BSC-M302 (CSE & IT)

Taylor's & Maclaurin's expansion

If $f(x,y)$ is a function which possesses continuous partial derivatives of order n in any domain of a point (a,b) and the domain is large enough to contain a point $(a+h, b+k)$ within it, then $\exists$ a positive no., $0<\theta<1$, such that

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots$$

$$+ \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n, \text{ where}$$

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),$$

$0<\theta<1$

is called the remainder after n terms and the theorem is Taylor's theorem with remainder or Taylor's expansion about the point (a,b).

If we put $a=b=0$; $h=x, k=y$, we get

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \cdots$$

$$+ \frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n-1} f(0,0) + R_n,$$

where $R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y)$, $0<\theta<1$, is called the Maclaurin's th. or Maclaurin's expansion.

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The theorem (Taylor's), can be stated in still another form,

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \cdots$$
$$+ \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n,$$

where

$$R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+(x-a)\theta,\ b+(y-b)\theta).$$

$0<\theta<1$,

called the Taylor's expansion of $f(x,y)$ about the point (a,b) in powers of $(x-a)$ and $(y-b)$.

Ex) Expand $x^4 + x^2y^2 - y^4$ about the point $(1,1)$ upto terms of the 2^{nd} degree. Find the form of R_3.

Soln Here $f(x,y) = x^4 + x^2y^2 - y^4$.

$f_x = 4x^3 + 2xy^2$ $\qquad f(1,1) = 1$

$f_y = 2x^2y - 4y^3$ $\qquad f_x(1,1) = 6$

$f_{xx} = 12x^2 + 2y^2$ $\qquad f_y(1,1) = -2$

$f_{yy} = 2x^2 - 12y^2$ $\qquad f_{xx} = 14$

$f_{xy} = 4xy$ $\qquad f_{xy} = 4,\ f_{yy} = -10.$

$\therefore\ x^4 + x^2y^2 - y^4 = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1)$

$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)\right]$

$+ R_3$

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$$= 1 + 6(x-1) - 2(y-1) + \frac{1}{2!}\left[(x-1)^2 \cdot 14 + (y-1)^2 \cdot (-10) + 2(x-1)(y-1)\cdot 4\right] + R_3$$

where $R_3 = \frac{1}{3!}\left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^3 f\big(1+(x-1)\theta,\ 1+(y-1)\theta\big)$

$0 < \theta < 1$.

Ex2 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$.

Soln Here $f(x,y) = x^2y + 3y - 2$

$f_x = 2xy$, $f_y = x^2+3$, $f_{xx} = 2y$, $f_{yy} = 0$,

$f_{xy} = 2x = f_{yx}$, $f_{xxx} = 0$, $f_{xyy} = 0$, $f_{xxy} = 2$, $f_{yyy} = 0$,

$f_{yxx} = 2$, $f_{yyx} = 0 = f_{xyy}$

$\cancel{\therefore f(1,-2) = (1^2\times(-2)) + (3\times(-2)) - 2 = -6}$

$f(1,-2) = 1^2 \times (-2) + 3\times(-2) - 2 = -10$

$\cancel{=-10}$, $f_x(1,-2) = -4$, $f_y(1,-2) = 4$

$f_{xx}(1,-2) = -4$, $f_{xy}(1,-2) = 2$, $f_{yy} = 0$.

$$\therefore x^2y + 3y - 2 = f(1,-2) + (x-1)\cdot(-4) + (y+2)\cdot(4) + \frac{1}{2!}\left[(x-1)^2\cdot(-4) + (y+2)^2\cdot(0) + 2(x-1)(y+2)\cdot 2\right]$$
$$+ \frac{1}{3!}\left[(x-1)^3\cdot 0 + 3(x-1)^2(y+2)\cdot 2 + 3(x-1)(y+2)^2\cdot 0 + (y+2)^3\cdot 0\right]$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$ Ans

Ex 3 Find the first 4 terms of the Maclaurin's expansion of $e^{ax}\cos by$.

Soln Here $f(x,y) = e^{ax}\cos by$

$f(0,0) = e^{0}\cos 0 = 1$

$f_x = a e^{ax}\cos by$, $f_x(0,0) = a$

$f_y = -b e^{ax}$ ~~cos by~~ $\sin by$, $f_y(0,0) = 0$

$f_{xx} = a^2 e^{ax}\cos by$, $f_{xx}(0,0) = a^2$

$f_{yy} = -b^2 e^{ax}\cos by$, $f_{yy}(0,0) = -b^2$

$f_{xy} = -ab e^{ax}\sin by$, $f_{xy}(0,0) = 0$.

$f_{xxx} = a^3 e^{ax}\cos by$, $f_{xxx}(0,0) = a^3$

$f_{yyy} = b^3 e^{ax}\sin by$, $f_{yyy}(0,0) = 0$

$f_{xyy} = -ab^2 e^{ax}\cos by$, $f_{xyy}(0,0) = -ab^2$

$f_{xxy} = -a^2 b e^{ax}\sin by$, $f_{xxy}(0,0) = 0$

$$\therefore f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0)$$

$$= 1 + ax + \frac{a^2x^2 - b^2y^2}{2!} + \frac{a^3x^3 + 3x^2y.0 - 3xy^2ab^2}{3!}$$

Ans