Probabilistic Inference

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CSCI 4525 / 5525

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Inference with Full Joint Dist.

Previously, we saw how to do inferences using the full joint probability distribution table.

For problems with many variables, these tables get very large. Inference also gets very expensive.

We need:

- A more compact way to represent the distributions
- More efficient inference methods

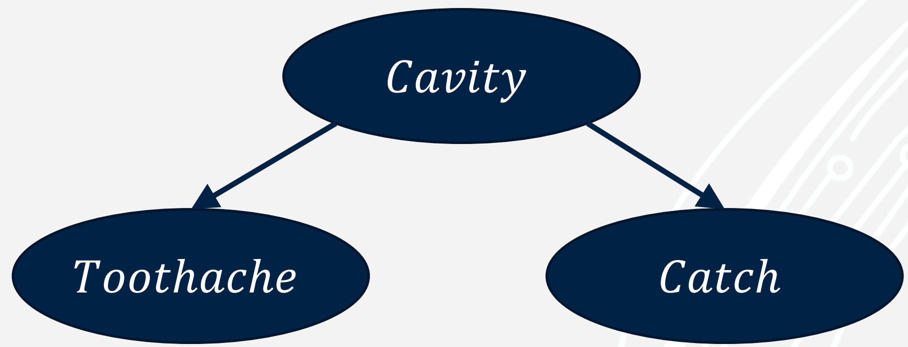
Bayesian Networks

A **Bayesian Network**, or Bayes Net for short, is a directed acyclic graph such that:

- Nodes represent random variables
- Directed edges go from parent to child (and ideally represent causality)
- Each node defines a probability distribution that quantifies the effect of its parents on its value

Bayes Nets

$P(cavity)$
0.20



$Weather$	$P(Weather)$
$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01

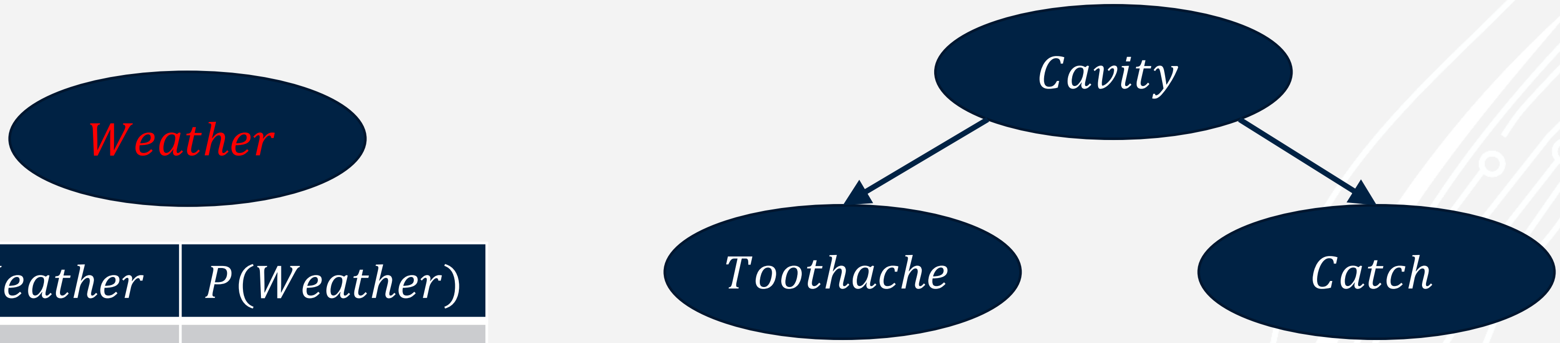
$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

Notice that *Weather* is disconnected from the rest of the network because it is not related to the other variables.

$P(cavity)$
0.20



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$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01

$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

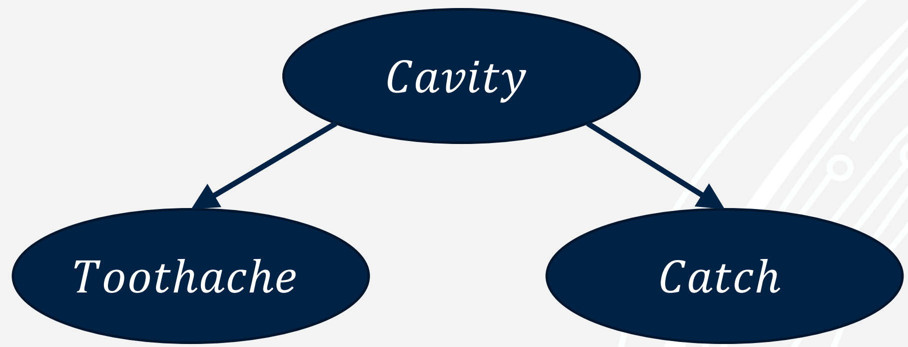
$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

The cells in this table sum to 1, but some of the cells are not shown because their values are implied.

$P(cavity)$
0.20

$Weather$	$P(Weather)$
$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01



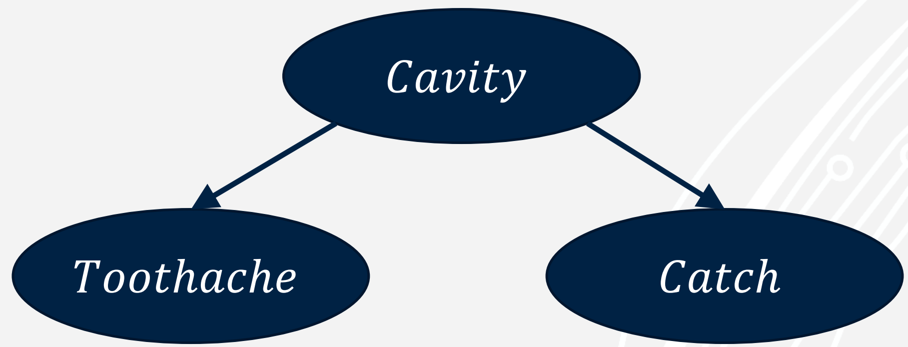
$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

This row represents $P(cavity) = 0.2$

$P(cavity)$
0.20



$Weather$	$P(Weather)$
$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01

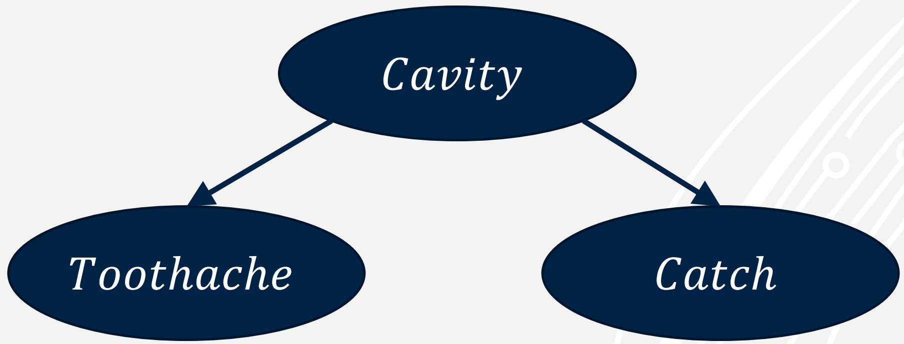
$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

