Geometry

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Objectives

- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
  - Line segments
  - Polygons

Basic Elements

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

Coordinate-Free Geometry

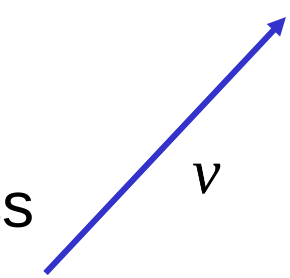
- When we learned simple geometry, most of us started with a Cartesian approach
  - Points were at locations in space **p**=(x,y,z)
  - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
  - Physically, points exist regardless of the location of an arbitrary coordinate system
  - Most geometric results are independent of the coordinate system
  - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical

Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

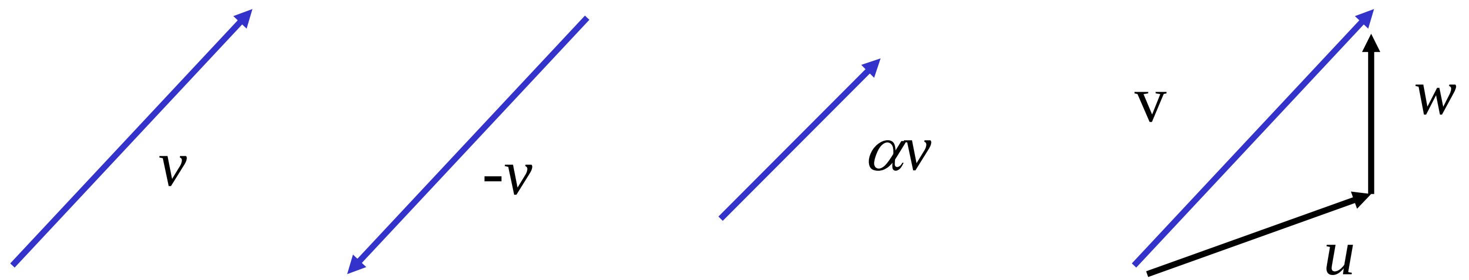
Vectors

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u=\alpha v$
  - Vector-vector addition: $w=u+v$
- Expressions such as

  $v=u+2w-3r$

Make sense in a vector space

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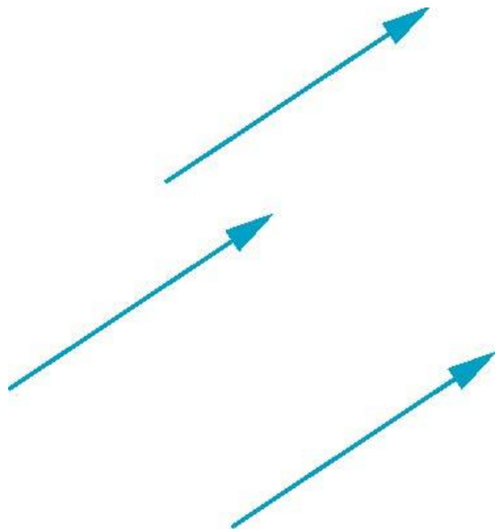
Vectors Lack Position

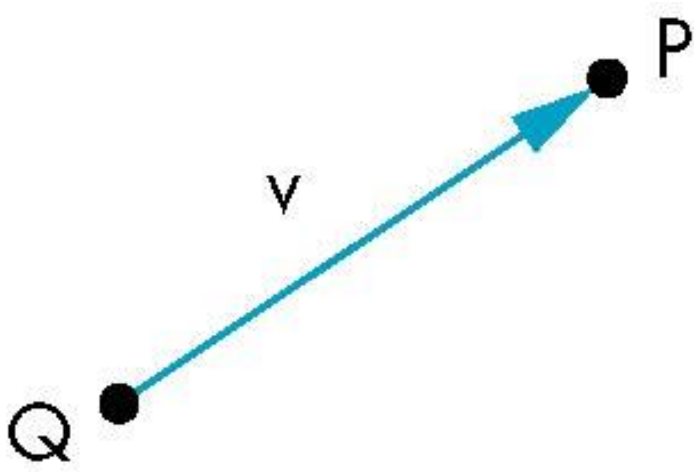
- These vectors are identical
  - Same direction and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



