Designing Parametric Cubic Curves

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Objectives

- Introduce the types of curves
  - Interpolating
  - Hermite
  - Bezier
  - B-spline
- Analyze their performance

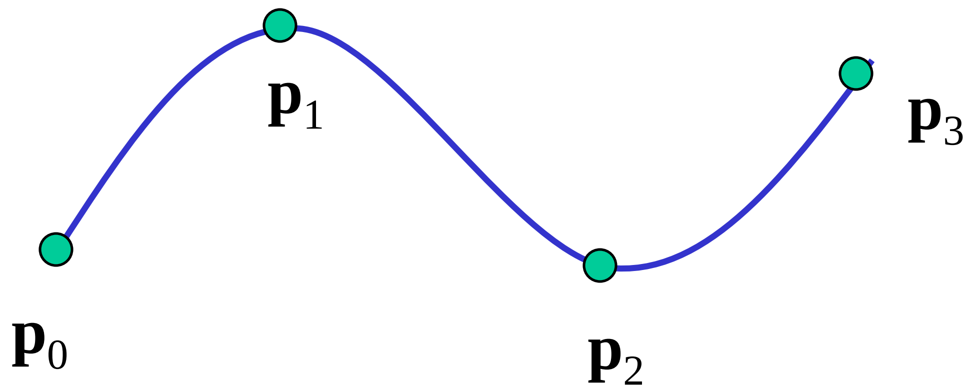
Matrix-Vector Form

$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

define $\quad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$

then $\quad \mathrm{p}(u) = \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$

Interpolating Curve



Given four data (control) points $\mathbf{p}_0$, $\mathbf{p}_1$,$\mathbf{p}_2$, $\mathbf{p}_3$ determine cubic $\mathbf{p}(u)$ which passes through them

Must find $\mathbf{c}_0$,$\mathbf{c}_1$,$\mathbf{c}_2$, $\mathbf{c}_3$

Interpolation Equations

apply the interpolating conditions at $u=0, 1/3, 2/3, 1$

$$p_0=p(0)=c_0$$
$$p_1=p(1/3)=c_0+(1/3)c_1+(1/3)^2c_2+(1/3)^3c_3$$
$$p_2=p(2/3)=c_0+(2/3)c_1+(2/3)^2c_2+(2/3)^3c_3$$
$$p_3=p(1)=c_0+c_1+c_2+c_3$$

or in matrix form with $\mathbf{p} = [p_0\ p_1\ p_2\ p_3]^T$

$$\mathbf{p}=\mathbf{A}\mathbf{c} \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Interpolation Matrix

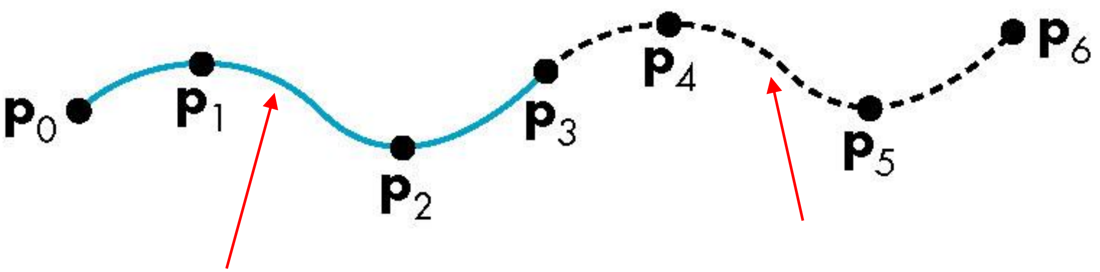
Solving for $\mathbf{c}$ we find the *interpolation matrix*

$$\mathbf{M}_I = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix}$$

$$\mathbf{c} = \mathbf{M}_I \mathbf{p}$$

Note that $\mathbf{M}_I$ does not depend on input data and can be used for each segment in x, y, and z

Interpolating Multiple Segments



use $\mathbf{p} = [p_0\ p_1\ p_2\ p_3]^T$

use $\mathbf{p} = [p_3\ p_4\ p_5\ p_6]^T$

Get continuity at join points but not continuity of derivatives

Blending Functions

Rewriting the equation for p(u)

