Lecture Notes: Dynamical Systems Theory

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1 Introduction

In our discussion of single-species population growth, all our models had the form of a recurrence relation

$$p_{n+1} = f\left(p_n\right) \tag{1.1}$$

where n is the time elapsed, p_n is the population density at time n, and

$$f\colon [0,\infty) \to [0,\infty)$$

is a reasonable function. The model (1.1) says that, given an initial population size p_0, we can determine p_n for any n by simple recursion. That is, we have a *deterministic* (non-random) rule for determining p_{n+1} using only p_n. This rule is a manifestation of a mathematical object called a **(discrete-time) dynamical system**. To get more insight into population models, then, we should learn a bit about dynamical systems theory.

To emphasize the breadth of dynamical systems theory, I take an abstract approach following [3, Ch. 1]. Of course, since I do not assume the reader has a background in general topology or even calculus, I avoid any discussion of *continuous-time* dynamical systems, and also I am forced to state several definitions informally.

2 Basic Definitions

2.1 State Spaces

Let X be any set of numbers, points in space, or symbols representing all possible **states** of some physical process we want to model. This set X is naturally called a **state space**.

Example 2.1. *For a model of paramecium growth, the "state" of the population at a given time is simply the population density. So, for this case, the state space is $X = [0,\infty)$.*

Before moving on to more complicated examples, we need to recall a simple set-theoretic definition.

Definition 2.2. *Let S_1, S_2 be any sets. We define their* ***Cartesian product*** $S_1 \times S_2$ *by*

$$S_1 \times S_2 = \{(x, y) \mid x \in S_1, \quad y \in S_2\}.$$

Example 2.3. *Consider a model of two competing species, say, a predator species and its prey species. The state of the two-species ecosystem is then the two population densities. The state space in this situation is*

$$X = [0, \infty) \times [0, \infty).$$

Example 2.4. *Consider a bead sliding on a long, straight wire. Newton's laws of motion tell us that the state of the bead at any given time is described by its position and velocity at that particular time. Since the wire is assumed to be very long, the set of its possible positions can be identified with the real line $\mathbb{R} = (-\infty, +\infty)$. Of course, its set of possible velocities is also $\mathbb{R}$. Therefore, the state space of the bead is*

$$X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2,$$

where $\mathbb{R}^2$ denotes the Cartesian plane.

Example 2.5. *Suppose the wire from the previuous example is bent so that its ends touch one another, forming a hoop. This transformation affects the positions, but not the velocities. The state space then becomes*

$$X = S^1 \times \mathbb{R}$$

where S^1 denotes the unit circle lying in the plane $\mathbb{R}^2$ (we are tacitly assuming that positions are measured so that the hoop has radius $= 1$).

Exercise 2.1. Suppose an ecologist wants to model the long-term effect of climate change on forest composition. The forest the ecologist is describing has five dominant tree species that may cooperate or compete for resources. Additionally, the ecologist has determined that temperature is the main climatic quantity they want to vary in their model. Write down a state space the ecologist can use for their model.

Exercise 2.2. Can you identify the state space from example 2.5 with a familiar geometric shape?

Exercise 2.3. Determine a state space modelling the positions of the two hands on a wall clock. Can you identify the shape of this state space with a particular pastry?

2.2 Evolution Operators

Having a state space X isn't enough to model a time-varying physial process: we need a rule for describing how a point in state space moves over time.

Definition 2.6. *Consider a family of functions on X indexed by $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$:*

$$\begin{aligned} \Phi_n : & X \to X, \\ & x \mapsto \Phi_n(x). \end{aligned}$$

We say this family $\{\Phi_n(x)\}_{n \in \mathbb{N}_0}$ is a ***family of evolution operators on*** *X if, for all $x \in X$, the following two conditions are satisfied:*

1. *$\Phi_0(x) = x$, and*
2. *(**autonomy**) for all $n, m \in \mathbb{N}_0$,*

$$\Phi_{n+m}(x) = \Phi_n(\Phi_m(x)).$$

This definition is pretty abstract, so we should stop to really internalize it. The first thing to recognize is that we should think of the index n as a time variable. The condition on Φ_0 then implies that the function corresponding to time zero "does nothing". This makes intuitive sense: if we don't let a process run for *any* amount of time, we shouldn't expect to see any change! Next, the autonomy condition is reasonable to understand as well: it says that if you move x for time m and then move the resulting output $\Phi_m(x)$ for time n, it's no different than moving x for the total time $n + m$. Put simply, the "law of motion" expressed by the evolution operators does not change with time.

