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Bezier and Spline Curves and Surfaces

Ed Angel

Professor Emeritus of Computer Science

University of New Mexico

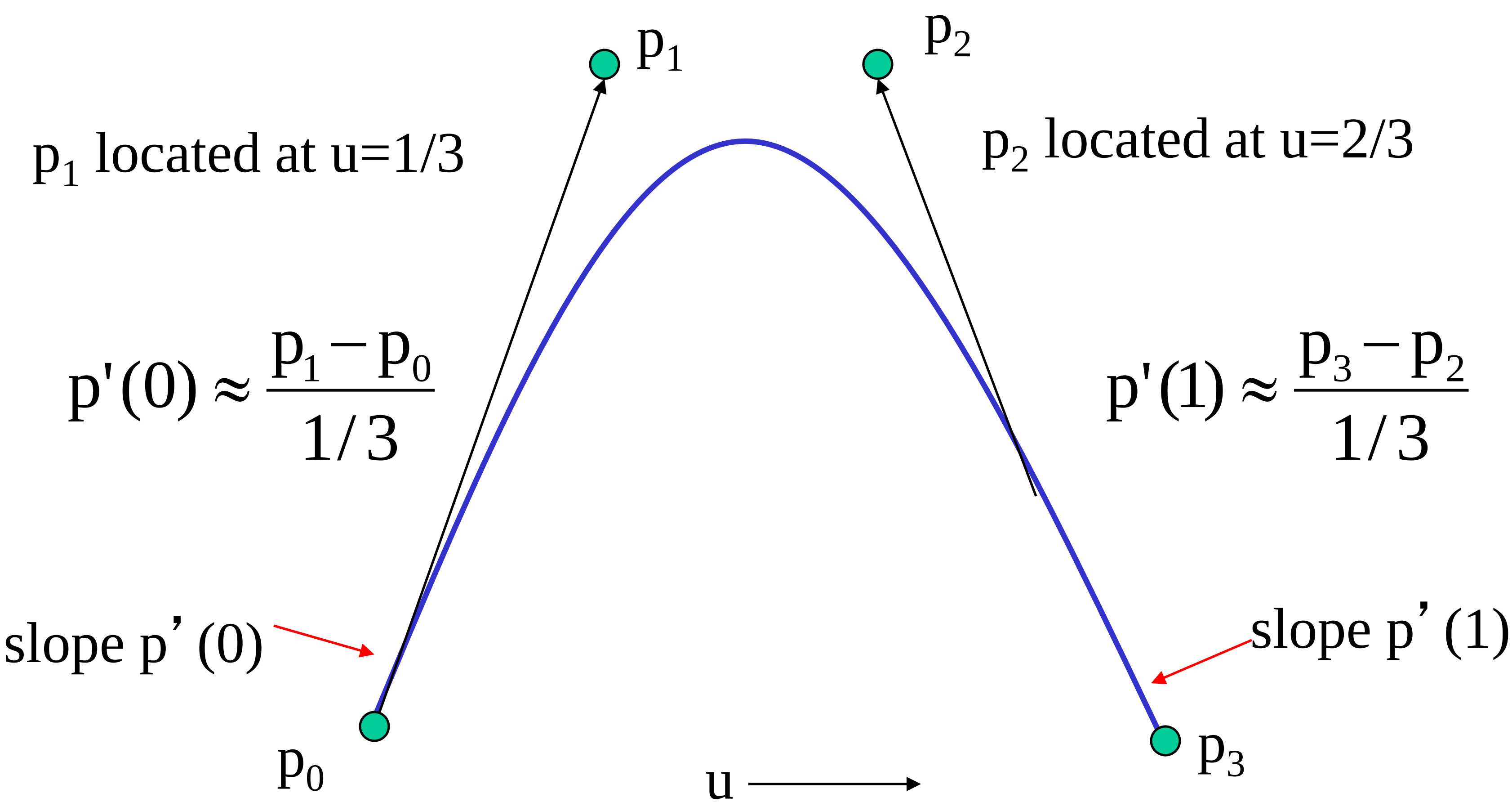
Objectives

- Introduce the Bezier curves and surfaces
- Derive the required matrices
- Introduce the B-spline and compare it to the standard cubic Bezier

