AESM1511 Assignment 4: Eigenfunctions and finite-difference modelling

The total number of points you can obtain is 20 and you need 12 to pass.

Assignment 4.1: Eigenfunctions of an LTI system

Let us input a signal to a system. If the output signal differs only by a constant from the input signal, then the input signal is said to be an *eigenfunction* of the system and the corresponding amplitude factor is known as the eigenvalue of the system. Note that the amplitude-scaling factor can be complex. If we consider the input signal to be complex exponential $f(t) = \exp(-st)$, with s a complex value, then the output of a linear time-invariant (LTI) system can be written as

$$p(t) = \int_{-\infty}^{\infty} g(\tau)\exp[s(t-\tau)]\mathrm{d}\tau, \tag{1}$$

where it can be observed that the part of the exponential depending on t can be taken outside of the integral, and we have

$$p(t) = G(s)\exp(st), \qquad G(s) = \int_{-\infty}^{\infty} g(\tau)\exp(-s\tau)\mathrm{d}\tau. \tag{2}$$

Hence, the complex exponential function, with complex constant s, is an eigenfunction and the s-dependent complex constant $G(s)$ is an eigenvalue of the system.

Suppose we have a periodic real signal with period T, where T is a positive non-zero value such that

$$f(t) = f(t-T) \text{ for all } t, \tag{3}$$

the fundamental period T_0 being the smallest positive value of T for which the equation is valid and the corresponding fundamental angular frequency is given by

$$\omega_0 = 2\pi/T_0. \tag{4}$$

The unit amplitude complex exponential $\exp(\mathrm{i}\omega_0 t)$ is the basis of a set of harmonic functions given by

$$\Phi_n(t) = \exp(\mathrm{i}n\omega_0 t), \qquad n = 0, \pm1, \pm2, \cdots. \tag{5}$$

With this function in mind, we can think of linear combinations of these harmonically related functions that are all periodic with fundamental period T_0:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n \exp(\mathrm{i}n\omega_0 t). \tag{6}$$

The first term, with $n = 0$, is known as the *DC*, or zero-frequency or constant, term. The two terms with $n = -1$ and $n = +1$ have the period equal to T_0 and are known as the fundamental or first harmonic components, while in general all other values of n lead to periods equal to T_0/n and are known as the $n-$th harmonic components of the signal $f(t)$. Looking at the above equation, we can see that a periodic signal can be represented exactly by a series of harmonic signals. Such a representation is known as a Fourier series representation. If we know the periodic signal (the harmonic signals are known), we can calculate the coefficients a_n as

$$a_n = \frac{1}{T_0}\int_{T_0} f(t)\exp(-\mathrm{i}n\omega_0 t)\mathrm{d}t. \tag{7}$$

We can see from this expression that the following properties hold for the coefficients:

$$a_n = a^*_{-n} \qquad \Im\{a_0\} = 0. \tag{8}$$

Hence, we can represent a periodic time function as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\{a_n\exp(in\omega_0 t) + a^*_n\exp(-in\omega_0 t)\} = a_0 + 2\sum_{n=1}^{\infty}\Re\{a_n\exp(\mathrm{i}n\omega_0 t)\}. \tag{9}$$

There are two real forms to represent a real periodic function in terms of real coefficients. First, let us write the complex coefficients as $a_n = b_n + \mathrm{i}c_n$ with both b_n ad c_n real. Then, using Euler's formula ($e^{ix} = cos(x) + i * sin(x)$), we find

$$f(t) = a_0 + 2\sum_{n=1}^{\infty}\{b_n\cos(n\omega_0 t) - c_n\sin(n\omega_0 t)\}. \tag{10}$$

If we now write $a_n = A_n\exp(-\mathrm{i}\theta_n)$, with both the amplitude A_n and the phase θ_n real, we obtain

$$f(t) = a_0 + 2\sum_{n=1}^{\infty} A_n\cos(n\omega_0 t + \theta_n). \tag{11}$$

In Fourier's time, other famous scientists, like Euler and Lagrange, did not believe that trigonometric series were useful. The paper Fourier wrote about this subject in 1807 was reviewed by four scientists. One of them was Lagrange, who insisted to reject the paper, and in fact the paper was never published. In 1822, Fourier published his book "Théorie analytique de la chaleur" (The analytical theory of heat), and seven years later it was Dirichlet who proved the precise conditions under which periodic signals can be represented by a Fourier series. The relevance and importance of Fourier series analysis in signal analysis is huge and it has become the roots of many disciplines in science, engineering, and mathematics.

