Lecture Notes: Dynamics on the Circle

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December 2, 2023

1 Math Review: Modular Arithmetic, Circle Geometry, and Trig Functions

1.1 Modular Arithmetic

Our class begins at 10 AM every Saturday. Even though each class takes place at a different point in time (measured, say, from the beginning of life on Earth), the time elapsed between each class is the same from week to week. This insight is the main idea of modular arithmetic.

We start the formal discussion with a definition.

Definition 1.1. *Let a, p, q be integers. We say that*

$$p = q \mod a$$

if there exists an integer m such that

$$p = ma + q.$$

In other words, two numbers p, q are equal mod a if q is the remainder resulting from the division p/a.

Example 1.2.

a) $p = 0 \mod p$ *(just take $m = 1$).*

b) $1 = 3 \mod 2$ *(again, take $m = 1$).*

c) $16 = 1 \mod 3$ *(this time, take $m = 5$).*

Note that, if you are asked to compute $p \mod a$, this means you should return the smallest integer q such that $q = p \mod a$.

Additionally, we can define equivalence mod 1 for *real* numbers, not just integers:

Definition 1.3. *Let x, y be real numbers. We say*

$$x = y \mod 1$$

if $x - y$ is an integer.

Example 1.4.

a) $1 = 0 \mod 1$.

b) $p = 0 \mod 1$ *for any integer p.*

c) $1.51 = 0.51 \mod 1$.

d) $9000.001 = 0.001 \mod 1$

e) $-1.51 = 0.49 \mod 1$.

So, two positive real numbers are equivalent mod 1 if and only if they have the same decimal part. If the numbers are allowed to have different signs, things are a little more complicated. Additionally, note that the class of all real numbers that are equivalent mod 1 to some fixed number $x \in [0, 1)$ can all be "represented" by x. So, it's useful to simplify $y \mod 1$ by finding $x \in [0, 1)$ such that $x = y \mod 1$.

We can generalize the above construction even more:

Definition 1.5. *Fix an integer a. Let x, y be real numbers. We say*

$$x = y \mod a$$

if $x - y$ is an integer that is divisible by a.

By taking $a = 12$ or $a = 24$ above, we obtain "clock arithmetic"!

Exercise 1.1. Compute the following:

a) $27 \mod 24$, and

b) $2^n \mod 3$ for any natural number n.

Exercise 1.2. Compute the following:

a) $1.1 \mod 1$,

b) $-0.1 \mod 1$,

c) $11.55 \mod 1$,

d) $34.17 \mod 24$ (also find approximately the time of day this corresponds to).

Exercise 1.3. Suppose $a, b \in \mathbb{R}$ with $a \neq 0$. Additionally, suppose we have

$$ab = 0 \mod 1.$$

Is it true that $b = 0 \mod 1$?

1.2 Circle Geometry

We have already discussed how the circle can be thought of as an interval with its endpoints glued together. We even made a craft in the first class to illustrate this! We can think of each point on the interval as parameterizing our angular displacement along the circle, measured from some fixed reference point. A natural question arises: what interval do we pick? Here are some common choices:

1. if we measure angles in degrees, the interval should be $[0, 360]$;
2. if we measure angles in radians, the interval should be $[0, 2\pi]$;
3. if we choose length units so that the circumference of our circle is 1, then the interval should be $[0.1]$;
4. if we want points on the circle to represent wall-clock times, then the interval should be $[0, 12]$ or $[0, 24]$ depending on if you're using a $12-$ or $24-$hour clock.

In these notes, we'll adhere to the third convention. That is, for our purposes,

$$\mathbb{S}^1 = [0,1]\,/\{0=1\}\,, \tag{1.1}$$

where "/..." means "enforce the equivalence ... " or "glue such that...".

We'll now argue why we're allowed to write

$$\mathbb{S}^1 = \mathbb{R} \mod 1. \tag{1.2}$$

This correspondence is of great importance for circle dynamics. Pick any real number y. Remember that the class of all real numbers that are equivalent to y mod 1 can all be represented by $x \in [0, 1)$ such that $x = y \mod 1$. Using our circle geometry convention, this point x corresponds to a particular point on the circle. By the same token, y corresponds to a point on the circle, namely the same point that x corresponds to. Reversing this logic, any point $x \in [0, 1)$ on the circle can be represented by any of the real numbers that are equivalent to x mod 1. This reflects that essentially any global coordinatization of the circle has some arbitrariness: going around the circle once, twice, or n times always leaves you where you ended off!

