Example

1. Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n - 6}{3}$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

Proof. We use induction on n.

Base Case: $n = 1$

$$y_{n+1} = y_2 = \frac{2y_1 - 6}{3} = 2 > -6$$

We prove the base case.

Inductive Step: Let $k \in \mathbb{N}, k \geq n$, suppose $y_k > -6$,

$$\begin{aligned} y_{k+1} &= \frac{2y_k - 6}{3} \\ &> \frac{2(-6) - 6}{3} \\ &> \frac{-18}{3} \\ &> -6 \end{aligned}$$

This proves the inductive step.

Conclude by Induction, the result holds. □

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, ...)$ is decreasing.

Proof. We use strong induction on n.

Base Case: $n = 1$ and $n = 2$

$$\begin{aligned} n = 1 &: y_1 = 6 \\ n = 2 &: y_2 = 2 < y_1 \end{aligned}$$

the base case holds.

Induction Step: Let $k \in \mathbb{N}, k \geq 2$ and suppose $y_n < y_{n-1} \forall n \leq k$, then we have

$$\begin{aligned} y_{k+1} &= \frac{2y_k - 6}{3}, y_k = \frac{2y_{k-1} - 6}{3} \\ \implies y_{k+1} - y_k &= \frac{(2y_k - 6) - (2y_{k-1} - 6)}{3} \\ y_k < y(k-1) &\implies (2y_k - 6) < (2y_{k-1} - 6) \\ &\implies y_{k+1} < y_k \end{aligned}$$

The inductive step holds. Prove by strong induction, the statement holds.

$\square$