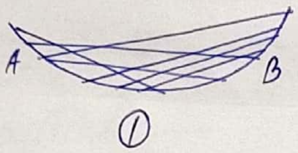
Concavity & Convexity :-
Points of Inflexion

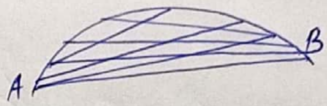
Def. of Convex Curve :- ①

Being a continuous function or part of a continuous function with the property that a line joining any two points on its graph lies on or above the graph.

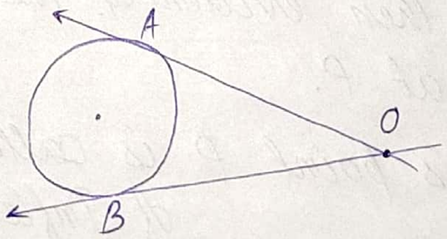
Concave Curve :- ②

— a line joing any two points on its graph lies below the graph.

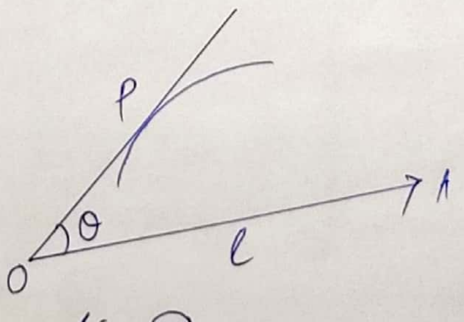




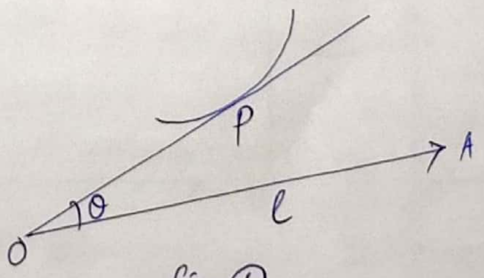
We all have intuitive concept of Concavity and Convexity; Any arc of a circle is concave to all points within the circle, whilst to a point without the circle, the portion lying between that point and the chord of contact of tangents drawn from the point is said to be convex and the remainder of the circumference concave.



Concavity and Convexity with respect to a line :-



fig→ⓐ



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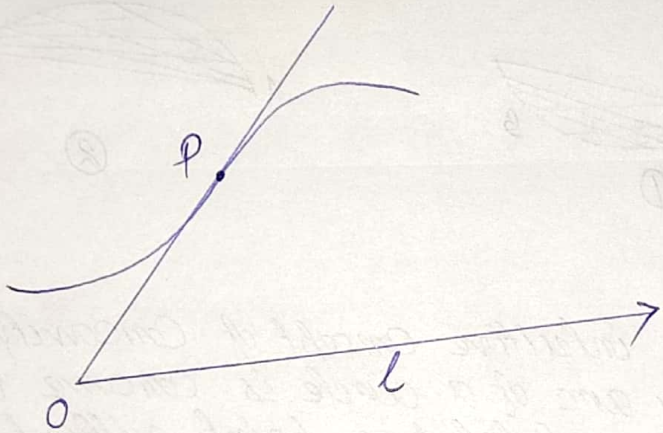
Let P be a given point on a plane curve. Let 'l' be a straight line not passing through P. Then the curve is —

a) Concave at P w.r.t. the line l if a sufficiently small arc containing P lies within the acute angle formed by l and the tangent to the curve at P

b) Convex at P w.r.t. the line l if a sufficiently small arc containing P lies without the acute angle formed by l and the tangent to the curve at P.



On the other hand if the curve is convex on one side of P and concave on the other, w.r.t. the line l, then evidently the curve crosses its tangent at P.

This point P is called the point of Inflexion.

Concave Upwards/Downwards : Convex Upwards/Downwards —

Consider a plane curve whose equation w.r.t. a given set of rectangular axes be $y = f(x)$.
Let P be a point on this curve. We assume that the tangent PT at the point P is not parallel to Y-axis. Then if the curve does not cross its tangent at P, it will, before and after the point P, be situated on the same side of the tangent PT in a small nbd of P.

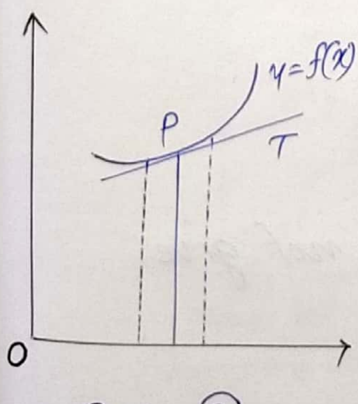
That is, $\exists$ a sufficiently small arc of the curve which —

a) lies entirely above the tangent PT, or
b) lies entirely below the tangent PT.