Bezier's Idea

- In graphics and CAD, we do not usually have derivative data
- Bezier suggested using the same 4 data points as with the cubic interpolating curve to approximate the derivatives in the Hermite form

Approximating Derivatives



$$p'(0) \approx \frac{p_1 - p_0}{1/3}$$

$$p'(1) \approx \frac{p_3 - p_2}{1/3}$$

Equations

Interpolating conditions are the same

$$p(0) = p_0 = c_0$$
$$p(1) = p_3 = c_0+c_1+c_2+c_3$$

Approximating derivative conditions

$$p'(0) = 3(p_1 - p_0) = c_0$$
$$p'(1) = 3(p_3 - p_2) = c_1+2c_2+3c_3$$

Solve four linear equations for $\mathbf{c}=\mathbf{M}_B\mathbf{p}$

Bezier Matrix

$$\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$\mathrm{p}(u) = \mathbf{u}^T \mathbf{M}_B \mathbf{p} = \mathbf{b}(u)^T \mathbf{p}$$

blending functions

Blending Functions

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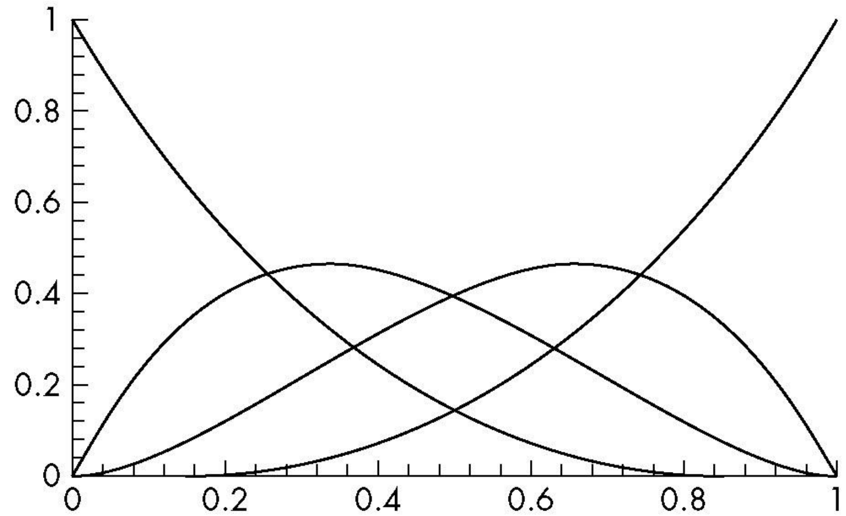
$$\mathbf{b}(u) = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}$$



Note that all zeros are at 0 and 1 which forces the functions to be smooth over (0,1)