1.3 Basics on Trig Functions

In basic trigonometry, you learned how to compute the sine and cosine of a given angle θ using figure 1:

1. draw a right triangle with a hypotenuse of length 1, with θ being one of the interior acute angles of the triangle;

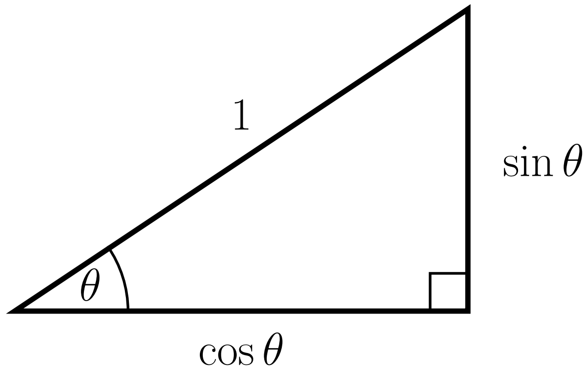

Figure 1: A right triangle that illustrates the definitions of sine and cosine.

2. the sine of θ, denoted $\sin\theta$, is given by the length of the side of the triangle *opposite* to θ;

3. the cosine of θ, denoted $\cos\theta$, is given by the length of the side of the triangle *adjacent* to θ.

Easy! Now, notice how $\sin\theta$ and $\cos\theta$ are functions of θ. We plot the graphs of these functions in figure 2. Something interesting emerges from these plots: owing to the arbitrariness of angles (that is, θ and $\theta + 2\pi$ should have the same sine and the same cosine), the graphs of $\sin\theta$ and $\cos\theta$ are **periodic**! To find the **periods** of these functions, we look at the distance between successive crests (highest points on the curve). From the graph, both $\sin\theta$ and $\cos\theta$ are 2π-periodic:

$$\sin(\theta) = \sin(\theta + 2\pi), \quad \cos(\theta) = \cos(\theta + 2\pi) \quad \forall\ \theta \in \mathbb{R}.$$

This is exactly what we expect in light of our earlier discussion. Additionally, notice how the graphs imply that both sine and cosine are always bounded above by $+1$ and below by -1. Geometrically, this is because the shorter legs of a right triangle cannot have lengths greater than that of the hypotenuse.

Exercise 1.4. Sketch the graphs of the functions below for $x \in [0, 3]$, and identify the period of each function:

a) $f_1(x) = \sin(2\pi x)$,

b) $f_2(x) = \sin(3\pi x)$, and

c) $f_3(x) = \sin(4\pi x)$.

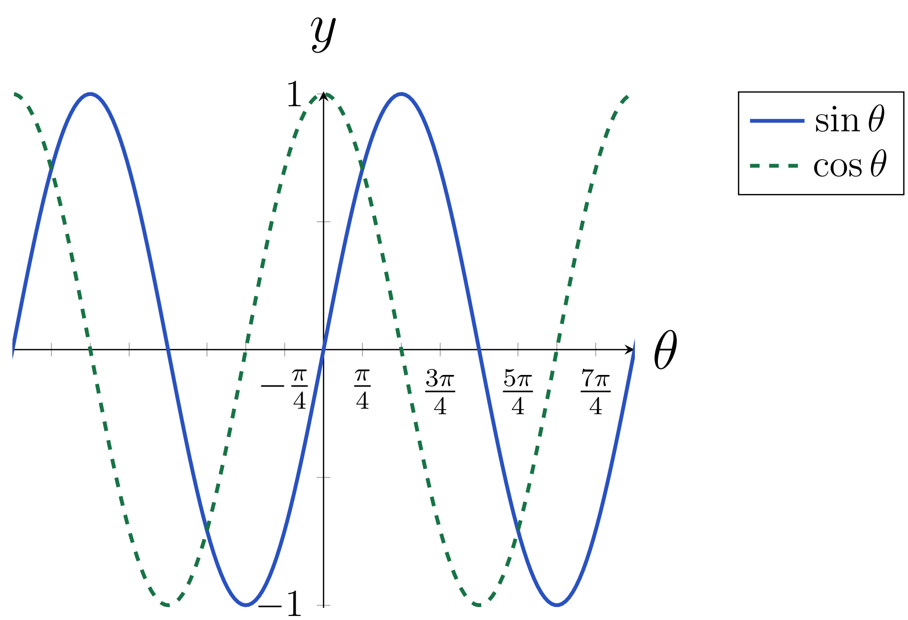

Figure 2: Plot of $\sin x$ and $\cos x$ for $x \in [-2\pi, 2\pi]$. Both functions have period 2π: look at the displacements between successive crests (or successive troughs).

