MCMC and Particle Filtering

- Single-move MCMC;
- Block-move MCMC;
- Bootstrap filter;
- Auxiliary Particle Filter;
- APS + parameter estimation

Stochastic Volatility Models

Stochastic volatility models

$$
\begin{aligned}
y_t &\sim N(0, exp(\lambda_t)) \\
\lambda_t &= \alpha + \phi\lambda_{t-1} + \omega_t \qquad \omega_t \sim N(0, \sigma^2)
\end{aligned}
$$

Priors

$$
\begin{aligned}
\lambda_1 &\sim N\left(\frac{\alpha}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right) \\
\alpha &\sim N(a_\alpha, b_\alpha) \\
\phi &\sim TN(a_\phi, b_\phi) \\
\sigma^2 &\sim IG(a_\sigma, b_\sigma)
\end{aligned}
$$

Single Move MCMC (Jacquier et al. 1994)

- Sampling one state at the time:

$$
\begin{aligned}
p(\lambda_t|\lambda_{(-t)},\Theta) &= p(\lambda_t|\lambda_{t-1},\lambda_{t+1},\Theta) \\
&\propto p(y_t|\lambda_t)p(\lambda_t|\lambda_{t-1})p(\lambda_{t+1}|\lambda_t)
\end{aligned}
$$

Density does not have standard form…

… accept/reject step (or possibly MH)

Any other complication?

FFBS (Kim, Shephard, Chib 98)

- Problem: How to filter forward?
  - Solution: Approximation through mixtures

$$\begin{aligned} log(y_t^2) &= \lambda_t + log(\nu_t^2) \\ \lambda_t &= \alpha + \phi\lambda_{t-1} + \omega_t \end{aligned}$$

$$log(\nu_t^2) \sim log(\chi^2) \approx \sum_{i=1}^{7} q_i N(a_i, b_i)$$

FFBS (Kim, Shephard, Chib 98)

- Sample the indicator variable

$$P(k_t = j | log(y_t^2), \lambda_t, \Theta) \propto q_j N(log(y_t^2) | a_j + \lambda_t, b_j)$$

- Forward-Filtering Backward Sampling (as usual).

$$p(\lambda_1, \dots, \lambda_T | D_T, \Theta) = p(\lambda_T | D_T, \Theta) \prod_{t=1}^{T-1} p(\lambda_t | \lambda_{t+1}, D_t, \Theta)$$

- Details in notes from STA214

Particle Filtering

- Observational model

$$p(y_t|x_t, \Theta)$$

- Markov evolution model

$$p(x_t|x_{t-1}, \Theta)$$

- Goal: sequentially update posteriors

$$\cdots \longrightarrow p(x_t, \Theta|D_t) \longrightarrow p(x_{t+1}, \Theta|D_{t+1}) \longrightarrow \ldots$$

Particle Filtering

- Example

$$
\begin{aligned}
y_t &= \frac{x_t^2}{20} + \nu_t \\
x_t &= \frac{1}{2}x_{t-1} + 25\frac{x_{t-1}}{(1 + x_{t-1}^2)} + 8cos(1.2t) + \omega_t
\end{aligned}
$$

Particle Filtering

- Prior

$$p(x_t|D_{t-1},\Theta) = \int p(x_{t-1}|D_{t-1},\Theta)p(x_t|x_{t-1},\Theta)dx_{t-1}$$

- Prediction

$$p(y_t|D_{t-1},\Theta) = \int p(y_t|x_t,D_{t-1},\Theta)p(x_t|D_{t-1},\Theta)dx_t$$

- Update

$$p(x_t|D_t,\Theta) \propto p(x_t|D_{t-1},\Theta)p(y_t|x_t,D_{t-1},\Theta)$$

Particle Filtering

- Possible solutions:
  - Extended Kalman-filters
  - Grid-based methods for integration
  - Piecewise linear approximations
  - Sequential importance sampling (particle filters)

Particle Filtering

- Numerical approximations based on **"particles"** and corresponding weights

$$\{x_t^{(j)} : j = 1, \ldots, N\} \qquad \{w_t^{(j)} : j = 1, \ldots, N\}$$

- Prior and posterior can be approximated by the following mixtures:

$$\hat{p}(x_{t+1}|D_t, \Theta) \;=\; \sum_{j=1}^{N} p(x_{t+1}|x_t^{(j)}, \Theta) w_t(j)$$

$$\hat{p}(x_{t+1}|D_{t+1}, \Theta) \;\propto\; p(y_{t+1}|x_{t+1}, D_t) \sum_{j=1}^{N} p(x_{t+1}|x_t^{(j)}, \Theta) w_t^{(j)}$$

Bayesian Bootstrap Filter (Gordon et al. 93)