Therefore, we know that $P(\neg cavity) = 0.8$

$P(cavity)$
0.20



$Weather$	$P(Weather)$
$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01

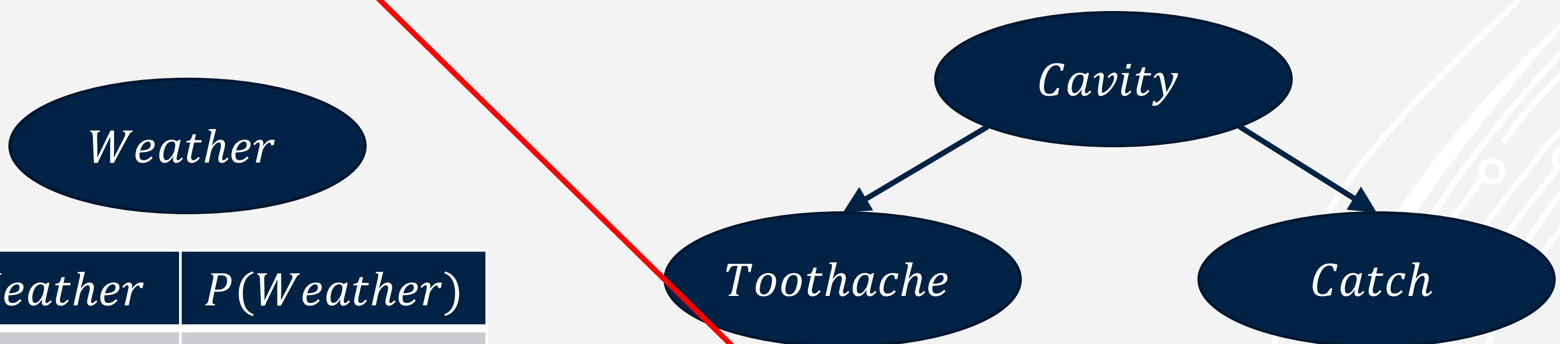
$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

This row represents
$P(toothache|cavity) = 0.6$

$P(cavity)$
0.20



$Weather$	$P(Weather)$
$sunny$	0.60
$rainy$	0.10
$cloudy$	0.29
$snowy$	0.01

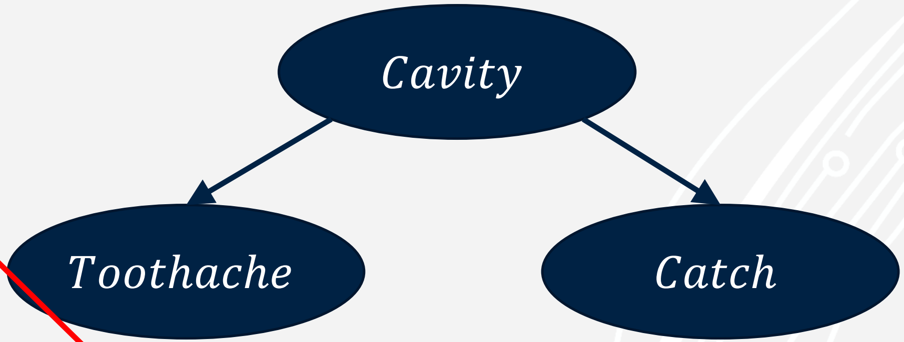
$Cavity$	$P(toothache)$
$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

Therefore, we know
$P(\neg toothache|cavity) = 0.4$

$P(cavity)$
0.20



$Weather$	$P(Weather)$
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$true$	0.60
$false$	0.10

$Cavity$	$P(catch)$
$true$	0.90
$false$	0.20

Bayes Nets

A Bayes Net is a compact* representation of a full joint probability distribution.

That is, any value you can calculate using the full join distribution can be calculated using a Bayes Net.

* The compactness of the network depends on how intelligently the edges have been drawn.

Bayes Nets Representation

We can express the full joint probability distribution of the dentistry domain as follows:

$$\mathbf{P}(Weather, Cavity, Toothache, Catch)$$

Bayes Nets Representation

We can express the full joint probability distribution of the dentistry domain as follows:

$$\mathbf{P}(Weather)\mathbf{P}(Cavity, Toothache, Catch)$$

$Weather$ is absolutely independent.

Bayes Nets Representation

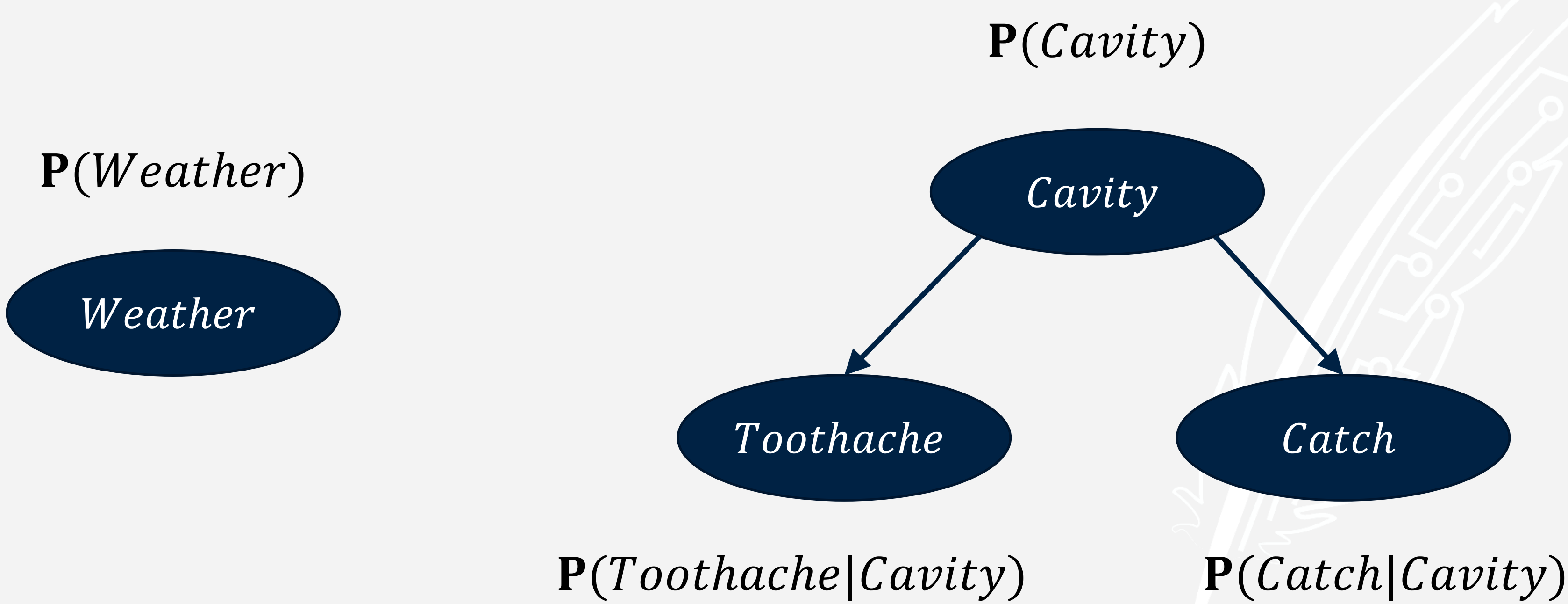
We can express the full joint probability distribution of the dentistry domain as follows:

$$\mathbf{P}(Weather)\mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$

$Toothache$ and $Catch$ are conditionally independent given the value of $Cavity$.

Bayes Nets

$$\mathbf{P}(Weather)\mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$



Inference in Bayes Nets

When we use a Bayes Net to calculate the probability of some event, we have three kinds of variables:

- We want to calculate the probability distribution for some **query variable** X
- We already know the values of some **evidence variables** $E_1 \dots E_n$
- The values of some **hidden variables** $Y_1 \dots Y_n$ are unknown

Alarm Domain

You have just installed a new burglary alarm system in your house, and your two neighbors have agreed to call you when they hear it go off. However:

- Sometimes earthquakes trigger the alarm.
- Sometimes your neighbor John confuses the phone ringing with the alarm and makes a false report
- Your neighbor Mary listens to loud music and sometimes does not hear the alarm.