If we feed the signal $f(t)$ into an LTI system, the output will have the same period as the input signal, and since the exponential function is an eigenfunction of the system with impulse-response function g(t), the output can be written directly as

$$p(t) = \sum_{n=-\infty}^{\infty} a_n G(n\omega_0)\exp(\mathrm{i}n\omega_0 t), \tag{12}$$

where the eigenvalues are obviously

$$G(n\omega_0) = \int_{-\infty}^{\infty} g(\tau) \exp(-\mathrm{i}n\omega_0\tau) \mathrm{d}\tau. \tag{13}$$

These so-called Fourier series representations exist if the series converges. This is subject to three conditions, termed Dirichlet conditions:

- $f(t)$ must be absolutely integrable over the period T_0, which means that $\int_{T_0} |f(t)| \mathrm{d}t < \infty$, which directly implies $|a_k| < \infty$.
- In any finite time interval, $f(t)$ is of bounded variation. This means that the total number of maximums and minimums is finite during one period of the signal.
- In any finite time interval, $f(t)$ can have only a finite number of finite-jump discontinuities.

Tasks and Questions (2 points):

Let us look at the rectangular wave (box function). It represents a function that is one (or, in general, a constant non-negative number) inside a time interval and zero outside this time interval. Let this time interval be of size $2T_1$ and let this function be periodic with period T_0. **Plot the box function using ones and zeros as a function of time for 201 points from $-T_0/2 < t < T_0/2$ for $T_0 = 4T_1$ and $T_0 = 4$.** Notice that for this choice of $T_0 = 4T_1$, the box function is exactly a symmetric square wave that has the unit value for half of the period and zero for the other half of the period.

The functional description of the box function is $B(t) = \sum_{k=-\infty}^{\infty} H(t - kT_0 + T_1) - H(t - kT_0 - T_1)$, for $T_0 > 2T_1$. **Does this function satisfy the three Dirichlet conditions? Explain why.**

The Fourier coefficients a_n are obtained from

$$a_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} B(t) \exp(-\mathrm{i}n\omega_0 t) \mathrm{d}t. \tag{14}$$

We find

$$a_0 = \frac{1}{T_0} \int_{-T_1}^{T_1} \mathrm{d}t = \frac{2T_1}{T_0}, \tag{15}$$

and this is the average value of $B(t)$. The other coefficients are obtained as

$$a_n = \frac{1}{T_0} \int_{-T_1}^{T_1} \exp(-\mathrm{i}n\omega_0 t) \mathrm{d}t = \frac{2}{n\omega_0 T_0} \left[\frac{\exp(\mathrm{i}n\omega_0 T_1) - \exp(-\mathrm{i}n\omega_0 T_1)}{2\mathrm{i}} \right] = \frac{\sin(n\omega_0 T_1)}{n\pi}, \tag{16}$$

and in our example of $T_0 = 4T_1$, you can use $\omega_0 T_1 = \pi/2$. **Compute the coefficients for $n = 0, 1, 2, \cdots, 80$. Now we approximate $B(t)$ by a finite number N, because we cannot compute until "infinity", as**