- At time t, suppose we have a set of random samples

$$\{x_t(j) : j = 1, \ldots, N\} \sim p(x_t | D_t, \Theta)$$

- We can **evolve** the particles through the system to obtain samples from the prior

$$\{x^*_{t+1}(j) : j = 1, \ldots, N\} \sim p(x_{t+1} | D_t, \Theta)$$

Bayesian Bootstrap Filter

- Using the prior as a importance density, the set of samples…

$$\{x^*_{t+1}(j) : j = 1, \ldots, N\}$$

- …with corresponding weights…

$$q_j \propto p(y_{t+1}|x^*_{t+1}(j), D_t, \Theta)$$

- …form a weighted sample from the posterior

$$p(x_{t+1}|D_{t+1}, \Theta)$$

Bayesian Bootstrap Filter

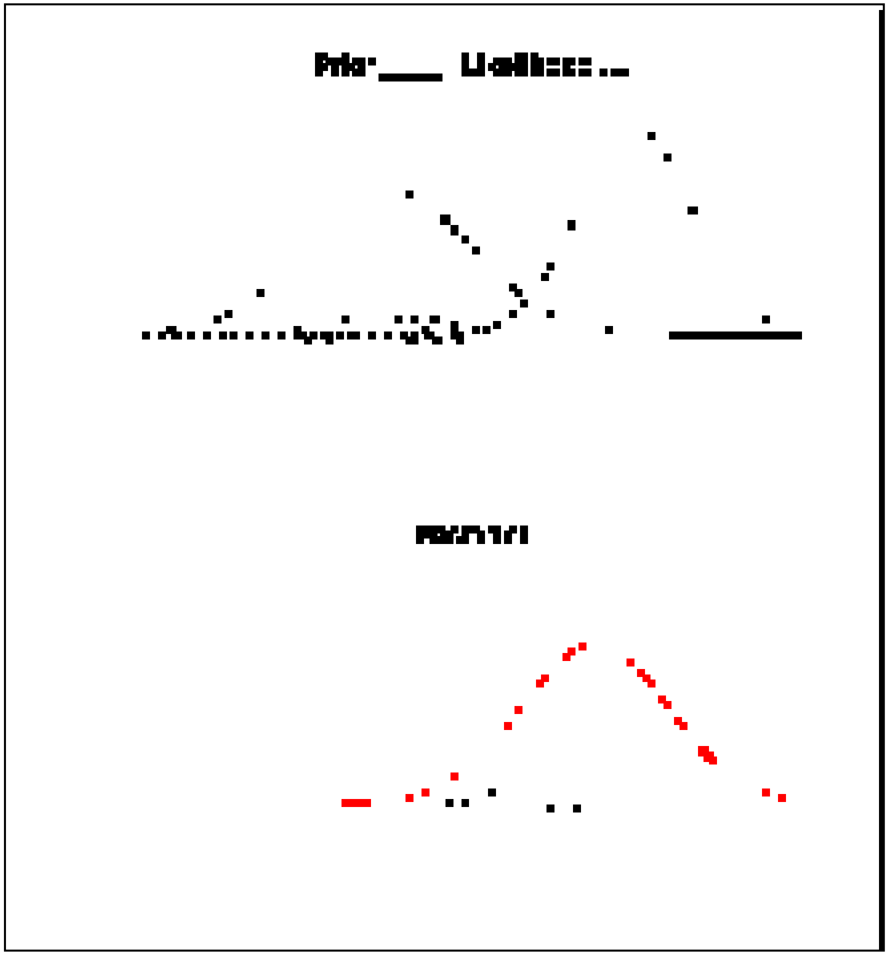
- Why? Sampling Importance Re-sampling (SIR)…

$$
\begin{aligned}
q_j &= \frac{p(x^*_{t+1}(j)|D_{t+1},\Theta)}{p(x^*_{t+1}(j)|D_t,\Theta)} \\
&\propto \frac{p(y_{t+1}|x^*_{t+1}(j),D_t,\Theta)p(x^*_{t+1}|D_t,\Theta)}{p(x^*_{t+1}|D_t,\Theta)} \\
&= p(y_{t+1}|x^*_{t+1}(j),D_t,\Theta)
\end{aligned}
$$

Key cancellation

Bayesian Bootstrap Filter

- Problem: degeneration of the filter



Auxiliary Particle Filter (Pitt & Shephard 99)

- The idea is to use the mixture approximation to facilitate computations while improving the importance function. The update step will be done by sampling form the following "auxiliary" posterior

$$p(x_{t+1}, k | D_{t+1}) \propto p(y_{t+1} | x_{t+1}, D_t) p(x_{t+1} | x_t^{(k)})$$

$$k = 1, \ldots, N$$

- Drawing from the above joint density and discarding the index k, produce a sample from the approximate posterior density. Again, SIR is used.