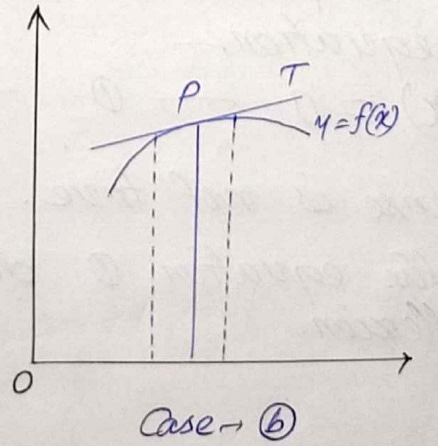
In case ⓐ, the curve is said to be concave upwards at P (or equivalently, convex downwards at P).

In case ⓑ, the curve is concave downwards at P (or equivalently, convex upwards at P)

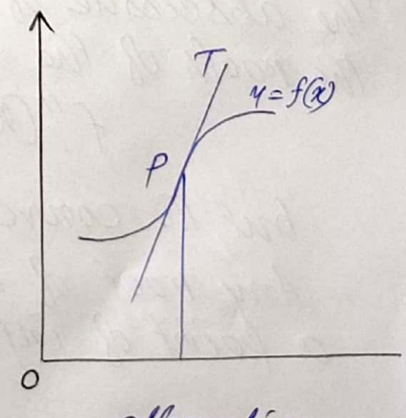
We note that the other alternative, namely, that the curve crosses the tangent at P, will mean that the curve has a point of inflexion at P.



Case→ ⓐ



Case→ ⓑ



alternative

In the first case, the curve has concavity at P towards the +ve side of the Y axis and in the case ⓑ, the curve has a concavity at P towards the negative side of the Y axis.

Thus the sense of concavity or convexity depends on the choice of axes which are fixed by convention. But this is not the case for a point of inflexion. The point where a curve crosses the tangent is an inflexional point so that its existance does not in any way depend on the choice of axes.

Criterion for Concavity or Convexity :-

Theorem 1 :-

Suppose the derivatives of first two order, $f'(x)$ and $f''(x)$ exists and are continuous in a small nbd of the point $P(x,y)$ and $f''(x) \neq 0$. Then the curve $y = f(x)$ turns its concavity upwards or downwards according as $f''(x)$ is positive or negative at the given point $P(x,y)$.

Points of Inflexion :-

From The above theorem, we can say if $f''(x)$ be continuous then such a point can only exist if $f''(x) = 0$.

The abscissae of the points of inflexion are, therefore the roots of the equation—

$$f''(x) = 0 \quad — ①$$

but the converse is not true.

Any root of the equation ① does not give a point of inflexion.

Theorem 2 :-

The points of inflexion of the curve $y = f(x)$ are those roots of $f''(x) = 0$, where $f''(x)$ has one sign in the left nbd and an opposite sign in the right nbd of the points under consideration.

Remark :-

In order that a root of $f''(x) = 0$ may give a point of inflexion, the first of the derivatives which does not vanish simultaneously with $f''(x)$, must be odd order.

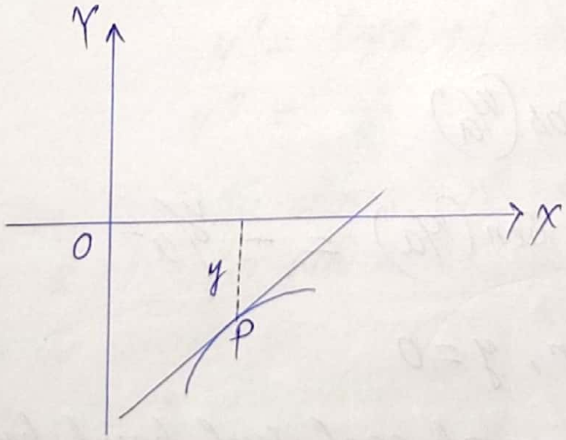
In case the first derivative that does not vanish is $f^n(x)$ where n is even, then the curve is concave upwards or downwards according as

$$f^n(x) > 0 \quad \text{or} \quad f^n(x) < 0 .$$

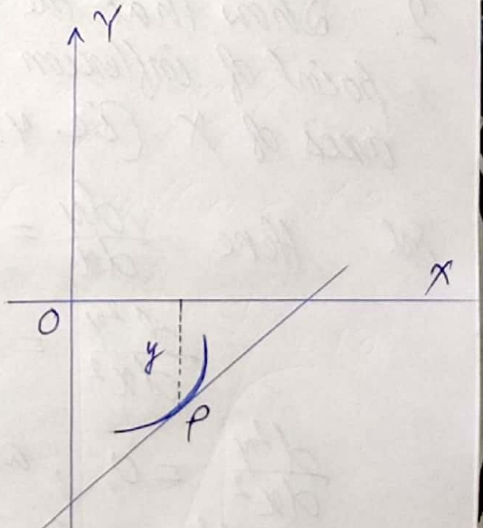
Test of Concavity or Convexity w.r.t. the axis of X :-

Rule :-

A curve $y = f(x)$ is convex or concave at $P(x, y)$ w.r.t. the axis of X according as $y \cdot \frac{d^2y}{dx^2}$ is positive or negative at P.



