Successive Differentiation

(n-th Derivative)

Introduction :-

If f has a derivative f' in a given interval and if f' is itself derivable, we denote the derivative of f' by f'' and call f'', the second order derivative of f.

Continuing in this manner, we obtain functions $f, f', f'', f''', f^{iv}, \ldots, f^{(n)}$, each of which is the derivative of the previous one. We call $f^{(n)}$, the n-th derivative of f or $f^{(n)}$ is the derivative of order n of the function f.

Notations ; $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x), \ldots \frac{d^ny}{dx^n} = f^n(x)$

$$\frac{dy}{dx} = y_1, \quad \frac{d^2y}{dx^2} = y_2, \ldots \quad \frac{d^ny}{dx^n} = y_n$$

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \ldots, \quad \frac{d^ny}{dx^n} = D^ny$$

Observation :-

In order that $f^n(x)$ may exist at a certain point x, it is clear that $f^{(n-1)}(x)$ must exist in a certain nbd of x and $f^{(n-1)}(x)$ must be derivable at x. Since $f^{(n-1)}$ must exist in a nbd of x, $f^{(n-2)}$ must be derivable in that nbd and so on.

Standard Results :-

Let $y = x^k$, $k \in \mathbb{R}$

$$y_1 = k x^{k-1}$$

$$y_2 = k(k-1) x^{k-2}$$

$$y_3 = k(k-1)(k-2) x^{k-3}$$

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We may infer that

$$y_n = k(k-1)(k-2) \cdots (k-(n-1)) x^{k-n}, \quad n \in \mathbb{N} \text{ +ve integer} \quad \text{-----} (1)$$

We can justify this result by Mathematical Induction.

Special Case :-

$$y = x^{k}$$

1. Let k be a +ve integer, then

$$y_k = k(k-1)(k-2)\cdots(k-k+1)\,x^{k-k}$$
$$= k(k-1)(k-2)\cdots 3\cdot 2\cdot 1$$
$$= \lfloor k$$

2. Let k be a +ve integer, but n is a +ve integer $> k$.
Then $y_n = 0$

3. k be a +ve integer

$$y_{k-1} = k(k-1)(k-2)\cdots(k-(k-1-1))\,x^{k-(k-1)}$$
$$= k(k-1)(k-2)\cdots 2\cdot x^{1}$$
$$= k(k-1)(k-2)\cdots 2\cdot 1\cdot x$$
$$= \lfloor k\cdot x$$

Similarly $y_{k-2} = \dfrac{\lfloor k}{\lfloor 2}\cdot x^{2}$

$$y_{k-3} = \frac{\lfloor k}{\lfloor 3}\,x^{3}$$

and so on.

4. Let k be a +ve real number so that $-k$ is a -ve real number.
Then if $y = x^{-k}$

so $y_n = -k(-k-1)(-k-2)\cdots(-k-n+1)\,x^{-k-n}$

$$= (-1)^{n}\,\frac{k(k+1)(k+2)\cdots(k+n-1)}{x^{n+k}}$$

If K be a +ve integer,

then $$y_n = (-1)^n \frac{\lfloor K+n-1}{\lfloor K-1 \; x^{n+K}}$$

5. If $y = x^{-1}$ i.e. $y = \frac{1}{x}$

we have $$y_n = \frac{(-1)^n \lfloor n}{x^{n+1}}$$, taking $K=1$

if $y = x^{-2}$, $$y_n = \frac{(-1)^n \lfloor n+1}{\lfloor 1 \cdot x^{n+2}}$$, taking $K=2$.

6. If $y = \log x \; (x>0)$;

now $y_1 = \frac{1}{x}$

Here y_n = n th derivative of $(\log x)$

= $(n-1)$ th derivative of $\frac{1}{x}$

$$= \frac{(-1)^{n-1} \lfloor n-1}{x^n}$$

7. If $y = \frac{1}{x-a}$

then $$y_n = \frac{(-1)^n \lfloor n}{(x-a)^{n+1}}$$

8. If $y = \frac{1}{ax+b}$

then $$y_n = \frac{(-1)^n \lfloor n}{(ax+b)^{n+1}} \cdot a^n.$$

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9. If $y = \frac{1}{x^2 - a^2}$, then find y_n.

