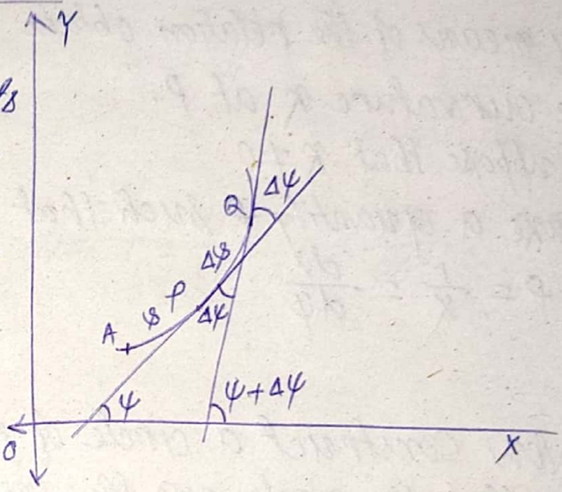
CURVATURE :-

The terms like flatness or sharpness are often used to describe the nature of bending of the curve at a particular point. Curvature of a curve at a particular point will give a definite numerical measure of bending which the curve undergoes at the point.

Measure of Bending : Definitions



Two tangents at two adjacent points P and Q, makes angles ψ and $\psi+\Delta\psi$ with OX. Suppose, arc $AP = s$, arc $AQ = s+\Delta s$, so that arc $PQ = \Delta s$

Then we construct the following def.

1. The angle $\Delta\psi$ through which the tangent turns as the point of contact travels from one end to another end of the arc PQ is called the total curvature of the arc PQ.

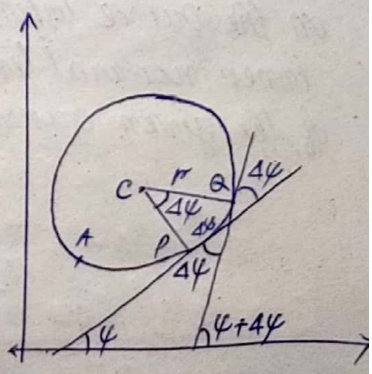
2. The mean or average curvature of the arc PQ is defined as $\Delta\psi/\Delta s$

3. The curvature (K) at a point P of the curve is defined as the limiting value of mean curvature when $\Delta s \to 0$

i.e. Curvature (K) at P $= \lim\limits_{\Delta s\to 0} \dfrac{\Delta\psi}{\Delta s} = \dfrac{d\psi}{ds}$

$$\boxed{K = \frac{d\psi}{ds}}$$

☐ **Let us consider a Circle**



Let the circle of radius r and Centre at C
arc $PQ = \Delta s$

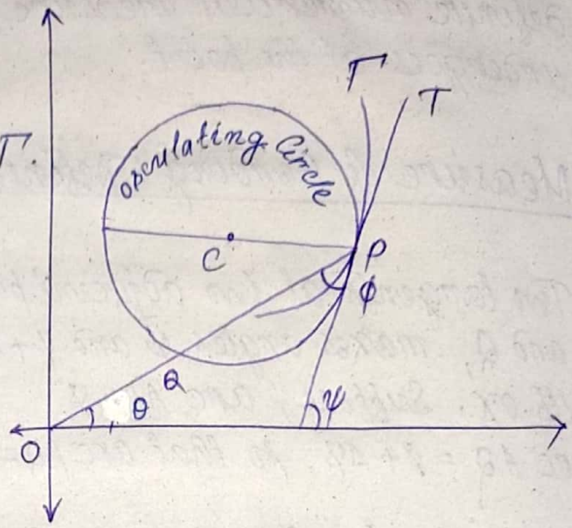
then we have $\Delta\psi = \dfrac{\Delta s}{r}$

or $\dfrac{\Delta\psi}{\Delta s} = \dfrac{1}{r}$

or $\lim\limits_{\Delta s\to 0} \dfrac{\Delta\psi}{\Delta s} = \dfrac{1}{r}$ or $\boxed{K = \dfrac{1}{r}}$

Thus in case of circle the curvature is the same at every point and is measured by the reciprocal of the radius.

