GOVERNMENT OF TAMILNADU

STANDARD TEN

MATHEMATICS

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Department of School Education

Untouchability is Inhuman and a Crime

CONTENTS

1 RELATIONS AND FUNCTIONS

"Mathematicians do not study objects, but relations between objects . . . Content to them is irrelevant: they are interested in form only" – Henri Poincare


Gottfried Wilhelm Leibniz
(1646 – 1716)

Gottfried Wilhelm Leibniz (also known as von Leibniz) was a prominent German mathematician, philosopher, physicist and inventor. He wrote extensively on 26 topics covering wide range of subjects among which were Geology, Medicine, Biology, Epidemiology, Paleontology, Psychology, Engineering, Philology, Sociology, Ethics, History, Politics, Law and Music Theory.

In a manuscript Leibniz used the word "function" to mean any quantity varying from point to point of a curve. Leibniz provided the foundations of Formal Logic and Boolean Algebra, which are fundamental for modern day computers. For all his remarkable discoveries and contributions in various fields, Leibniz is hailed as "The Father of Applied Sciences".

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Learning Outcomes

- To define and determine cartesian product of sets.
- To define a relation as a subset of cartesian product of sets.
- To understand function as a special relation.
- To represent a function through an arrow diagram, a set of ordered pairs, a table, a rule or a graph.
- To classify functions as one-one, many-one, onto, into and bijection.
- To study combination of functions through composition operation.
- To understand the graphs of linear, quadratic, cubic and reciprocal functions.

1.1 Introduction

The notion of sets provides the stimulus for learning higher concepts in mathematics. A set is a collection of well-defined distinguishable objects. This means that a set is merely a collection of something which we may recognize. In this chapter, we try to extend the concept of sets in two forms called Relations and Functions. For doing this, we need to first know about cartesian products that can be defined between two non-empty sets.

It is quite interesting to note that most of the day-to-day situations can be represented mathematically either through a relation or a function. For example, the distance travelled by a vehicle in given time can be represented as a function. The price of a commodity can be expressed as a function in terms of its demand. The area of polygons and volume of common objects like circle, right circular cone, right circular cylinder, sphere can be expressed as a function with one or more variables.

In class IX, we had studied the concept of sets. We have also seen how to form new sets from the given sets by taking union, intersection and complementation.

Now we are about to study a new set called "cartesian product" for the given sets A and B.

1.2 Ordered Pair

Observe the seating plan in an auditorium (Fig.1.1). To help orderly occupation of seats, tokens with numbers such as (1,5), (7,16), (3,4), (10,12) etc. are issued. The person who gets (4,10) will go to row 4 and occupy the 10th seat. Thus the first number denotes the row and the second number, the seat. Which seat will the visitor with token (5,9) occupy? Can he go to 9th row and take the 5th seat? Do (9,5) and (5,9) refer to the same location? No, certainly! What can you say about the tokens (2,3), (6,3) and (10,3)?

Fig. 1.1

This is one example where a pair of numbers, written in a particular order, precisely indicates a location. Such a number pair is called an ordered pair of numbers. This notion is skillfully used to mathematize the concept of a "Relation".

RWCCEK

1.3 Cartesian Product

Illustration 1

Let us consider the following two sets.

A is the set of 3 vegetables and B is the set of 4 fruits. That is,

A = {carrot, brinjal, ladies finger} and B = {apple, orange, grapes, strawberry}

What are the possible ways of choosing a vegetable with a fruit? (Fig.1.2)

Vegetables (A)	Fruits (B)
Carrot (c)	Apple (a)
Brinjal (b)	Orange (o)
Ladies finger (l)	Grapes (g)
	Strawberry (s)

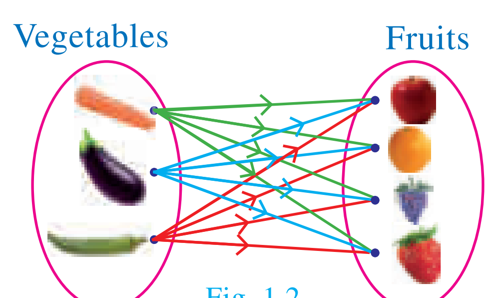

Fig. 1.2

We can select them in 12 distinct pairs as given below.

$(c, a), (c, o), (c, g), (c, s), (b, a), (b, o), (b, g), (b, s), (l, a), (l, o), (l, g), (l, s)$

This collection represents the cartesian product of the set of vegetables and set of fruits.