Alarm Problem

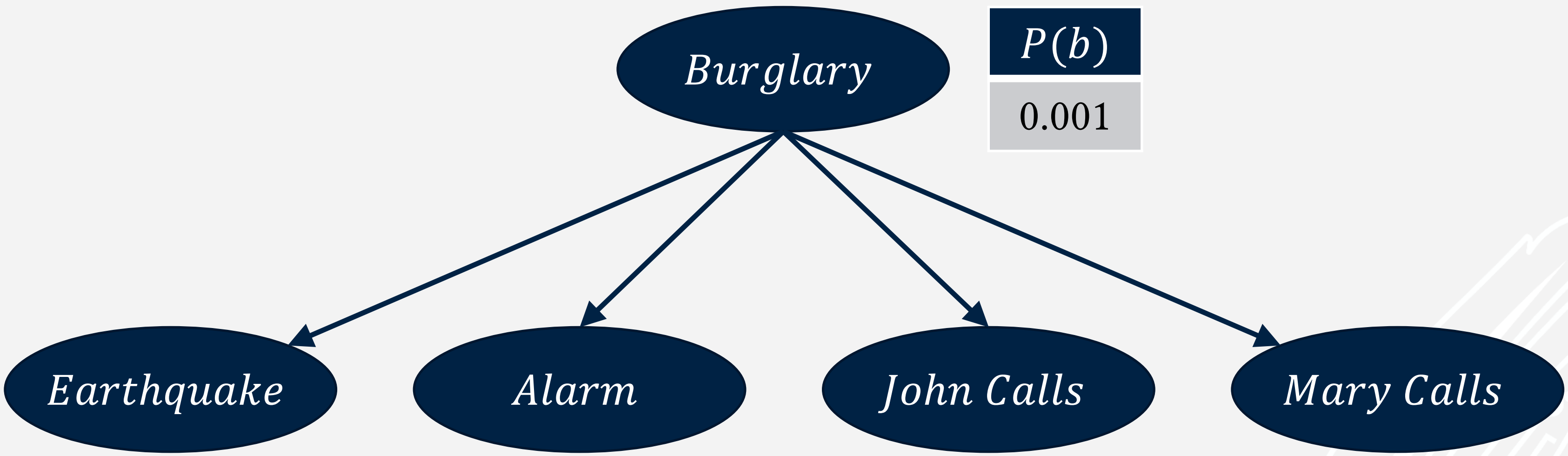
Given that both John and Mary have called, how likely is it that a burglary has occurred?

- How should we construct the network?
- Given the network, how can we answer the query?

Naïve Bayes

One simple solution for constructing the network is to assume that all evidence variables are conditionally independent given the query.

This is called the Naïve Bayes assumption.



$P(b)$
0.001

B	$P(e)$
$true$	0.002
$false$	0.002

B	$P(a)$
$true$	0.945
$false$	0.291

B	$P(j)$
$true$	0.850
$false$	0.310

B	$P(m)$
$true$	0.665
$false$	0.214

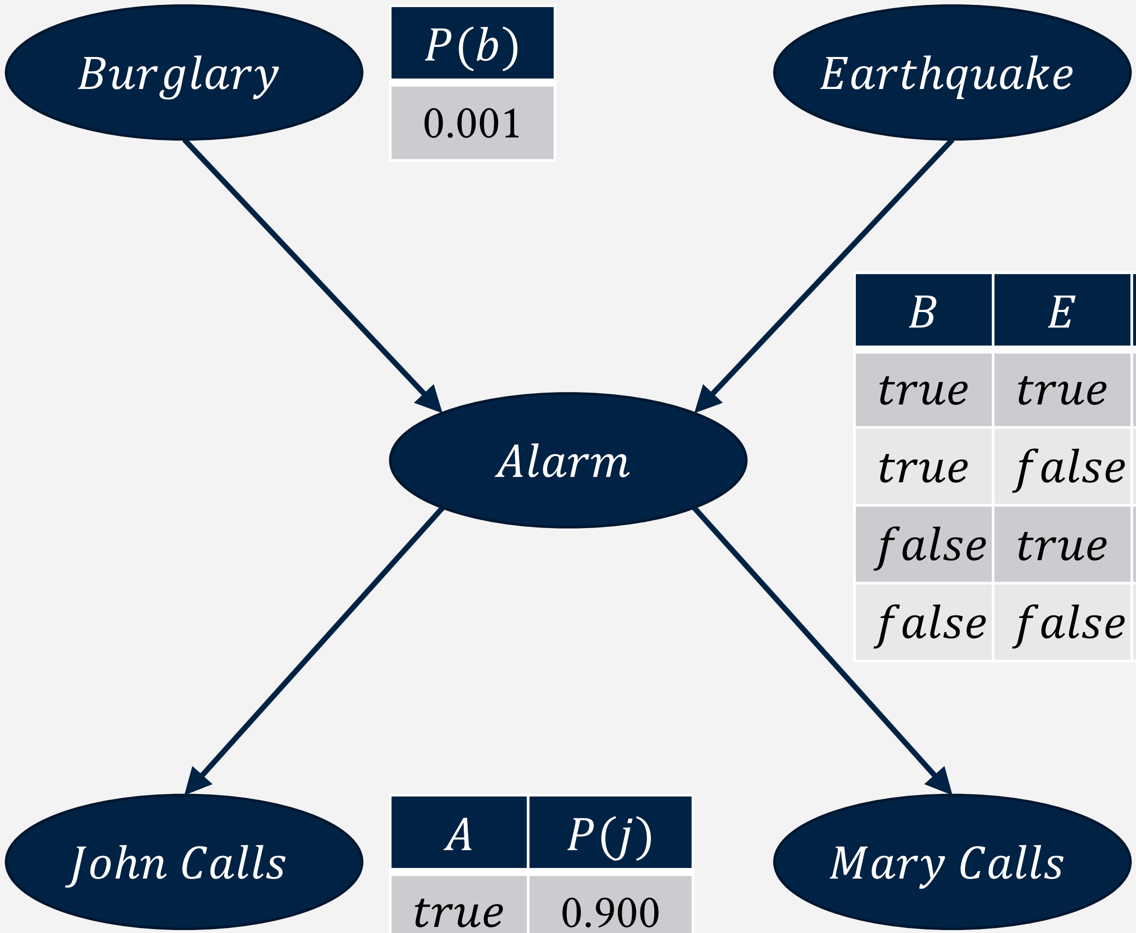
Naïve Bayes

Naïve Bayes assumes that all evidence is conditionally independent. This is rarely true, but these networks make calculations very easy and they are often surprisingly accurate given their simplicity.

Full Bayes Net

Let's consider a better solution based on what we know about the causal relationships in the problem.

- Burglaries can cause the alarm to go off.
- Earthquakes can cause the alarm to go off.
- The alarm can cause John to call.
- The alarm can cause Mary to call.



$P(b)$
0.001

$P(e)$
0.002

B	E	$P(a)$
$true$	$true$	0.950
$true$	$false$	0.940
$false$	$true$	0.290
$false$	$false$	0.001

A	$P(j)$
$true$	0.900
$false$	0.050

A	$P(m)$
$true$	0.700
$false$	0.010

Alarm Problem

Given that both John and Mary have called, how likely is it that a burglary has occurred?

Query Variable: B

Evidence Variables: $J = true, M = true$

Hidden Variables: E, A

Summing Out

While using a Bayes Net to calculate the exact probability of some query, we need to remove all the hidden variables from the equation because we don't know their values.

In other words, we need to consider all the possible values of each hidden variable. This process is called **summing out**.

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

We want to know the probability distribution for some query variable X, given some evidence e.

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

There is some hidden variable Y whose value we don't know.

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

So, we consider every possible value of Y, which is $\{y_1, y_2, \ldots, y_n\}$.
If Y is Boolean, when $\{true, false\}$.
If Y is something like $Weather$, then $\{sunny, rainy, cloudy, snowy\}$.

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

Because we are considering all possible worlds for Y, we need to add the values of all those worlds together.

Summing Out

$$\mathbf{P}(X|e) = \alpha \sum_{y} \mathbf{P}(X, e, y)$$

And finally we need to normalize the result.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = ?$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha\mathbf{P}(b,j,m)$$

Consider those possible worlds in which there was a burglary, John Called, and Mary called.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha\mathbf{P}(b,j,m,E,A)$$

We don't know the values of the hidden variables. We need to consider worlds in which there was and was not an earthquake. We need to consider worlds in which the alarm did and did not go off.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \mathbf{P}(b,j,m,e,A)$$

Sum out the hidden variable E ($Earthquake$).