Example 2.7. *Consider the state space $X = \mathbb{R}$. Fix any $a \in \mathbb{R}$, then define a family of functions on $\mathbb{R}$ by*

$$\Phi_n(x) = a^n(x). \tag{2.1}$$

We show below that $\{\Phi_n(x)\}_{n \in \mathbb{N}_0}$ is a family of evolution operators by checking that the conditions of definition 2.6 are met. Pick any $x \in \mathbb{R}$.

- *First,*

$$\Phi_0(x) = a^0 x = x,$$

so the first condition is satisfied (in the case $a = 0$, we are using the combinatorial convention that $0^0 = 1$).

- *As for the second condition (autonomy), pick any $n, m \in \mathbb{N}_0$. We then have*

$$\begin{aligned} \Phi_{n+m}(x) &= a^{n+m} x \\ \Phi_n(\Phi_m(x)) &= \Phi_n(a^m x) = a^n(a^m x) = a^n a^m x = a^{n+m} x, \end{aligned}$$

so $\Phi_{n+m}(x) = \Phi_n(\Phi_m(x))$ as desired.

Suppose we let $a = 1 + r$ for $r > 0$, and pick some $p_0 \geq 0$. We then define

$$p_n = \Phi_n(p_0) = (1 + r)^n p_0.$$

In particular, we have the recurrence relation

$$p_{n+1} = (1 + r)\, p_n.$$

So, this example generalizes the Malthusian difference equation model we encountered earlier! Said differently, running the Malthusian recurrence relation gives a family of evolution operators. In particular, we can draw state evolution curves in the n-p_n plane by computing

$$\Phi_1 (p_0) = p_1, \Phi_2 (p_0) = p_2, ...$$

Example 2.8. *Consider again $X = \mathbb{R}$, and let $f\colon X \to X$ be any nice function. Recall that the symbol $\circ$ denotes function composition (for instance, $f \circ g(x) = f(g(x))$). We define a time-indexed family of functions $\Phi_n\colon \mathbb{R} \to \mathbb{R}$ by*

$$\Phi_n(x) = \begin{cases} x & \text{if} \quad n = 0 \\ \underbrace{f \circ f \circ \cdots \circ f}_{n \ \text{times}}(x) & \text{else.} \end{cases} \tag{2.2}$$

Since function composition is ***associative****, autonomy is satisfied and therefore $\{\Phi_n\}_{n\in\mathbb{N}_0}$ is a family of evolution operators. This example allows us to relate all our earlier population models*

$$p_{n+1} = f(p_n) \tag{2.3}$$

to evolution operators. To see how the concrete and abstract notation are related, let's pick any initial state p_0. Suppose $p_1, p_2, ...$ are computed using the recurrence relation (2.3) above. We have

$$\begin{aligned} \Phi_1(p_0) &= f(p_0) = p_1. \\ \Phi_2(p_0) &= f(f(x)) = f(p_1) = p_2 \\ &\vdots \\ \Phi_n(p_0) &= p_n. \end{aligned}$$

That is, applying the recurrence relation in (2.3) to p_n is the same as applying Φ_1 to p_n. All in all, we have seen that all difference equations of the form (2.3) generate a family of evolution operators. In particular, the iteration map $f(x)$ can be identified with the obviously named ***time-one map*** *$\Phi_1(x)$. Note that this result is actually true regardless of the state space X, and we just focused on $X = \mathbb{R}$ for concreteness.*

Example 2.9. *We now establish the converse to the above example: that is, we show that every family of evolution operators $\{\Phi_n(x)\}_{n\in\mathbb{N}_0}$ on a given state space X gives rise to a recurrence relation*

$$x_{n+1} = f(x_n)$$

for well-chosen $f(x)$. In light of the previous discussion, the correct $f(x)$ is easily seen to be

$$f(x) = \Phi_1(x),$$

the time-one map.