2 First Steps with Circle Dynamics

2.1 Basic Ideas

Most interesting dynamical systems on the circle begin life as dynamical systems on $\mathbb{R}$. Consider a general dynamical system on the line, $\mathcal{S} = \{\mathbb{R}, f\}$:

$$T_{n+1} = f\left(T_n\right). \tag{2.1}$$

How does this system "descend" to a system on the circle? Since our model for the circle is $[0, 1]$ with endpoints identified, we can go from the real line to the circle simply by considering the dynamics mod 1 (or, equivalently, by taking (1.2) seriously!). If we define

$$\varphi_n = T_n \mod 1,$$

then we have

$$\varphi_{n+1} = f\left(T_n\right) \mod 1.$$

There is a small problem here: the right-hand side depends on T_n, *not* φ_n, so it doesn't look like a dynamical system on the circle. Fortunately, if we define a circle map $\widetilde{f}\colon \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\widetilde{f}\,(T \bmod 1) = f(T) \bmod 1 \tag{2.2}$$

then we can write

$$\varphi_{n+1} = \widetilde{f}\left(\varphi_n\right).$$

Now we've got a dynamical system on the circle!

Example 2.1. *Consider the simple dynamical system*

$$T_{n+1} = T_n.$$

So, $f(T) = T$ here. We then have

$$\widetilde{f}(\varphi) = \varphi$$

as well, so the associated circle system is

$$\varphi_{n+1} = \varphi_n.$$

There is one problem with the above "descent via mod 1" construction: $\widetilde{f}$ may not be uniquely defined by (2.2)! The following example illustrates when this might happen.

Example 2.2. *Consider the dynamical system*

$$T_{n+1} = T_n^3, \quad T_n \in \mathbb{R}.$$

So, $f(T) = T^3$. If we try to define an associated circle map $\widetilde{f}$ via (2.2)*, we get*

$$\widetilde{f}\,(T \bmod 1) = T^3 \bmod 1. \tag{2.3}$$

Now, $-0.1 = 0.9 \mod 1$. So, we <u>should</u> have

$$\widetilde{f}\,(-0.1 \bmod 1) = \widetilde{f}\,(0.9 \bmod 1)\,.$$

However, (2.3) *gives*

$$\begin{aligned}\widetilde{f}\,(-0.1 \bmod 1) &= (-0.1)^3 \mod 1 = 0.999,\\ \widetilde{f}\,(0.9 \bmod 1) &= (0.9)^3 \mod 1 = 0.729.\end{aligned}$$

So, $\widetilde{f}$ does not make sense as a function: it isn't single-valued.

Therefore, only certain dynamical systems on $\mathbb{R}$ descend to dynamical systems on $\mathbb{S}^1$. If $f(T)$ is such that (2.2) uniquely defines a map $\widetilde{f}$ on $\mathbb{S}^1 \to \mathbb{S}^1$, we say that $\widetilde{f}$ (and therefore the associated circle dynamics) is **well-defined**.

Remark. *Note that $\widetilde{f}$, when it exists, is defined so that the diagram below* ***commutes****:*

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "\mathrm{mod}\ 1"'] & \mathbb{R} \arrow[d, "\mathrm{mod}\ 1"] \\ \mathbb{S}^1 \arrow[r, "\widetilde{f}"] & \mathbb{S}^1 \end{tikzcd}$$

This means that we can start in the top-left corner of the square, and no matter what path we take we end with the same result in the bottom-right corner as long as we follow the arrows.

Exercise 2.1. Consider the state space $\mathbb{R}$. Does the dynamical system

$$T_{n+1} = T_n^2, \quad T_n \in \mathbb{R}$$

descend to a well-defined dynamical system on the circle?

Exercise 2.2. Suppose $T_0 \in \mathbb{N}$ is not necessarily an equilibrium for

$$T_{n+1} = f(T_n),$$

but $\varphi_0 = T_0 \mod 1$ is an equilibrium for

$$\varphi_{n+1} = f(\varphi_n) \mod 1.$$

This means $T_n - T_0$ must always be an integer. Does this always imply that

$$T_n - T_0 = n(T_1 - T_0)?$$

2.2 The Shift System

Now that we know how to construct dynamical systems on the circle from dynamical systems on the line, we turn to a detailed study of some concrete examples. As always, we should start simply. For a fixed constant $\tau \in \mathbb{R}$, the associated **shift system** is defined by the recurrence relation

$$T_{n+1} = T_n + \tau. \tag{2.4}$$

This trivially descends to a well-defined dynamical system on the circle:

$$\varphi_{n+1} = \varphi_n + \tau \qquad \mod 1. \tag{2.5}$$

The nomenclature here is obvious: the dynamics takes a point in state space and *shifts* it by a fixed displacement at each time step.