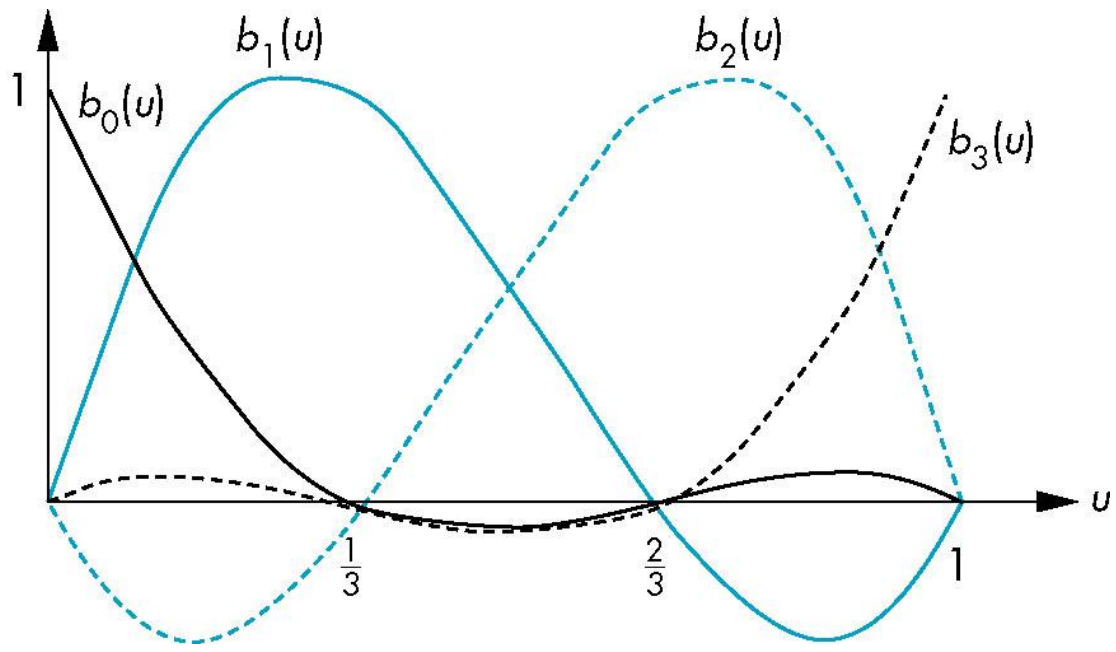
$$p(u)=\mathbf{u}^T\mathbf{c}=\mathbf{u}^T\mathbf{M}_I\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$$

where $b(u) = [b_0(u)\ b_1(u)\ b_2(u)\ b_3(u)]^T$ is an array of *blending polynomials* such that $p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$

$$b_0(u) = -4.5(u-1/3)(u-2/3)(u-1)$$
$$b_1(u) = 13.5u\,(u-2/3)(u-1)$$
$$b_2(u) = -13.5u\,(u-1/3)(u-1)$$
$$b_3(u) = 4.5u\,(u-1/3)(u-2/3)$$

Blending Functions

- These functions are not smooth
  - Hence the interpolation polynomial is not smooth

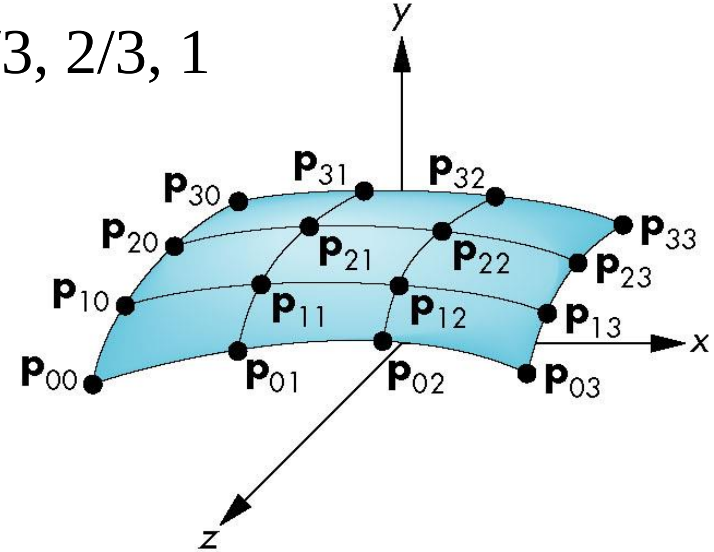


Interpolating Patch

$$p(u,v)=\sum_{i=o}^{3}\sum_{j=0}^{3}c_{ij}u^i v^j$$

Need 16 conditions to determine the 16 coefficients c_{ij}

Choose at u,v = 0, 1/3, 2/3, 1



Matrix Form

Define $\mathbf{v} = [1\ v\ v^2\ v^3]^T$

$\mathbf{C} = [c_{ij}]$ $\quad$ $\mathbf{P} = [p_{ij}]$

$p(u,v) = \mathbf{u}^T C \mathbf{v}$

If we observe that for constant u (v), we obtain interpolating curve in v (u), we can show