Our examples have shown that we can just as well identify a family of evolution operators $\{\Phi_n(x)\}_{n\in\mathbb{N}_0}$ with its time-one map $f(x) = \Phi_1(x)$.

Exercise 2.4. Let X be any state space.

a) Suppose we have a family of time-dependent functions

$$f(n,x)\colon \mathbb{N}_0 \times X \to X.$$

Can the difference equation

$$x_{n+1} = f\left(n, x_n\right) \tag{2.4}$$

be associated to a family of evolution operators?

b) Given an example of a simple biological process that could be modelled by a recurrence of the form (2.4).

2.3 Dynamical Systems Defined

With all the machinery we've built up so far, defining dynamical systems is straightforward.

Definition 2.10. *Let X be a state space and let $\{\Phi_n(x)\}_{n\in\mathbb{N}_0}$ be a family of evolution operators on X. We then say that the pair*

$$\mathcal{S} = \left\{X, \{\Phi_n(x)\}_{n\in\mathbb{N}_0}\right\}$$

*constitutes a **dynamical system**.*

Though the notation is a little heavy, the core idea is simple: a dynamical system consists of a physically interesting state space together with a rule for how points in that state space move over time.

In practice, we are allowed to identify a given dynamical system $\mathcal{S}$ with its state space and its time-one map $f(x)$, as opposed to the full family of evolution operators $\{\Phi_n(x)\}_{n\in\mathbb{N}_0}$; that is, we can write

$$\mathcal{S} = \{X, f\}\,.$$

Since our ultimate goal is to understand difference equations, we will mostly adhere to this convention in the sequel.

Example 2.11. *Given a state space X and any nice function $f(x)$ mapping X to itself, we know from example 2.8 how to obtain a family of evolution operators. Thus, the pair $\mathcal{S} = \{X, f\}$ forms a dynamical system. In particular, every difference equation of the form* (2.3) *corresponds to a dynamical system. For a concrete example, the dynamical system*

$$\mathcal{S}_{Malthus} = \{\mathbb{R}, x \mapsto ax\}$$

corresponds to generalized Malthusian growth or decay.

Remark. Since we have learned that dynamical systems can be characterized by their time-one maps just as well as their evolution operators, you may ask the following question: why bother with the heavy abstract definition of an evolution operator in the first place? The primary reason I have chosen to start from evolution operators rather than time-one maps in the general theory is to emphasize the importance of autonomy. If I had just focused on time-one maps, we would have gotten autonomy for free and would not have focused on it as an interesting property. Instead, we saw that demanding autonomy implies that it suffices to focus on time-one maps. This sort of top-down approach is very popular in contemporary mathematics (perhaps more popular than I would personally like it to be), and now is a good time to start getting used to it.

3 Equilibria and their Stability

3.1 Equilibria Defined. Examples.

Given a dynamical system $\mathcal{S} = \{X, f\}$, what points in X are "special"? Well, since the point of defining dynamical systems is to studying the movement of points in state space under the rule of the time-one map, an obvious answer is this: any point that does not move is "special". In the context of single-species population modelling, we saw two such special points, namely the trivial case where no organisms were present, and the very interesting case where we were at the environment's carrying capacity. All this leads to a new definition.

