Task

① Show that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{\lfloor n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right)$

② If $y = a\cos(\log x) + b\sin(\log x)$, then prove that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$$

③ If $y = (x^2-1)^n$, then show that

$$(x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$$

④ If $y = 2\cos x(\sin x - \cos x)$, show that $(y_{10})_0 = 2^{10}$

⑤ If $y = \frac{x}{x+1}$, show that $y_5(0) = \lfloor 5$.

Problem

① If $P_n = D^n(x^n \log x)$, prove that $P_n = nP_{n-1} + \lfloor n-1$

Given $P_n = D^n(x^n \log x)$

$$= D^{n-1}\left(x^n \cdot \frac{1}{x} + n x^{n-1}\log x\right)$$

$$= D^{n-1}(x^{n-1}) + n D^{n-1}(x^{n-1}\log x)$$

$$= \lfloor n-1 + n P_{n-1}$$

$$= n P_{n-1} + \lfloor n-1 \quad \text{proved}$$

② If $f(x) = x^n$, prove that

$$f(1) + \frac{f'(1)}{\lfloor 1} + \frac{f''(1)}{\lfloor 2} + \frac{f'''(1)}{\lfloor 3} + \cdots + \frac{f^n(1)}{\lfloor n} = 2^n$$

③ If $f(x) = \tan x$ and n is a +ve integer, prove with the help of Leibnitz's theorem that

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) - \cdots = \sin\left(\frac{n\pi}{2}\right)$$

Sol. Here $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$\Rightarrow f(x)\cdot\cos x = \sin x$$

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now applying Leibnitz's theorem, we have

$$f^{n}(x)\cos x + {}^{n}C_1 f^{n-1}(x)\cdot(-\sin x) + {}^{n}C_2 f^{n-2}(x)(-\cos x)$$
$$+ {}^{n}C_3 f^{n-3}(x)\cdot\sin x + {}^{n}C_4 f^{n-4}(x)\cdot\cos x + \cdots = \sin\left(\frac{n\pi}{2}+x\right)$$

putting $x=0$ on both sides, we get

$$f^{n}(0) - {}^{n}C_2 f^{n-2}(0) + {}^{n}C_4 f^{n-4}(0) - \cdots = \sin\left(\frac{n\pi}{2}\right)$$

proved

④ If $y = \cos(10\cos^{-1}x)$, then show that

$$(1-x^2)y_{12} = 21\,x\,y_{11}$$

⑤ If $y = \cosh(\sin^{-1}x)$, prove that, ⓘ $(1-x^2)y_2 - xy_1 - y = 0$

ⓘⓘ $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)y_n$

Also find the value of y_n when $x=0$.

Sol. Determination of $(y_n)_0$:- we have the results —

$$y = \cosh(\sin^{-1}x) \quad , \quad y_1 = \frac{\sinh(\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$(1-x^2)y_2 - xy_1 - y = 0 \quad , \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)y_n$$

putting $x=0$ in each of the above four relations —

$$(y)_0 = \cosh(\sin^{-1}0) = \cosh 0 = 1 \,, \quad (y_1)_0 = \frac{\sinh(\sin^{-1}0)}{\sqrt{1-0}} = 0$$

$$(y_2)_0 = (y)_0 = 1 \,, \quad \text{finally } (y_{n+2})_0 = (n^2+1)(y_n)_0 \,.$$

In the last relation, putting $n = 1, 3, 5, 7, \cdots$

$$(y_3)_0 = (1^2+1)(y_1)_0 = 0 \,; \quad (y_5)_0 = (3^2+1)(y_3)_0 = 0, \text{ etc.}$$

thus $(y_{2n+1})_0 = 0$, for all +ve integer values of n.

Next we put $n=2$, we get

$$(y_4)_0 = (2^2+1)(y_2)_0 = (2^2+1)\cdot 1$$

putting $n=4$, we get

$$(y_6)_0 = (4^2+1)(y_4)_0 = (4^2+1)(2^2+1)\cdot 1 \text{ , etc.}$$

thus

$$(y_{2n})_0 = \{(2n-2)^2+1\}\{(2n-4)^2+1\}\cdots(4^2+1)(2^2+1)\cdot 1$$

we conclude

$$(y_n)_0 = \{(n-2)^2+1\}\{(n-4)^2+1\}\cdots\cdots(4^2+1)(2^2+1)\cdot 1$$

if n be an even.

$= 0$ if n be odd.

⑥ If $y = \tan^{-1}x$, then prove that

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$$

Find also the value of $(y_n)_0$.

$\begin{cases} = 0, \text{ even} \\ = (-1)^{\frac{n-1}{2}}\lfloor n-1, \text{ odd.} \end{cases}$

⑦ If $x+y=1$, prove that the n th derivative of $x^n y^n$ is $\lfloor n\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}\cdot x^2 - ({}^nC_3)^2 y^{n-3}x^3 + \cdots + (-1)^n x^n\}$.