Auxiliary Particle Filter

- At time t, suppose we have a set of random samples and weights

$$\{x_t^{(k)}, w_t^{(k)} : k = 1, \ldots, N\}$$

- For each k, set the “estimates” and weights

$$\begin{aligned} \mu_{t+1}^{(k)} &= E(x_{t+1} | x_t^{(k)}) \\ g_{t+1}^{(k)} &\propto w_t^{(k)} p(y_{t+1} | \mu_{t+1}^{(k)}) \end{aligned}$$

- Sample the auxiliary variable j with probability given by $g_{t+1}^{(j)}$ followed by

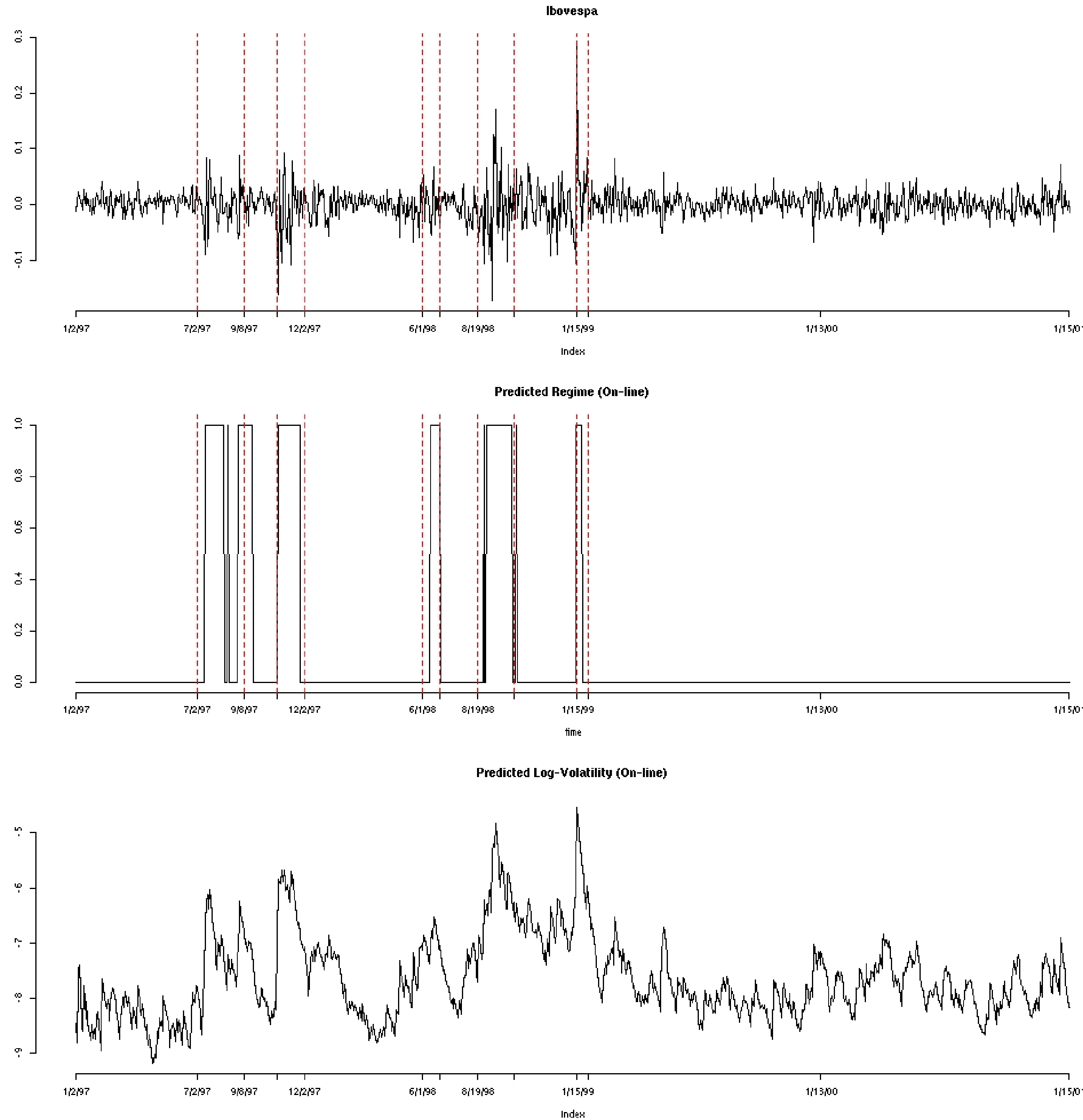
$$x_{t+1}^{(j)} \sim p(x_{t+1} | x_t^{(j)})$$