Definition 3.1. *A point $x_* \in X$ is called an **equilibrium** of the dynamical system $\mathcal{S} = \{X, f\}$ if*

$$f(x_*) = x_*.$$

Said differently, equilibria are **fixed points** of the time-one map. Using autonomy, it's trivial to show

Lemma 3.2. *For all $n \in \mathbb{N}_0$,*

$$\Phi_n\left(x_*\right) = \left(x_*\right).$$

Proof. Let $x_0 = x_*$, and define $x_n = \Phi_n\left(x_*\right)$. By autonomy, the sequence x_n obeys the recurrence relation

$$x_{n+1} = f\left(x_n\right).$$

To prove the lemma, it suffices to prove the claim $x_n = x_*$ for all n. The proof proceeds by induction. For $n = 0$,

$$x_1 = f(x_0) = f\left(x_*\right) = x_*,$$

so the claim holds in thise case. Now, assume the claim has been proven up to some index n. We then have

$$x_{n+1} = f\left(x_n\right) = f\left(x_*\right) = x_*.$$

The claim then holds for $n+1$. Since n was arbitrary, the principle of induction implies that the claim holds for all $n \in \mathbb{N}_0$. $\square$

Example 3.3. *Fix $a \in \mathbb{R}$ and consider the dynamical system*

$$\mathcal{S}_{Malthus} = \{\mathbb{R}, x \mapsto ax\}.$$

Here, $f(x) = ax$. To find the equilibria for this dynamical system we need to solve the algebraic equation

$$x = ax \Rightarrow (a-1)x = 0.$$

In the case $a \neq 1$, the only solution is

$$x_* = 0.$$

On the other hand, if $a = 1$, then every *$x \in \mathbb{R}$ is an equilibrium! This makes sense: the $a = 1$ case has a dynamical update rule that always says "do nothing".*

Example 3.4. *Let $r, K > 0$ be real parameters, then define*

$$f(x) = x + a\,(K - x)\,x.$$

Consider the discrete logistic dynamical system

$$\mathcal{S}_{DLM} = \{\mathbb{R}, f(x)\}.$$

What are the equilibria for this system? We have to solve the algebraic equation

$$x = f(x) = x + a\,(K - x)\,x.$$

We immediately see that $x_* = 0$ is a solution to this equation, corresponding to the trivial "no organisms" case. Additionally, we have the nontrivial equilibrium $x_* = K$, which we know corresponds to a carrying capacity.

Exercise 3.1.

a) Find all equilibria for the Beverton-Holt model.

b) Find all equilibria for the Ricker model.

Exercise 3.2. Find all equilibria for

$$\mathcal{S} = \left\{(0,\infty)\,,\ x \mapsto \frac{1}{x}\right\}.$$

Exercise 3.3. [Computer Exercise] Find all equilibria for

$$\mathcal{S} = \{\mathbb{R},\ x \mapsto \cos(x)\}.$$

Give a numerical answer that is accurate to twelve decimal places.

3.2 Notions of Stability. Basic Examples.

Now, if we start at an equilibrium, we stay there for all time. If we *don't* start at an equilibrium, however, what might happen? Do we eventually reach the equilibrium, or at least get very close to it? Do we instead go far away from the equilibrium, as if pushed away from it by a force field? Do any of these possibilities depend on how far away from the equilibrium we start? In our discussion of logistic growth, we saw that these questions have interesting answers: even with a small amount of initial organisms present, the population quickly rose to the carrying capacity of its environment. This represents *repulsion* away from the zero equilibrium and *attraction* towards the carrying capacity. So, asking the questions stated above for different equilibria *in the same dynamical system* can lead to different answers! This is our first step towards developing **stability theory** for equilibria.

To develop stability theory properly, we must define several different notions of stability or attraction. Unfortunately, the definition below is a little bit imprecise, but for the purposes of a survey course this won't cause serious trouble.

Definition 3.5 (Informal). *Let x_* be an equilibrium for a dynamical system $\mathcal{S} = \{X, f\}$.*

1. *We say that x_* is **Lyapunov stable** if*

$$x_0 \approx x_*$$

   implies

$$\Phi_n(x_0) \approx x_* \quad \forall\, n \in \mathbb{N}_0.$$

2. *We say that x_* is **attracting** if, for all $x_0 \approx x_*$ we have*

$$\lim_{n\to\infty} \Phi_n(x_0) = x_*.$$

3. *We say that x_* is **asymptotically stable** if it is both Lyapunov stable and attracting.*

4. *We say that x_* is **unstable** if it is not Lyapunov stable.*

Exercise 3.4. *Consider the Malthus dynamical system*

$$\mathcal{S}_{Malthus} = \{\mathbb{R}, x \mapsto ax\}.$$

Recall from example 3.3 that $\mathcal{S}_{Malthus}$ has a unique equilibrium $x_* = 0$ as long as $a \neq 1$, and if $a = 1$ every real number is an equilibrium. Let's understand how the stability of $x_* = 0$ depends on a.