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \sum_{a} \mathbf{P}(b,j,m,e,a)$$

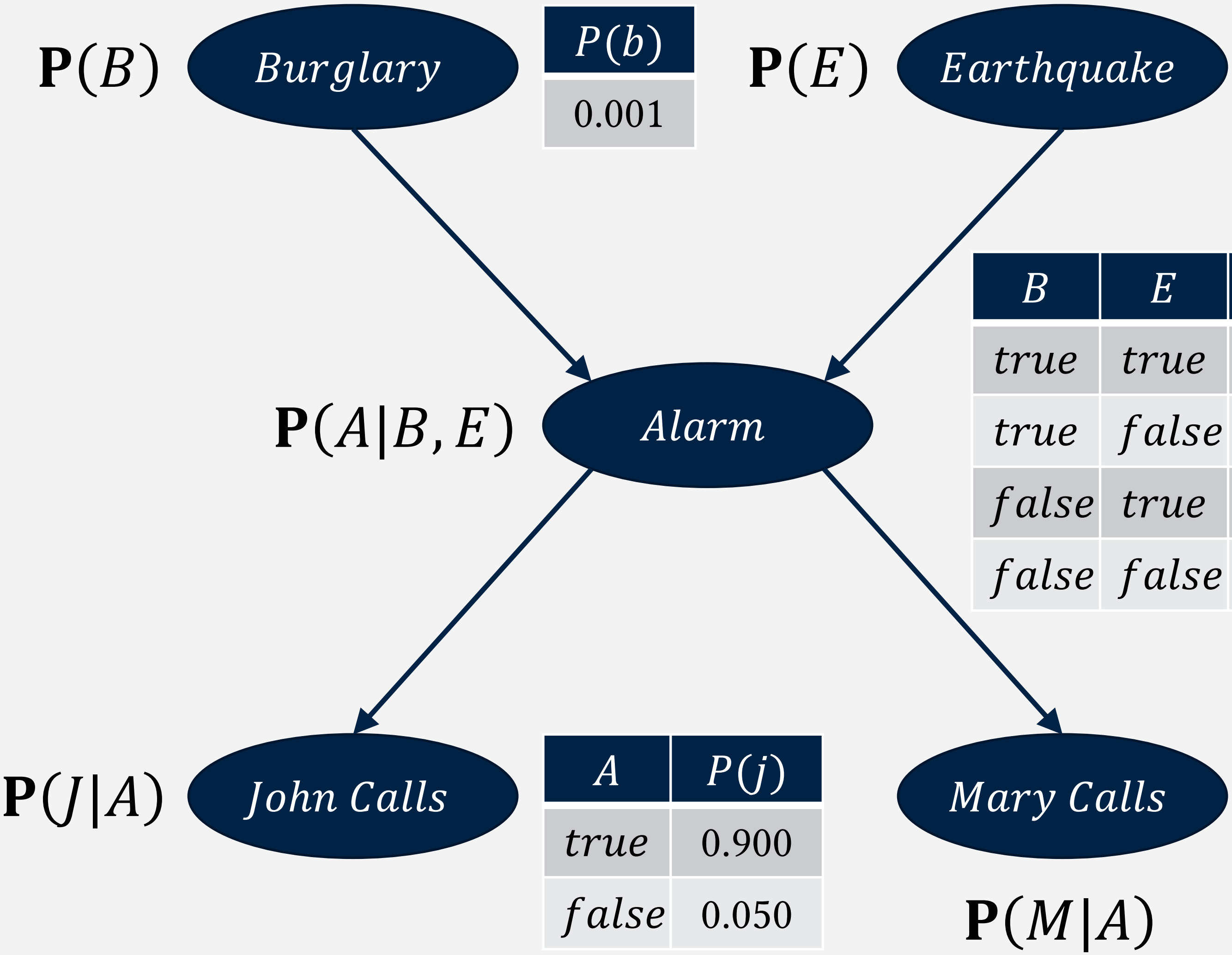
Sum out the hidden variable A ($Alarm$).

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \sum_{a} \mathbf{P}(b,j,m,e,a)$$

Now we need to break this term down into smaller parts based on the structure of our Bayes Net.



$P(b)$
0.001

$P(e)$
0.002

B	E	$P(a)$
true	*true*	0.950
true	*false*	0.940
false	*true*	0.290
false	*false*	0.001

A	$P(j)$
true	0.900
false	0.050

A	$P(m)$
true	0.700
false	0.010

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \sum_{a} \mathbf{P}(b,j,m,e,a)$$

Now we need to break this term down into smaller parts based on the structure of our Bayes Net.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \sum_{a} \mathbf{P}(b)\mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e)$$

Now we need to break this term down into smaller parts based on the structure of our Bayes Net.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \sum_{a} \mathbf{P}(b)\mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e)$$

Now we need to calculate the value of this equation.

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) =$$

$$\alpha \sum_{e} \sum_{a} \mathbf{P}(b)\mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e)$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) =$$
$$\alpha \sum_{e} \sum_{a} 0.001 \cdot \mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e)$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = \alpha \sum_{e} \begin{matrix} 0.001 \cdot \mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e) + \\ 0.001 \cdot \mathbf{P}(j|\neg a)\mathbf{P}(m|\neg a)\mathbf{P}(e)\mathbf{P}(\neg a|b,e) \end{matrix}$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) =$$
$$\alpha \cdot$$
$$\big(0.001 \cdot \mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(e)\mathbf{P}(a|b,e)\big) +$$
$$\big(0.001 \cdot \mathbf{P}(j|\neg a)\mathbf{P}(m|\neg a)\mathbf{P}(e)\mathbf{P}(\neg a|b,e)\big) +$$
$$\big(0.001 \cdot \mathbf{P}(j|a)\mathbf{P}(m|a)\mathbf{P}(\neg e)\mathbf{P}(a|b,\neg e)\big) +$$
$$\big(0.001 \cdot \mathbf{P}(j|\neg a)\mathbf{P}(m|\neg a)\mathbf{P}(\neg e)\mathbf{P}(\neg a|b,\neg e)\big)$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$$\mathbf{P}(b|j,m) = 479.53 \cdot (0.001 \cdot 0.900 \cdot 0.700 \cdot 0.002 \cdot 0.950) + (0.001 \cdot 0.050 \cdot 0.010 \cdot 0.002 \cdot 0.050) + (0.001 \cdot 0.900 \cdot 0.700 \cdot 0.998 \cdot 0.940) + (0.001 \cdot 0.050 \cdot 0.010 \cdot 0.998 \cdot 0.060)$$

Alarm Problem

Given that John and Mary have called, what are the chances that there was a burglary?

$\mathbf{P}(b|j,m) \approx 0.284$

There is about a 28% chances that there was a burglary.

Advanced Bayes Nets

- The straight-forward process of calculating probabilities by enumeration is correct, but it is time-consuming and often repeats calculations
- Improvement: Avoid repeated calculations (e.g. Variable Elimination algorithm)
- Improvement: Value estimation in networks too large for exact inference

Estimating Values via Sampling

- Problem: Calculating exact values may be too time-consuming in large networks.
- Solution: Sample randomly from the space of possible values to estimate actual values.

Algorithms which run in a fixed amount of time and use random sampling to estimate calculations (with some margin of error) are called **Monte Carlo** algorithms, named for the famous casino.

Random Sampling

Say you have a coin and want to test whether or not it is fair.

$$\mathbf{P}(Coin) = \langle heads, tails \rangle$$

After flipping the coin 1 time:

$$\mathbf{P}(Coin) = \langle 1{,}0 \rangle$$

Random Sampling

Say you have a coin and want to test whether or not it is fair.

$\mathbf{P}(Coin) = \langle heads, tails \rangle$

After flipping the coin 10 times:
$\mathbf{P}(Coin) = \langle 7,3 \rangle$

Random Sampling

Say you have a coin and want to test whether or not it is fair.

$\mathbf{P}(Coin) = \langle heads, tails \rangle$

After flipping the coin 100 times:

$\mathbf{P}(Coin) = \langle 59{,}41 \rangle$

Random Sampling

Say you have a coin and want to test whether or not it is fair.

$\mathbf{P}(Coin) = \langle heads, tails \rangle$

After flipping the coin 1000 times:
$\mathbf{P}(Coin) = \langle 508{,}492 \rangle$

Random Sampling

Say you have a coin and want to test whether or not it is fair.

