Ex — Show that $\lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right) = 0.$

Soln. Let $\epsilon > 0$ be given.

Now, $\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$< |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$

$< |x| + |y| \le \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$

$= 2\sqrt{x^2+y^2} < \epsilon$

This is true if

or, $x^2+y^2 < \frac{\epsilon^2}{4}.$

$x^2 < \frac{\epsilon^2}{8}, \; y^2 < \frac{\epsilon^2}{8}$

or, $|x| < \frac{\epsilon}{2\sqrt{2}} = \delta, \quad |y| < \frac{\epsilon}{2\sqrt{2}} = \delta.$

$\therefore \forall \epsilon > 0, \exists \delta > 0$ such that

$\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| < \epsilon$ when $|x| < \delta, |y| < \delta.$

$\Rightarrow \lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right) = 0.$

Continuity.

A fn. $f(x,y)$ is said to be continuous at (a,b) of its domain of definition if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Non-existence of limit

Ex' Let us consider the fn.

$f(x,y) = \frac{xy}{x^2+y^2}.$

Check whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not.

~~If we put y = mx, then~~

If we approach (0,0) along any axis (i.e. $x=0$ or $y=0$) then

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

If we approach (0,0) along $y = mx$, then

$$f(x,y) = \frac{x \cdot mx}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{m}{1+m^2}$$

which is different for different values of m. (but limit of a fn. is unique whatever may be the approach)

∴ The limit doesn't exists.

Ex 2: Show that

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} \text{ doesn't exist.}$$

Ex 3. $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Ex 6. $$\lim \frac{x^3+y^3}{x-y}$$

Ex 4. $$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Ex 5. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$$

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Ex. Show that the following limits exist.

(1) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

(2) $$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}.$$

Ex 6. (3) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$$ doesn't exist.

Soln. If we approach (0,0) along x-axis (i.e. $y=0$)

then $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to0} x^2 = 0.$$

If we approach (0,0) along y-axis (i.e. $x=0$)

then $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y} = \lim_{y\to0} -y^2 = 0.$$

If we approach (0,0) along $y=mx$,

then $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$$

$$= \lim_{x\to0} \frac{x^3+m^3x^3}{x-mx} = \lim_{x\to0} \frac{x^2(1+m^3)}{1-m}$$

$$= 0.$$

If we approach (0,0) along $y = x-mx^3$,

then

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y} = \lim_{x\to0} \frac{x^3+(x-mx^3)^3}{mx^3}$$

$$= \lim_{x\to0} \frac{x^3\{1+(1-mx^2)^3\}}{mx^3} = \frac{2}{m}$$

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which is different for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$ doesn't exist.

Investigate the continuity of the following functions at $(0,0)$.

Ex1. $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y)=0 \end{cases}$

Ex2. $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Ex3. $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & x^4+y^2 \neq 0 \\ 0, & (x,y)=(0,0) \end{cases}$

Soln Ex1. If we approach $(0,0)$ along $y=mx$,

then $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

$= \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2}$

$= \frac{1-m^2}{1+m^2}$ which is different for different values of m. So, the limit doesn't exist.

Hence the fn. $f(x,y)$ is discontinuous at $(0,0)$.

Ex 2. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln. Let $\epsilon > 0$.

Now, $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right|$

$$< |x||y| \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$< \sqrt{x^2+y^2} \cdot \cancel{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \sqrt{x^2+y^2} < \epsilon$$

$$\text{or, } x^2+y^2 < \epsilon^2$$

This is true if $x^2 < \epsilon^2/2$; $y^2 < \epsilon^2/2$

$$\text{or, } |x| < \frac{\epsilon}{\sqrt{2}} = \delta, \quad |y| < \frac{\epsilon}{\sqrt{2}} = \delta$$

Thus $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ such that

$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon$ when $|x| < \delta$, $|y| < \delta$.

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Ex3. ~~If we eq~~

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & \cancel{(x,y) \neq (0,0)}\ (x^4+y^2 \neq 0) \\ 0, & (x,y) = (0,0) \end{cases}$$

If we approach the origin $(0,0)$ along the path $y = mx^2$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

$$= \lim_{x\to 0} \frac{x^2 \cdot mx^2}{x^4+m^2x^4} = \frac{m}{1+m^2},$$ which

is different for different values of m.

$\therefore$ The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ doesn't exist.

Hence the fn. is not continuous at $(0,0)$.

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