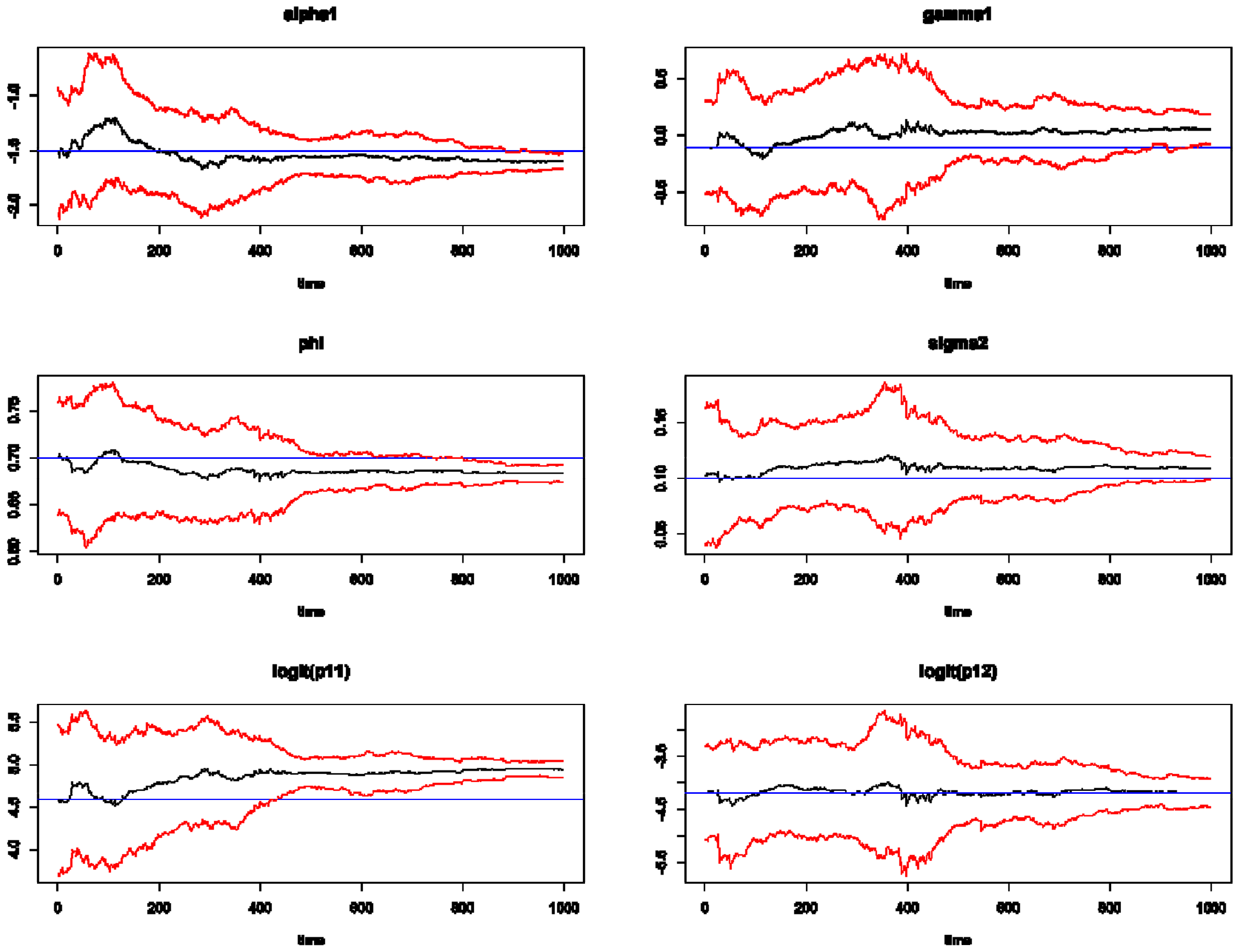
Auxiliary Particle Filter

- Compute the new weights

$$w_{t+1}^{(j)} \propto \frac{p(y_{t+1}|x_{t+1}^{(j)})}{p(y_{t+1}|\mu_{t+1}^{(j)})}$$

Back to SVM


Ibovespa
0.3
0.2
0.1
0.0
-0.1
1/2/97
7/2/97
9/8/97
12/2/97
6/1/98
8/19/98
1/15/99
1/13/00
1/15/01
Index
Predicted Regime (On-line)
1.0
0.8
0.6
0.4
0.2
0.0
1/2/97
7/2/97
9/8/97
12/2/97
6/1/98
8/19/98
1/15/99
1/13/00
1/15/01
time
Predicted Log-Volatility (On-line)
-5
-6
-7
-8
-9
1/2/97
7/2/97
9/8/97
12/2/97
6/1/98
8/19/98
1/15/99
1/13/00
1/15/01
Index


alpha1
gamma1
phi
sigma2
logit(p11)
logit(p12)
time