With these preliminaries we proceed to give definitions for the circle of curvature and its associated terms.



Consider a point P on the curve Γ. By means of the relation obtain the curvature κ at P. Suppose that $\kappa \neq 0$. Take a quantity ρ such that

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

Now construct a circle of radius ρ and centre at C so that the circle and the curve Γ have the same tangent at P; the circle is drawn in such a way that it lies on the same side of the tangent as the curve. (Osculating Circle) The circle has the same curvature as the given curve at P. We call this circle as the circle of curvature at P. Its centre C is the centre of curvature for the curve at P, its radius $CP (= \rho)$, normal to the curve, is the radius of curvature of the curve at P.

Osculating Circle :-

The osculating circle of a sufficiently smooth plane curve at a given point P on the curve has been traditionally defined as the circle passing through P and a pair of additional points on the curve infinitesimally close to P. Its centre lies on the inner normal line and its curvature defines the curvature of the given curve at that point.

Radius of Curvature :- Cartesian Form

Let the equation of the curve be $y = f(x)$ or $x = f(y)$

then $\tan\psi = \frac{dy}{dx} = y_1$

therefore $\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds} = \frac{d^2y}{dx^2} \cdot \cos\psi$

now $\rho = \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{3/2}}{y_2}, (y_2 \neq 0)$.

so $$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \; y_2 \neq 0}$$ ⊛

Polar Equation :- $r = f(\theta)$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2 - rr_2}, \quad \text{where } r_1 = \frac{dr}{d\theta},\; r_2 = \frac{d^2r}{d\theta^2}$$

Example (1.)

In the catenary, $y = a\cosh(x/a)$

we have $y_1 = \sinh(x/a)$, $y_2 = \frac{1}{a}\cosh(x/a)$

then $1+y_1^2 = 1+\sinh^2(x/a) = \cosh^2 x/a$

therefore $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = a\cosh^2(x/a) = y^2/a$.

so $\rho \propto y^2$. also.

Pedal Equation :- $p = f(r)$

$$\rho = r\frac{dr}{dp}$$

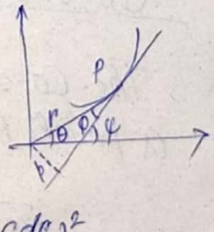
pedal Equation

$p = r\sin\phi$

$\psi = \theta + \phi$

$\tan\phi = r\frac{d\theta}{dr}$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.



For Parametric Equations :-

$x = \phi(t)$, $y = \psi(t)$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$x' = \frac{d\phi}{dt}$, $y' = \frac{d\psi}{dt}$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = y'/x'$

and $\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^3}$.

Example (2)

In the ellipse $\frac{a^2b^2}{p^2} = a^2+b^2-r^2 \rightarrow$ (pedal equation of ellipse)

differentiation w.r.t. p gives

$$-2\frac{a^2b^2}{p^3} = -2r\frac{dr}{dp}$$

therefore $\rho = r.\frac{dr}{dp}$

$$\Rightarrow \rho = \frac{a^2b^2}{p^3} \quad \text{Ans.}$$

Ex(3). Show that the radius of curvature at any point on the cardiode

$r = a(1-\cos\theta)$ is $\frac{2}{3}\sqrt{2ar}$.

Here $r_1 = a\sin\theta$, $r_2 = a\cos\theta$

then
$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\{a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)\}^{3/2}}{a^2(1-2\cos\theta+\cos^2\theta)+2a^2\sin^2\theta-a^2(\cos\theta-\cos^2\theta)}$$

$$= \frac{\{2a^2(1-\cos\theta)\}^{3/2}}{a^2(1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta)}$$

$$= \frac{(2ar)^{3/2}}{3(a^2)(1-\cos\theta)} = \frac{2}{3}\sqrt{2ar} \quad \text{proved.}$$

Co-ordinates of Centre of Curvature :- (Circle of Curvature)

The Co-ordinates of Centre of Curvature corresponding to a point $P(x,y)$ of a curve $y=f(x)$ are

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2} \quad (y_2 \neq 0)$$

Corollary :- If x_1, x_2 are respectively the first and second derivatives of x w.r.t. y, then these formula may be transformed into—

$$\bar{x} = x + \frac{1+x_1^2}{x_2}, \quad \bar{y} = y - \frac{x_1(1+x_1^2)}{x_2}.$$

Circle of Curvature :-

Equation $\Rightarrow (x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$.