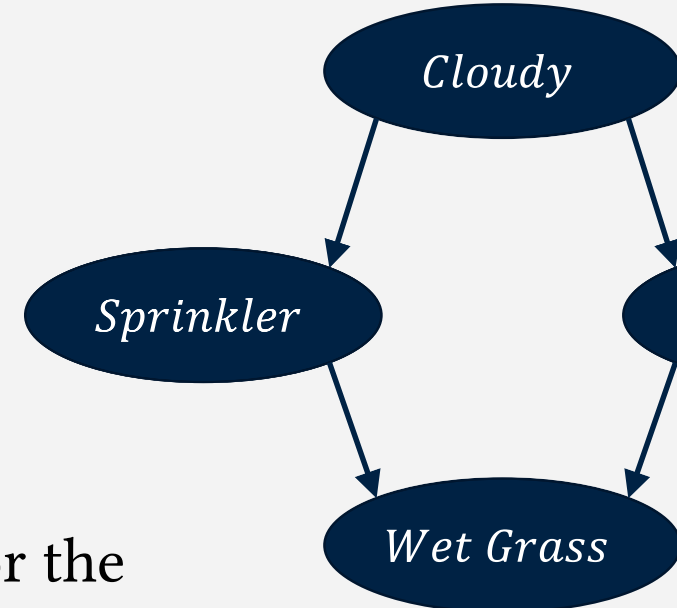
$\mathbf{P}(Coin) = \langle heads, tails \rangle$

After flipping the coin ∞ times:
$\mathbf{P}(Coin) = \langle 0.5, 0.5 \rangle$

Lawn Domain

- The weather is cloudy about half the time.
- When it is cloudy, it is more likely to rain.
- When it is not cloudy, I am more likely to need to turn on my sprinkler to water my grass.
- Both my sprinkler and the rain make my grass wet.

Lawn Domain



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

Let us suppose, for the sake of some examples, that this network is too large for exact inference.

Lawn Query

Query: $\mathbf{P}(Rain|cloudy, wet) = ?$

"Given that it is cloudy outside and my grass is wet, what are the chances that it rained today?"

Query Variable: $Rain$

Evidence Variables: $Cloudy = true, Wet\ Grass = true$

Hidden Variable: $Sprinkler$

Lawn Query

Query: $\mathbf{P}(Rain|cloudy, wet) = ?$

"Given that it is cloudy outside and my grass is wet, what are the chances that it rained today?"

Query Variable: $Rain$

Evidence Variables: $cloudy, wet$

Hidden Variable: $Sprinkler$

Lawn Query

Query: $\mathbf{P}(R|c,w) = ?$

"Given that it is cloudy outside and my grass is wet, what are the chances that it rained today?"

Query Variable: R

Evidence Variables: c, w

Hidden Variable: S

Sampling in Bayes Nets

Value estimation via direct sampling:

- Rejection Sampling
- Likelihood Weighting

Rejection Sampling

```
Generate the desired number of samples this way:
    For each node in topological order:
        Randomly sample a value for the variable.
Remove all samples inconsistent with the evidence.
Add up the remaining samples to estimate the query.
```

Rejection Sampling

```
Generate the desired number of samples this way:
    For each node in topological order:
        Randomly sample a value for the variable.
Remove all samples inconsistent with the evidence.
Add up the remaining samples to estimate the query.
```

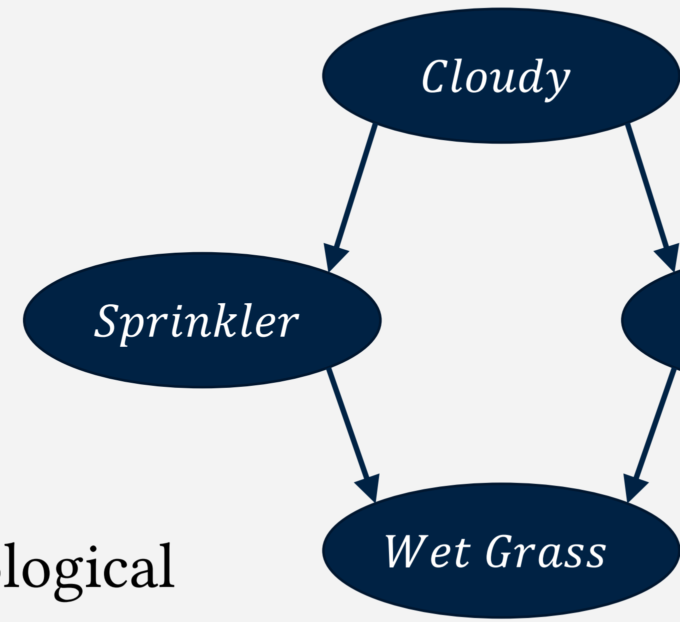
An order such that every node is guaranteed to come after all of its parents.

Rejection Sampling

```
Generate the desired number of samples this way:
    For each node in topological order:
        Randomly sample a value for the variable.
Remove all samples inconsistent with the evidence.
Add up the remaining samples to estimate the query.
```

Roll a die with as many sides as values, weighted appropriately for that value.

Rejection Sampling



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

One possible topological order: C, S, R, W

Rejection Sampling

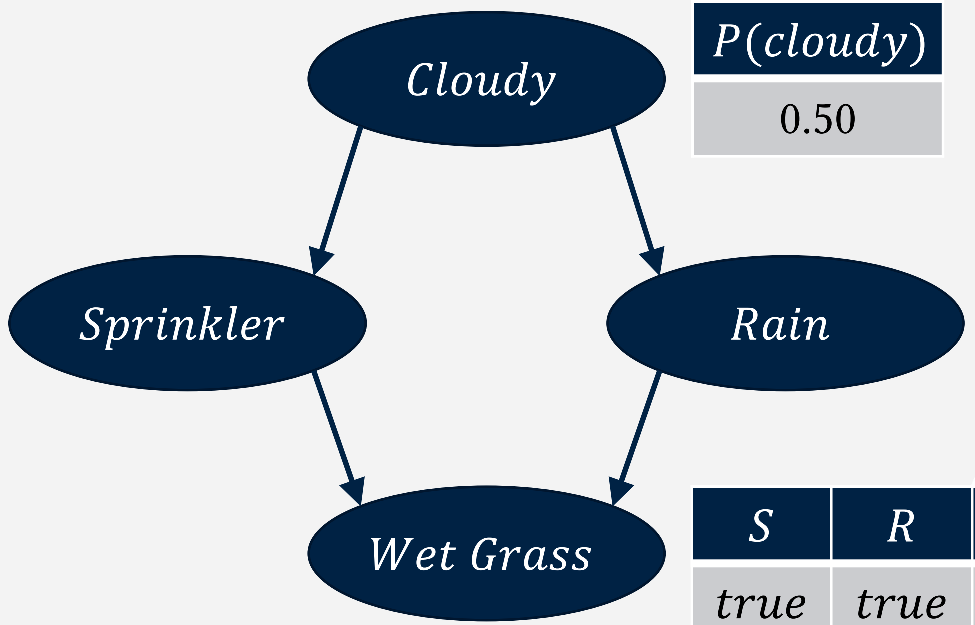
To generate a sample:

- Sample $\mathbf{P}(C) = ?$

Samples generated:

1. [?,?,?,?]

Rejection Sampling



$P(cloudy)$
0.50

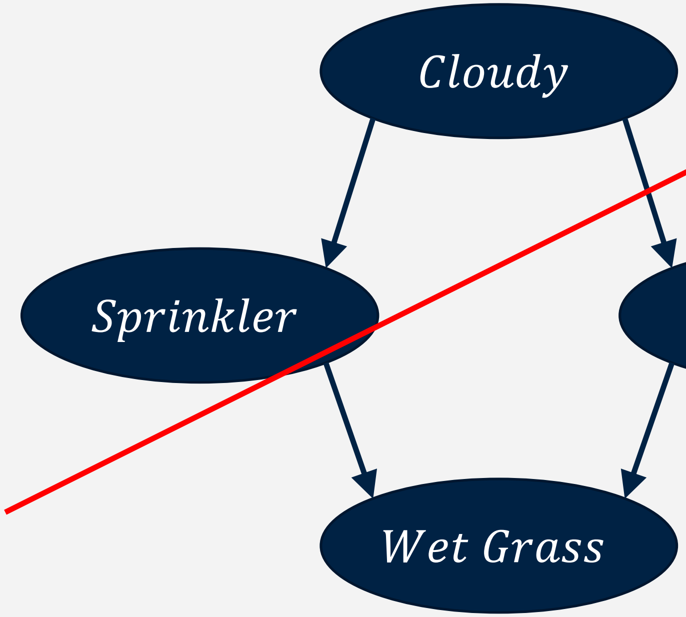
C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(C) = ?$$

Rejection Sampling



$P(cloudy)$
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$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(C) = \langle 0.5, 0.5 \rangle$$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle$

Samples generated:

1. [?,?,?,?]

