Ackermann function is not primitive recursive*

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In this entry, we show that the Ackermann function $A(x,y)$, given by

$$A(0,y)=y+1, \qquad A(x+1,0)=A(x,1), \qquad A(x+1,y+1)=A(x,A(x+1,y))$$

is not primitive recursive. We will utilize the properties of A listed in this entry.

The key to showing that A is not primitive recursive, is to find a properties shared by all primitive recursive functions, but not by A. One such property is in showing that A in some way "grows" faster than any primitive recursive function. This is formalized by the notion of "majorization", which is explained here.

Proposition 1. *Let $\mathcal{A}$ be the set of all functions majorized by A. Then $\mathcal{PR} \subseteq \mathcal{A}$.*

Proof. We break this up into three parts: show all initial functions are in $\mathcal{A}$, show $\mathcal{A}$ is closed under functional composition, and show $\mathcal{A}$ is closed under primitive recursion. The proof is completed by realizing that $\mathcal{PR}$ is the smallest set satisfying the three conditions.

In the proofs below, $\boldsymbol{x}$ denotes some tuple of non-negative integers $(x_1,\dots,x_n)$ for some n, and $x=\max\{x_1,\dots,x_n\}$. Likewise for $\boldsymbol{y}$ and y.

1. We show that the zero function, the successor function, and the projection functions are in $\mathcal{A}$.

    - $z(n)=0<n+1=A(0,n)$, so $z\in\mathcal{A}$.
    - $s(n)=n+1<n+2=A(1,n)$, so $s\in\mathcal{A}$.
    - $p_m^k(x_1,\dots,x_k)=x_m\le x<x+1=A(0,x)$, so $p_m^k\in\mathcal{A}$.

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2. Next, suppose $g_1, \ldots, g_m$ are k-ary, and h is m-ary, and that each g_i, and h are in $\mathcal{A}$. This means that $g_i(\boldsymbol{x}) < A(r_i, x)$, and $h(\boldsymbol{y}) < A(s, y)$. Let

   $$f = h(g_1, \ldots, g_m), \quad \text{and} \quad g_j(\boldsymbol{x}) = \max\{g_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.$$

   Then $f(\boldsymbol{x}) < A(s, g_j(\boldsymbol{x})) < A(s, A(r_j, x)) < A(s + r_j + 2, x)$, showing that $f \in \mathcal{A}$.

3. Finally, suppose g is k-ary and h is $(k+2)$-ary, and that $g, h \in \mathcal{A}$. This means that $g(\boldsymbol{x}) < A(r, x)$ and $h(\boldsymbol{y}) < A(s, y)$. We want to show that f, defined by primitive recursion via functions g and h, is in $\mathcal{A}$.

   We first prove the following claim:

   $$f(\boldsymbol{x}, n) < A(q, n + x), \quad \text{for some } q \text{ not depending on } x \text{ and } n.$$

   Pick $q = 1 + \max\{r, s\}$, and induct on n. First, $f(\boldsymbol{x}, 0) = g(\boldsymbol{x}) < A(r, x) < A(q, x)$. Next, suppose $f(\boldsymbol{x}, n) < A(q, n+x)$. Then $f(\boldsymbol{x}, n+1) = h(\boldsymbol{x}, n, f(\boldsymbol{x}, n)) < A(s, z)$, where $z = \max\{x, n, f(\boldsymbol{x}, n)\}$. By the induction hypothesis, together with the fact that $\max\{x, n\} \leq n+x < A(q, n+x)$, we see that $z < A(q, n+x)$. Thus, $f(\boldsymbol{x}, n+1) < A(s, z) < A(s, A(q, n+x)) \leq A(q-1, A(q, n+x)) = A(q, n+1+x)$, proving the claim.

   To finish the proof, let $z = \max\{x, y\}$. Then, by the claim, $f(\boldsymbol{x}, y) < A(q, x+y) \leq A(q, 2z) < A(q, 2z+3) = A(q, A(2, z)) = A(q+4, z)$, showing that $f \in \mathcal{A}$.

Since $\mathcal{PR}$ is by definition the smallest set containing the initial functions, and closed under composition and primitive recursion, $\mathcal{PR} \subseteq \mathcal{A}$. $\square$

As a corollary, we have

Corollary 1. *The Ackermann function A is not primitive recursive.*

Proof. Otherwise, $A \in \mathcal{A}$, which is impossible. $\square$