Bernstein Polynomials

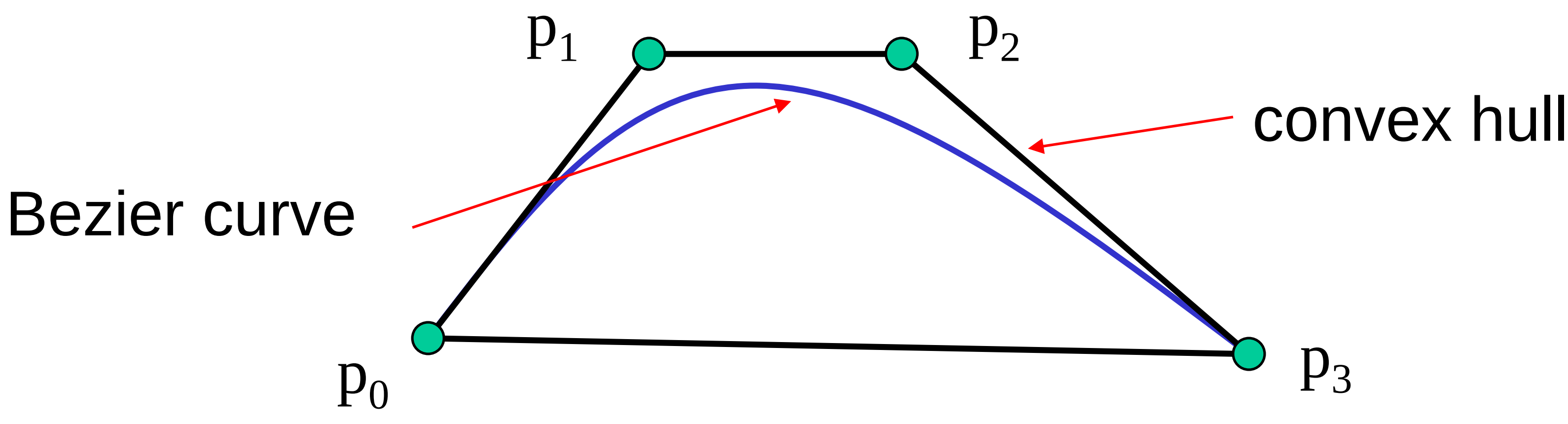
- The blending functions are a special case of the Bernstein polynomials

$$b_{\mathrm{kd}}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

- These polynomials give the blending polynomials for any degree Bezier form
  - All zeros at 0 and 1
  - For any degree they all sum to 1
  - They are all between 0 and 1 inside (0,1)

Convex Hull Property

- The properties of the Bernstein polynomials ensure that all Bezier curves lie in the convex hull of their control points
- Hence, even though we do not interpolate all the data, we cannot be too far away

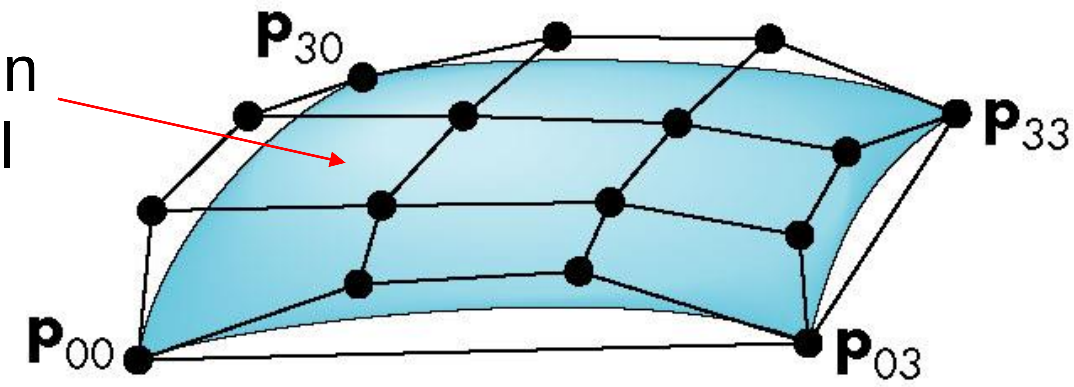


Bezier Patches

Using same data array $\mathbf{P}$=[p_{ij}] as with interpolating form

$$p(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} b_i(u)\, b_j(v)\, p_{ij} = u^T \mathbf{M}_B \mathbf{P} \mathbf{M}_B^T v$$

Patch lies in convex hull



Analysis

- Although the Bezier form is much better than the interpolating form, we have the derivatives are not continuous at join points
- Can we do better?
  - Go to higher order Bezier
    - More work
    - Derivative continuity still only approximate
    - Supported by OpenGL
  - Apply different conditions
    - Tricky without letting order increase