* Continuation

-: CURVATURE :-

Example - ④

Find the radius of curvature of $x^3 + y^3 = 3axy$, at the point $\left(\frac{3}{2}a, \frac{3}{2}a\right)$.

Sol.

We have $x^3 + y^3 = 3axy$

differentiating the equation we get —

$$x^2 + y^2 \frac{dy}{dx} = ay + ax\frac{dy}{dx}$$

$\therefore \frac{dy}{dx}$ at the given point $\left(\frac{3a}{2}, \frac{3a}{2}\right) = -1$

differentiating once more

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

which gives $\frac{d^2y}{dx^2} = -\frac{32}{3a}$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

and therefore $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-\frac{32}{3a}}$$

i.e. $\rho = \frac{3a}{16}\sqrt{2}$. Ans.

(numerically)

⑤ Find radius of curvature for the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Sol. Here parametric equations are given

$x' = a(1+\cos\theta)$, $y' = a\sin\theta$

$x'' = -a\sin\theta$, $y'' = a\cos\theta$.

Therefore,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

putting all the values and simplifying we get

$\rho = 4a\cos\theta/2$ Ans.

Otherwise, obtain $\frac{dy}{dx} = \frac{y'}{x'} = \tan\theta/2$

and $\frac{d^2y}{dx^2} = \frac{1}{4a\cos^4\theta/2}$

and putting in $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ we have

$\rho = 4a\cos\theta/2$ Ans

Q.① Find the radius of curvature at any point on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

(ans → $4a\sin\theta/2$)

② Find the radius of curvature at any point (x, y) of ① $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(iii) $y^2 = 4ax$

(iv) $x^2 + y^2 = a^2$

(v) $xy = c^2$

(vi) $y = mx + c$

③ Find radius of curvature of ① $r = ae^{\theta\cot\alpha}$ at any point θ.

(ii) $r = \frac{l}{1+e\cos\theta}$ at $\theta = \pi$.

(Ans → ① $r/\sin\alpha$ (ii) l.)

Concept of Curvature :- Newton's Approach

① If a curve passes through the origin and the axis of x is tangent at the origin, then

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$ gives the radius of curvature at the origin.

⑪ On the other hand if a curve passes through the origin and Y axis is the tangent at the origin, then the radius of curvature at the origin is

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}.$$

⑪ For a curve, passing through the origin, if $ax+by=0$ be the tangent at the origin, then the radius of curvature at $(0,0)$ is

$$\boxed{\rho = \frac{1}{2}\sqrt{a^2+b^2} \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2+y^2}{ax+by}}$$

Example ①

Find the radius of curvature at the origin for the curve $x^3+y^3-2x^2+6y=0$ — ①

Sol.

Here $y=0$, i.e X axis is the tangent at the origin.

So $\rho = \frac{1}{2}\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y}$ $\Rightarrow 2\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y}$

Now dividing the equation ① by y we get

$$x\cdot\frac{x^2}{y} + y^2 - 2\cdot\frac{x^2}{y} + 6 = 0$$

Taking limit as $x\to 0$, $y\to 0$ we get

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(x \cdot \frac{x^2}{y}\right) + \lim_{\substack{x\to 0\\ y\to 0}} y^2 - 2 \cdot \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y} + 6 = 0$$

$$\Rightarrow 0 \cdot 2\rho + 0 - 2 \cdot 2\rho + 6 = 0$$

$$\Rightarrow 4\rho = 6$$

$$\Rightarrow \rho = 3/2.$$

② Find the radius of curvature at the origin of the conic

$$y - x = x^2 + 2xy + y^2 \quad \text{--- ①}$$

Sol.

Here $y - x = 0$ is the tangent at the origin for the given conic.