Definition

If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that $a \in A,\ b \in B$ is called the Cartesian Product of A and B, and is denoted by $A \times B$.

Thus, $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

Note

- $A \times B$ is the set of all possible ordered pairs between the elements of A and B such that the first coordinate is an element of A and the second coordinate is an element of B.
- $B \times A$ is the set of all possible ordered pairs between the elements of A and B such that the first coordinate is an element of B and the second coordinate is an element of A.
- If $a = b$, then $(a, b) = (b, a)$.
- The "cartesian product" is also referred as "cross product".

Illustration 2

Let A = {1, 2, 3} and B = {a, b}. Write $A \times B$ and $B \times A$?

$A \times B = \{1,2,3\} \times \{a,b\} = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$ (as shown in Fig.1.3)

$B \times A = \{a,b\} \times \{1,2,3\} = \{(a,1), (a,2), (a,3),(b,1), (b,2), (b,3)\}$ (as shown in Fig.1.3)

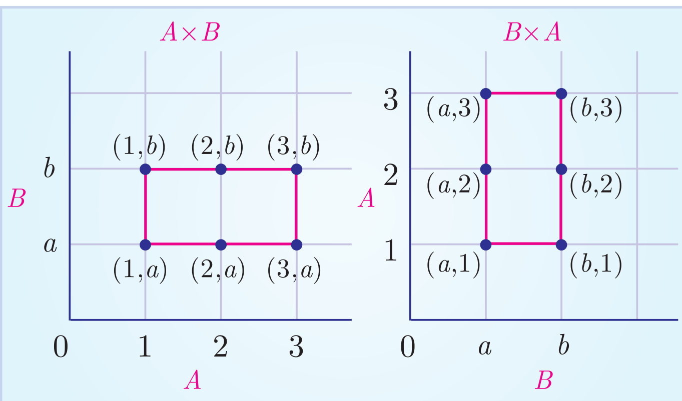

Fig. 1.3

Thinking Corner

When will $A \times B$ be equal to $B \times A$?

Note

- In general $A \times B \neq B \times A$, but $n(A \times B) = n(B \times A)$
- $A \times B = \phi$ if and only if $A = \phi$ or $B = \phi$
- If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$

Recall of standard infinite sets

Natural Numbers $\mathbb{N} = \{1,2,3,4\ldots\}$; Whole Numbers $\mathbb{W} = \{0,1,2,3, \ldots\}$;

Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$; Rational Numbers $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$;

Real Numbers $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$, where $\mathbb{Q}'$ is the set of all irrational numbers.

Illustration 3

For example, let A be the set of numbers in the interval [3, 5] and B be the set of numbers in the interval [2,3]. Then the Cartesian product $A \times B$ corresponds to the rectangular region shown in the Fig. 1.4. It consists of all points (x, y) within the region.

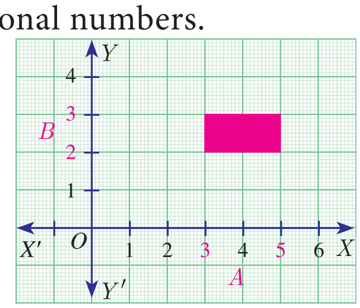

Fig. 1.4

Progress Check

1. For any two non-empty sets A and B, $A \times B$ is called as _____.
2. If $n(A \times B) = 20$ and $n(A) = 5$ then $n(B)$ is _____.
3. If $A = \{-1,1\}$ and $B = \{-1,1\}$ then geometrically describe the set of points of $A \times B$.
4. If A, B are the line segments given by the intervals $(-4, 3)$ and $(-2, 3)$ respectively, represent the cartesian product of A and B.

Note

- The set of all points in the cartesian plane can be viewed as the set of all ordered pairs (x, y) where x, y are real numbers. In fact, $\mathbb{R}\times\mathbb{R}$ is the set of all points which we call as the cartesian plane.

Activity 1

Let $A=\{x \mid x\in\mathbb{N}, x\le 4\}$, $B=\{y \mid y\in\mathbb{N}, y<3\}$

Represent $A\times B$ and $B\times A$ in a graph sheet. Can you see the difference between $A\times B$ and $B\times A$?