Convex at P w.r.t. axis of X
(Concave downwards)



Concave at P w.r.t. the X axis.
(Concave upwards)

Note In case of the curve $x = f(y)$ test by the quantity $x \cdot \frac{d^2x}{dy^2}$, instead of $y \frac{d^2y}{dx^2}$.

Example

① Show that the curve $y = x^3$ has a point of inflexion at $x = 0$.

Sol.

Here $\frac{dy}{dx} = 3x^2$ and $\frac{d^2y}{dx^2} = 6x$

at $x = 0$, $\frac{d^2y}{dx^2} = 0$

when $x < 0$ (sufficiently near zero) $\frac{d^2y}{dx^2}$ remains $-ve$ so that the curve is concave downwards there. But when $x > 0$ (sufficiently near zero), $\frac{d^2y}{dx^2}$ becomes $+ve$ so that the curve is concave upwards there.

Hence $x = 0$ is a point of inflexion of the curve.

2. Show that the curve $y = \sin(x/a)$ has a point of inflexion whenever the curve crosses the axis of x (i.e. $y = 0$)

Sol. Here $\frac{dy}{dx} = \frac{1}{a}\cos(x/a)$

$$\frac{d^2y}{dx^2} = -\frac{1}{a^2}\sin(x/a) = -y/a^2$$

$\frac{d^2y}{dx^2} = 0$, whenever, $y = 0$

$\frac{d^2y}{dx^2}$ changes sign at each such point. hence etc.

3. Show that the curve $y = \log x$ $(x > 0)$ is everywhere convex upwards. Discuss concavity or convexity w.r.t. the axis of x. What can you say about the curve $y = x \log x$ $(x > 0)$?

Sol. ⊛ For the curve $y = \log x$,

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}$$

So y'' is negative for all values of x for which the function is defined (i.e $x > 0$)

Hence the curve is convex upwards (or concave downwards) at all points where $x > 0$.

We know that $y = \log x$ is negative or positive according as $0 < x < 1$ or $x > 1$

Thus for $0 < x < 1$, $y \frac{d^2y}{dx^2}$ is +ve

and for $x > 1$, $y \frac{d^2y}{dx^2}$ is −ve

Hence the curve is convex w.r.t. the axis of x if $0 < x < 1$ and concave w.r.t. the axis of x when $x > 1$

⊛ For the curve $y = x \log x$

$$y' = \log x + 1$$

$$y'' = \frac{1}{x}$$

See that the domain of definition of the function is $x > 0$, for which y'' is always +ve

So the curve is everywhere concave upwards (or convex downwards).

(You can verify, drawing the graphs for both the cases.)

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④ Find the points of inflexion, if any, of the curve

$$y = \frac{x^3}{a^2 + x^2}$$

Sol.

$$\frac{dy}{dx} = \frac{x^2(3a^2 + x^2)}{(a^2 + x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2a^2x(3a^2 - x^2)}{(a^2 + x^2)^3}$$

and

$$\frac{d^3y}{dx^3} = \frac{6a^2\{(x^2 - 3a^2)^2 - 9a^4 + a^2\}}{(a^2 + x^2)^4}$$

thus $\frac{d^2y}{dx^2} = 0$ if $x = 0, +\sqrt{3}a, -\sqrt{3}a$

Check: $\frac{d^3y}{dx^3} \neq 0$, for each such values of x

Hence we conclude that these are the points of inflexion.

⑤ Find the points of inflexion, if any, of the curve $x = (\log y)^3$.

Sol. Here

$$\frac{dx}{dy} = \frac{3(\log y)^2}{y}$$

$$\frac{d^2x}{dy^2} = \frac{3\log y}{y^2}(2 - \log y)$$

$$\frac{d^3x}{dy^3} = \frac{6(\log y)^2 - 18(\log y) + 6}{y^3}$$

Now $\frac{d^2x}{dy^2} = 0$ at $y = 1$ and $y = e^2$

at each such points $\frac{d^3x}{dy^3} \neq 0$ (check)

therefore $(0, 1)$ and $(8, e^2)$ are the two points of inflexion of the curve.

6. Find the range of values of x for which

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$

is concave upwards or downwards.
Find also its points of inflexion, if any.

Sol. $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$

$$\frac{d^2y}{dx^2} = 12(x^2 - 3x + 2) = 12(x-1)(x-2)$$

For $-\infty < x < 1$, $\frac{d^2y}{dx^2} > 0$, hence concave upwards in this range.

At $x = 1$, $\frac{d^2y}{dx^2} = 0$

For $1 < x < 2$, $\frac{d^2y}{dx^2} < 0$, hence Concave downwards in this range.

Clearly, then $x = 1$ is a point of inflexion.

at $x = 2$, $\frac{d^2y}{dx^2} = 0$

For $2 < x < \infty$, $\frac{d^2y}{dx^2} > 0$, hence Concave upwards in this range.

clearly $x = 2$ is a point of inflexion.

Q.1. Find the points of inflexion, if any of $y = e^{-x^2}$.

Q.2. Find the range of values of x for which $y = 3x^5 - 40x^3 + 3x - 20$ is Concave upwards or downwards. Find also points of inflexion if any.