v=P-Q

P=v+Q

Affine Spaces

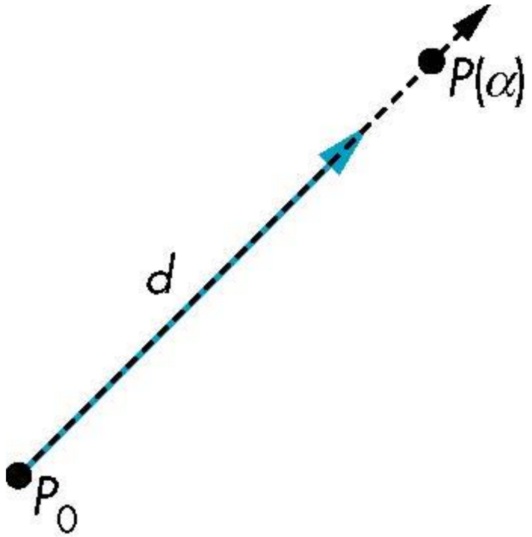
- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
  - Scalar-point operations
- For any point define

$1 \bullet P = P$

$0 \bullet P = \mathbf{0}$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha \mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$



Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$
$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

Rays and Line Segments

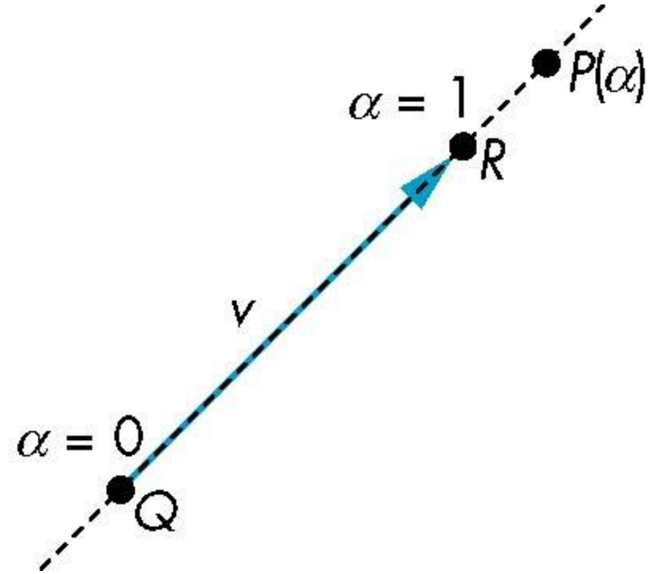
- If $\alpha >= 0$, then $P(\alpha)$ is the *ray* leaving P_0 in the direction $\mathbf{d}$