Example 1.1 If $A = \{1,3,5\}$ and $B = \{2,3\}$ then (i) find $A\times B$ and $B\times A$.
(ii) Is $A\times B = B\times A$? If not why? (iii) Show that $n(A\times B) = n(B\times A) = n(A)\times n(B)$

Solution Given that $A = \{1,3,5\}$ and $B = \{2,3\}$

(i) $A\times B = \{1,3,5\} \times \{2,3\} = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\}$ ...(1)

$B\times A = \{2,3\} \times \{1,3,5\} = \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$ ...(2)

(ii) From (1) and (2) we conclude that $A\times B \neq B\times A$ as $(1,2) \neq (2,1)$ and $(1,3) \neq (3,1)$, etc.

(iii) $n(A)=3$; $n(B) = 2$.

From (1) and (2) we observe that, $n(A\times B) = n(B\times A) = 6$;

we see that, $n(A)\times n(B) = 3 \times 2 = 6$ and $n(B)\times n(A) = 2\times 3 = 6$

Hence, $n(A\times B) = n(B\times A) = n(A)\times n(B) = 6$.

Thus, $n(A\times B) = n(B\times A) = n(A)\times n(B)$.

Example 1.2 If $A\times B = \{(3,2), (3,4), (5,2), (5,4)\}$ then find A and B.

Solution $A\times B = \{(3,2), (3,4), (5,2), (5,4)\}$

We have $A = \{$set of all first coordinates of elements of $A\times B\}$. Therefore, $A = \{3,5\}$

$B = \{$set of all second coordinates of elements of $A\times B\}$. Therefore, $B = \{2,4\}$

Thus $A = \{3,5\}$ and $B = \{2,4\}$.

Example 1.3 Let $A=\{x\in\mathbb{N} \mid 1<x<4\}$, $B=\{x\in\mathbb{W} \mid 0\le x<2\}$ and $C=\{x\in\mathbb{N} \mid x<3\}$.
Then verify that

(i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$ (ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

Solution $A=\{x\in\mathbb{N} \mid 1<x<4\}=\{2,3\}$, $B=\{x\in\mathbb{W} \mid 0\le x<2\}=\{0,1\}$,
$C=\{x\in\mathbb{N} \mid x<3\}=\{1,2\}$

(i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

$B\cup C = \{0,1\}\cup\{1,2\} = \{0,1,2\}$

$A\times(B\cup C) = \{2,3\}\times\{0,1,2\} = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2)\}$ ...(1)

$A\times B = \{2,3\}\times\{0,1\} = \{(2,0),(2,1),(3,0),(3,1)\}$

$A\times C = \{2,3\}\times\{1,2\} = \{(2,1),(2,2),(3,1),(3,2)\}$

$$(A\times B)\cup(A\times C) = \{(2,0),(2,1),(3,0),(3,1)\}\cup\{(2,1),(2,2),(3,1),(3,2)\}$$
$$= \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2)\} \quad \text{...(2)}$$

From (1) and (2), $A\times(B\cup C)=(A\times B)\cup(A\times C)$ is verified.

(ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

$$B\cap C = \{0,1\}\cap\{1,2\} = \{1\}$$
$$A\times(B\cap C) = \{2,3\}\times\{1\} = \{(2,1),(3,1)\} \quad \text{... (3)}$$
$$A\times B = \{2,3\}\times\{0,1\} = \{(2,0),(2,1),(3,0),(3,1)\}$$
$$A\times C = \{2,3\}\times\{1,2\} = \{(2,1),(2,2),(3,1),(3,2)\}$$
$$(A\times B)\cap(A\times C) = \{(2,0),(2,1),(3,0),(3,1)\}\cap\{(2,1),(2,2),(3,1),(3,2)\}$$
$$= \{(2,1),(3,1)\} \quad \text{... (4)}$$

From (3) and (4), $A\times(B\cap C)=(A\times B)\cap(A\times C)$ is verified.

Note

- The above two verified properties are called distributive property of cartesian product over union and intersection respectively. In fact, for any three sets A, B, C we have (i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$ (ii) $A\times(B\cap C)=(A\times B)\cap(A\times C)$.