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle$

There is a 50% chance of the weather being cloudy, so just flip a coin.

Samples generated:

1. [?,?,?,?]

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$

Samples generated:

1. $[true, ?, ?, ?]$

Rejection Sampling

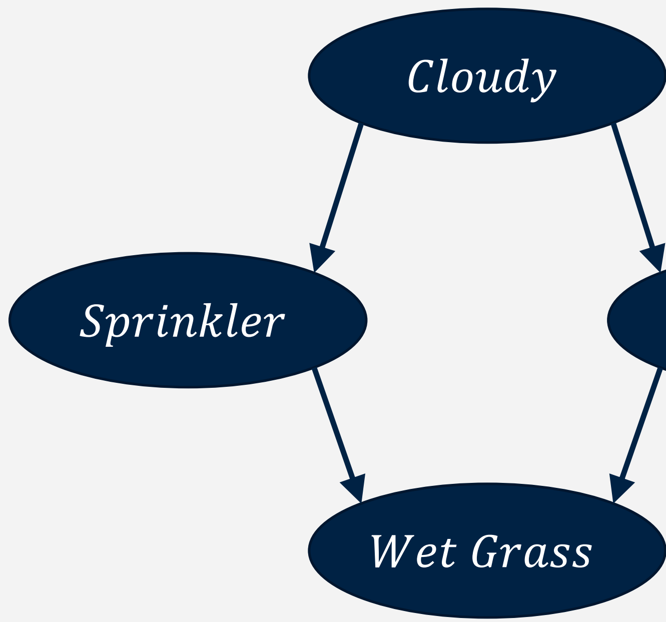
To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = ?$

Samples generated:

1. $[true, ?, ?, ?]$

Rejection Sampling



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C	$P(rain)$
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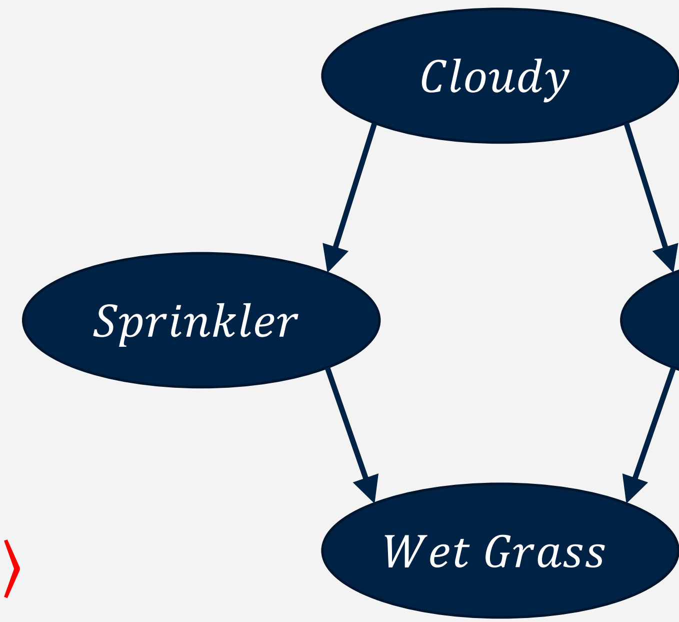
S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(S|c) = ?$$

Rejection Sampling



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle$

Samples generated:

1. $[true, ?, ?, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle$

The value has a 10% chances of being *true* and a 90% chance of being *false*.

Samples generated:

1. $[true, ?, ?, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle$

Roll a 10 sided die. If it reads 1, the value is $true$. If it reads $2 - 10$, the value is $false$.

Samples generated:

1. $[true, ?, ?, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5,0.5\rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1,0.9\rangle = false$

Samples generated:

1. $[true, false, ?, ?]$

Rejection Sampling

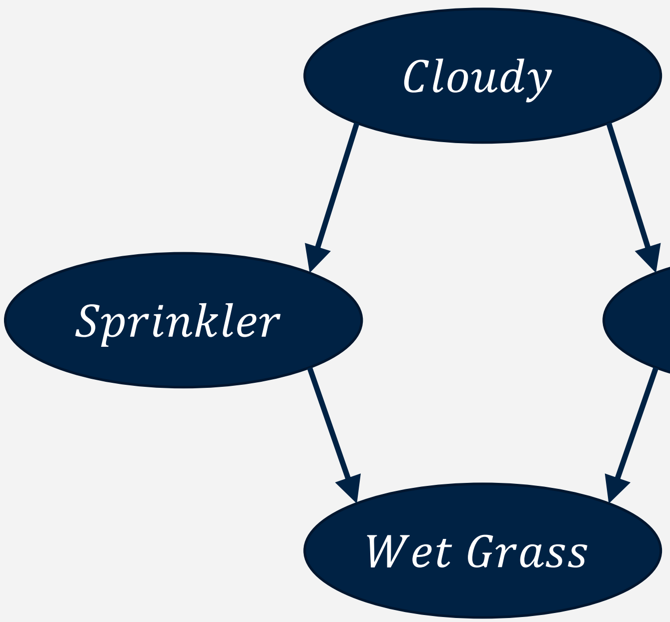
To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = ?$

Samples generated:

1. $[true, false, ?, ?]$

Rejection Sampling



$P(cloudy)$
0.50

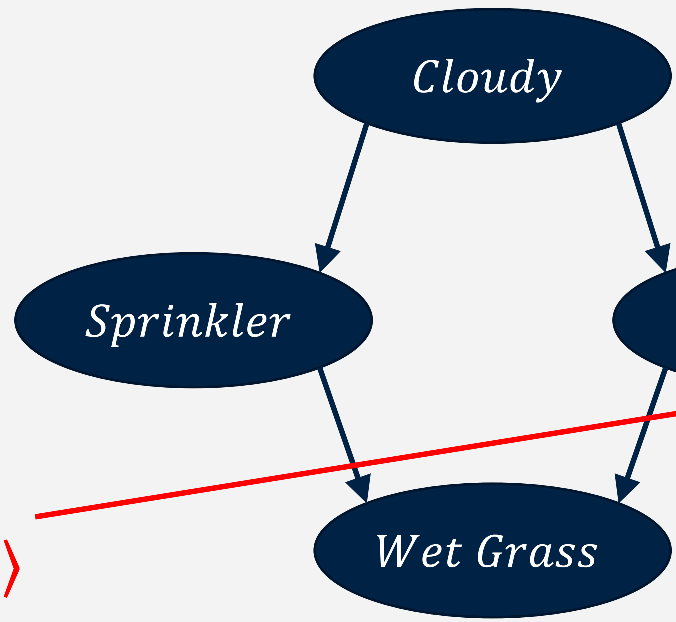
C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(R|c) = ?$$

Rejection Sampling



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle$$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle$

Samples generated:

1. $[true, false, ?, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle$

The value has an 80% chances of being $true$ and a 20% chance of being $false$.

Samples generated:

1. $[true, false, ?, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle$

Roll a 10 sided die. If it reads 1 - 8, the value is $true$. If it reads 9 – 10, the value is $false$.