- *If $a = 1$, then x_* is Lyapunov stable but not asymptotically stable. To see why, pick any nonzero x_0 with $|x_0| \ll 1$, so $x_0 \approx x_* = 0$. Then,*

$$\Phi_n(x_0) = a^n x_0 = x_0 \quad \forall\, n$$

  Accordingly, $\Phi_n(x_0) \approx 0$ for all n, so we're Lyapunov stable. However, $\lim_{n\to\infty} \Phi_n = x_0 \neq 0 = x_*$. So, x_* cannot be attracting, hence it can't be asymptotically stable.

- *Similar arguments show that if $a = -1$ then x_* is Lyapunov stable hut not asymptotically stable. This case is a little more interesting than the previous one because now x_* is the only equilibrium point.*

- *Suppose $|a|< 1$. We shall show that $x_* = 0$ is asymptotically stable. To prove this, first pick any $x_0 \in \mathbb{R}$. We have for any n that*

$$\Phi_n (x_0) = a^n x_0.$$

Therefore,

$$|\Phi_n (x_0)| = |a|^n |x_0|< |x_0| \quad (\textit{since } |a|< 1)\,.$$

If $|x_0|\ll 1$, then the above implies $|\Phi_n (x_0)| \ll 1$ for all n, so x_* is Lyapunov stable. Since $|a|< 1$, the above also shows that

$$\lim_{n\to\infty} \Phi_n (x_0) = 0 = x_*,$$

so the equilibrium is also attracting. We conclude that $x_* = 0$ is asymptotically stable if $|a|< 1$.

- *If $|a|> 1$, then x_* is unstable. To see why, we again turn to*

$$\Phi_n (x_0) = a^n x_0.$$

Pick N to be large enough so that

$$|a^n x_0| = |a|^n |x_0|\geq 10^{32} \quad \forall\; n \geq N;$$

this is doable since $|a|> 1$. Thus, for $n \geq N$ we have

$$|\Phi_n (x_0)| \gg 1,$$

meaning Lyapunov stability fails. This implies that $x_* = 0$ is unstable if $|a|> 1$.

We summarize this discussion with a lemma:

Lemma 3.6 (Stability of the Zero Equilibrium for Malthus Dynamics)**.** *For a fixed $a \in \mathbb{R}$, consider the Malthus dynamical system*

$$\mathcal{S}_{\mathit{Malthus}} = \{\mathbb{R}, x \mapsto ax\}\,.$$

The equilibrium $x_* = 0$ is

1. *Lyapunov stable but not asymptotically stable if $|a|= 1$,*

2. *asymptotically stable if $|a|< 1$, and*

3. *unstable if $|a|> 1$.*

$\square$

Before moving on to general techniques for establishing (in)stability, we discuss the stability of a biologically significant dynamical system: the Beverton-Holt model of logistic growth.

Proposition 3.7 (Asymptotic Stability for Beverton-Holt)**.** *Given two real parameters $b > 1, L > 0$, consider the Beverton-Holt dynamical system*

$$p_{n+1} = \frac{b}{1 + \frac{b-1}{L} p_n} p_n$$

defined on the state space $X = [0, \infty)$. The nontrivial equilibrium

$$p_* = L$$

is asymptotically stable.

Proof. The proof relies on developing a simple, exact formula for p_n in terms of p_0. Having such an easy expression for the evolution operators of a dynamical system is very rare, so don't count on it in general! This argument just goes to show how special the Beverton-Holt system really is.