If we use two points to define v, then

$P(\alpha) = Q + \alpha (R-Q) = Q + \alpha v$

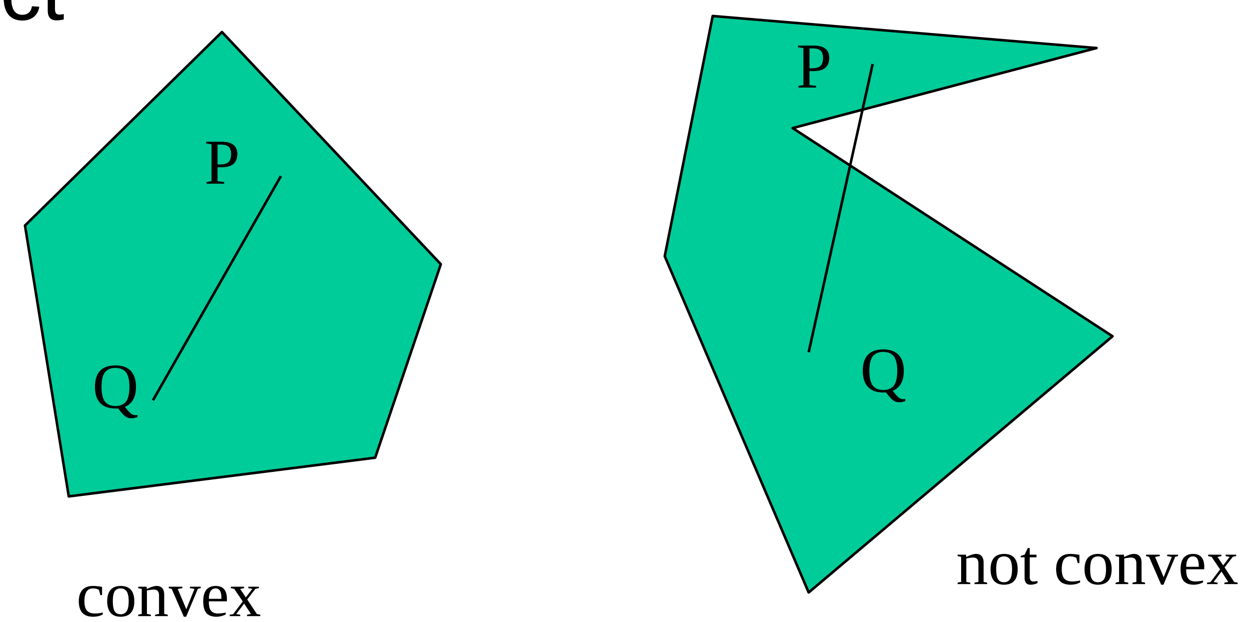
$= \alpha R + (1-\alpha) Q$

For $0 <= \alpha <= 1$ we get all the points on the *line segment* joining R and Q



Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



Affine Sums

- Consider the "sum"

$P=\alpha_1 P_1+\alpha_2 P_2+\ldots\ldots+\alpha_n P_n$

Can show by induction that this sum makes sense iff

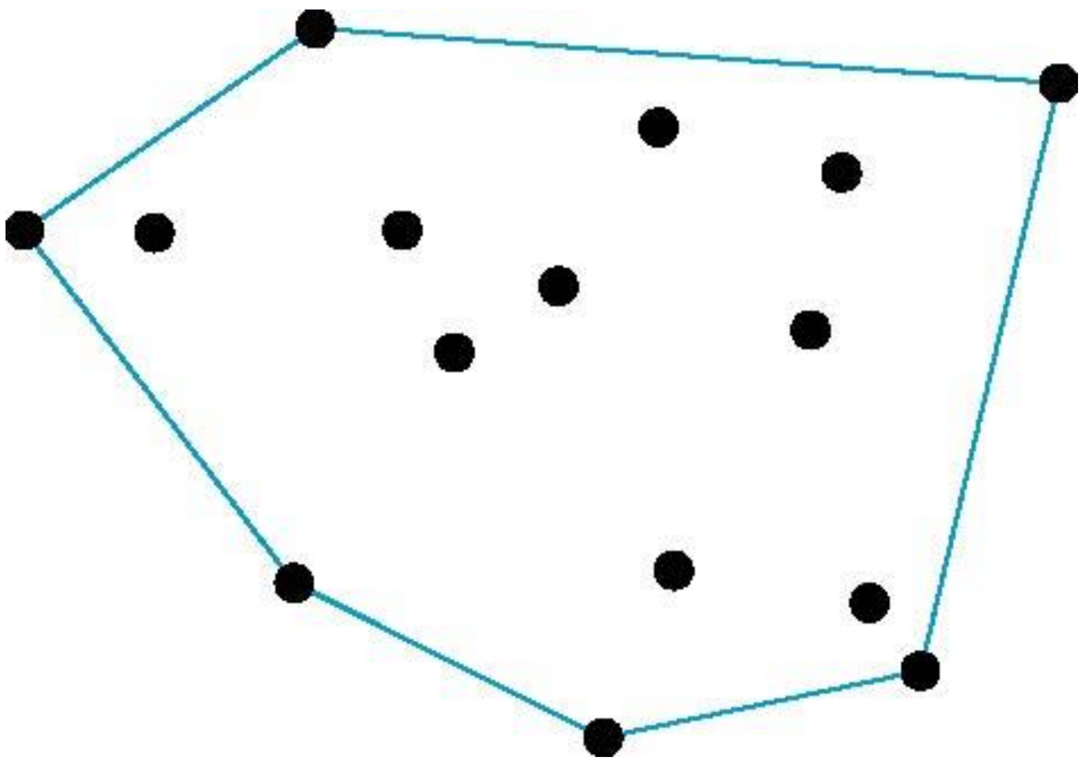
$\alpha_1+\alpha_2+\ldots\ldots\alpha_n=1$

