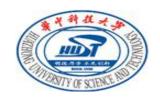
# 导数与偏导数







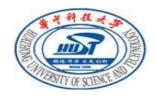
一元函数导数: 
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

左导数: 
$$f'_{-}(x_0) = \lim_{x \to x_0^{-}} \frac{f(x) - f(x_0)}{x - x_0}$$

右导数: 
$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

# 高阶导数





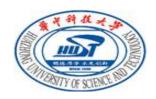
# 多元函数偏导数

全微分

方向导数,梯度



## 基本计算



- 1. 导数的四则运算
- 2. 复合函数求导数或偏导数
- 3. 参数方程确定的函数求导数或偏导数
- 4. 隐函数求导数或偏导数
- 5. 相应地求高阶导数或偏导数
- 6. 求某一固定点的高阶导数
- 7. 方向导数的计算





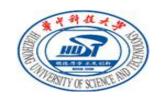
1. (7th final) 设f(t)二阶连续可导,且 $f(t) \neq 0$ ,若

$$egin{cases} x = \int_0^t f(s) \, \mathrm{d} \, s, & \mathrm{d}^2 \, y \ y = f(t), & \mathrm{d}^2 \, x^2 = \underline{\qquad}. \end{cases}$$

分析: 
$$\frac{dy}{dt} = f'(t)$$
  $\frac{dx}{dt} = f(t)$   $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{f(t)}$ 

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}} = \frac{\frac{f''(t)f(t) - \left(f'(t)\right)^2}{f^2(t)}}{f(t)} = \frac{f''(t)f(t) - \left(f'(t)\right)^2}{f^3(t)}$$





$$2.(6^{th})$$
 设 $y = y(x)$ 由 $x = \int_1^{y-x} \sin^2\left(\frac{\pi t}{4}\right) dt$ 所确定,则 
$$\left.\frac{dy}{dx}\right|_{x=0} = \underline{\qquad}.$$

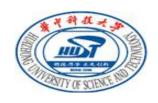
分析: 隐函数求导数

$$1 = \sin^2\left(\frac{\pi(y-x)}{4}\right)(y'-1)$$

$$y(0) = 1$$

$$y'(0) = 3$$

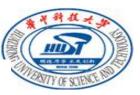




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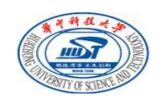
(注意考虑问题的全面性)





4. (8<sup>th</sup>)设
$$f(x) = e^x \sin 2x$$
,则 $f^{(4)}(0) =$ \_\_\_\_\_

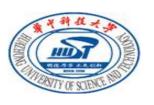




**5.(2nd)** 第三题: (15 分)设
$$y = f(x)$$
由参数方程 
$$\begin{cases} x = 2t + t^2 \\ y = \psi(t) \end{cases}$$

所确定. 且 
$$\frac{\mathrm{d}^2 y}{\mathrm{d} \, x^2} = \frac{3}{4(1+t)}$$
 , 其中 $\psi(t)$  具有二阶导数,曲线  $y = \psi(t)$  与  $y = \int_1^{t^2} e^{-u^2} \, \mathrm{d} \, u + \frac{3}{2e}$  在  $t = 1$  处相切. 求函数 $\psi(t)$  .





$$6.(8^{th})$$
 3. 设 $f(x)$ 有连续导数,且 $f(1)=2$ . 记 $z=f(e^xy^2)$ ,若

$$\frac{\partial z}{\partial x} = z$$
,  $f(x)$ 在 $x > 0$ 的表达式为\_\_\_\_\_\_

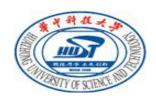




7.(2nd) (4) 设
$$f(t)$$
有二阶连续导数,  $r = \sqrt{x^2 + y^2}$ ,  $g(x,y) = f\left(\frac{1}{r}\right)$ ,

求
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$
.





