BSC-M 302 (CSE, IT)

Functions of several variables

Explicit & Implicit function.

Let us consider a function of n independent variables

$$z = f(x, y, ---, t) \quad —①$$

z is the dependent variable.

In case of several variables it is quite unlikely to express one of the variables explicitly in terms of the others. In such a case we say that it is an implicit function. But eqn ① is an explicit fn. whereas $\phi(x, y, ---, t) = 0$ is an implicit fn.

Let us consider a fn. of z (dependent variable) of two independent variables x and y.

$$\therefore z = f(x, y)$$

The collection of all such pts (x, y) is called the domain of the fn.

If the domain is bdd. by a closed curve c, it is called closed if f is defined $\forall$ pts within and on c and open when f is defined $\forall$ pts within c but not on c.

Neighbourhood of a pt.

Suppose (a, b) is any point. The set of values pts (x, y) is said to be in the nbd. of (a, b)

if $|x-a| < \delta$, $|y-b| < \delta$

where δ is an arbitrary small positive number.

So, $(a-\delta, a+\delta; b-\delta, b+\delta)$ is the nbd. (square nbd).

we may consider the ~~nbd~~ pts. inside ~~of~~ the circle $x^2+y^2=\delta^2$ as the nbd of the point $(0,0)$. (circular nbd.)

__Limit of a function.__

$$\lim_{(x,y)\to(a,b)} f(x,y) = l.$$

if fn f is said to tend to a limit l as (x,y) tends to (a,b) if corresponding to every $\epsilon\ (>0)$, $\exists$ a positive number δ, such that

$|f(x,y)-l| < \epsilon$ $\forall (x,y)$ which satisfy

$|x-a| < \delta$, $|y-b| < \delta$.

l is called the double/simultaneous limit of f when $(x,y) \to (a,b)$.

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Ex. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0.$$

1st method. Let $|f(x,y) - l| < \epsilon$ is given.

i.e. $\left|\frac{xy(x^2-y^2)}{x^2+y^2} - 0\right| < \epsilon$ is given

we have to find a suitable $\delta(\epsilon) > 0$ which lies in the nbd of $(0,0)$.

we know that

$|x| \leq \sqrt{x^2+y^2}$, $|y| < \sqrt{x^2+y^2}$

& $\left|\frac{x^2-y^2}{x^2+y^2}\right| < 1$.

$\therefore$ L.H.S $= \left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| \leq |x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|$

$\leq \sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}\cdot 1$

$= x^2+y^2$

Now, $\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| \leq x^2+y^2 < \epsilon$.

or, $x^2+y^2 < \epsilon$

we choose $\epsilon = \delta^2$.

$\therefore$ $x^2+y^2 < \epsilon = \delta^2$ is a circular nbd,

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$\therefore$ $\forall \epsilon > 0$, $\exists$ a $\delta(\epsilon) > 0$ such that
$\left| xy \frac{(x^2-y^2)}{x^2+y^2} \right| < \epsilon \ \forall \ x^2+y^2 < \delta^2$.
which implies that
$$\lim_{(x,y) \to (0,0)} f(x,y) = 0,$$
where $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$.

2nd Method - ~~test~~ Put $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \left| xy \frac{x^2-y^2}{x^2+y^2} - 0 \right| = \left| r^2 \sin\theta\cos\theta . \cos 2\theta \right|$$
$$= \left| \frac{r^2}{2} \sin 2\theta \cos 2\theta \right|$$
$$= \left| \frac{r^2}{4} 2\sin 2\theta \cos 2\theta \right|$$
$$= \left| \frac{r^2}{4} \sin 4\theta \right|$$
$$\le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \epsilon = \epsilon/_2 + \epsilon/_2$$

or if $\frac{x^2}{4} < \epsilon/_2$, $\frac{y^2}{4} < \epsilon/_2$

or, if $|x| < \sqrt{2\epsilon} = \delta$, $|y| < \sqrt{2\epsilon} = \delta$

$\therefore$ $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ such that
$\left| xy \frac{(x^2-y^2)}{x^2+y^2} - 0 \right| < \epsilon$ when $|x| < \delta$, $|y| < \delta$.

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Ex 2: Prove that

$$\lim_{(x,y)\to(1,2)} (x^2+2y) = 5.$$

Soln: Let $\epsilon > 0$ be given. We have to a a δ which lies in the nbd. of $(1,2)$

$\therefore |f(x,y) - l| < \epsilon$ (given)

or, $|x^2+2y-5| < \epsilon$

or, $|x^2-1+2y-4| < \epsilon$

or, $|(x-1)(x+1)+2(y-2)| < \epsilon$

Now,

$$|x^2+2y-5| = |(x-1)(x+1)+2(y-2)|$$

$$\leq |x-1||x+1| + 2|y-2|$$

$$\leq 2|x-1| + 2|y-2|$$

$$< \epsilon = \epsilon/2 + \epsilon/2$$

if $2|x-1| < \epsilon/2$, $2|y-2| < \epsilon/2$

or, $|x-1| < \epsilon/4 = \delta$, $|y-2| < \epsilon/4 = \delta$,

$\therefore \forall \epsilon > 0, \exists$ a $\delta > 0$ such that

$|x^2+2y-5| < \epsilon$ when $|x-1| < \delta$, $|y-2| < \delta$

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Ex: Show that $\lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right) = 0$.

Soln. Let $\epsilon > 0$ be given.

Now, $\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$< |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$

$< |x| + |y| \le \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$

$= 2\sqrt{x^2+y^2} < \epsilon$

This is true if

or, $x^2+y^2 < \frac{\epsilon^2}{4}$.

$x^2 < \frac{\epsilon^2}{8}$, $y^2 < \frac{\epsilon^2}{8}$

or, $|x| < \frac{\epsilon}{2\sqrt{2}} = \delta$, $|y| < \frac{\epsilon}{2\sqrt{2}} = \delta$.

$\therefore \forall \epsilon > 0$, $\exists \delta > 0$ such that

$\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| < \epsilon$ when $|x| < \delta$, $|y| < \delta$.

$\Rightarrow \lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right) = 0$.

Continuity.

A fn. $f(x,y)$ is said to be continuous at (a,b) of its domain of definition if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Non-existence of limit

Ex: Let us consider the fn.

$f(x,y) = \frac{xy}{x^2+y^2}$.

Check whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not.