To begin, call $c = \frac{b-1}{L}$, and $u_n = p_n^{-1}$. A routine calculation then shows.

$$u_{n+1} = \frac{1}{b} u_n + \frac{c}{b}. \tag{3.1}$$

Thus, our system has been transformed into a linear one. Finding an explicit expression for u_n is easy using exercise 3.6:

$$u_n = \left(u_0 - \frac{c}{b-1} \right) b^{-n} + \frac{c}{b-1}.$$

If this doesn't satisfy you, please stop and check that the above satisfies the recurrence relation from (3.1). Transforming back to $p_n = u_n^{-1}$ gives an explicit expression for p_n:

$$p_n = \frac{b^n}{1 + \frac{b^n - 1}{L} p_0} p_0.$$

With this solution in hand, we first establish Lyapunov stability. Pick $p_0 \approx L$. Then, using our formula for p_n, we have

$$p_n - L = \frac{b^n p_0 - \left(1 + \frac{b^n - 1}{L} p_0\right) L}{1 + \frac{b^n - 1}{L} p_0} = \frac{p_0 - L}{1 + \frac{b^n - 1}{L} p_0}.$$

Since $b > 1$ and $L, p_0 > 0$, we have $1 + \frac{b^n - 1}{L} p_0 \geq 1$. The above then implies

$$|p_n - L| \leq |p_0 - L| \,.$$

Since $|p_0 - L| \ll 1$, we conclude that $|p_n - L| \ll 1$ so we're Lyapunov stable.

To complete the proof, we need to show $p_* = L$ is attracting. We simply compute a limit assuming $p_0 \neq 0$ (that is, we are "in a neighbourhood" of $p_* = L > 0$):

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty} \frac{b^n p_0}{1 + \frac{b^n - 1}{L} p_0} \\
&= \lim_{n\to\infty} \frac{p_0}{b^{-n} + \frac{1-b^{-n}}{L} p_0} \\
&= \lim_{n\to\infty} \frac{p_0}{\frac{p_0}{L} + b^{-n}\left(1 - \frac{p_0}{L}\right)} \\
&= \frac{p_0}{\frac{p_0}{L}} \quad (\text{since } b > 1) \\
&= p_{\mathrm{eq}},
\end{aligned}$$

so we're all done.

□

In all the situations we've seen so far, discussing stability was relatively simple. In general, one needs a little more theoretical machinery to understand the stability properties of a given dynamical system rigorously (of course, numerical computation can be used to help formulate conjectures on stability). The two main techniques for systematically establishing stability are

1. drawing **cobweb diagrams**, and
2. performing a **linearization** of the time-one map $f(x)$.

Both these methods have their advantages and disadvantages, and we'll discuss these very soon. Since I do not assume you are familiar with calculus, I can't discuss the powerful tool of linearization in complete detail here, but we can still see it in action in a few simple circumstances.

Exercise 3.5. Consider the dynamical system from exercise 3.2. Determine the stability properties of each equilibrium point.

Exercise 3.6. Let $a, b \in \mathbb{R}$ be any fixed real numbers. Consider the linear difference equation

$$x_{n+1} = ax_n + b. \tag{3.2}$$

a) Show that we can write x_n explicitly as

$$x_n = \begin{cases} x_0 + nb & \text{if} \quad a = 1 \\ \\ a^n x_0 + b\frac{a^n-1}{a-1} & \text{if} \quad a \neq 1. \end{cases} \tag{3.3}$$

Hint: if done correctly, you will have to sum a geometric series!

b) Assuming $b \neq 0$, find the unique equilibrium x_* for (3.2). Your answer will depend on a and b.

c) By making the change of variables

$$y_n = x_n - x_*,$$

determine the stability properties of x_* as a function of a *only*.

