极限

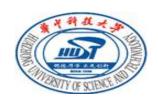






- 基本概念清楚
- 基本定理掌握
- 基本计算熟悉





极限是一种思想,是有限运算到无限运算的推广从有穷到无限,从理想到一般,由已知探究未知的方式

例如 定积分概念的形成

规则图形面积 —— 不规则图形面积

除了思想,其余都是中等数学!





• 在实际计算极限时,多关注局部与整体的数值变化的依赖关系

例如
$$\lim_{n\to+\infty}\sum_{k=1}^n\frac{k^n}{n^n}$$

$$\frac{k^n}{n^n} = \left(\frac{k}{n}\right)^n \rightarrow ?$$





 $k \ll n$ 时

$$\frac{k^n}{n^n} = \left(\frac{k}{n}\right)^n \longrightarrow 0$$

k可以远到什么程度 使得这个极限依然成立

k靠近n时,即k = n - i

$$\frac{k^n}{n^n} = \left(\frac{k}{n}\right)^n = \left(1 - \frac{i}{n}\right)^n \rightarrow e^{-i}$$





猜测

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{k^n}{n^n} = 1 + e^{-1} + e^{-2} + \dots = \frac{e}{e-1}$$





$$\lim_{n\to+\infty}x_n=a \Longleftrightarrow \forall \varepsilon>0, \exists N, n>N, |x_n-a|$$

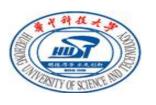
$$\lim_{x\to x_0} f(x) = a \iff \forall \varepsilon > 0, \exists \delta, 0 < |x-x_0| < \delta, |f(x)-a| < \varepsilon$$

左,右极限…

- 唯一性
- 局部有界性
- 保不等号性
- 单调有界收敛性



极限计算的常见方法



- 1. 定义及等价无穷小
- 2. L'Hospital法则
- 3. 夹逼定理,微分中值定理
- 4. Stolz定理
- 5. Taylor展开





$$\lim_{x\to x_0} f(x) = f(x_0) \iff \forall \varepsilon > 0, \exists \delta, |x-x_0| < \delta, |f(x)-f(x_0)| < \varepsilon$$

左连续, 右连续

间断点及其分类

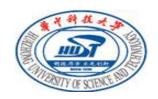


连续函数的整体性质



- 介值性质
- 最值性质
- 一致连续性质





1.(11th)
$$\lim_{x\to 0} \frac{\ln\left(e^{\sin x} + \sqrt[3]{1-\cos x}\right) - \sin x}{\arctan(4\sqrt[3]{1-\cos x})} = \underline{\qquad}$$

Hints: L'Hospital法则不太合适

分母可以考虑先用等价无穷小

化简分子



历届试题分析
$$\lim_{x\to 0} \frac{\ln\left(e^{\sin x} + \sqrt[3]{1-\cos x}\right) - \sin x}{\arctan(4\sqrt[3]{1-\cos x})} = \underline{\hspace{1cm}}$$



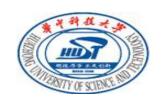
$$\ln(e^{\sin x} + \sqrt[3]{1 - \cos x}) - \sin x = \ln(1 + e^{-\sin x} \sqrt[3]{1 - \cos x})$$
$$\sim e^{-\sin x} \sqrt[3]{1 - \cos x}$$

$$\arctan\left(4\sqrt[3]{1-\cos x}\right) \sim 4\sqrt[3]{1-\cos x}$$

$$\lim_{x \to 0} \frac{\ln(e^{\sin x} + \sqrt[3]{1 - \cos x}) - \sin x}{\arctan(4\sqrt[3]{1 - \cos x})} = \lim_{x \to 0} \frac{e^{-\sin x} \sqrt[3]{1 - \cos x}}{4\sqrt[3]{1 - \cos x}}$$

$$=\frac{1}{4}$$





$$2.$$
 (8th) 若 $f(1) = 0, f'(1)$ 存在,则极限
$$I = \lim_{x \to 0} \frac{f\left(\sin^2 x + \cos x\right)\tan 3x}{\left(e^{x^2} - 1\right)\sin x} = \underline{\qquad}$$

Hints: L' Hospital并不合适,原因?