Samples generated:

1. $[true, false, ?, ?]$

NIL

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$

Samples generated:

1. $[true, false, true, ?]$

Rejection Sampling

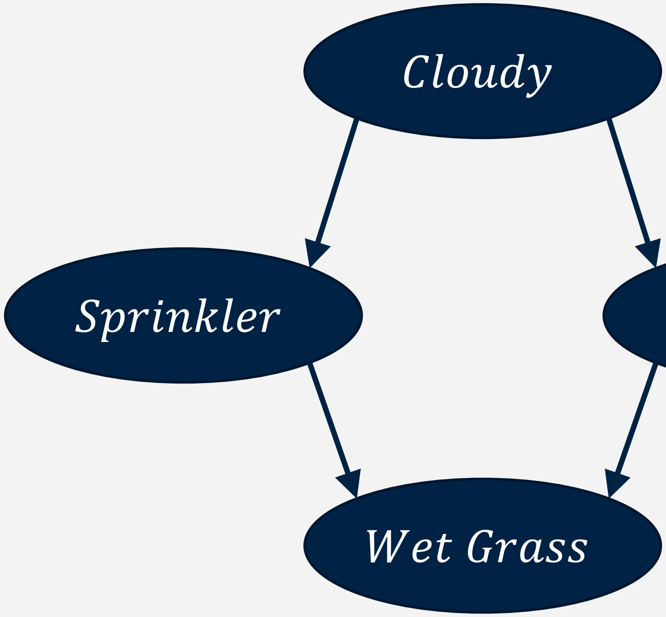
To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- Sample $\mathbf{P}(W|\neg s, r) = ?$

Samples generated:

1. $[true, false, true, ?]$

Rejection Sampling



$P(cloudy)$
0.50

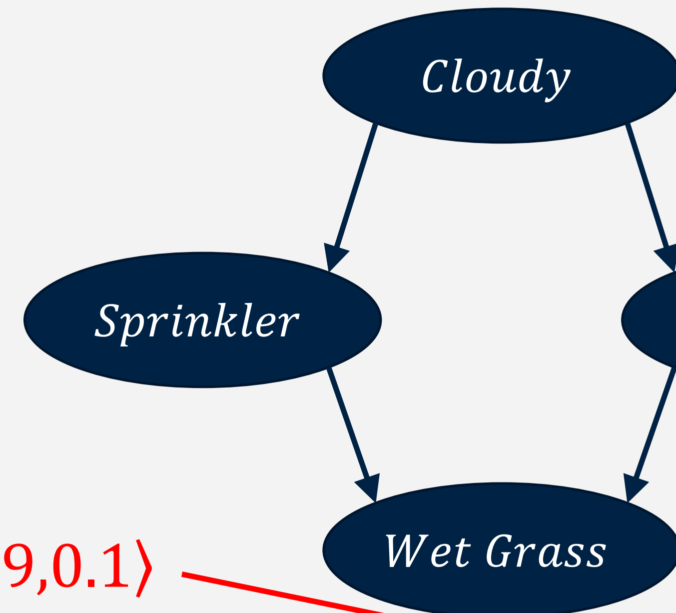
C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(W|\neg s, r) = ?$$

Rejection Sampling



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

$$\mathbf{P}(W|\neg s, r) = \langle 0.9, 0.1 \rangle$$

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- Sample $\mathbf{P}(W|\neg s, r) = \langle 0.9, 0.1 \rangle$

Samples generated:

1. $[true, false, true, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- Sample $\mathbf{P}(W|\neg s, r) = \langle 0.9, 0.1 \rangle$

Roll a 10 sided die. If it reads 1 - 9, the value is $true$. If it reads 10, the value is $false$.

Samples generated:

1. $[true, false, true, ?]$

Rejection Sampling

To generate a sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- Sample $\mathbf{P}(W|\neg s, r) = \langle 0.9, 0.1 \rangle = true$

Samples generated:

1. $[true, false, true, true]$

Rejection Sampling

To generate another sample:

Samples generated:

1. $[true, false, true, true]$
2. [?,?,?,?]

Rejection Sampling

To generate another sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = false$

Samples generated:

1. $[true, false, true, true]$
2. $[false, ?, ?, ?]$

NIL

Rejection Sampling

To generate another sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = false$
- Sample $\mathbf{P}(S|\neg c) = \langle 0.5, 0.5 \rangle = true$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, ?, ?]$

Rejection Sampling

To generate another sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = false$
- Sample $\mathbf{P}(S|\neg c) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(R|\neg c) = \langle 0.2, 0.8 \rangle = false$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, ?]$

NIL

Rejection Sampling

To generate another sample:

- Sample $\mathbf{P}(C) = \langle 0.5, 0.5 \rangle = false$
- Sample $\mathbf{P}(S|\neg c) = \langle 0.5, 0.5 \rangle = true$
- Sample $\mathbf{P}(R|\neg c) = \langle 0.2, 0.8 \rangle = false$
- Sample $\mathbf{P}(W|s, \neg r) = \langle 0.9, 0.1 \rangle = true$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$

Rejection Sampling

Generate 8 more samples.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$\mathbf{P}(R|c,w) = \langle ?\,, ?\,\rangle$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$\mathbf{P}(R|c,w) = \langle ?\,,\,?\,\rangle$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$\mathbf{P}(R|c, w) = \langle ?, ? \rangle$

Reject those samples not consistent with the evidence.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$$\mathbf{P}(R|c,w) = \left\langle \frac{?}{5}, \frac{?}{5} \right\rangle$$

Count those samples in which the query holds.

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$$\mathbf{P}(R|c,w) = \left\langle \frac{4}{5}, \frac{1}{5} \right\rangle$$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

Count samples to estimate:

$\mathbf{P}(R|c,w) = \langle 0.8{,}0.2 \rangle$

Samples generated:

1. $[true, false, true, true]$
2. $[false, true, false, true]$
3. $[true, false, false, true]$
4. $[true, true, true, true]$
5. $[false, true, true, true]$
6. $[false, true, false, true]$
7. $[true, false, true, true]$
8. $[false, true, false, true]$
9. $[true, false, true, true]$
10. $[true, false, true, false]$

Rejection Sampling

The problem with rejection sampling is that some of the effort we put into generating samples is wasted when those samples are rejected.

The more evidence variables that are given, the more samples we will reject and the less reliable our final estimation will be.

Can we avoid wasting work?

Likelihood Weighting

- Also generates random samples, but only samples in which the evidence holds.
- Samples are also generated with a weight to describes how likely they are.
  - Samples in which the evidence holds (but it is unlikely that it would happen that way) have low weight.
  - Samples in which the evidence holds (and it is likely that it would happen that way) have high weight.

Likelihood Weighting

```
Generate the desired number of samples this way:
    Let weight = 1.
    For every variable V in topological order:
        If V is an evidence variable with value v:
            weight = weight · P(V = v|parents(V))
        Else:
            Randomly sample from P(V|parents(V)).
    Return the event and the weight.
Add up the weighted samples to estimate the query.
```

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.

Samples generated:

1. [?,?,?,?] $w = ?$

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c)$

Samples generated:

1. $[true,?,?,?]$ $w = ?$

NIL

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$

Samples generated:

1. $[true,?,?,?]$ $w = ?$

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

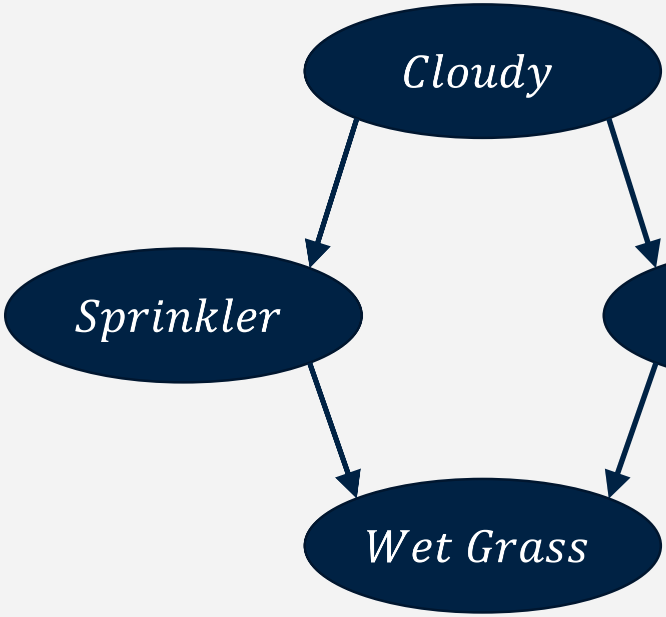
  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = ?$