$$B(t) \approx \sum_{n=-N}^{N} a_n \exp(\mathrm{i}n\omega_0 t). \tag{17}$$

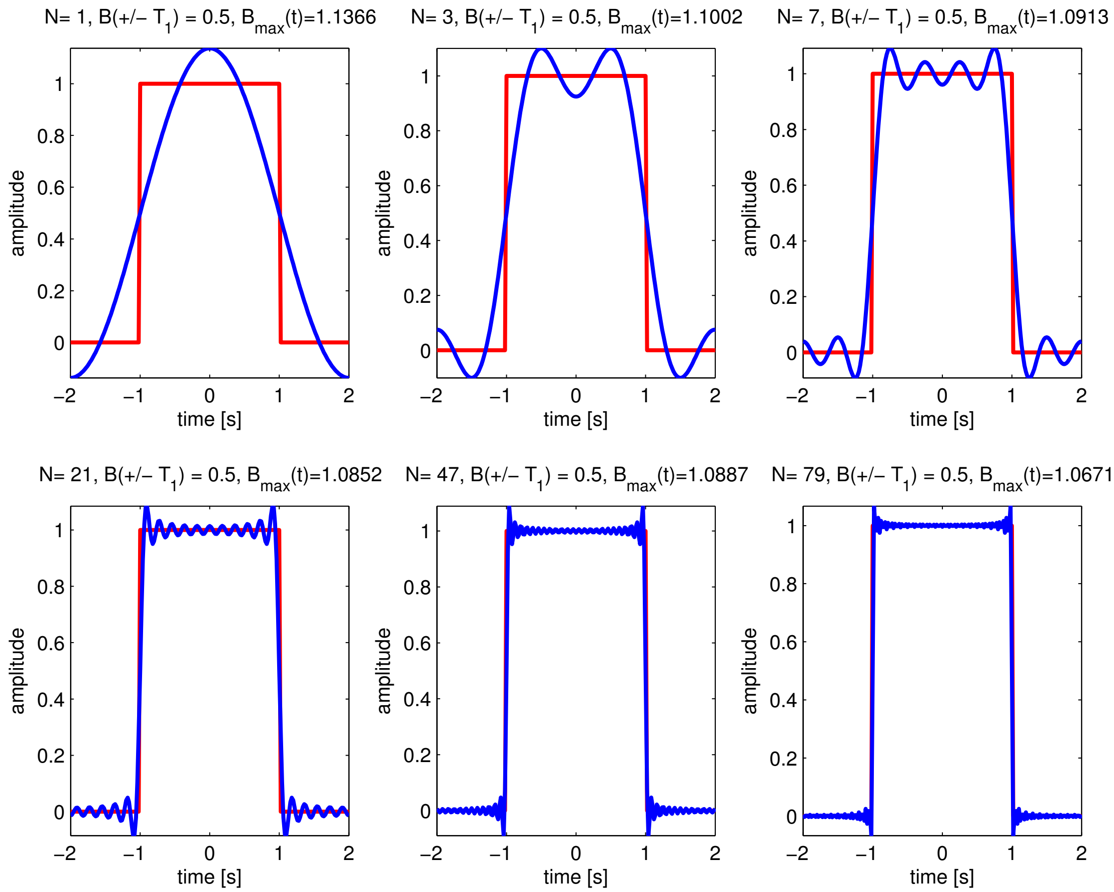

Figure 1: The first period of the square wave function plotted with its Fourier series representation with maximum end points $\{1, 3, 7, 21, 47, 79\}$.

Plot the exact box function and the approximate B(t) results for $N = 1, 3, 7, 21, 47, 79$ **for each value of** N **in a different plot, as shown in Figure 1. What is the value of your result at the locations** $t = \pm T_1$**, does it change with** N**? What is the peak value, does it change with** N**? Show these answers in the plotted figure.**

The occurrence of the ripples is known as Gibbs phenomenon and plays an important role in geophysics (and in signal theory in general). **Compute the sum of the squares of the local differences between the numerical approximate Fourier series representation** $B(t)$ **and the exact box function as a function of** N**. This sum of squared differences is called the energy in the difference. Use this notion to explain what happens to your results when** N **changes. Show this change in a figure.**

What happens when you would take $T_0 = 6T_1$ **in comparison to the case when** $T_0 = 4T_1$**? Explain this also by showing figures like for** $T_0 = 4T_1$**.**

Make a flowchart of the code (of course, before you start coding).