对相应因子考虑等价无穷小替换

f的处理,回归导数定义



历届试题分析
$$I = \lim_{x \to 0} rac{f\left(1\right) = 0, f'\left(1\right)$$
存在,则极限 $I = \lim_{x \to 0} rac{f\left(\sin^2 x + \cos x\right) \tan 3x}{\left(e^{x^2} - 1\right) \sin x} = \underline{\qquad}$



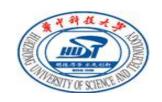
$$\tan 3x \sim 3x$$

$$\tan 3x \sim 3x \qquad e^{x^2} - 1 \sim x^2 \qquad \sin x \sim x$$

$$\frac{f(\sin^2 x + \cos x) - f(1)}{\sin^2 x + \cos x - 1} \left(\sin^2 x + \cos x - 1\right) \sim f'(1) \left(\sin^2 x + \cos x - 1\right)$$

$$I = \lim_{x \to 0} \frac{f'(1)(\sin^2 x + \cos x - 1) \cdot 3x}{x^2 \cdot x} = \frac{3}{2}f'(1)$$





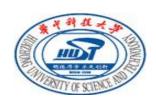
3.(9th final) 极限
$$\lim_{x\to 0} \frac{\tan x - \sin x}{x \ln \left(1 + \sin^2 x\right)} = \underline{\qquad}$$

Hints: 分母可以考虑先做等价无穷小替换

分子可以先化简,再做等价无穷小替换



历届试题分析
$$\frac{\operatorname{ken} x - \sin x}{x \ln \left(1 + \sin^2 x\right)} = \underline{\hspace{1cm}}$$

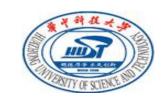


$$\ln(1+\sin^2 x) \sim \sin^2 x \sim x^2$$

$$\tan x - \sin x = \frac{\sin x (1 - \cos x)}{\cos x} \sim \frac{1}{2}x^3$$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x \ln(1 + \sin^2 x)} = \frac{1}{2}$$





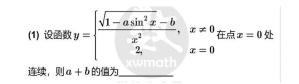
设函数
$$y=egin{cases} rac{\sqrt{1-a\sin^2x-b}}{x^2}, & x
eq 0 \ \hline x^2, & x=0 \end{cases}$$
 在点 $x=0$ 处

连续,则a+b的值为_

Hints: 唯一知道的是连续点,自然需要用连续的定义

两个量,要么可以将a + b作为一个整体出现,要么解出来,需要两个方程.







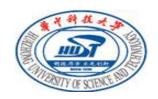
$$\lim_{x\to 0}\frac{\sqrt{1-a\sin^2x}-b}{x^2}=2$$

$$\sqrt{1-a\sin^2 x}-b\to 0\ (x\to 0)\ \Rightarrow b=1$$

$$\lim_{x\to 0} \frac{\sqrt{1-a\sin^2 x}-1}{x^2} = \lim_{x\to 0} \frac{\sqrt{1-a\sin^2 x}-1}{-a\sin^2 x} \cdot \frac{-a\sin^2 x}{x^2} = -\frac{1}{2}a$$

$$\Rightarrow a = -4 \Rightarrow a + b = -3$$





5.(10th)
$$\lim_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2} = \underline{\qquad}.$$

Hints: L' Hospital法则有点繁琐

熟悉等价无穷小关系
$$1 - \cos x \sim \frac{1}{2}x^2$$

在分子中插入一些项,分别计算极限





$$1 - \cos x \sqrt{\cos 2x} \sqrt[3]{\cos 3x} = 1 - \cos x + \cos x \left(1 - \sqrt{\cos 2x} \sqrt[3]{\cos 3x}\right)$$

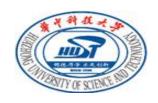
$$1 - \sqrt{\cos 2x} \sqrt[3]{\cos 3x} = 1 - \sqrt{\cos 2x} + \sqrt{\cos 2x} \left(1 - \sqrt[3]{\cos 3x}\right)$$

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x} \sqrt[3]{\cos 3x}}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} + \lim_{x \to 0} \cos x \frac{1 - \sqrt{\cos 2x}}{x^2}$$

$$+\lim_{x\to 0}\cos x\sqrt{\cos 2x}\frac{1-\sqrt[3]{\cos 3x}}{x^2}$$

$$=\frac{1}{2}+1+\frac{3}{2}=3$$



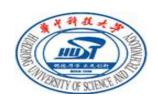


$$6.(9 ext{th})$$
 设 $f(x)$ 具有二阶连续导数,且
$$f(0)=f'(0)=0, f''(0)=6\,,$$
 则 $\lim_{x o 0} rac{f\left(\sin^2 x
ight)}{x^4}=$ _______。