Therefore from Newtonian method

$$\rho = \frac{1}{2}\sqrt{(-1)^2 + (1)^2} \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 + y^2}{y - x} \qquad (\text{as } a = -1,\ b = 1)$$

$$= \frac{1}{2}\sqrt{2} \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 + y^2}{x^2 + 2xy + y^2} \qquad (\text{from ①})$$

$$= \frac{1}{\sqrt{2}} \lim_{\substack{x\to 0\\ y\to 0}} \frac{1 + (y/x)^2}{1 + 2\cdot(y/x) + (y/x)^2} \qquad (\text{dividing by } x^2)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1+1}{1+2+1} \qquad \left(\text{as } y - x = 0,\ y/x = 1\right)$$

So, $\rho = \dfrac{1}{2\sqrt{2}}$.

Task Q. ① Show that the radii of curvature of the curve $y^2 = x^2(a+x)/(a-x)$ at the origin are $\pm a\sqrt{2}$.

(ii) Find the radius of curvature at the vertex (origin) of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ $(4a)$

(iii) Find radius of curvature at origin of the curve $x^2 + 6y^2 + 2x - y = 0$ $(\sqrt{5}/10)$

(iv) Find radius of curvature at the origin of the curve $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$ $\left(-\sqrt{2}, \frac{5\sqrt{5}}{2}\right)$

__Example__ (Circle of curvature)

(1) Find the equation of circle of curvature of $2xy + x + y = 4$ at the point $(1,1)$.

__Sol.__

Here $2xy + x + y = 4$

differentiating w.r.t. x

$$2y + 2x \cdot y_1 + 1 + y_1 = 0$$

$$\therefore (y_1)_{(1,1)} = -1$$

differentiating once more, we have

$$4y_1 + 2xy_2 + y_2 = 0$$

$$\therefore (y_2)_{(1,1)} = 4/3$$

we have now $\rho = \dfrac{\{1 + (-1)^2\}^{3/2}}{4/3} = \dfrac{3}{\sqrt{2}}$

and the centre of curvature corresponding to $(1,1)$

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$= 1 - \frac{(-1)(1 + (-1)^2)}{4/3} = 5/2$$

similarly $\bar{y} = 5/2$

Hence the equation of circle of curvature is

$$\left(x - 5/2\right)^2 + \left(y - 5/2\right)^2 = \left(3/\sqrt{2}\right)^2$$

$$\Rightarrow 4x^2 + 4y^2 - 20x - 20y + 32 = 0 \quad \text{Ans.}$$

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Q.1. Find the equation of circle of curvature at the indicated points —