Here $y = \frac{1}{x^2 - a^2}$

$= \frac{1}{(x+a)\cdot(x-a)}$

$= \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

So $y_n = \frac{1}{2a} (-1)^n \lfloor n \left\{ \frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} \right\}$

Q. If $y = \frac{1}{x^2 + a^2}$ then find y_n.

10. If $y = \frac{x^n}{x-1}$, then find y_n.

Here $y = \frac{x^n}{x-1}$

$= \frac{x^n - 1 + 1}{x-1}$

$= \frac{x^n - 1}{x-1} + \frac{1}{x-1}$

$= x^{n-1} + x^{n-2} + \cdots + 1 + \frac{1}{x-1}$

then $y_n = 0 + 0 + \cdots + 0 + (-1)^n \lfloor n \frac{1}{(x-1)^{n+1}}$

$= \frac{(-1)^n \lfloor n}{(x-1)^{n+1}}$

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11. If $y = \frac{1}{a-x}$

then $y_1 = (-1)(-1)(a-x)^{-2}$

$y_2 = (-1)(-2)(-1)^2(a-x)^{-3}$

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$y_n = (-1)(-2)(-3)\cdots(-n)(-1)^n(a-x)^{-1-n}$

$= 1\cdot 2\cdot 3\cdots n\,(-1)^{2n}(a-x)^{-1-n}$

$= \frac{\lfloor n}{(a-x)^{n+1}}$ (as $(-1)^{2n}=1$).

Q. (i) If $y = (a-bx)^m$, find y_n, $m \in R$

(ii) If $y = \log(ax+b)^p$, find y_n.

(iii) If $y = \sqrt{x}$, find y_n.

12. If $y = e^{ax}$ then $y_n = a^n e^{ax}$

13. If $y = \sin ax$

then $y_1 = a\cos ax = a\sin(\pi/2 + ax)$

$y_2 = -a^2\sin ax = a^2\sin(2\cdot\pi/2 + ax)$

$y_3 = -a^3\cos ax = a^3\sin(3\cdot\pi/2 + ax)$

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Finally $y_n = a^n\sin(n\pi/2 + ax)$

14. Similarly if $y = \cos ax$ then $y_n = a^n\cos(n\pi/2 + ax)$

We note that

$$D^{2n}(\sin ax) = a^{2n}\sin(2n.\pi/2 + ax)$$
$$= a^{2n}\sin(n\pi + ax)$$
$$= a^{2n}(\sin ax \cos n\pi + \cos ax \sin n\pi)$$
$$= a^{2n}(-1)^n \sin ax \qquad \because \sin n\pi = 0, \ \cos n\pi = (-1)^n$$
$$= (-a^2)^n \sin ax$$

and also $D^{2n}(\cos ax) = (-a^2)^n \cos ax$.

Q. If $y = \sin^2 x$ then find y_n.

Here $y = \sin^2 x$

$$= \frac{1}{2}(1 - \cos 2x)$$

So $y_n = -\frac{1}{2}\cdot 2^n \cos(n\pi/2 + 2x)$

$$= -2^{n-1}\cos(2x + n\pi/2).$$

Q. If $y = \sin kx + \cos kx$

then prove that $y_n = k^n\{1 + (-1)^n \sin 2kx\}^{1/2}$.

Sol. $y_n = k^n\{\sin(n\pi/2 + kx) + \cos(n\pi/2 + kx)\}$

$$= k^n\left[\{\sin(n\pi/2 + kx) + \cos(n\pi/2 + kx)\}^2\right]^{1/2}$$
$$= k^n\{\sin^2(n\pi/2 + kx) + \cos^2(n\pi/2 + kx) + 2\sin(n\pi/2 + kx)\cos(n\pi/2 + kx)\}^{1/2}$$
$$= k^n\{1 + \sin(n\pi + 2kx)\}^{1/2}$$
$$= k^n[1 + (-1)^n \sin 2kx]^{1/2}$$

15. If $y = e^{ax}\sin bx$

$y_1 = ae^{ax}\sin bx + be^{ax}\cos bx$

$= e^{ax}(a\sin bx + b\cos bx)$

let $a = r\cos\theta$, $b = r\sin\theta$, so $\theta = \tan^{-1}(b/a)$

then $y_1 = e^{ax}(r\cos\theta\sin bx + r\sin\theta\cos bx)$

$= re^{ax}\sin(bx+\theta)$

similarly $y_2 = r^2e^{ax}\sin(bx+2\theta)$

so $y_n = r^n e^{ax}\sin(bx+n\theta)$.