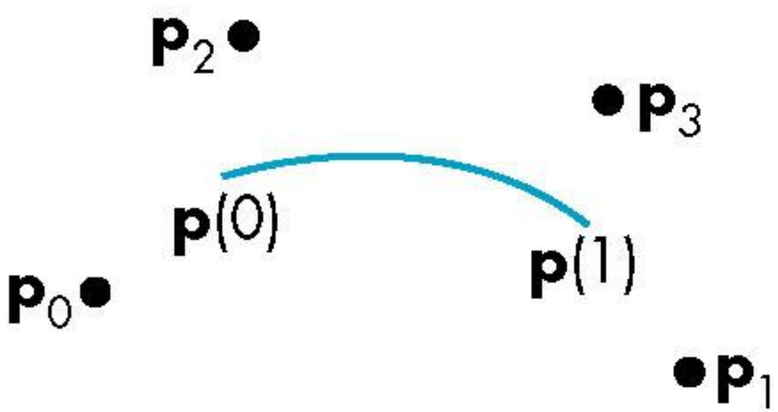
B-Splines

- Basis splines: use the data at $\mathbf{p}=[p_{i-2}\ p_{i-1}\ p_i\ p_{i+1}]^T$ to define curve only between p_{i-1} and p_i
- Allows us to apply more continuity conditions to each segment
- For cubics, we can have continuity of function, first and second derivatives at join points
- Cost is 3 times as much work for curves
  - Add one new point each time rather than three
- For surfaces, we do 9 times as much work

Cubic B-spline

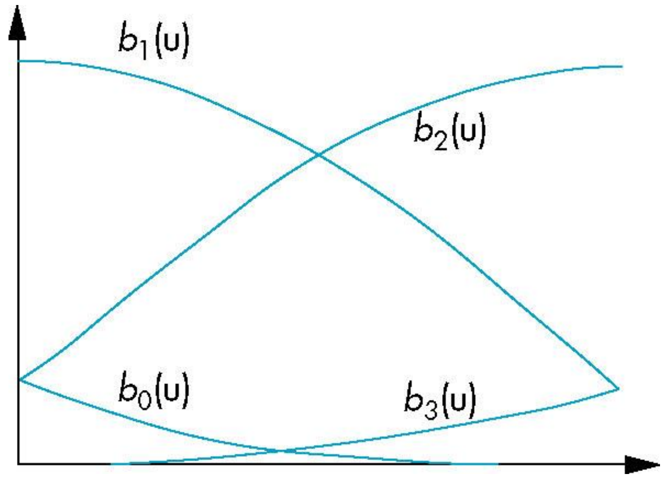
$$p(u) = \mathbf{u}^T\mathbf{M}_S\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$$

$$\mathbf{M}_S = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

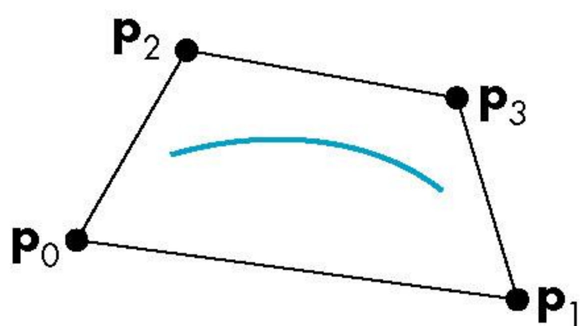


Blending Functions

$$\mathbf{b}(u) = \frac{1}{6}\begin{bmatrix} (1-u)^3 \\ 4-6u^2+3u^3 \\ 1+3u+3u^2-3u^2 \\ u^3 \end{bmatrix}$$



