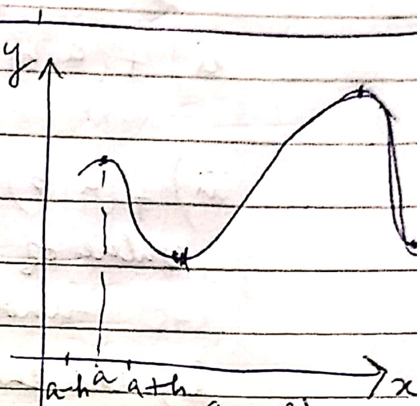
Maxima & Minima for functions of 2 variables

A point (a,b) is called a local (or relative) maximum point of $f(x,y)$ if $\exists$ a region surrounding the point (a,b) in which $f(x,y) < f(a,b)$ $\forall$ points (x,y) except (a,b) in this region.

A point (a,b) is called a local or relative minimum point of $f(x,y)$ if $\exists$ a region surrounding the point (a,b) for which $f(x,y) > f(a,b)$ $\forall$ points (x,y) except (a,b) in this region.



If a be a (local) maximum then $f(a+h) < f(a)$ for small h.
If a be a (local) minimum then $f(a+h) > f(a)$ & $f(a-h) > f(a)$ for a fn. of single variable

Necessary condition for extrema

Th: If a function $z = f(x,y)$ has a maximum or minimum point at (a,b), then $f_x(a,b) = 0 = f_y(a,b)$, provided these partial derivatives exist.

Th: Conditions of Extrema

Let $z = f(x,y)$ be a continuous fn. having 2^{nd} order partial derivatives and (a,b) be a point satisfying the equations $f_x = 0 = f_y$ ie

$f_x(a,b)=0$, $f_y(a,b)=0$, If H is defined by $H(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$

then

i) $f(a,b)$ is maximum if $H(a,b) > 0$ & $f_{xx}(a,b) < 0$ (or $f_{yy}(a,b) < 0$)

ii) $f(a,b)$ is a minimum if $H(a,b) > 0$ & $f_{xx}(a,b) > 0$ (or $f_{yy}(a,b) > 0$)

iii) $f(a,b)$ is neither a max. nor a min. value of $f(x,y)$ at (a,b) if $H(a,b) < 0$ and

iv) the case is doubtful and need further investigation if $H(a,b) = 0$.

Saddle point:

A point (a,b) is called a saddle point of $f(x,y)$ if it is neither a max. nor a min. at (a,b) though $f_x(a,b)=0=f_y(a,b)$.

[in this case $H(x,y)$ above is ≤ 0]

Stationary (or critical point)

A point (a,b) is called a stationary point of $f(x,y)$ if $f_x(a,b)=0=f_y(a,b)$.

Working Rules.

Given a fn $f(x,y)$

step 1. Solve $f_x = 0 = f_y$ to find the stationary point (a,b).

step 2. calculate $H(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \{f_{xy}(a,b)\}^2$

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classmate

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Step 3 (A) If $H(a,b) > 0$, then $f(x,y)$ has

i) max at (a,b) $f_{xx}(a,b) < 0$ (or $f_{yy}(a,b) < 0$)

ii) Min. at (a,b) if $f_{xx}(a,b) > 0$ (or $f_{yy}(a,b) > 0$)

(B) If $H(a,b) < 0$, then $f(x,y)$ has neither a max. nor a min. at (a,b).

Here (a,b) is a saddle point.

(C) If $H(a,b) = 0 \Rightarrow$ case is doubtful & needs further investigation.

Ex 1: Find the maxima & minima of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Soln. Here $f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$, $f_{xx} = 6x$,

$f_{yy} = 6y$, $f_{xy} = 0 = \frac{\partial}{\partial x}(f_y)$

For max or min

$$f_x = 0 = f_y$$

$\therefore$ $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

& $3y^2 - 12 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$.

$\therefore$ The stationary points are

$(1,2)$, $(-1,-2)$, $(-1,2)$, $(1,-2)$

At $(1,2)$,

$$H(x,y) = \left[f_{xx} f_{yy} - (f_{xy})^2\right]_{(1,2)}$$

$$= (6x \times 6y - 0)_{(1,2)}$$

$$= 72 > 0$$

& $f_{xx}(1,2) = (6x)_{(1,2)} = 6 > 0$

$\therefore$ $(1,2)$ is a minima.