1.3.1 Cartesian Product of three Sets

If A, B, C are three non-empty sets then the cartesian product of three sets is the set of all possible ordered triplets given by

$$A\times B\times C = \{(a,b,c) \text{ for all } a\in A, b\in B, c\in C\}$$

Illustration for Geometrical understanding of cartesian product of two and three sets

Let $A=\{0,1\}$, $B=\{0,1\}$, $C=\{0,1\}$

$$A\times B = \{0,1\}\times\{0,1\} = \{(0,0),(0,1),(1,0),(1,1)\}$$

Representing $A\times B$ in the xy - plane we get a picture shown in Fig. 1.5.

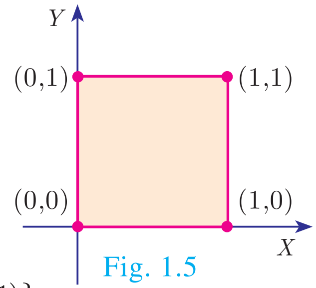

Fig. 1.5

$$(A\times B)\times C = \{(0,0),(0,1),(1,0),(1,1)\}\times\{0,1\}$$
$$= \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1)(1,1,0),(1,1,1)\}$$

Representing $A\times B\times C$ in the xyz - plane we get a picture as shown in Fig. 1.6.

Thus, $A\times B$ represent vertices of a square in two dimensions and $A\times B\times C$ represent vertices of a cube in three dimensions.

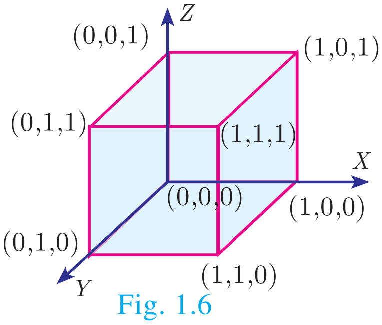

Fig. 1.6

Note

- In general, cartesian product of two non-empty sets provides a shape in two dimensions and cartesian product of three non-empty sets provide an object in three dimensions.

Exercise 1.1

1. Find $A\times B$, $A\times A$ and $B\times A$

   (i) $A=\{2,-2,3\}$ and $B=\{1,-4\}$ (ii) $A=B=\{p,q\}$ (iii) $A=\{m,n\}$; $B=\phi$

2. Let $A=\{1,2,3\}$ and $B=\{x \mid x$ is a prime number less than $10\}$. Find $A\times B$ and $B\times A$.

3. If $B\times A=\{(-2,3),(-2,4),(0,3),(0,4),(3,3),(3,4)\}$ find A and B.

4. If $A=\{5,6\}$, $B=\{4,5,6\}$, $C=\{5,6,7\}$, Show that $A\times A=(B\times B)\cap(C\times C)$.

5. Given A=$\{1,2,3\}$, $B=\{2,3,5\}$, $C=\{3,4\}$ and $D=\{1,3,5\}$, check if $(A\cap C)\times(B\cap D)=(A\times B)\cap(C\times D)$ is true?

6. Let $A=\{x\in\mathbb{W} \mid x<2\}$, $B=\{x\in\mathbb{N} \mid 1<x\leq 4\}$ and $C=\{3,5\}$. Verify that

   (i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$ (ii) $A\times(B\cap C)=(A\times B)\cap(A\times C)$

   (iii) $(A\cup B)\times C=(A\times C)\cup(B\times C)$

7. Let A = The set of all natural numbers less than 8, B = The set of all prime numbers less than 8, C = The set of even prime number. Verify that

   (i) $(A\cap B)\times C=(A\times C)\cap(B\times C)$ (ii) $A\times(B-C)=(A\times B)-(A\times C)$

1.4 Relations

Many day-to-day occurrences involve two objects that are connected with each other by some rule of correspondence. We say that the two objects are related under the specified rule. How shall we represent it? Here are some examples,

Relationship	Expressing using the symbol R	Representation as ordered pair
New Delhi is the capital of India	New Delhi R India	(New Delhi, India)
Line AB is perpendicular to line XY	line AB R line XY	(line AB, line XY)
–1 is greater than –5	–1 R –5	(–1, –5)
ℓ is a line of symmetry for ΔPQR	ℓ R ΔPQR	(ℓ, ΔPQR)

How are New Delhi and India related? We may expect the response, "New Delhi is the capital of India". But there are several ways in which 'New Delhi' and 'India' are related. Here are some possible answers.

- New Delhi is the capital of India.
- New Delhi is in the northern part of India.
- New Delhi is one of the largest cities of India etc.