Assignment 4.2: Scattering from a multilayered earth – a numerical solution by finite differences

In a one-dimensional world, the electric field E_y and the magnetic field H_x can be calculated using the equations

$$\partial_z H_x(z,t) = (\sigma(z) + \varepsilon(z)\partial_t)E_y(z,t) + W(z,t), \tag{18}$$

$$\partial_z E_y(z,t) = \mu(z)\partial_t H_x(z,t), \tag{19}$$

where σ is electric conductivity (S/m), ε is electric permittivity (F/m), μ is magnetic permeability (H/m), ∂_z denotes differentiation with respect to the vertical coordinate z, and ∂_t denotes differentiation with respect to time. Let us assume that we have an arbitrary electric current source, non-zero only in the upper half space $\mathcal{D}_1; z < 0$. The domain $\mathcal{D}_2; 0 < z < d$ is the domain in which scattering can occur, and can have any finite length and contain an arbitrary heterogeneous medium. Finally, the domain $\mathcal{D}_3; z > d$ describes a homogeneous half space.

We solve the above two equations numerically. For that, we have to discretize the domain in the vertical and horizontal direction and then compute the derivaties using the definition of a partial derivative. To discretize the space and time, we divide the space and time into equal intervals:

$$z_m = m\Delta z; \tag{20}$$

$$t_n = n\Delta t, \tag{21}$$

with $\{m,n\} \in \mathbb{Z}$.

The spatial derivatives we have to take are then best evaluated in between the discrete points in z at $(m+1/2)\Delta z$ and central differences are taken:

$$\partial_z H_x(z,t)\Big|_{z=m\Delta z} \approx \frac{H_x(z_{m+1/2},t) - H_x(z_{m-1/2},t)}{\Delta z}. \tag{22}$$

Similarly, we can approximate the time derivatives of the electric field as

$$\partial_t E_y(z,t)\Big|_{t=(n+1/2)\Delta t} \approx \frac{E_y(z,t_{n+1}) - E_y(z,t_n)}{\Delta t}. \tag{23}$$

We have, therefor, the first discrete equation:

$$\begin{aligned}[H_x(m+1/2,n+1/2) - H_x(m-1/2,n+1/2)] = \{&\sigma(m)E_y(m,(n+1/2))\Delta z \\ &+\varepsilon(m)[E_y(m,n+1) - E_y(m,n)]\frac{\Delta z}{\Delta t}\},\end{aligned} \tag{24}$$

where the discrete evaluation of the electric field for the conductivity is at a point where we do not want to sample the electric field. Since we have accepted an error of $\mathcal{O}[(\Delta t)^2]$, in the limit of $\Delta t \downarrow 0$, in the approximation of the time derivative associated with the electric conductivity we introduce a linear interpolation for this electric field by taking it as

$$E_y(m,(n+1/2)\Delta t) \approx [E_y(m,n+1)+E_y(m,n)]/2, \tag{25}$$

which is of the same order of accuracy. We can, therefor, write the discrete equation as

$$\begin{aligned} E_y(m,n+1) = {} & \frac{\varepsilon(m)-\sigma(m)\Delta t/2}{\varepsilon(m)+\sigma(m)\Delta t/2} E_y(m,n) + \frac{1}{\varepsilon(m)+\sigma(m)\Delta t/2}\frac{\Delta t}{\Delta z} \\ & \times [H_x(m+1/2,n+1/2) - H_x(m-1/2,n+1/2)]. \end{aligned} \tag{26}$$

In this equation, we have considered only source-free domains and we will treat the effect of the source separately. For the second Maxwell equation, we find a similar discrete expression but because there is no magnetic conductivity, it is much simpler:

$$H_x(m+1/2,n+1/2) = H_x(m+1/2,n-1/2) + \frac{\Delta t}{\mu(m+1/2)\Delta z}[E_y(m+1,n)-E_y(m,n)]. \tag{27}$$