Hints: L' Hospital法则

Taylor展开



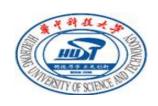


1)
$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(\xi)x^2 = \frac{1}{2}f''(\xi)x^2$$

$$f(\sin^2 x) = \frac{1}{2}f''(\xi)(\sin^2 x)^2$$

$$\lim_{x\to 0} \frac{f(\sin^2 x)}{x^4} = \lim_{x\to 0} \frac{1}{2} f''(\xi) \frac{\sin^4 x}{x^4} = \frac{1}{2} f''(0) = 3$$



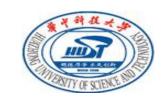


2)
$$\lim_{x\to 0} \frac{f(\sin^2 x)}{x^4} = \lim_{x\to 0} \frac{f'(\sin^2 x) \cdot 2\sin x \cos x}{4x^3}$$

$$=\lim_{x\to 0}\frac{f'(\sin^2x)}{2x^2}$$

$$= \lim_{x\to 0} \frac{f'(\sin^2 x)}{\sin^2 x} \cdot \frac{\sin^2 x}{2x^2} = 3$$





7.(3rd)
$$\lim_{x\to 0} \frac{\left(1+x\right)^{\frac{2}{x}}-e^2\left(1-\ln\left(1+x\right)\right)}{x}.$$

Hints: 若用L'Hospital, 可预见到比较繁琐

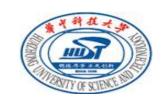
第一个函数结构并不友好,可以考虑Taylor展开

若注意到这两点,就能有新思路:1. $ln(1+x) \sim x$;

2.
$$(1+x)^{\frac{2}{x}} \rightarrow e^2$$



历届试题分析
$$\lim_{x \to 0} \frac{\left(1+x\right)^{\frac{2}{x}} - e^2\left(1-\ln\left(1+x\right)\right)}{x}$$
.



1)
$$(1+x)^{\frac{2}{x}} = e^{\frac{2}{x}\ln(1+x)} = e^{\frac{2}{x}\left(x-\frac{1}{2}x^2+o(x^2)\right)} = e^2e^{-x+o(x)}$$

 $= e^2(1-x+o(x))$
 $e^2(1-\ln(1+x)) = e^2(1-x+o(x))$

$$\lim_{x\to 0} \frac{(1+x)^{\frac{2}{x}} - e^2(1-\ln(1+x))}{x} = \lim_{x\to 0} \frac{o(x)}{x} = 0$$



历届试题分析
$$\lim_{x\to 0} \frac{\left(1+x\right)^{\frac{2}{x}}-e^2\left(1-\ln\left(1+x\right)\right)}{x}$$
.



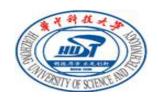
2)
$$(1+x)^{\frac{2}{x}} - e^2(1-\ln(1+x)) = (1+x)^{\frac{2}{x}} - e^2 + e^2\ln(1+x)$$

$$(1+x)^{\frac{2}{x}}-e^2=e^{2\left(e^{2\left(\frac{1}{x}\ln(1+x)-1\right)}-1\right)}\sim 2e^{2\left(\frac{1}{x}\ln(1+x)-1\right)}$$

$$\sim 2e^2\left(-\frac{1}{2}x\right)$$

$$\lim_{x \to 0} \frac{(1+x)^{\frac{2}{x}} - e^2(1 - \ln(1+x))}{x} = \lim_{x \to 0} \frac{(1+x)^{\frac{2}{x}} - e^2}{x} + \lim_{x \to 0} \frac{e^2 \ln(1+x)}{x}$$





8.(2ed) 求
$$\lim_{x\to\infty} e^{-x} \left(1 + \frac{1}{x}\right)^{x^2}$$
.

Hint: 最直接也许也是最合理的想法是将后面的函数具体化

解:由

$$\left(1+\frac{1}{x}\right)^{x^2}=e^{x^2\ln\left(1+\frac{1}{x}\right)}=e^{x^2\left(\frac{1}{x}-\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)}=e^{x-\frac{1}{2}+o(1)}$$



历届试题分析
$$^{\dag}\lim_{x \to \infty} e^{-x} \left[1 + \frac{1}{x}\right]^{x^2}$$

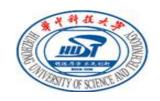


可得

$$\lim_{x \to \infty} e^{-x} \left(1 + \frac{1}{x} \right)^{x^2} = \lim_{x \to \infty} e^{-x} e^{x - \frac{1}{2} + o(1)}$$

$$=e^{-\frac{1}{2}}$$





9.(1st) 第二题: 求极限
$$\lim_{x\to 0}\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^x$$
 , 其中 n 是

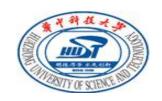
给定的正整数.