$$\mathbf{C} = M_I P M_I$$

$$p(u,v) = u^T M_I P M_I^T v$$

Blending Patches

$$p(u,v) = \sum_{i=o}^{3} \sum_{j=0}^{3} b_i(u)\, b_j(v)\, p_{ij}$$

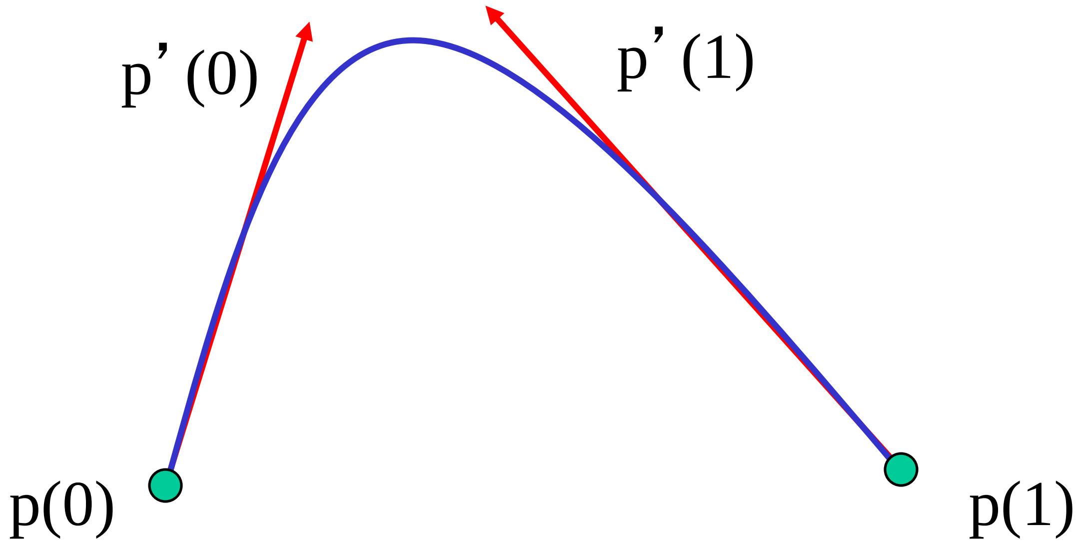
Each $b_i(u)b_j(v)$ is a blending patch

Shows that we can build and analyze surfaces from our knowledge of curves

Other Types of Curves and Surfaces

- How can we get around the limitations of the interpolating form
  - Lack of smoothness
  - Discontinuous derivatives at join points
- We have four conditions (for cubics) that we can apply to each segment
  - Use them other than for interpolation
  - Need only come close to the data

Hermite Form



Use two interpolating conditions and two derivative conditions per segment

Ensures continuity and first derivative continuity between segments

Equations

Interpolating conditions are the same at ends

$$p(0) = p_0 = c_0$$
$$p(1) = p_3 = c_0+c_1+c_2+c_3$$

Differentiating we find $p'(u) = c_1+2uc_2+3u^2c_3$

Evaluating at end points

$$p'(0) = p'_0 = c_1$$
$$p'(1) = p'_3 = c_1+2c_2+3c_3$$

Matrix Form

$$\mathbf{q} = \begin{bmatrix} p_0 \\ p_3 \\ p'_0 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c}$$

Solving, we find $\mathbf{c}=\mathbf{M}_H\mathbf{q}$ where $\mathbf{M}_H$ is the Hermite matrix

$$\mathbf{M}_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

Blending Polynomials

$$p(u) = \mathbf{b}(u)^T\mathbf{q}$$

$$\mathbf{b}(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

Although these functions are smooth, the Hermite form is not used directly in Computer Graphics and CAD because we usually have control points but not derivatives

However, the Hermite form is the basis of the Bezier form

Parametric and Geometric Continuity

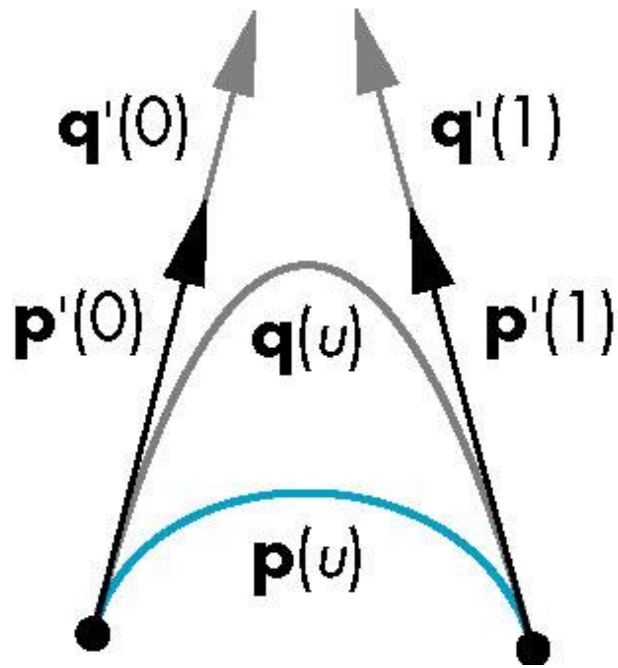
- We can require the derivatives of x, y,and z to each be continuous at join points (*parametric continuity*)
- Alternately, we can only require that the tangents of the resulting curve be continuous (*geometry continuity*)
- The latter gives more flexibility as we have need satisfy only two conditions rather than three at each join point

Example

- Here the p and q have the same tangents at the ends of the segment but different derivatives
- Generate different

  Hermite curves
- This techniques is used

in drawing applications



Higher Dimensional Approximations

- The techniques for both interpolating and Hermite curves can be used with higher dimensional parametric polynomials
- For interpolating form, the resulting matrix becomes increasingly more ill-conditioned and the resulting curves less smooth and more prone to numerical errors
- In both cases, there is more work in rendering the resulting polynomial curves and surfaces