Since we have assumed a source in the upper half space and a finite-size domain where heterogeneities exist, we can define the domain by thickness d of the part with the heterogeneities; thus, we discretize de domain of interest d in M steps such that $\Delta z = d/M$. For $z > d$ we only have a wave propagating away from the scattering domain, which is the transmitted wave. Since the conductivity is non-zero in the homogeneous half space, we make an approximation, called high-frequency approximation, for the wave equation for the electric field:

$$[\partial_z^2 - \frac{1}{c_3^2}\partial_t^2 + \sigma_3\mu_3\partial_t + \frac{\sigma_3 Z_3}{4}]E_y(z,t) = 0, \tag{28}$$

in which $Z_3 = \sqrt{\mu_3/\varepsilon_3}$ and where the term in the left-hand side without derivatives involves the approximation. The operator in the square brackets can be split in two one-way wave-equation terms as

$$[\partial_z - \frac{1}{c_3}(\partial_t - \frac{\sigma_3}{2\varepsilon_3})][\partial_z + \frac{1}{c_3}(\partial_t - \frac{\sigma_3}{2\varepsilon_3})]E_y(z,t) = 0, \tag{29}$$

and the transmitted field satisfies the one-way equation

$$[\partial_z + \frac{1}{c_3}(\partial_t - \frac{\sigma_3}{2\varepsilon_3})]E_y^t(z,t) = 0, \tag{30}$$

which is now used to make a boundary condition at $z = d$. First, we observe that both the electric and magnetic fields are continuous across $z = d$ and that the time derivative of the magnetic field must also be continuous, which leads to

$$\lim_{z\uparrow d}\partial_t H_x(z,t) = \lim_{z\downarrow d}\partial_t H_x(z,t) \Longleftrightarrow \lim_{z\uparrow d}\frac{1}{\mu_2(d)}\partial_z E_y(z,t) = \lim_{z\downarrow d}\frac{1}{\mu_3}\partial_z E_y(z,t). \tag{31}$$

Since we will compute the vertical derivative of the electric field inside the scattering domain, we use it in the one-way equation to obtain

$$\frac{\mu_3}{\mu_2(d)} \lim_{z\uparrow d} \partial_z E_y(z,t) + \frac{1}{c_3}(\partial_t - \frac{\sigma_3}{2\varepsilon_3})E_y(z,t) = 0. \tag{32}$$

We now apply the backward difference scheme in both the spatial and temporal derivative to obtain

$$\begin{aligned} &\frac{\mu_3}{\mu_2(d)\Delta z}[3E_y(M,n)/2 - 2E_y(M-1,n) + E_y(M-2,n)/2] \\ &+\frac{1}{c_3\Delta t}[3E_y(M,n)/2 - 2E_y(M,n-1) + E_y(M,n-1)/2] - \frac{1}{2}\sigma_3 Z_3 E_y(M,n) = 0. \end{aligned} \tag{33}$$

The above equation is rewritten in terms of an explicit expression of the electric field at the end point:

$$\begin{aligned} \left(\frac{\mu_3}{\mu_2(d)} + \frac{\Delta z}{c_3\Delta t} - \sigma_3 Z_3 \Delta z/3\right) E_y(M,n) = &\frac{\mu_3}{3\mu_2(d)}[4E_y(M-1,n) - E_y(M-2,n)] \\ &+\frac{\Delta z}{3c_3\Delta t}[4E_y(M,n-1) - E_y(M,n-2)]. \end{aligned} \tag{34}$$

This solves the boundary at $z = d$. Now, we must solve for the boundary at $z = 0$. The upper half space is a lossless space and the incident field, denoted $E_y^i(z,t)$, is assumed to be known and generated by the source, while the upgoing wave has an unknown amplitude and satisfies the one-way equation

$$[\partial_z - \frac{1}{c_1}\partial_t]E_y^r(z,t) = 0. \tag{35}$$

Applying the same operator on the total field, we obtain

$$[\partial_z - \frac{1}{c_1}\partial_t]E_y(z,t) = -\frac{2}{c_1}E_y^i(z,t), \tag{36}$$

since $E_y^i(z,t) = E_y^i(t - z/c_1)$. Applying the same, but now forward, difference scheme for space and time derivatives we obtain

$$\begin{aligned} &\left(\frac{\mu_1}{\mu_2(0)} + \frac{\Delta z}{c_1\Delta t}\right) E_y(0,n) = \frac{\mu_1}{3\mu_2(0)}[4E_y(1,n) - E_y(2,n)] \\ &\quad+\frac{\Delta z}{3c_1\Delta t}[4E_y(0,n-1) - E_y(0,n-2)] + \frac{2\Delta z}{3c_1}E_y^{i,\prime}(0,n), \end{aligned} \tag{37}$$

where $E_y^{i,\prime}(0,n)$ denotes the time derivative of the incident field, provided it is known in closed form, otherwise a finite-difference value can be used.

