classmate

Date
Page

$$= \frac{4}{x+y}\left[\frac{-(x+y)^2 + 2(x^2+2y^2)}{x+y}\right]$$

$$= 4\frac{[x^2+y^2-2xy]}{(x+y)^2}$$

$$= 4\left(\frac{x-y}{x+y}\right)^2 = \text{L.H.S. (proved)}.$$

<u>Ex4.</u> If $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

then show that at the origin $f_{xy} \neq f_{yx}$.

<u>Soln:</u> we know that

$$f_{xy}(0,0) = \left(\frac{\partial}{\partial x} f_y\right)_{(0,0)}$$

$$= \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}.$$

Here $f_y(h,0) = \lim_{k\to 0} \dfrac{f(h,k) - f(h,0)}{k}$

$$= \lim_{k\to 0} \frac{\frac{h\cancel{k}(h^2-k^2)}{h^2+k^2} - 0}{\cancel{k}}$$

$$= h$$

$$f_y(0,0) = \underset{k\to 0}{Lt} \frac{f(0,k) - f(0,0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{0-0}{k} = 0.$$

$$\therefore f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{h-0}{h} = 1$$

Again,

$$f_{yx}(0,0) = \left(\frac{\partial}{\partial y}(f_x)\right)_{(0,0)}$$

$$= \underset{k\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k}$$

Now,

$$f_x(0,k) = \underset{h\to 0}{Lt} \frac{f(h,k) - f(0,k)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\cancel{h}k\frac{(h^2-k^2)}{h^2+k^2} - 0}{\cancel{h}}$$

$$= -k.$$

$$f_x(0,0) = \underset{h\to 0}{Lt} \frac{f(h,0) - f(0,0)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{0-0}{h} = 0.$$

$$\therefore f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{-k-0}{k}$$

$$= -1.$$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0).$

Derivative of a composite function.

Functions of functions

So far we have considered fns. of the form $z = f(x, y, ---)$, where variables $x, y,$ are independent variables. We now consider functions $z = f(x, y, ---)$ where $x, y, ----$ are not independent variables, but are functions of other variables $u, v, ---$ so that $x = g(u, v, ---)$, $y = h(u, v, ---)$.

If $z = f(x, y)$ is a differentiable fn. of x, y & $x = g(u, v)$, $y = h(u, v)$ are themselves functions of independent variables u, v then z is a differentiable fn. of u, v and

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

☆ If $z = f(x, y)$ is a fn. of x & y & x, y be differentiable fns of a single variable t ie $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

☆ If $z = f(x, y)$ be a fn. of x & y & x, y be differentiable fns of 2 independent variables u & v, i.e. $x = g(u, v)$, $y = h(u, v)$, then z possesses continuous partial derivatives w.r.t u & v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Total derivative

If $u = f(x,y)$, where $x = \varphi(t)$ & $y = \psi(t)$, then we can express u as a fn. of t alone by substituting the values of x & y in $f(x,y)$. Thus we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u to distinguish it from the partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Change of variable.

If $u = f(x,y)$, $x = \varphi(s,t)$, $y = \psi(s,t)$, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t}.$$

(known as chain rule)

Ex 1. If $u = u(y-z, z-x, x-y)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln. Let $r = y-z$, $s = z-x$, $t = x-y$, so that $u = u(r,s,t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot 0 + \frac{\partial u}{\partial s} \cdot (-1) + \frac{\partial u}{\partial t} \cdot 1$$

$$= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}\cdot(-1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}. \quad \text{--- (3)}$$

Adding (1), (2) & (3), we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \text{ (proved)}$$

Ex. If $\phi(v^2-x^2, v^2-y^2, v^2-z^2) = 0$, where v is a fn. of x, y, z, show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}.$$

Soln. Let $a = v^2-x^2$, $b = v^2-y^2$, $c = v^2-z^2$

$$\therefore \frac{\partial \phi}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial \phi}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial \phi}{\partial c}\cdot\frac{\partial c}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial a}\left(2v\frac{\partial v}{\partial x} - 2x\right) + \frac{\partial \phi}{\partial b}\left(2v\frac{\partial v}{\partial x} - 0\right) + \frac{\partial \phi}{\partial c}\left(2v\frac{\partial v}{\partial x} - 0\right) = 0$$

classmate
Date
Page

or, $\frac{v\,\partial\varphi}{\partial a}\cdot\frac{\partial v}{\partial x} - x\frac{\partial\varphi}{\partial a} + \frac{v\,\partial\varphi}{\partial b}\cdot\frac{\partial v}{\partial x} + \frac{v\,\partial\varphi}{\partial c}\cdot\frac{\partial v}{\partial x} = 0$

or, $\cancel{\frac{v\partial\varphi}{\partial x}}\ \frac{v\,\partial v}{\partial x}\left(\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b} + \frac{\partial\varphi}{\partial c}\right) = x\frac{\partial\varphi}{\partial a}$.

or, $\frac{v}{x}\cdot\frac{\partial v}{\partial x}\left(\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b} + \frac{\partial\varphi}{\partial c}\right) = \frac{\partial\varphi}{\partial a}$.

or, $\frac{v}{x}\cdot\frac{\partial v}{\partial x} = \frac{\frac{\partial\varphi}{\partial a}}{\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b} + \frac{\partial\varphi}{\partial c}}$

Similarly, $\frac{v}{y}\cdot\frac{\partial v}{\partial y} = \frac{\frac{\partial\varphi}{\partial b}}{\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b} + \frac{\partial\varphi}{\partial c}}$

& $\frac{v}{z}\cdot\frac{\partial v}{\partial z} = \frac{\frac{\partial\varphi}{\partial c}}{\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b} + \frac{\partial\varphi}{\partial c}}$.

$\therefore\ \frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$ (proved).

Ex. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Soln. $u = f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{x} - \frac{1}{z}\right)$

Let $a = \frac{1}{x} - \frac{1}{y}$, $b = \frac{1}{x} - \frac{1}{z}$.

$\therefore u = f(a, b)$. diff. w.r.t x, we get,

$\therefore \frac{\partial u}{\partial x} = \frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial x}$

Scanned with CamScanner

$$= \frac{\partial f}{\partial a}\cdot\left(-\frac{1}{x^2}\right) + \frac{\partial f}{\partial b}\cdot\left(-\frac{1}{x^2}\right).$$

or, $x^2\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b}$

diff. w.r.t. y, we get,

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial y}$$

$$= \frac{\partial f}{\partial a}\cdot\left(\frac{1}{y^2}\right) + \frac{\partial f}{\partial b}\cdot 0$$

or, $y^2\frac{\partial u}{\partial y} = \frac{\partial f}{\partial a}.$

diff. w.r.t. z, we get,

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial z}.$$

$$= \frac{\partial f}{\partial a}\cdot 0 + \frac{\partial f}{\partial b}\cdot\frac{1}{z^2}.$$

or, $z^2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial b}.$

$\therefore$ $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$

$$= \cancel{-\frac{\partial f}{\partial a}} - \cancel{\frac{\partial f}{\partial b}} + \cancel{\frac{\partial f}{\partial a}} + \cancel{\frac{\partial f}{\partial b}}$$

$= 0$ (proved).