So, when we wish to specify a particular relation, providing only one ordered pair

(New Delhi, India) it may not be practically helpful. If we ask the relation in the following set of ordered pairs,

{(New Delhi, India), (Washington, USA), (Beijing, China), (London, U.K.), (Kathmandu, Nepal)} then specifying the relation is easy.

Progress Check

Let A = {1, 2, 3, 4} and B = {a, b, c}.

1. Which of the following are relations from A to B?	2. Which of the following are relations from B to A?
(i) { (1, b), (1, c), (3, a), (4, b) }	(i) { (c, a), (c, b), (c, 1) }
(ii) { (1, a), (b, 4), (c, 3) }	(ii) { (c, 1), (c, 2), (c, 3), (c, 4) }
(iii) { (1, a), (a, 1), (2, b), (b, 2) }	(iii) { (a, 4), (b, 3), (c, 2) }

Illustration 4

Students in a class	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
Heights (in feet)	4.5	5.2	5	4.5	5	5.1	5.2	5	4.7	4.9

Let us define a relation between heights of corresponding students. (Fig.1.7)

R = {(heights, students)}

$$R=\{(4.5,S_1),\ (4.5,S_4),\ (4.7,S_9),\ (4.9,S_{10}),\ (5,S_3),\ (5,S_5),(5,S_8),\ (5.1,S_6),\ (5.2,S_2),\ (5.2,S_7)\}$$

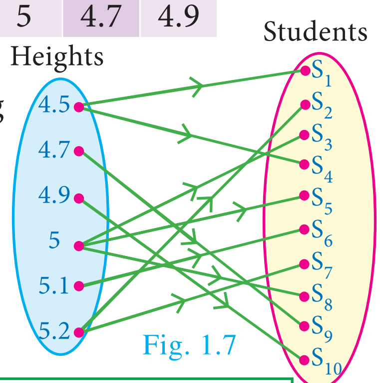

Fig. 1.7

Definition

Let A and B be any two non-empty sets. A 'relation' R from A to B is a subset of $A \times B$ satisfying some specified conditions. If $x \in A$ is related to $y \in B$ through R , then we write it as $x\,\mathrm{R}\,y$. $x\,\mathrm{R}\,y$ if and only if $(x,y) \in \mathrm{R}$.

The domain of the relation $\mathrm{R} = \{x \in A \mid x\,\mathrm{R}\,y$, for some $y \in B\}$

The co-domain of the relation R is B

The range of the relation $\mathrm{R} = \{y \in B \mid x\mathrm{R}y$, for some $x \in A\}$

From these definitions, we note that domain of $\mathrm{R} \subseteq A$, co-domain of $\mathrm{R} = B$ and range of $\mathrm{R} \subseteq B$.

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Illustration 5

Let A = {1,2,3,4,5} and B = {Mathi, Arul, John}

A relation R between the above sets A and B can be represented by an arrow diagram (Fig. 1.8).

Then, domain of R = {1,2,3,4}

range of R= {Mathi, Arul, John} = co-domain of R .

Note that domain of R is a proper subset of A.

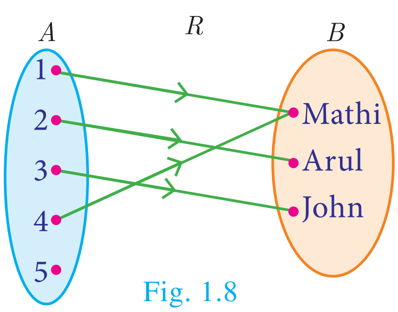


Fig. 1.8

Activity 2

Let A and B be the set of lines in xy-plane such that A consists of lines parallel to X-axis. For $x \in A$, $y \in B$, let R be a relation from A to B defined by $x\text{R}y$ if x is perpendicular to y. Find the elements of B using a graph sheet.

Illustration 6

Let A = {1,3,5,7} and B = {4,8}. If R is a relation defined by "is less than" from A to B, then 1R4 (since 1 is less than 4). Similarly, it is observed that 1R8, 3R4, 3R8, 5R8, 7R8

Equivalently R= {(1,4), (1,8), (3,4), (3,8), (5,8), (7,8)}

Note

- In the above illustration $A \times B$ = {(1,4), (1,8), (3,4), (3,8), (5,4), (5,8), (7,4),(7,8)} R = {(1,4), (1,8), (3,4), (3,8), (5,8), (7,8)} We see that R is a subset of $A \times B$.