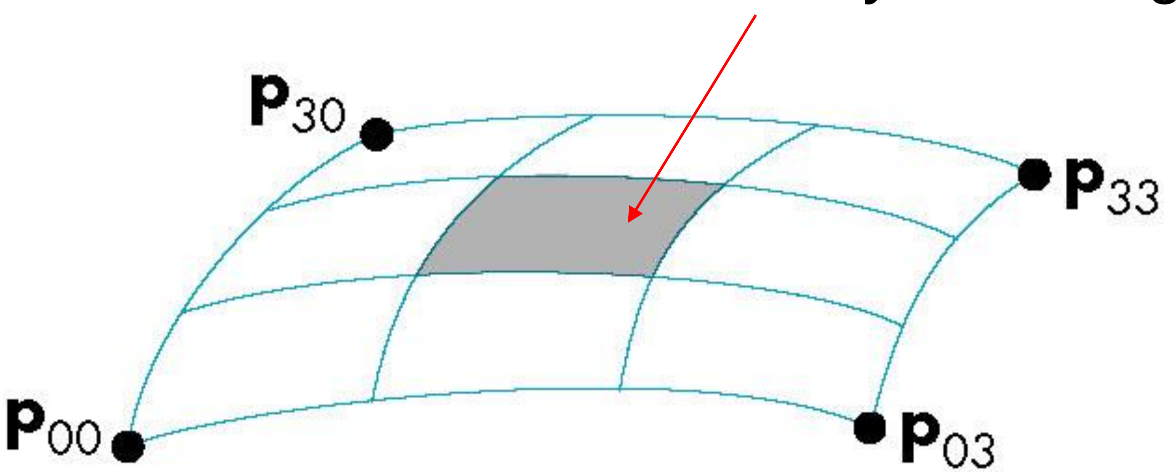
convex hull property



B-Spline Patches

$$p(u,v)=\sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)\, b_j(v)\, p_{ij} = u^T \mathbf{M}_S \mathbf{P} \mathbf{M}_S^T v$$

defined over only 1/9 of region



Splines and Basis

- If we examine the cubic B-spline from the perspective of each control (data) point, each interior point contributes (through the blending functions) to four segments
- We can rewrite p(u) in terms of the data points as

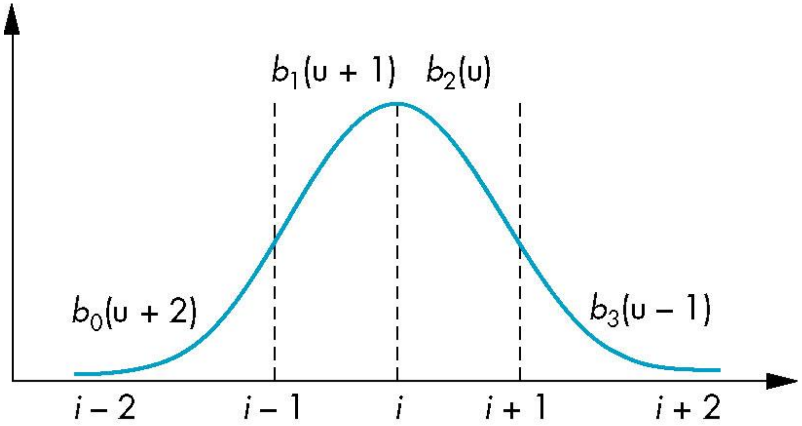
$$p(u) = \sum B_i(u)\, p_i$$

defining the basis functions $\{B_i(u)\}$

Basis Functions

In terms of the blending polynomials

$$B_i(u) = \begin{cases} 0 & u < i-2 \\ b_0(u+2) & i-2 \le u < i-1 \\ b_1(u+1) & i-1 \le u < i \\ b_2(u) & i \le u < i+1 \\ b_3(u-1) & i+1 \le u < i+2 \\ 0 & u \ge i+2 \end{cases}$$



Generalizing Splines

- We can extend to splines of any degree
- Data and conditions to not have to given at equally spaced values (the *knots*)
  - Nonuniform and uniform splines
  - Can have repeated knots
    - Can force spline to interpolate points
- Cox-deBoor recursion gives method of evaluation

NURBS

- Nonuniform Rational B-Spline curves and surfaces add a fourth variable w to x,y,z
  - Can interpret as weight to give more importance to some control data
  - Can also interpret as moving to homogeneous coordinate
- Requires a perspective division
  - NURBS act correctly for perspective viewing
- Quadrics are a special case of NURBS