$= (a^2+b^2)^{n/2} e^{ax}\sin(bx + n\tan^{-1} b/a)$

Q. (i) If $y = e^{2x}\sin x$ then find y_n.

(ii) If $y = e^x\cos^2 3x$ then find y_n.

(iii) If $y = 10^{5-3x}$ then find y_n.

(iv) If $y = e^{ax}\cos^2 bx$ then find y_n.

<u>Use of Partial Fraction in Finding nth Derivative</u> :-

If $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$, find y_n

Here $y = \dfrac{x^2+x-1}{x(x+3)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$

$\therefore x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$

putting $x = 0, 2, -3$

$A = 1/6$, $B = 1/3$, $C = 1/2$

so $y = \dfrac{1}{6x} + \dfrac{1}{3(x+3)} + \dfrac{1}{2(x-2)}$

therefore $y_n = (-1)^n \underline{n} \left\{ \dfrac{1}{6}\dfrac{1}{x^{n+1}} + \dfrac{1}{3}\dfrac{1}{(x+3)^{n+1}} + \dfrac{1}{2}\dfrac{1}{(x-2)^{n+1}} \right\}$

Q.(i) If $y = \dfrac{x^2-6}{x^3-x^2-2x}$ then find y_n.

(ii) If $y = \dfrac{x-2}{x^3-x^2-5x-3}$ then find y_n.

(iii) If $y = \dfrac{13x}{(x-3)(x^2+x+1)}$ then find y_n.

(iv) Evaluate $D^n\left(\dfrac{b+cx}{a+2bx+cx^2}\right)$.

Ex-(a) If $y = x^{2n}$, where n being a +ve integer, then show that $y_n = 2^n\{1.3.5\cdots(2n-1)\}x^n$

Sol.

$$y = x^{2n}$$

$$y_1 = 2n\,x^{2n-1}$$

$$y_2 = 2n(2n-1)\,x^{2n-2}$$

$$\cdots\cdots\cdots\cdots$$

$$y_n = 2n(2n-1)\cdots(2n-n+1)\,x^{2n-n}$$

$$= 2n(2n-1)\cdots(n+1)\,x^n$$

$$= \frac{2n(2n-1)\cdots(n+1)n(n-1)(n-2)\cdots 3\cdot2\cdot1}{n(n-1)(n-2)\cdots 3\cdot2\cdot1}\,x^n$$

$$= \frac{\{2n(2n-2)(2n-4)\cdots 6\cdot4\cdot2\}\{(2n-1)(2n-3)\cdots 3\cdot1\}\,x^n}{\underline{|n}}$$

$$= \frac{2^n\{n(n-1)(n-2)\cdots 3\cdot2\cdot1\}\{(2n-1)(2n-3)\cdots 5\cdot3\cdot1\}\,x^n}{\underline{|n}}$$

$$= 2^n\{1\cdot3\cdot5\cdots(2n-1)\}\,x^n. \quad \text{proved}$$

Q. If $y = x^{n-1}\log x$, then prove that $y_n = \dfrac{\underline{|n-1}}{x}$.

Q. If $y = \sin x \sin 2x \sin 3x$, then find y_n.

Q. Find y_n, where $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

{ Hints → put $x = \tan\theta$
$\Rightarrow \theta = \tan^{-1}x$

it will reduce to $y = \theta/2 = \frac{1}{2}\tan^{-1}x$

then $y_1 = \frac{1}{2}\frac{1}{1+x^2}$. }

Q. Find n th derivative of (i) $\frac{x^2}{x-1}$ (ii) $\tan^{-1}\frac{1+x}{1-x}$

(iii) 2^{-7x} (iv) $\frac{\log x}{x}$ (v) $\tan^{-1}(x/a)$