Illustration 7

In a particular area of a town, let us consider ten families A, B, C, D, E, F, G, H, I and J with two children. Among these, families B, F, I have two girls; D, G, J have one boy and one girl; the remaining have two boys. Let us define a relation R by $x\text{R}y$, where x denote the number of boys and y denote the family with x number of boys. Represent this situation as a relation through ordered pairs and arrow diagram.

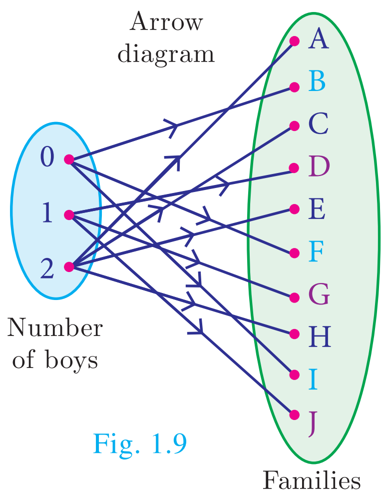


Fig. 1.9

Since the domain of the relation R is concerned about the number of boys, and we are considering families with two children, the domain of R will consist of three elements given by {0,1,2}, where 0, 1, 2 represent the number of boys say no, one, two boys respectively. We note that families with two girls are the ones with no boys. Hence the relation R is given by

$$\text{R} = \{(0,B),(0,F),(0,I),(1,D),(1,G),(1,J),(2,A),(2,C),(2,E),(2,H)\}$$

This relation is shown in an arrow diagram (Fig.1.9).

Example 1.4 Let $A = \{3,4,7,8\}$ and $B = \{1,7,10\}$. Which of the following sets are relations from A to B?

(i) R_1 ={(3,7), (4,7), (7,10), (8,1)} (ii) R_2= {(3,1), (4,12)}

(iii) R_3= {(3,7), (4,10), (7,7), (7,8), (8,11), (8,7), (8,10)}

Solution $A \times B$ = {(3,1), (3,7), (3,10), (4,1), (4,7), (4,10), (7,1), (7,7), (7,10), (8,1), (8,7), (8,10)}

(i) We note that, $\text{R}_1 \subseteq A \times B$. Thus, R_1 is a relation from A to B.

(ii) Here, $(4,12) \in \text{R}_2$, but $(4,12) \notin A \times B$. So, R_2 is not a relation from A to B.

(iii) Here, $(7,8) \in \text{R}_3$, but $(7,8) \notin A \times B$. So, R_3 is not a relation from A to B.

Note

- A relation may be represented algebraically either by the roster method or by the set builder method.
- An arrow diagram is a visual representation of a relation.

Example 1.5 The arrow diagram shows (Fig.1.10) a relationship between the sets P and Q. Write the relation in (i) Set builder form (ii) Roster form (iii) What is the domain and range of R.

Solution

(i) Set builder form of R $= \{(x,y) \mid y = x-2,\ x \in P, y \in Q\}$

(ii) Roster form R $= \{(5,3),(6,4),(7,5)\}$

(iii) Domain of R = {5,6,7} and range of R $= \{3,4,5\}$

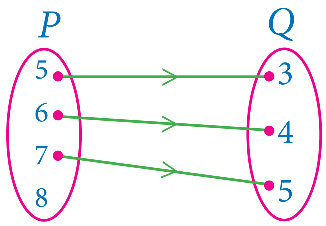


Fig. 1.10

'Null relation'

Let us consider the following examples. Suppose $A = \{-3,-2,-1\}$ and $B = \{1,2,3,4\}$. A relation from A to B is defined as $a - b = 8$ i.e., there is no pair(a,b) such that $a - b = 8$. Thus R contain no element and so $R = \phi$.

A relation which contains no element is called a "Null relation".

Do you know?

If $n(A) = p$, $n(B) = q$, then the total number of relations that exist between A and B is 2^{pq} .

Exercise 1.2

1. Let $A = \{1,2,3,7\}$ and $B = \{3,0,-1,7\}$, which of the following are relation from A to B?

   (i) $R_1 = \{(2,1), (7,1)\}$ (ii) $R_2 = \{(-1,1)\}$

   (iii) $R_3 = \{(2,-1), (7,7), (1,3)\}$ (iv) $R_4 = \{(7,-1), (0,3), (3,3), (0,7)\}$

2. Let $A=\{1,2,3,4,...,45\}$ and R be the relation defined as "is square of" on A. Write R as a subset of $A \times A$. Also, find the domain and range of R.