解:
$$\lim_{x\to 0} \frac{e}{x} \ln \left(\frac{e^x(e^{nx}-1)}{n(e^x-1)} - 1 + 1 \right) = \lim_{x\to 0} \frac{e}{x} \left(\frac{e^x(e^{nx}-1)}{n(e^x-1)} - 1 \right)$$

$$= \lim_{x\to 0} e^{\frac{e^{(n+1)x}-(n+1)e^x+n}{nx^2}} = \frac{n+1}{2}$$

$$\lim_{x \to 0} \left(\frac{e^x + e^{2x} + \dots + e^{nx}}{n} \right)^{\frac{e}{x}} = \lim_{x \to 0} e^{\frac{e}{x}} \ln \left(\frac{e^x (e^{nx} - 1)}{n(e^x - 1)} - 1 + 1 \right) = e^{\frac{n+1}{2}}$$



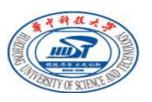


10.(9th final) 求极限:
$$\lim_{n\to\infty} \left[n+\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right]$$
.

Hint: 直接用Stirling公式 $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ 替代并不适合,因为涉及到了差运算。 基本想法是将差化成积,将指数幂数统一成指数形式。

解:
$$^{n+1}\sqrt{(n+1)!} - ^n\sqrt{n!} = ^n\sqrt{n!}\left(\frac{^{n+1}\sqrt{(n+1)!}}{^n\sqrt{n!}} - 1\right)$$





$$\sqrt[n]{n!} \sim \frac{n}{e} (n \to +\infty)$$

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1 = e^{\frac{1}{n+1}\sum_{k=1}^{n+1} \ln k - \frac{1}{n}\sum_{k=1}^{n} \ln k} - 1$$

$$=e^{\frac{1}{n+1}\ln(n+1)-\frac{1}{n(n+1)}\sum_{k=1}^{n}\ln k}-1$$

$$\sim \frac{1}{n+1}\ln(n+1) - \frac{1}{n(n+1)} \sum_{k=1}^{n} \ln k$$





$$\lim_{n \to \infty} \sqrt[n]{n!} \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1 \right) = \lim_{n \to \infty} \frac{n}{e} \left(\frac{1}{n+1} \ln(n+1) - \frac{1}{n(n+1)} \sum_{k=1}^{n} \ln k \right)$$

$$=\lim_{n\to\infty}\frac{1}{e}\left(\ln n-\frac{1}{n}\sum_{k=1}^n\ln k\right)$$

$$=-\lim_{n\to\infty}\frac{1}{e}\sum_{k=1}^n\frac{1}{n}\ln\frac{k}{n}$$

$$=-\frac{1}{e}\int_0^1 \ln x \, dx = \frac{1}{e}$$





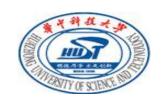
Remarks:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^n f\left(a+\frac{k}{n}(b-a)\right)=\int_a^b f(x)dx$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$$





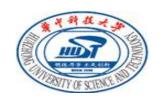
11.(4th final) 计算
$$\lim_{x\to 0+} \left[\ln(x \ln a) \cdot \ln\left(\frac{\ln ax}{\ln(x/a)}\right) \right], (a>1).$$

解:
$$\frac{\ln ax}{\ln \frac{x}{a}} \to 1$$

$$\ln \frac{\ln ax}{\ln \frac{x}{a}} = \ln \left(1 + \left(\frac{\ln ax}{\ln \frac{x}{a}} - 1 \right) \right) \sim \frac{\ln ax - \ln \frac{x}{a}}{\ln \frac{x}{a}} = \frac{2\ln a}{\ln \frac{x}{a}}$$

$$\lim_{x\to 0+} \left[\ln(x\ln a) \cdot \ln\frac{\ln ax}{\ln \frac{x}{a}}\right] = \lim_{x\to 0+} \ln(x\ln a) \cdot \frac{2\ln a}{\ln \frac{x}{a}} = 2\ln a$$





12.(8th final) 第六题: 设
$$a_n = \sum_{k=1}^n \frac{1}{k} - \ln n$$
.

(1)证明:极限 $\lim_{n \to \infty} a_n$ 存在;

(2)记
$$\lim_{n \to \infty} a_n = C$$
 ,讨论级数 $\sum_{n=1}^{\infty} \left(a_n - C \right)$ 的敛散性.