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At $(-1,-2)$,

$$H(-1,-2) = \left[6x \times 6y - (0)^2\right]_{(-1,-2)}$$

$$= 72 > 0$$

& $f_{xx}(-1,-2) = (6x)_{(-1,-2)}$

$$= -6 < 0$$

$\therefore$ $f(x,y)$ has a maxima at $(-1,-2)$.

At $(-1,2)$,

$$H(-1,2) = (36xy)_{(-1,2)}$$

$$= -72 < 0$$

&

At $(1,-2)$,

$$H(1,-2) = (36xy)_{(1,-2)}$$

$$= -72 < 0$$

$\therefore$ $f(x,y)$ has neither max nor min at $(1,-2)$ & $(-1,2)$ (saddle points)

Ex 2: Find the maxima and minima of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Soln.

Here $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$\therefore$ $f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$.

The stationary points are given by

$f_x = 3x^2 + 3y^2 - 30x + 72 = 0$ ——— (i)

$f_y = 6y(x-5) = 0$. ——— (ii)

From (ii) either $y=0$ or $x=5$.

Putting $y=0$ in ①, we get

$3x^2 - 30x + 72 = 0$

or, $x^2 - 10x + \cancel{8}24 = 0$

or, $x^2 - 6x - 4x + 24 = 0$

or, $x(x-6) - 4(x-6) = 0$

or, $(x-4)(x-6) = 0$

either $x = 4$ or $x = 6$.

Putting $x=5$ in (i) we get,

$75 + 3y^2 - 150 + 72 = 0$

or, $y^2 = 1$

$\therefore y = \pm 1$.

$\therefore$ The stationary points are ~~(0,4), (0,6)~~ (4,0), (6,0), (5,1), (5,-1).

Now,

$f_{xx} = 6x - 30$, $f_{yy} = \cancel{6y}\, 6x - 30$,

$f_{xy} = 6y$.

So, $H(x,y) = f_{xx} f_{yy} - \{f_{xy}\}^2$

$= 36(x-5)^2 - 36y^2$

$= 36\{(x-5)^2 - y^2\}$.

a) at $(4,0)$, $H(4,0) = 36 > 0$ & $f_{xx}(4,0) = -6 < 0 \Rightarrow (4,0)$ is a maxima & the max. value is $f(4,0) = 4^3 + 0 - 15 \times 4^2 - 0 + 72 \times 4 = 112$.

b) at $(6,0)$, $H(6,0) = 36 > 0$, $f_{xx}(6,0) = 6 > 0 \Rightarrow (6,0)$ is a minimum & the min. value is $f(6,0) = 6^3 + 0 - 15 \times 6^2 - 0 + 72 \times 6 = 108$.

c) at $(5,1)$, $H(5,1) = -36 < 0$, so $(5,1)$ is a saddle point i.e. the fn has neither max nor min. at $(5,1)$.

d) at $(5,-1)$, $H(5,-1) = -36 < 0$, so $(5,-1)$ is also a saddle point.

Ex3: Find the maximum and minimum values of the function $f(x,y) = x^3 + y^3 - 3axy$.

Soln: $f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

The critical points are given by

$3x^2 - 3ay = 0$ —(i) $3y^2 - 3ax = 0$ —(ii)

(i) − (ii) gives,

$x^2 - y^2 + a(x-y) = 0$

or, $(x-y)(x+y+a) = 0$

∴ either $x = y$ or $x + y = -a$.

If $x = y$, then (i) gives,

$x = 0$ or a

and hence in this case solutions are

$(0,0)$, (a,a);

putting $y=-x-a$ in (i) we get,

$x^2+ax+a^2=0$, which ~~has~~ has no real solution.

Now, $f_{xx}=6x$, $f_{yy}=6y$, $f_{xy}=-3a$

$\therefore H(x,y)=f_{xx}f_{yy}-(f_{xy})^2=36xy-9a^2$

Now,

a) At $(0,0)$, $H(0,0)=-9a^2<0$, so $f(x,y)$ has a saddle point at $(0,0)$ [neither max nor min]

b) At (a,a), $H(a,a)=36a^2-9a^2$
$=27a^2>0$

and $f_{xx}=6a \gtrless 0$ according as $a \gtrless 0$.

$\therefore$ when $a>0$, $f_{xx}>0$ and hence $f(x,y)$ has a max. & f_{max} at (a,a) $=a^3+a^3-3a^3$
$=-a^3$.

when $a<0$, $f_{xx}<0$ and hence $f(x,y)$ has a min. at (a,a) & $f_{min}=-a^3$.

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