Method of Lagrange's Multiplier.

Let $u = f(x, y, z)$ be a function of 3 variables which are connected by the relation $\phi(x, y, z) = 0$.

working Rule.

1. construct $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$
2. Obtain the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0 = \frac{\partial F}{\partial z}$.
3. Solve the above (3) equations along with the equation $\phi(x, y, z) = 0$.

The values of x, y, z so obtained will give the stationary values of $f(x, y, z)$

Ex 1. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1,1,1)$ using the method of Lagrange's multipliers.

Ans. Let $P(x, y, z)$ be any point on $x + 2y + 3z = 13$.
The distance b/w $P(x, y, z)$ & $(1,1,1)$ is

$$\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}.$$

we take the square of the distance as $f(x, y, z)$.

$\therefore$ $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$
subject to the constraint

$$x + 2y + 3z = 13$$

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Lagrange's function is

$F(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2+\lambda(x+2y+3z-13)$.

λ is Lagrange's multiplier.

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= 2(x-1)+\lambda=0 \Rightarrow x=-\lambda/2+1.\\ \frac{\partial F}{\partial y} &= 2(y-1)+2\lambda=0 \Rightarrow y=-\lambda+1\\ \frac{\partial F}{\partial z} &= 2(z-1)+3\lambda=0 \Rightarrow z=-\frac{3}{2}\lambda+1.\end{aligned}\right\} \text{(i)}$$

& $x+2y+3z-13=0.$ — (ii)

substituting the values of x,y,z in (ii) we get,

$$\left(-\lambda/2+1\right)+2(-\lambda+1)+3\left(-\frac{3\lambda}{2}+1\right)=13.$$

or, $\lambda=-1$.

& the stationary point is $\left(3/2, 2, 5/2\right)$.

Now, $x+2y+3z=13$

or, $z=\frac{13-x-2y}{3}$

$\therefore \frac{\partial z}{\partial x}=-\frac{1}{3}, \quad \frac{\partial z}{\partial y}=-\frac{2}{3}$.

From $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$,

we get, $\frac{\partial f}{\partial x}=2(x-1)+2(z-1)\frac{\partial z}{\partial x}$

$=2(x-1)-\frac{2}{3}(z-1)$.

$\frac{\partial f}{\partial y}=2(y-1)+2(z-1)\frac{\partial z}{\partial y}=2(y-1)-\frac{4}{3}(z-1)$.

$f_{xx}=2-\frac{2}{3}\frac{\partial z}{\partial x}=2+\frac{2}{9}=\frac{20}{9}$

$f_{yy}=2-\frac{4}{3}\frac{\partial z}{\partial y}=2+\frac{8}{9}=\frac{26}{9}$.

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$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4}{3}\frac{\partial z}{\partial x} = \frac{4}{9}.$$

$$H = f_{xx}f_{yy} - (f_{xy})^2$$

$$= \frac{20}{9} \times \frac{26}{9} - \frac{16}{81} = \frac{1}{81}(520 - 16) > 0.$$

$$\& \; f_{xx} = \frac{20}{9} > 0.$$

$\therefore$ The square of the distance is minimum at $(3/2, 2, 5/2)$ (& hence the distance is also minimum) & the min. distance is $\left\{(3/2 - 1)^2 + (2-1)^2 + (5/2 - 1)^2\right\}^{\frac{1}{2}} = \frac{\sqrt{14}}{2}$ units

Ex 2. If $xyz = c^3$, a constant, using Lagrange's multiplier method, evaluate the minimum value of $f(x, y, z) = xy + yz + zx$.

Soln: The Lagrangian function is

$$L(x, y, z, \lambda) = xy + yz + zx + \lambda(xyz - c^3)$$

where λ is Lagrange's multiplier

Now,

$$\frac{\partial L}{\partial x} = y + z + \lambda yz = 0$$

$$\text{or, } -\lambda = \left(\frac{1}{y} + \frac{1}{z}\right) \quad \text{— (i)}$$

$$\frac{\partial L}{\partial y} = x + z + \lambda xz = 0$$

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or, $-\lambda = \left(\frac{1}{x}+\frac{1}{z}\right)$ —(ii)

$$\frac{\partial L}{\partial z} = y + x + \lambda xy = 0$$

or, $-\lambda = \left(\frac{1}{x}+\frac{1}{y}\right)$ —(iii)

& we have $xyz = c^3$. —(iv)

∴ we have

$$-\lambda = \frac{1}{y}+\frac{1}{z} = \frac{1}{z}+\frac{1}{x} = \frac{1}{x}+\frac{1}{y} \text{ —(v)}$$

Adding these eqns., we get,

$$2\left(\frac{1}{y}+\frac{1}{x}+\frac{1}{z}\right) = -3\lambda \text{ — (vi)}$$

(vi) $- 2\times$(i) gives,

$2\times\frac{1}{x} = -\lambda$ or, $x = -\frac{2}{\lambda}$.

Similarly, $y = z = -\frac{2}{\lambda} = x$.

∴ (iv) gives, $x = y = z = c$.

∴ The stationary point is (c, c, c)

Putting $z = \frac{c^3}{xy}$ (from iv) we get,

$$f(x,y,z) = xy + yz + zx$$

$$= xy + y\frac{c^3}{xy} + x\cdot\frac{c^3}{xy}$$

$$= xy + \frac{c^3}{x} + \frac{c^3}{y} = g(x,y) \text{ (say)}$$

Now, $g_x = y - \frac{c^3}{x^2}$, $g_y = x - \frac{c^3}{y^2}$

$g_{xx} = \frac{2c^3}{x^3}$, $g_{yy} = \frac{2c^3}{y^3}$, $g_{xy} = 1$

$$\therefore H = \left[g_{xx}g_{yy} - (g_{xy})^2\right]_{(c,c,c)}$$

$$= \frac{2c^3}{c^3} \times \frac{2c^3}{c^3} - 1$$

$$= 4 - 1 = 3 > 0$$

$$\& \left[g_{xx}\right]_{(c,c,c)} = 2 > 0.$$

$\therefore f(x,y,z) = xy + yz + zx$ is minimum at (c,c,c) and the min. value is $\left[f(x,y,z)\right]_{(c,c,c)} = 3c^2$. Ans.

Ex 3. Using Lagrange's multiplier method, find the maximum and minimum values of the following functions subject to the given conditions:

i) x^2+y^2 subject to $3x+2y=6$

ii) x^2y^2 subject to $x+y=1$

iii) shortest distance of $x^2+y^2+z^2=36$ from the point $(1,2,2)$.

Drawback — The drawback of Lagrange's method is that we cannot determine the nature of the stationary point always. Sometimes, it can be determined from physical consideration of the problem.

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$$\frac{\partial L}{\partial z} = y + x + \lambda xy = 0$$

or, $-\lambda = \cancel{\sigma}\left(\frac{1}{x}+\frac{1}{y}\right)$ — (iii)

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$$\therefore H = \left[g_{xx} g_{yy} - (g_{xy})^2\right]_{(c,c,c)}$$

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