It's easy to see that there are no equilibria for (2.4) if τ is not zero (if it is zero, then every point is an equilibrium). Also, if τ is a nonzero integer, then the circle system (2.5) fixes every point on the circle. All in all, the equilibria of the shift system are not very interesting. Periodic orbits, however, have a much more fun structure:

Lemma 2.3. *The dynamical system* (2.5) *has a periodic orbit if and only if τ is rational.*

Proof. Suppose a P-periodic orbit $\{\varphi_0, ..., \varphi_{P-1}\}$ for the circle system exists. From (2.5), we know

$$\varphi_n = \varphi_0 + n\tau \mod 1 \quad \forall\, n \in \mathbb{N}_0.$$

Plugging in $n = P$ gives

$$P\tau = 0 \mod 1,$$

which implies $P\tau = M$ for some integer M. In turn,

$$\tau = \frac{M}{P},$$

which is the definition of a rational number.

Next, suppose τ is rational. This means there exist relatively prime integers M, P such that

$$\tau = \frac{M}{P}.$$

We then have

$$\varphi_n = \varphi_0 + \frac{nM}{P} \mod 1 \quad \forall\, n \in \mathbb{N}_0.$$

By choosing $n = P$ in the above equation, we get

$$\varphi_P = \varphi_0 + M \mod 1 = \varphi_0 \mod 1.$$

Therefore, the orbit of φ_0 is periodic with period at most P. □

In fact, we've shown that if τ is rational then *all* orbits of the shift system are periodic!

Exercise 2.3. Suppose T_n is defined by (2.4) with initial state T_0, and τ is such that the circle orbit $\mathcal{O}_{T_0 \mod 1}$ is P-periodic. Show that, for any natural number m,

$$T_{mP} - T_{(m-1)P} = T_P - T_0.$$

Exercise 2.4. Sharpen the conclusion of lemma 2.3 by showing that, if $\tau = M/P$ with M, P relatively prime, then all orbits of (2.5) are P-periodic.

2.3 Rotation Number

I now introduce an orbit-characterizing number that is of great importance in circle dynamics.

Definition 2.4. *Consider a dynamical system of the form* (2.1) *that descends to a well-defined system on the circle. We define the* ***rotation number*** *of the orbit* $\mathcal{O}_{T_0}$ *by*

$$\rho = \lim_{n\to\infty} \frac{T_n - T_0}{n} \in [-\infty, \infty]\,.$$

What does ρ mean in plain language? Even though the rotation number is defined in terms of a line system, it's most useful for describing the associated descended system on the circle. For many circle maps of interest, there will eventually be an n such that $T_n = T_0 \mod 1$; this could occur due to (quasi-)periodicity or chaotic oscillations. The integer $T_n - T_0$ tells us how many rotations around the circle are required for this "recurrence" to happen: in other words, it gives the number of times $\widetilde{f}$ completely rotates circle when we go from T_n to T_0. A factor of $\frac{1}{n}$ is necessary to normalize $T_n - T_0$ so it doesn't just keep growing monotonically. Therefore, the rotation number can roughly be thought of as a measure of the average number of times $\widetilde{f}$ winds the circle around itself.

The best way to understand ρ is to work out some examples:

Example 2.5. *Suppose T_* is an equilibrium for*

$$T_{n+1} = f\left(T_n\right).$$

Then, the rotation number associated to $\mathcal{O}_{T_*}$ is simply 0. *This makes sense in light of our discussion above: in equilibrium, the circle is wound about itself zero times.*

Example 2.6. *Let's compute the rotation number associated to the shift system* (2.4). *First, for any n we may write*

$$T_n = T_0 + n\tau.$$

Therefore,

$$\frac{T_n - T_0}{n} = \tau \quad \forall\; n \in \mathbb{N}_0.$$

We conclude that

$$\rho = \tau$$

for any orbit of the shift system. In this way, we can use lemma 2.3 to say that rationality of ρ detects periodicity of the underlying orbit: this is an insight we'll return to later on.

