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BSC-M302 (CSE/IT)

Homogeneous function (Euler's theorem)

Any function $f(x,y)$ which can be expressed in the form $x^n \phi(y/x)$ is called a homogeneous fn. of order n in x and y e.g. if

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_n y^n$$

$$= x^n \left[a_0 + a_1 y/x + a_2 (y/x)^2 + \cdots + a_n (y/x)^n \right]$$

in which every term is of the n^{th} degree is called a homogeneous fn. of order n.

Euler's theorem on homogeneous fn (for two variables)

If $f(x,y)$ be a homogeneous function of x and y of degree n having continuous partial derivatives, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Extension of Euler's theorem (for two variables.)

If $f(x,y)$ be a homogeneous fn of x and y of degree n having second order continuous partial derivatives, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y)$$

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Ex 1: If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, then prove that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

Soln: Let $z = \sin u$

$$= \frac{x^2+y^2}{x+y}$$

$$= \frac{x^2\left[1+(y/x)^2\right]}{x\left[1+y/x\right]}$$

$$= x\frac{\left[1+(y/x)^2\right]}{1+y/x}$$

$$= x^1\phi(y/x).$$

$\therefore$ $z = \sin u$ is a homogeneous fn. of order 1 in x and y.

$\therefore$ By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1\cdot z = z$$

or, $x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u.$

or, $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$

or, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$ (proved)

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Ex 2. If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$,

then prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$.

Soln:- Here

$$f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}.$$

$$= \frac{1}{x^2}\left[1 + \frac{1}{y/x} + \frac{-\log(y/x)}{1+(y/x)^2}\right]$$

$$= x^{-2}\left[1 + \frac{1}{y/x} - \frac{\log(y/x)}{1+(y/x)^2}\right]$$

$$= x^{-2}\phi(y/x).$$

$\therefore$ $f(x,y)$ is a homogeneous fn. of order -2 and hence by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0. \text{ (proved)}$$

Ex 3- Verify Euler's theorem for the function $f(x,y) = \sin^{-1}\frac{x}{y} + \tan^{-1}y/x$.

Soln. $f(x,y) = x^0 \cdot \sin^{-1}\frac{1}{y/x} + x^0 \tan^{-1} y/x$.

$\therefore$ $f(x,y)$ is a homogeneous fn. of order 0 & hence by Euler's th.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 . f(x,y) = 0 \quad \text{--- (1)}$$

Now, $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right)$

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$$= \frac{1}{\sqrt{1-\cancel{(x/y)}^2}}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} - \frac{1}{x^2+y^2} \cdot (y)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\therefore x\frac{\partial f}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{--- (2)}$$

Also, $$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \left(-\frac{x}{y^2}\right)$$

$$+ \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial f}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{--- (3)}$$

Adding (2) & (3) we get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0. \text{ (verified)}$$

Ex 4. If $u = x f(y/x) + g(y/x)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x f(y/x)$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

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soln. Let $u = v + w$ — (1)
where $v = x f(y/x) \rightarrow$ homogeneous fn. of order 1.
$w = x^0 g(y/x) \rightarrow$ homogeneous fn. of order 0.

$\therefore$ By Euler's th. (for v),
$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1.v = v \quad \text{— (2)}$$

(for w)

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.w = 0. \quad \text{— (3)}$$

Adding (2) & (3) we get,
$$x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = v + 0$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x f(y/x) \quad \text{(proved)} \quad (4)$$

Differentiating both sides of (4) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = f(y/x) + x f'(y/x)\left(-\frac{y}{x^2}\right)$$
$$= f(y/x) - \frac{y}{x} f'(y/x) \quad \text{— (5)}$$

Again differentiating both sides of (4) partially w.r.t y, we get,

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = x f'(y/x)\left(\frac{1}{x}\right)$$
$$= f'(y/x) \quad \text{— (6)}$$

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we assume here that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Multiplying ⑤ by x and ⑥ by y and adding, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$+ y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = x f(y/x)$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0.$$

proved

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