The derivative of the incident field is given by

$$E_y^{i,\prime}(0,t) = -\frac{Z_0\partial_t W(t - |z^S|/c_1)}{2}, \tag{38}$$

with $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ and the wavelet itself is a so-called causal Ricker wavelet given by

$$W(t) = (1 - 4\pi^2(t/\tau - 1)^2)\exp[-2\pi^2(t/\tau - 1)^2], \tag{39}$$

where $\tau = \sqrt{2}/f_c$ and f_c denotes the centre frequency of the wavelet. It can be seen that the causal Ricker wavelet is defined as the negative second-order time derivative of a shifted Gaussian pulse. The maximum frequency that you should consider is $f_{\text{max}} = 4f_c$ and the time step should be $\Delta t \leq 1/(2f_{\text{max}})$.

The corresponding stability criterion is to use a step size $\Delta z \geq c_{\text{max}}\Delta t$, where c_{max} denotes the maximum velocity in the scattering domain and Δt should be such that the frequency spectrum is well-sampled.

Necessary numbers are the free-space values for velocity c_0, magnetic permeability μ_0, and electric permittivity ε_0 given by

$$c_0 = 299792458\text{m/s}, \mu_0 = 4\pi \times 10^{-7}\text{H/m}, \varepsilon_0 = 1/(\mu_0 c_0^2),$$

and $\varepsilon_i = \varepsilon_{r,i}\varepsilon_0$, where ε_i is the electric permittivity of layer i and $\varepsilon_{r,i}$ is the relative permittivity of layer i.

The model you should use is

	ε_r	$\sigma(S/m)$	μ_r	$d(m)$
upper half space	1	0	1	0.24;
layer 1	6	1e-3	1	0.10;
layer 2	2	1e-4	1	0.003;
layer 3	16	1e-2	1	0.05;
layer 4	6	1e-3	1	0.15;
lower half space	9	1e-3	1	0;

,

where it is observed that the thickness, d, of the lower half space is irrelevant and put to zero here, while the thickness of the upper half space actually denotes the location of the source $|z^S|$ above the first boundary. Notice that the relative values for ε and μ are given and the actual values are obtained by taking $\varepsilon_m = \varepsilon_{r;m}\varepsilon_0$ and $\mu_m = \mu_{r;m}\mu_0$ in each layer m.

Tasks and Questions

For all questions, where it is relevant, you should use three centre frequencies – 500 MHz, 1 GHz, and 2 GHz – for the wavelet defined in equation (39).

1. (2 points) Make a flowchart of the code.

2. (2 points) Layers 1 to, and including, 4 are part of the domain $\mathcal{D}_2$ that you must discretize. Your time step has a maximum that is fixed by the centre-frequency value, the total time window for which you should compute the electric and magnetic fields is $t_w = 12$ ns. From the model values in the above table, compute the maximum velocity in the grid and find a maximum step size in z. Make sure, by writing an explicit check in the code, this step is smaller than the smallest layer thickness in your model

otherwise the thin layer is not sampled and the result will be wrong. Make a discrete model in space and time and define the values of ε, σ, and μ for each grid point and store them in arrays.

3. (2 points) Precompute the incident electric field for all discrete times.

4. (6 points) Based on equations (26)-(27) and the boundary conditions (34) and (37), and using the above model parameters, make a code and compute three accurate time-domain solutions, using the wavelet with each of the three centre frequencies defined above, for the electric field and magnetic field everywhere in $\mathcal{D}_2$. Use explicit for-loops for points in time and points in space.

5. (2 points) Write the code without an explicit loop over points in space.