Samples generated:

1. $[true,?,?,?]$ $w = ?$

Likelihood Weighting



$P(cloudy)$
0.50

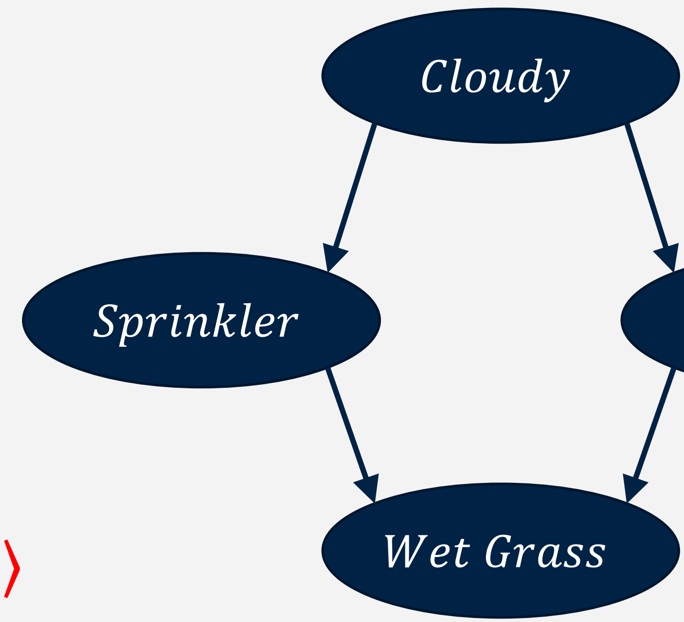
C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$$\mathbf{P}(S|c) = ?$$

Likelihood Weighting



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

$$\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle$$

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1,0.9 \rangle$

Samples generated:

1. $[true,?,?,?]$ $w = ?$

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$

- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1,0.9 \rangle$

Roll a 10 sided die. If it reads 1, the value is $true$. If it reads 2 - 10, the value is $false$.

Samples generated:

1. $[true,?,?,?]$ $w = ?$

Likelihood Weighting

Topological order: C, S, R, W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$

Samples generated:

1. $[true, false, ?, ?]$ $w = ?$

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1,0.9 \rangle = false$
- R is not an evidence variable.

  Sample $\mathbf{P}(R|c) = \langle 0.8,0.2 \rangle$

Samples generated:

1. $[true,false,?,?]$ $w = ?$

Likelihood Weighting

Topological order: C, S, R, W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- R is not an evidence variable.

  Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$

Samples generated:

1. $[true, false, true, ?]$ $w = ?$

NIL

Likelihood Weighting

Topological order: C,S,R,W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- R is not an evidence variable.

  Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- W is an evidence variable with value w.

  $w = w \cdot P(w|\neg s, r)$

Samples generated:

1. $[true, false, true, true]$ $w = ?$

NIL

Likelihood Weighting

Topological order: C, S, R, W

To generate a weighted sample:

- Initially, $w = 1$.
- C is an evidence variable with value c.

  $w = w \cdot P(c) = 1 \cdot 0.5 = 0.5$
- S is not an evidence variable.

  Sample $\mathbf{P}(S|c) = \langle 0.1, 0.9 \rangle = false$
- R is not an evidence variable.

  Sample $\mathbf{P}(R|c) = \langle 0.8, 0.2 \rangle = true$
- W is an evidence variable with value w.

  $w = w \cdot P(w|\neg s, r) = 0.5 \cdot 0.9 = 0.45$

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$

Given the evidence, this particular event has weight of 0.45 that reflects its likelihood.

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$

Likelihood Weighting

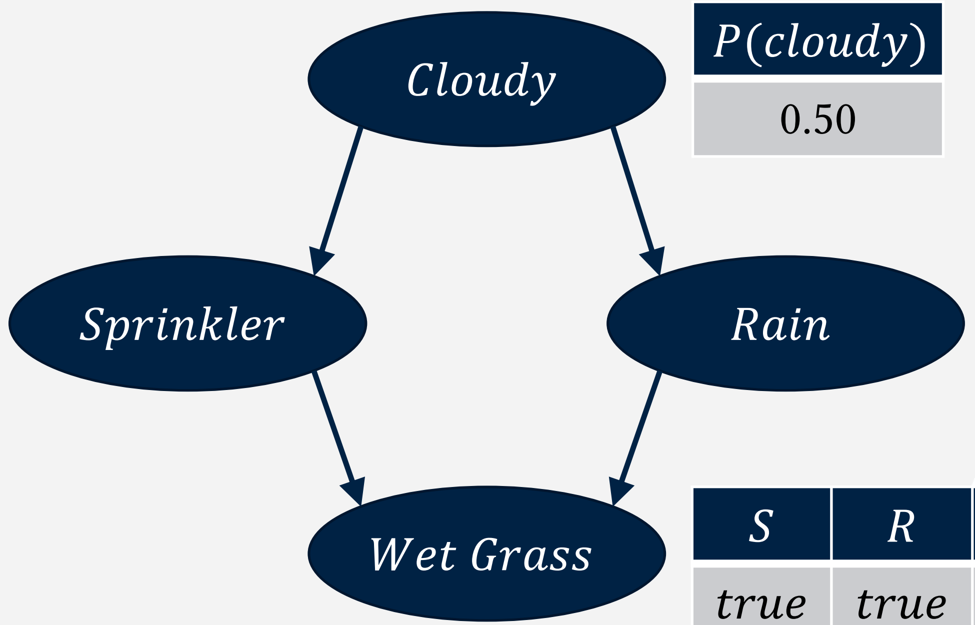
Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]\ w = 0.45$
2. $[true, true, true, true]\ w = 0.495$
3. $[true, true, true, true]\ w = 0.495$
4. $[true, false, false, true]\ w = 0$

A weight of 0 means that this event can never happen (and thus should count for nothing in the final tally).

Likelihood Weighting



$P(cloudy)$
0.50

C	$P(sprinkler)$
$true$	0.10
$false$	0.50

C	$P(rain)$
$true$	0.80
$false$	0.20

S	R	$P(wet)$
$true$	$true$	0.99
$true$	$false$	0.90
$false$	$true$	0.90
$false$	$false$	0.00

$\mathbf{P}(w|\neg s, \neg r) = 0$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]\ w = 0.45$
2. $[true, true, true, true]\ w = 0.495$
3. $[true, true, true, true]\ w = 0.495$
4. $[true, false, false, true]\ w = 0$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$

Likelihood Weighting

Generate 9 more samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Count weighted samples to estimate:
$\mathbf{P}(R|c, w) = \langle ?, ? \rangle$

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Count weighted samples to estimate:
$\mathbf{P}(R|c,w) = \langle ?\,,?\,\rangle$

Note how the evidence holds in all samples. We avoid the waste of rejecting any samples.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Count weighted samples to estimate:

$\mathbf{P}(R|c,w)$
$= \langle 0.45 + 0.495 + 0.495 + 0.495 + 0.495 + 0.45 + 0.495, 0 + 0.45 + 0.45 \rangle$

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Count weighted samples to estimate:
$\mathbf{P}(R|c,w) = \langle 3.375, 0.9 \rangle$

Finally, normalize the result.

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$

Likelihood Weighting

Count weighted samples to estimate:
$\mathbf{P}(R|c,w) \approx \langle 0.79, 0.21 \rangle$

Samples generated:

1. $[true, false, true, true]$ $w = 0.45$
2. $[true, true, true, true]$ $w = 0.495$
3. $[true, true, true, true]$ $w = 0.495$
4. $[true, false, false, true]$ $w = 0$
5. $[true, true, true, true]$ $w = 0.495$
6. $[true, true, false, true]$ $w = 0.45$
7. $[true, true, true, true]$ $w = 0.495$
8. $[true, false, true, true]$ $w = 0.45$
9. $[true, true, false, true]$ $w = 0.45$
10. $[true, true, true, true]$ $w = 0.495$