Before concluding this section, I mention the following useful "re-definition" of the rotation number:

Lemma 2.7 (Integral Form of Rotation Number)**.** *Define $\Delta T_k = T_k - T_{k-1}$. Then,*

$$\rho = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \Delta T_k. \tag{2.6}$$

Proof. Observe that

$$\begin{aligned}
\sum_{k=1}^{n} \Delta T_k &= \sum_{k=1}^{n} \left(T_k - T_{k-1}\right) \\
&= (T_1 - T_0) + (T_2 - T_1) + \cdots + (T_n - T_{n-1}) \\
&= T_n - T_0,
\end{aligned}$$

so we're done. □

Exercise 2.5. Suppose $a > 0$. Compute all possible rotation numbers for the dynamical system

$$T_{n+1} = aT_n.$$

Your answer may depend on a!

3 The Arnol'd System

For the remainder of these notes, we focus on a complex, beautiful example. We consider the **Arnol'd dynamical system** defined in terms of two parameters $b, \tau \in \mathbb{R}$:

$$T_{n+1} = T_n + \tau + b \sin(2\pi T_n). \tag{3.1}$$

In exercise 3.1 you are asked to show that this descends to a well-defined system on the circle. Here, we harmlessly call both the line and circle systems "the" Arnol'd system. Additionally, τ is called the **shift** and b is called the **coupling strength**.

Notice that (3.1) is a shift system with an added state-dependent periodic forcing (the sine term). For this reason, (3.1) is used in biology as a basic model for cyclic phenomena coupled to other cyclic phenomena: for instance, a generalized Arnol'd system was used by Mosheiff et al. [2] to model the effect of circadian rhythms on the time between reproduction in certain single-celled organisms. For more on the applications of Arnol'd-type systems in biology, see Glass and Mackey [1].

Now, we can undertake some theoretical discussion of the Arnol'd system. First, notice that

Lemma 3.1. *For any n, if T_n is defined by (3.1) then*

$$T_n = T_0 + n\tau + b\sum_{k=0}^{n-1} \sin(2\pi T_k).$$

□

Exercise 3.1. Verify that the Arnol'd system in (3.1) descends to a well-defined dynamical system on the circle.

3.1 Equilibria

Next, we determine the equilibria for Arnol'd.

Proposition 3.2.

- *If $b = 0$ or $\left|\frac{\tau}{b}\right| > 1$ then the Arnol'd system has no equilibria.*
- *If $\left|\frac{\tau}{b}\right| \leq 1$ then the Arnol'd system has infinitely many isolated equilibria. These equilibria descend to two distinct equilibria on the circle:*

$$\varphi_a = -\frac{1}{2\pi}\sin^{-1}\left(\frac{\tau}{b}\right) \mod 1 \tag{3.2a}$$

$$\varphi_b = \frac{1}{2\pi}\left[\pi + \sin^{-1}\left(\frac{\tau}{b}\right)\right] \mod 1, \tag{3.2b}$$

where we use the convention that $\operatorname{range}\sin^{-1} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Proof. The equilibria for (3.1) obey

$$T = T + \tau + b \sin(2\pi T) \Rightarrow -\tau = b \sin(2\pi T).$$

If $b = 0$ there are clearly no equilibria, so this case is done. If $b \neq 0$, then we can write

$$-\frac{\tau}{b} = \sin(2\pi T).$$

Since the range of the sine function is $[-1, 1]$, this equation has no solutions if $\left|\frac{\tau}{b}\right| > 1$. On the other hand, if $\left|\frac{\tau}{b}\right| \leq 1$, then owing to the periodicity of sine there are infinitely many solutions.

It remains to prove that these solutions descend to the points on the circle given by (3.2). By periodicity of sine, we know that we can represent the two circle equilibria using two points on the real line:

- some T_a satisfying $2\pi T_a \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and
- some T_b satisfying $2\pi T_b \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Using the aforementioned convention for $\sin^{-1}$, we have that

$$T_a = -\frac{1}{2\pi} \sin^{-1}\left(\frac{\tau}{b}\right) = \varphi_a \mod 1$$

as desired. As for T_b, notice that

$$\pi - 2\pi T_b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and recall that

$$\sin\theta = \sin(\pi - \theta).$$

We can therefore write

$$-\frac{\tau}{b} = \sin(\pi - 2\pi T_b).$$

This implies

$$\pi - 2\pi T_b = \sin^{-1}\left(-\frac{\tau}{b}\right),$$

which gives

$$T_b = \frac{1}{2\pi}\left[\pi + \sin^{-1}\left(\frac{\tau}{b}\right)\right] = \varphi_b \mod 1$$

after a bit of algebra. □

We leave the stability analysis of these equilibria to the numerical experiments.