6. (2 points) Every time you halve the step size in space the solution should improve. Halve the step size two times. Check whether you need to halve simultaneously the time step as well. Calculate the difference between the original values and the values with the halved step and the difference between the values with the halved step and with the quartered step. To calculate the difference, use the energy in the differences as described in Assignment 4.1 (normalize this value by the number of points used in the calculation, i.e., time samples times depth samples). If the difference between two discretization levels should be less than 1%, how many points per smallest wavelength must you use?

7. (2 points) Plot in one figure the electric and magnetic fields for all points in $\mathcal{D}_2$ (and only for those points) and for all time points as a 2D image plot with axis labels and annotations. Do this for all frequencies. Indicate in the title the frequency and whether the figure shows the electric or magnetic field using font size 16 and font name Arial, add a color bar and scale the color axis to 25% of maximum and minimum of the amplitude values. An example is given in Figure 2.

Submitting your results

Submit your complete solution before 23.59 on October 29, 2020, in Brightspace under Assignments at the top of the courses webpage. Submit the results in the form of a zip file containing an executable .m file (or files if applicable) and a file (or files if applicable) with the flowchart. The name of the zip file should be

AESM1511-2020-matlab-a4-surname-studentnumber.zip

or

AESM1511-2020-matlab-a4-surname1-surname2-surname3.zip

if you are working in a group. Working in groups is encouraged. Still, working in groups of more than 3-4 people is not effective and not encouraged.

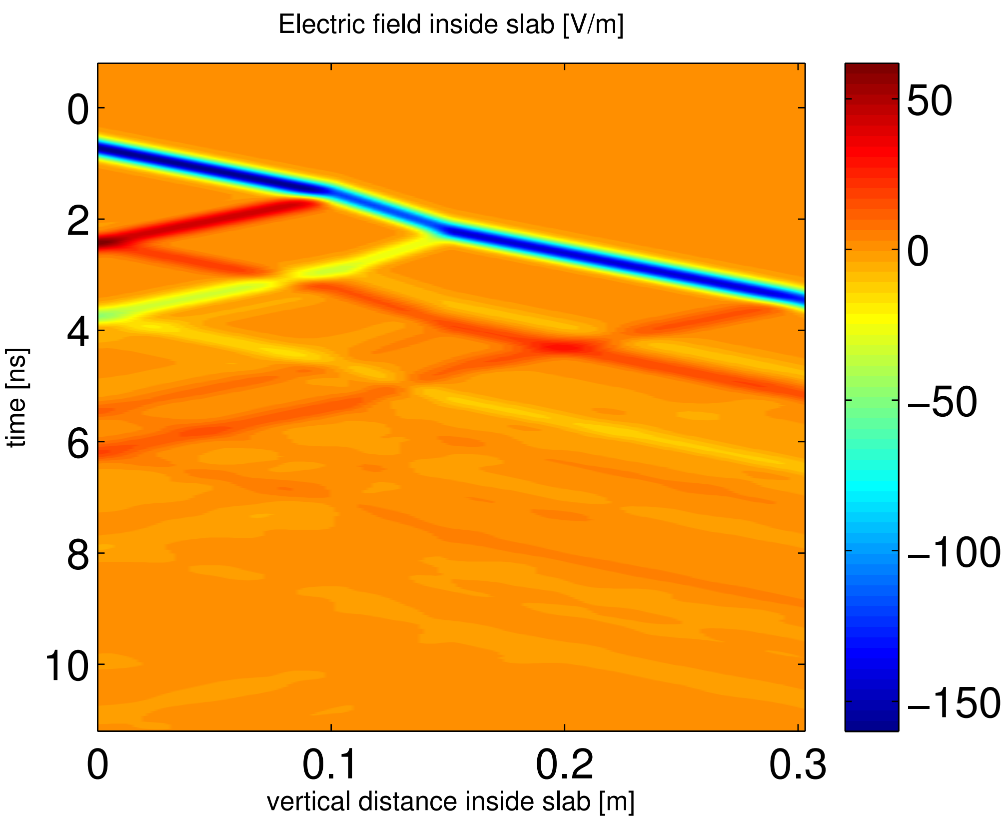

Figure 2: The electric field inside the layered medium as a function of time.