8.(3<sup>rd</sup>) 第五题: (15分)设z = z(x,y)是由方程

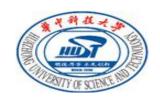
$$F\left(z+\frac{1}{x},z-\frac{1}{y}\right)=0$$

确定的隐函数,且具有连续的二阶偏导数,求证:

$$x^{2} \frac{\partial z}{\partial x} - y^{2} \frac{\partial z}{\partial y} = 1,$$

$$x^{3} \frac{\partial^{2} z}{\partial x^{2}} + xy (x - y) \frac{\partial^{2} z}{\partial x \partial y} - y^{3} \frac{\partial^{2} z}{\partial y^{2}} + 2 = 0.$$



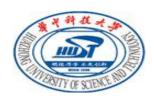


9.(4<sup>th</sup> ) 3. 已知函数 
$$z=u(x,y)e^{ax+by}$$
,且 $\dfrac{\partial^2 u}{\partial x\partial y}=0$ ,确定常数

a,b,使函数z=z(x,y)满足方程

$$rac{\partial^2 z}{\partial x \partial y} - rac{\partial z}{\partial x} - rac{\partial z}{\partial y} + z = 0.$$





$$10.(9^{ ext{th}})$$
 3.设 $w=fig(u,vig)$ 具有二阶连续偏导数,且

$$u = x - cy, v = x + cy,$$





11.(3rd final) 3. 设函数 f(x,y) 有二阶连续偏导数,满足  $f_y \neq 0$  且

$$f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx} = 0$$
 ,

$$y=yig(x,zig)$$
是由方程 $z=f(x,y)$ 所确定的函数,求 $rac{\partial^2 y}{\partial x^2}$ .



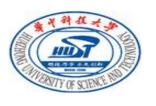


12.(4th final) 2、设f(u,v)具有连续偏导数,且满足

$$f_u(u,v) + f_v(u,v) = uv,$$

求 $y(x) = e^{-2x} f(x,x)$ 所满足的一阶微分方程,并求其通解.

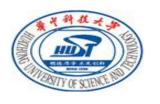




13.(10<sup>th</sup> final) (4) 设函数 z = z(x,y) 由方程 F(x-y,z) = 0 确定,其中

$$F(u,v)$$
具有连续二阶偏导数,则 $\dfrac{\partial^2 z}{\partial x \partial y}=$ \_\_\_\_\_\_





14.(5<sup>th</sup> final) 3. 设
$$F(x,y,z)$$
, $G(x,y,z)$ 有连续偏导数, $\frac{\partial (F,G)}{\partial (x,z)} \neq 0$ ,

曲线 
$$\begin{cases} F\left(x,y,z\right) = 0, \\ G\left(x,y,z\right) = 0 \end{cases}$$
 过点  $P_0\left(x_0,y_0,z_0\right)$ . 记  $\Gamma$  在  $xOy$  面上

的投影曲线为S. 求S 上过点 $\left(x_{0},y_{0}\right)$ 的切线方程.





15.(6<sup>th</sup> final) 二、(本题 12 分) 设 $\vec{l}_j$ ,  $j=1,2,\cdots,n$  是平面上点 $P_n$ 处的

 $n \geq 2$ 各方向向量,相邻两个向量之间的夹角为 $\dfrac{2\pi}{n}$ . 若函数

$$fig(x,yig)$$
在点 $P_0$ 有连续偏导,证明:  $\sum_{j=1}^nrac{\partial fig(P_0ig)}{\partial ec{l}_i}=0.$ 





16. 求证:在实轴上不存在可微函数f(x),使其满足

$$f(f(x)) = -x^3 + x^2 + 1$$

证明: 若存在这样的函数f(x)。则有

$$f(-x^3 + x^2 + 1) = f(f(f(x))) = -f^3(x) + f^2(x) + 1$$

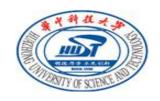
取x = 1,有

$$f(1) = -f^3(1) + f^2(1) + 1$$

易得f(1) = 1。

对
$$f(f(x)) = -x^3 + x^2 + 1$$
 两边求导





$$f'(f(x))f'(x) = -3x^2 + 2x$$

令x=1,则有

$$f'(f(1))f'(1) = f'(1)f'(1) = -1$$

矛盾!证毕。





17. 设
$$y = 1/\sqrt{1-x^2}$$
 arcsin  $x$ .

(1) 证明: 
$$(1-x^2)y'-xy-1=0$$
;

(2) 证明: 
$$(1-x^2)y^{(n+1)} - (2n+1)xy^{(n)} - n^2y^{(n-1)} = 0$$
;

(3) 求
$$y^{(n)}(0)$$
.