in which case we have the *affine sum* of the points $P_1,P_2,\ldots\ldots P_n$

- If, in addition, $\alpha_i>=0$, we have the *convex hull* of $P_1,P_2,\ldots\ldots P_n$

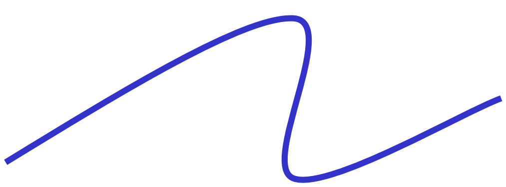
Convex Hull

- Smallest convex object containing $P_1, P_2, \ldots.. P_n$
- Formed by "shrink wrapping" points

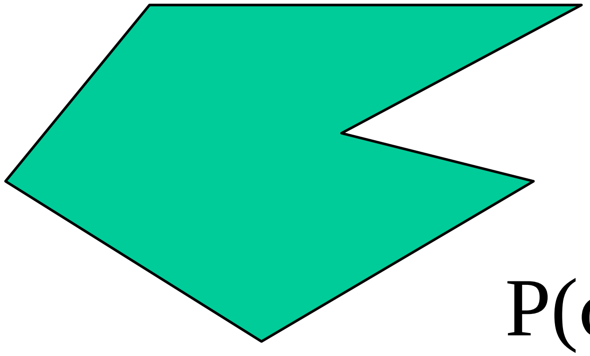


Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
  - Linear functions give planes and polygons



