U, V be two functions of x

then $(UV)_1 = U_1 V + U V_1$

where the suffixes denotes the order of differentiation.

then $(UV)_2 = (U_1 V + U V_1)_1$

$= (U_1 V)_1 + (U V_1)_1$

$= U_2 V + U_1 V_1 + U_1 V_1 + U V_2$

$= U_2 V + 2 U_1 V_1 + U V_2$

$= U_2 V + {}^2C_1\, U_1 V_1 + U V_2$

Leibnitz's Theorem on Successive Derivatives :-

If u and v be two functions of x, both derivable at least upto n times, then $y = uv$ is derivable n times and the n th derivatives of y is $y_n = (uv)_n$ is given by—

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n .$$

where the suffixes denotes the order of differentiation.

proof By direct differentiation, we get

$(uv)_1 = u_1 v + u v_1$

$(uv)_2 = u_2 v + 2u_1 v_1 + u v_2$

$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$

thus the theorem is true for $n=1$ and $n=2$

Let us assume that the theorem is true for a certain +ve integer m $(m < n)$, say.

so, $(uv)_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots$
$$\cdots + {}^mC_r u_{m-r} v_r + \cdots + {}^mC_m u v_m .$$

now differentiating both sides once more we get —

$$(uv)_{m+1} = \{u_{m+1} v + u_m v_1\} + {}^mC_1 \{u_m v_1 + u_{m-1} v_2\} + \cdots$$
$$\cdots + {}^mC_r \{u_{m-r+1} v_r + u_{m-r} v_{r+1}\} + \cdots + {}^mC_m \{u_1 v_m + u v_{m+1}\}$$

$$= u_{m+1} v + (1 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2 + \cdots$$
$$\cdots + ({}^mC_{r-1} + {}^mC_r) u_{m-r+1} v_r + \cdots + ({}^mC_{m-1} + {}^mC_m) u_1 v_m$$
$$+ u v_{m+1}$$

$$= u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \cdots + {}^{m+1}C_r u_{m-r+1} v_r$$
$$+ \cdots + {}^{m+1}C_m u_1 v_m + u v_{m+1}$$

(using the formula: ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$)

Thus this theorem is true for $n = m+1$, if it is true for $n = m$

so by the principle of Mathematical Induction, we say that the theorem is true for all positive integers n.

Ex-①

If $y = \tan^{-1}x$ then prove that —

① $(1+x^2)y_1 = 1$

② $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

Sol. ① Here differenting 1st time

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow \underbrace{(1+x^2)}_{v}\underbrace{y_1}_{u} = 1 \quad \text{---- ①} \quad \text{proved}$$

② Now applying Leibnitz's theorem to differentiate n times the equation ①

$$y_{n+1}(1+x^2) + {}^nC_1 y_n \cdot 2x + {}^nC_2 y_{n-1} \cdot 2 + 0 = 0$$

$$\Rightarrow (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$$

proved

Q.2. If $y = e^{x^2}$, prove that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$

③ If $y = \sin(m\sin^{-1}x)$, then show that
$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$.

Sol. Here $y = \sin(m\sin^{-1}x)$

$$y_1 = m\cos(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

(Squaring)

or $y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x)$
$= m^2(1-y^2)$

or $y_1^2(1-x^2) + m^2y^2 - m^2 = 0$

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again differentiating, we get

$$2y_1 y_2(1-x^2) - 2xy_1^2 + 2yy_1 m^2 = 0$$

$$\Rightarrow y_2(1-x^2) - xy_1 + m^2 y = 0 \quad \text{----} \quad (1)$$

Now differentiating (1) n times by Leibnitz's theorem we get, –

$$\left[(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2)\right] - \left[xy_{n+1} + {}^nC_1 y_n \cdot 1\right] + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n \quad \text{proved}$$

4. If $y = e^{\tan^{-1}x}$, then show that
$$(1+x^2)y_{n+2} + \{(2n+1)x - 1\}y_{n+1} + n(n+1)y_n = 0.$$

5. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

6. If $y = \cosh(\sin^{-1}x)$, then prove that –

(i) $(1-x^2)y_2 - xy_1 - y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)y_n$.

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