3. A Relation R is given by the set $\{(x,y) / y = x+3,\ x \in \{0,1,2,3,4,5\}\}$. Determine its domain and range.

4. Represent each of the given relations by (a) an arrow diagram, (b) a graph and (c) a set in roster form, wherever possible.

   (i) $\{(x,y) | x = 2y,\ x \in \{2,3,4,5\},\ y \in \{1,2,3,4\}$

   (ii) $\{(x,y) | y = x+3,\ x, y$ are natural numbers < 10}

5. A company has four categories of employees given by Assistants (A), Clerks (C), Managers (M) and an Executive Officer (E). The company provide ₹10,000, ₹25,000, ₹50,000 and ₹1,00,000 as salaries to the people who work in the categories

A, C, M and E respectively. If A_1, A_2, A_3, A_4 and A_5 were Assistants; C_1, C_2, C_3, C_4 were Clerks; M_1, M_2, M_3 were managers and E_1, E_2 were Executive officers and if the relation R is defined by xRy, where x is the salary given to person y, express the relation R through an ordered pair and an arrow diagram.

1.5 Functions

Among several relations that exist between two non-empty sets, some special relations are important for further exploration. Such relations are called "Functions".

Illustration 8

A company has 5 employees in different categories. If we consider their salary distribution for a month as shown by arrow diagram in Fig.1.11, we see that there is only one salary associated for every employee of the company.

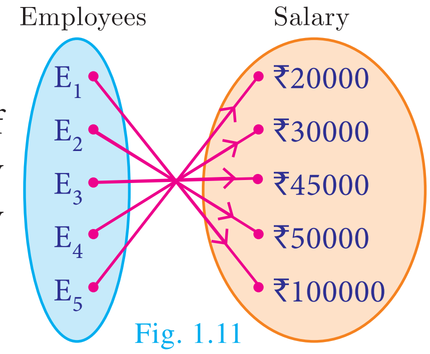


Fig. 1.11

Here are various real life situations illustrating some special relations:

1. Consider the set A of all of your classmates; corresponding to each student, there is only one age.
2. You go to a shop to buy a book. If you take out a book, there is only one price corresponding to it; it does not have two prices corresponding to it. (of course, many books may have the same price).
3. You are aware of Boyle's law. Corresponding to a given value of pressure P, there is only one value of volume V.
4. In Economics, the quantity demanded can be expressed as $Q = 360 - 4P$, where P is the price of the commodity. We see that for each value of P, there is only one value of Q. Thus the quantity demanded Q depend on the price P of the commodity.

We often come across certain relations, in which, for a given element of a set A, there is only one corresponding element of a set B. Such relations are called functions. We usually use the symbol f to denote a functional relation.

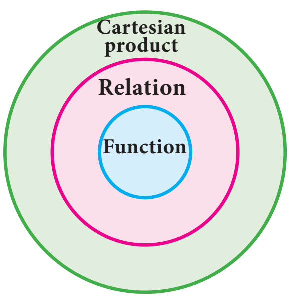


Fig. 1.12(a)

Definition

A relation f between two non-empty sets X and Y is called a function from X to Y if, for each $x \in X$ there exists only one $y \in Y$ such that $(x, y) \in f$.

That is, $f = \{(x,y) \mid \text{for all } x \in X, y \in Y\}$.

A function f from X to Y is written as $f : X \to Y$.

Comparing the definitions of relation and function, we see that every function is a relation. Thus, functions are subsets of relations and relations are subsets of cartesian product. (Fig.1.12(a))

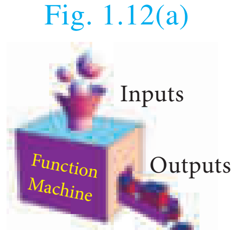


Fig. 1.12(b)

A function f can be thought as a mechanism (or device) (Fig.1.12(b)), which gives a unique output $f(x)$ to every input x.

DO YOU KNOW?

A function is also called as a mapping or transformation.