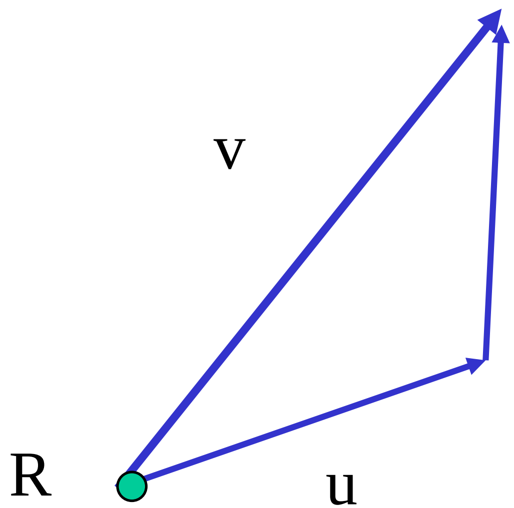
$P(\alpha)$

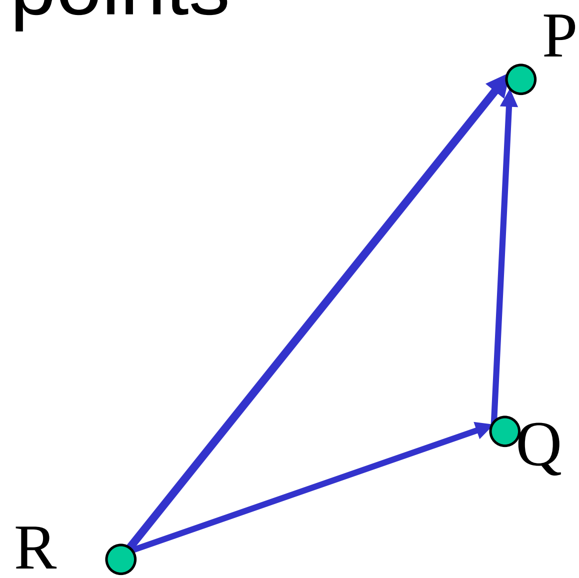


$P(\alpha, \beta)$

Planes

- A plane can be defined by a point and two vectors or by three points





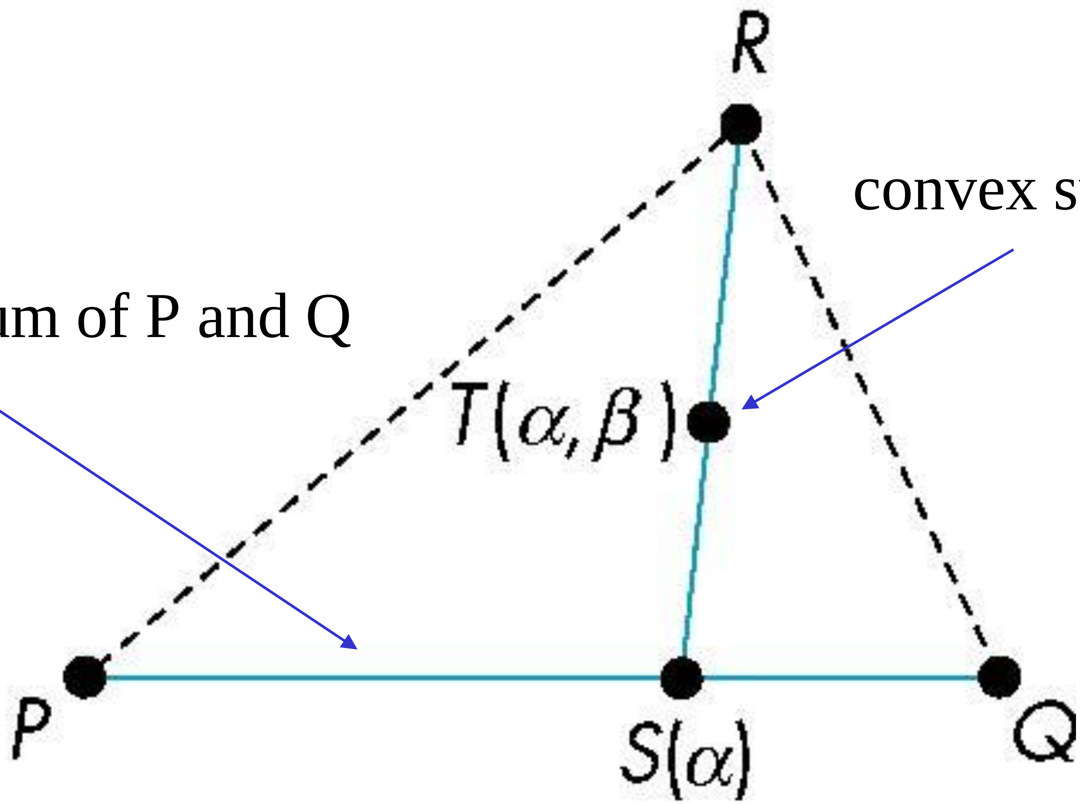
$P(\alpha,\beta)=R+\alpha u+\beta v$

$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$

Triangles

convex sum of $S(\alpha)$ and R

convex sum of P and Q



for $0 <= \alpha, \beta <= 1$, we get all points in triangle

Barycentric Coordinates

Triangle is convex so any point inside can be represented as an affine sum

$P(\alpha_{1,}\, \alpha_{2,}\, \alpha_3)=\alpha_1 P+\alpha_2 Q+\alpha_3 R$

where

$\alpha_1 + \alpha_2 + \alpha_3 = 1$

$\alpha_i >= 0$

The representation is called the **barycentric coordinate** representation of P

Normals

- Every plane has a vector n normal (perpendicular, orthogonal) to it
- From point-two vector form $P(\alpha,\beta)=R+\alpha u+\beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

$$(P(\alpha)-P) \cdot n=0$$