Here $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

$$\therefore \frac{\partial r}{\partial x} = \frac{1 \cdot 2x}{2\sqrt{x^2+y^2}}, \quad \frac{\partial r}{\partial y} = \frac{1 \cdot 2y}{2\sqrt{x^2+y^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2}}, \quad = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}. \text{ Ans.}$$

Note. Here $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1.$

Ex 2. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$,

then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln $$\frac{\partial u}{\partial x} = \frac{(1-xy).1-(x+y).(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)\cdot 1 - (x+y)\cdot(-x)}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= 0. \quad \text{Ans.}$$

Ex 3. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.
then show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$

Soln. Here $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

$$\therefore \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ -\frac{x_2 x_3}{x_1^2} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_3 x_2 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_2^2 x_1^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$$= -1(-2) + 1(2) = 2 + 2 = 4. \text{ Ans.}$$

Properties of Jacobians.

1. If $u(x,y)$ & $v(x,y)$ are functions of 2 independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

Proof: Let $u = f(x,y)$, $v = g(x,y)$

suppose on solving for x and y, we get,

$x = \phi(u,v)$, $y = \psi(u,v)$.

Then
$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \text{ (proved)}$$

Prop. (Chain Rule)

(2) If u, v are functions of r, s and r, s are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}.$$

Proof:-

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

classmate

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Note. If u, v, w are functions of r, s, t and r, s, t are functions of x, y, z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$$

Prop.

(3) If the functions u, v, w of independent variables x, y, z are not independent, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Proof. Since u, v, w are not independent, therefore there exists a relation $f(u,v,w) = 0$. —(1)

Differentiating both sides w.r.t x, y, z, we get,

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x} = 0.$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y} = 0.$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z} = 0$$

or,

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial w} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ are not zero simultaneous (because if $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial w} = 0$ then f is independent of u, v, w which contradi our assumption).

therefore

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

Interchanging rows & columns, we have

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0.$$

or, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

Note. If the functions $u(x,y)$, $v(x,y)$ of two independent variables x, y are not independent, then

$$\frac{\partial(u,v)}{\partial(x,y)} = 0.$$

Ex 4 If $x = uv$, $y = \dfrac{u-v}{u+v}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Soln. Here $\dfrac{\partial x}{\partial u} = v$, $\dfrac{\partial x}{\partial v} = u$, $\dfrac{\partial y}{\partial u} = \dfrac{(u+v).1-(u-v).1}{(u+v)^2}$

$$= \frac{2v}{(u+v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{(u+v).(-1)-(u-v).1}{(u+v)^2} = \frac{-2u}{(u+v)^2}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{2v}{(u+v)^2} & \frac{-2u}{(u+v)^2} \end{vmatrix} = \frac{-4uv}{(u+v)^2}$$

using the ~~...~~ prop. ① $\left(\text{ie } \dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1\right)$,

we get, $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}} = -\dfrac{(u+v)^2}{4uv}$

Ex 5: If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$, determine $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. (can use prop. 1)

Ex 6. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2 - y^2$,

$v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$.

Soln. Here $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$
$= 4r^2$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$ (chain rule)

$= 4r^2 . r = 4r^3.$

Ex 7. If $u = xy - yz - zx$, $v = x^2+y^2+z^2$, and $w = x+y-z$, then determine whether they (u,v,w) are dependent or not. If so, then find the relationship between them.

Soln: $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y-z & x-z & -y-x \\ 2x & 2y & 2z \\ 1 & 1 & -1 \end{vmatrix}$

$= (y-z)(-2y-2z) - (x-z)(-2x-2z) - (x+y)(2x-2y)$

$= -2(y^2-z^2) + 2(x^2-z^2) - 2(x^2-y^2) = 0.$

$= \cancel{-2y^2} + \cancel{2z^2} + \cancel{2x^2} - \cancel{2z^2} - \cancel{2x^2} + \cancel{2y^2}$

$\therefore$ u, v, w are not independent ie they are dependent.

Now,

$$w^2 = (x+y-z)^2$$
$$= x^2+y^2+z^2+2(xy-xz-yz).$$
$$= v + 2u.$$

or, $w^2 - v - 2u = 0$. Ans.

Ex. 8: Verify whether the following functions are functionally dependent and if so, find it,

$$u = \frac{x-y}{1+xy}, \quad v = \tan^{-1}x - \tan^{-1}y$$