Note

If $f : X \to Y$ is a function then

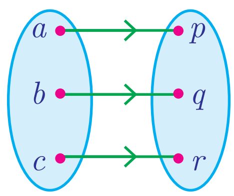
- The set X is called the domain of the function f and the set Y is called its co-domain.
- If $f(a) = b$, then b is called 'image' of a under f and a is called a 'pre-image' of b.
- The set of all images of the elements of X under f is called the 'range' of f.
- $f : X \to Y$ is a function only if
  - (i) every element in the domain of f has an image.
  - (ii) the image is unique.
- If A and B are finite sets such that $n(A) = p$, $n(B) = q$ then the total number of functions that exist between A and B is q^p.
- In this chapter we always consider f to be a real valued function.
- Describing domain of a function
  - (i) Let $f(x) = \dfrac{1}{x+1}$. If $x = -1$ then $f(-1)$ is not defined. Hence f is defined for all real numbers except at $x = -1$. So domain of f is $\mathbb{R} - \{-1\}$.
  - (ii) Let $f(x) = \dfrac{1}{x^2 - 5x + 6}$; If $x = 2, 3$ then $f(2)$ and $f(3)$ are not defined. Hence f is defined for all real numbers except at $x = 2$ and 3. So domain of $f = \mathbb{R} - \{2, 3\}$.

Progress Check

1. Relations are subsets of ____. Functions are subsets of ____.
2. True or False: All the elements of a relation should have images.
3. True or False: All the elements of a function should have images.
4. True or False: If $\mathrm{R} : A \to B$ is a relation then the domain of $\mathrm{R} = A$.
5. If $f : \mathbb{N} \to \mathbb{N}$ is defined as $f(x) = x^2$ the pre-image(s) of 1 and 2 are ____ and ____.
6. The difference between relation and function is ____.
7. Let A and B be two non-empty finite sets. Then which one among the following two collection is large?
   (i) The number of relations between A and B.
   (ii) The number of functions between A and B.

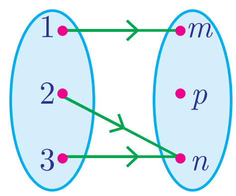
Illustration 9 - Testing for functions

Representation by Arrow diagram



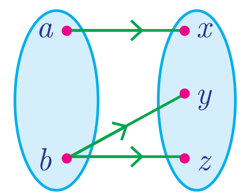
This represents a function. Each input corresponds to a single output.

Fig. 1.13(a)



This represents a function. Each input corresponds to a single output.

Fig. 1.13(b)



This is not a function. One of the input b is associated with two outputs.

Fig. 1.13(c)

Functions play very important role in the understanding of higher ideas in mathematics. They are basic tools to convert from one form to another form. In this sense, functions are widely applied in Engineering Sciences.

Note

- The range of a function is a subset of its co-domain.

Example 1.6 Let $X = \{1,2,3,4\}$ and $Y = \{2,4,6,8,10\}$ and R $= \{(1,2),(2,4),(3,6),(4,8)\}$. Show that R is a function and find its domain, co-domain and range?

Solution Pictorial representation of R is given in Fig.1.14. From the diagram, we see that for each $x \in X$, there exists only one $y \in Y$. Thus all elements in X have only one image in Y. Therefore R is a function. Domain $X = \{1,2,3,4\}$; Co-domain $Y = \{2,3,6,8,10\}$; Range of $f = \{2,4,6,8\}$.

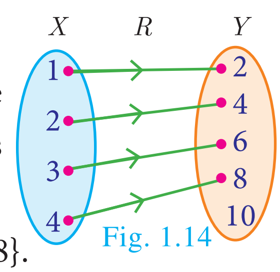


Fig. 1.14

Example 1.7 A relation 'f' is defined by $f(x) = x^2 - 2$ where, $x \in \{-2,-1,0,3\}$

(i) List the elements of f (ii) Is f a function?

Solution $f(x) = x^2 - 2$ where $x \in \{-2,-1,0,3\}$

(i) $$f(-2) = (-2)^2 - 2 = 2\,; \quad f(-1) = (-1)^2 - 2 = -1$$
$$f(0) = (0)^2 - 2 = -2\,; \quad f(3) = (3)^2 - 2 = 7$$

Therefore, $$f = \{(-2,2),(-1,-1),(0,-2),(3,7)\}$$

(ii) We note that each element in the domain of f has a unique image. Therefore f is a function.

Thinking Corner

Is the relation representing the association between planets and their respective moons a function?

Example 1.8 If $X = \{-5,1,3,4\}$ and Y = $\{a,b,c\}$, then which of the following relations are functions from X to Y?