3.2 Periodic Orbits and Phase Locking

In the numerical experiments, we saw that periodic orbits for the Arnol'd circle system definitely exist for certain values of τ, b. Let's take some time to understand periodic orbits a little more deeply. First, we explain how to detect periodicity using the rotation number.

Proposition 3.3. *Suppose T_0 evolves according to (3.1). Assume that the circle orbit $\mathcal{O}_{T_0 \mod 1}$ is P-periodic. Then, the rotation number ρ associated to $\mathcal{O}_{T_0}$ is*

$$\rho = \frac{T_P - T_0}{P}.$$

In particular, ρ is rational.

Proof. TODO: complete this proof! $\square$

Note that a rational rotation number does not necessarily imply a periodic orbit. In practice, however, we like to think that a rational ρ implies we're almost periodic.

We now use this result to explain how the Arnol'd system displays the phenomenon of **phase locking**. A bit of concreteness helps with this explanation greatly. Let's suppose we're modelling the circadian influence on cell reproduction: the "cyclic behaviour" under consideration is the cycle of cell reproduction. We start with a single newborn cell and call that cell "generation 0". After some time, this cell splits into two daughter cells, and we call these daughters "generation 1". This process repeats on and on to generation n. The variables of interest are

$$T_n = \text{time (measured in days from the start of the experiment) when a cell in generation } n \text{ was born,}$$
$$\varphi_n = \text{time of day (measured in } [0,1]) \text{ when a cell in generation } n \text{ was born.}$$

By convention we take $T_0 = 0$, so perhaps it is clearer to call φ_n the "relative" time of day. We model the evolution of T_n over successive generations via Arnol'd dynamics prescribed by (3.1). In particular, the forcing term oscillates once per day.

Let's suppose $\mathcal{O}_{T_0}$ descends to a P-periodic orbit on the circle. In biological terms, this means that generation 0 and generation P were born at the same time of day. As the circle system completes a cycle (that is, makes it all the way through a periodic orbit once), the state of the line system changes according to $T_P - T_0 = M \in \mathbb{N}_0$ (see exercise 3.2). This corresponds to M days passing or, equivalently, M cycles of the forcing term taking place. How many cell cycles take place during this time? By definition of the index n as a generation number, a total of P cell cycles elapse during the M days in question. Therefore, in this case the forcing term is "locked" to the cell cycle: for every M forcing cycles there are P cell cycles. This strong relationship between the driving and driven oscillators in question is called **phase locking**, and it is of great importance in chronobiology and physiology. In practice, we like to think of phase locking as a generalized version of synchronization.

Notice that, in the language of the above paragraph, the ratio of forcing cycles to cell cycles is equal to M/P. By proposition 3.3, this ratio is precisely the rotation number ρ! We have therefore shown that

$$\text{phase locking} \approx \text{(quasi-)periodic circle orbits} \approx \rho \in \mathbb{Q}.$$

Therefore, if one could find all the regions in $\tau - b$ space where the orbits of $T_0 = 0$ had rational, one could find all parameter pairs that resulted in phase locking! In the numerical experiments, we show a plot of (approximate) rotation numbers computed for various parameter pairs. The shapes of the **level-sets** $\rho =$ constant are extremely complicated. In particular, phase locking can be robust (that is, insensitive to small changes in τ, b) or unstable, depending on the particular phase-locking ratio.

Exercise 3.2. Suppose T_n is defined by (3.1) with initial state T_0, and b, τ are such that the circle orbit $\mathcal{O}_{T_0 \mod 1}$ is P-periodic. Show that, for any natural number m,

$$T_{mP} - T_{(m-1)P} = T_P - T_0.$$

References

[1] Leon Glass and Michael C. Mackey, *From clocks to chaos: the rhythms of life*, Princeton University Press, Princeton, NJ, 1988.

[2] Noga Mosheiff, Bruno M. C. Martins, Sivan Pearl-Mizrahi, Alexander Grünberger, Stefan Helfrich, Irina Mihalcescu, Dietrich Kohlheyer, James C. W. Locke, Leon Glass, and Nathalie Q. Balaban, *Inheritance of cell-cycle duration in the presence of periodic forcing*, Phys. Rev. X **8** (May 2018), 021035.