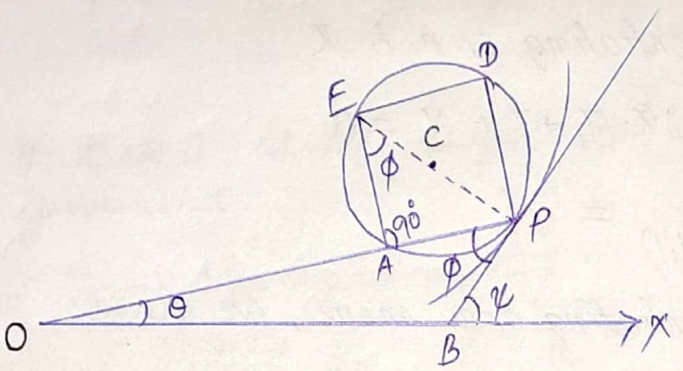
(i) $xy = 12$: $(3,4)$

(ii) $y = 3x^3 + 2x^2 - 3$: $(0,-3)$

(iii) $y = x^2 - 6x + 10$: $(3,1)$

(iv) $x + y = ax^2 + by^2 + cx^3$: $(0,0)$

Chord of Curvature through Pole (Origin) :-



$PE = 2\rho$
$\angle APE = 90^\circ - \phi$

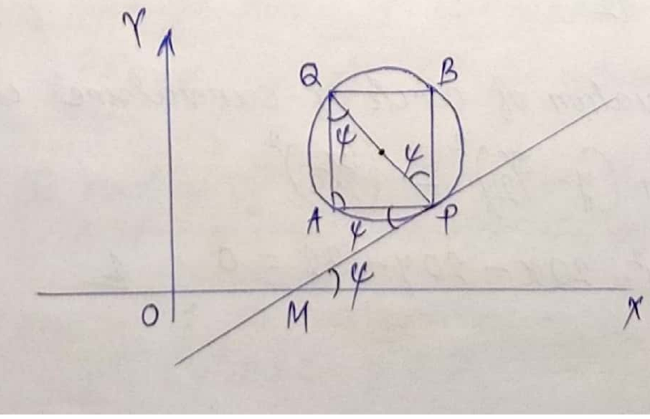
Let ρ be the radius of curvature and ϕ be the angle between radius vector and tangent at P. Then the chord of curvature through origin

$$\boxed{AP = 2\rho \sin\phi}$$

similarly chord of curvature perpendicular to radius vector

$$\boxed{DP = 2\rho\cos\phi}$$

Chord of curvature Parallel to Co-ordinate axes :-



$QP = 2\rho$

If ρ be the radius of curvature and ψ be the angle which tangent makes with x axis then —

Chord parallel to x-axis is $\boxed{AP = 2\rho \sin\psi}$

and chord parallel to y axis is $\boxed{BP = 2\rho \cos\psi}$

Example 1

Find the chord of curvature through the pole of the curve $r = ae^{m\theta}$.

Sol. Here $r = ae^{m\theta}$

$$r_1 = \frac{dr}{d\theta} = mae^{m\theta} = mr$$

$$r_2 = \frac{d^2r}{d\theta^2} = m^2 r$$

We have

$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

then $\tan\phi = \dfrac{r}{mr} = \dfrac{1}{m}$

therefore $\sin\phi = \dfrac{1}{\sqrt{1+m^2}}$

Now
$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2 - rr_2}$$
$$= \frac{(r^2+m^2r^2)^{3/2}}{r^2+2m^2r^2 - rm^2r}$$
$$= (r^2+m^2r^2)^{1/2} = r\sqrt{1+m^2}$$

Hence the required chord of curvature through pole

$$= 2\rho\sin\phi$$
$$= 2\cdot r\sqrt{1+m^2}\cdot\frac{1}{\sqrt{1+m^2}} = 2r \quad \text{Ans.}$$

Q. ① Find the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$.

② Find the chord of curvature parallel to y axis at the origin for the curve $y = mx + \left(\frac{x^2}{a}\right)$.

Sol. Given curve $y = mx + \frac{x^2}{a}$

$$\frac{dy}{dx} = m + \frac{2x}{a}$$

$$\frac{d^2y}{dx^2} = 2/a$$

We know,

$$\tan\psi = \frac{dy}{dx}$$

$$= m + \frac{2x}{a} = \left(\frac{2x + ma}{a}\right) \quad —①$$

Also

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1 + \left(\frac{2x+ma}{a}\right)^2\right\}^{3/2}}{2/a}$$

$$= \frac{\left\{a^2 + (2x + am)^2\right\}^{3/2}}{2a^2}$$

at the origin $(0,0)$, $\rho = \frac{\left\{a^2 + (am)^2\right\}^{3/2}}{2a^2}$

$$= \frac{1}{2} a (1 + m^2)^{3/2}$$

also at $(0,0)$, $\tan\psi = m$ from ①

so $\cos\psi = \frac{1}{\sqrt{1+m^2}}$

∴ chord of curvature parallel to y axis

$$= 2\rho\cos\psi$$

$$= 2 \cdot \frac{1}{2} a (1+m^2)^{3/2} \cdot \frac{1}{\sqrt{1+m^2}} = a(1+m^2) \text{ Ans}$$

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(3) Find the chord of curvature of the curve $r = a(1+\cos\theta)$ at the pole. $\left(\frac{4}{3} \cdot r\right)$

(4) Find the chord of curvature of the curve $y = a \log \sec(x/a)$ parallel to y axis. $(2a)$.