Exercise 3.7. [NOTE: I recommend you do exercise 3.6 before trying this problem] Let us measure time n in years. Suppose the population p_n of tuna in the Mediterranean sea grows exponentially with growth rate $r > 0$ (so $\Delta p_n = rp_n$), and that m tuna are harvested each year by fishermen.

a) Write down a difference equation modelling the evolution of p_n over time.

b) Show that your difference equation model has a unique equilibrium p_*.

c) Determine the stability properties of p_*.

d) Let p_0 denote the population of tuna in the Mediterranean on January 1, 2024. Is the fishing program sustainable if $p_0 < p_*$? What about if $p_0 \geq p_*$?

Exercise 3.8. Explain why, in principle, an attracting equilibrium may not be Lyapunov stable. You don't have to give an explicit example (it would be nice if you do though), just indicate the main "bad possibility" you think can happen.

3.3 Cobweb Diagrams

Our first systematic tool for determining stability is drawing **cobweb diagrams**. These are best learned by working through examples. The examples below are incomplete, and will be filled in during lecture.

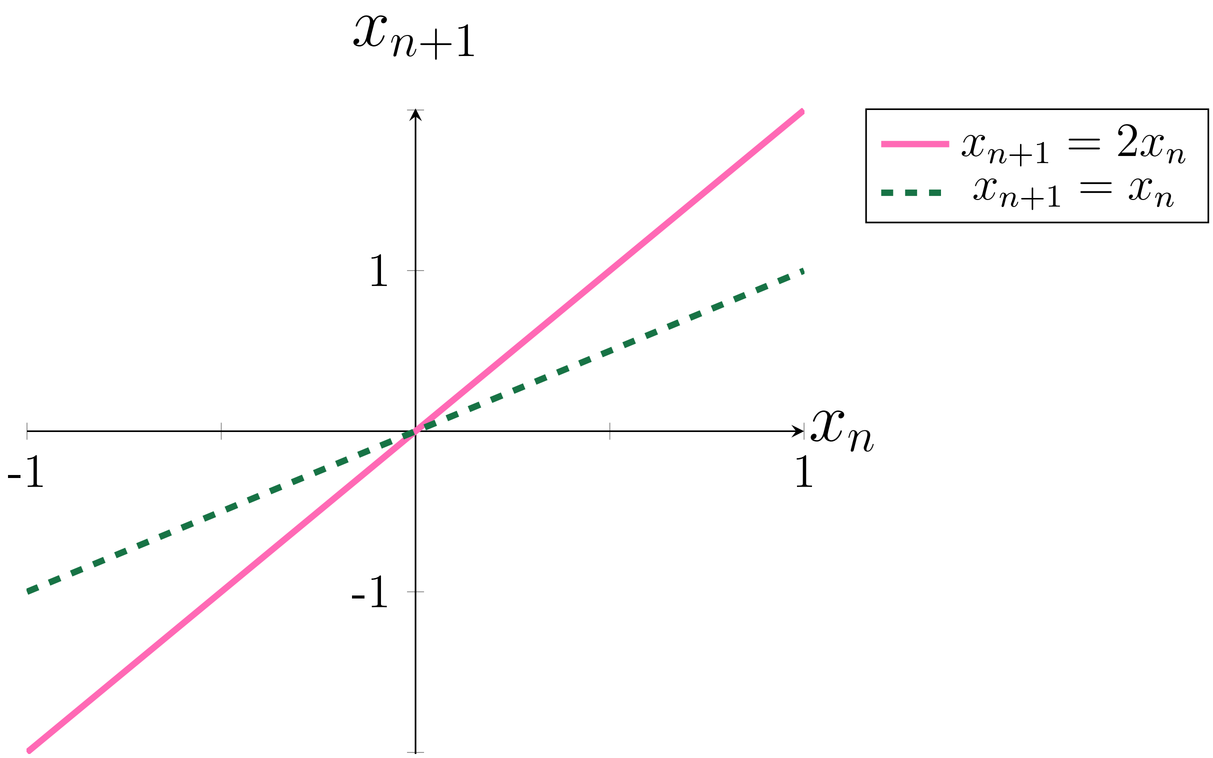

Figure 1: Setting up the cobweb diagram for $x_{n+1} = 2x_n$.

Example 3.8.

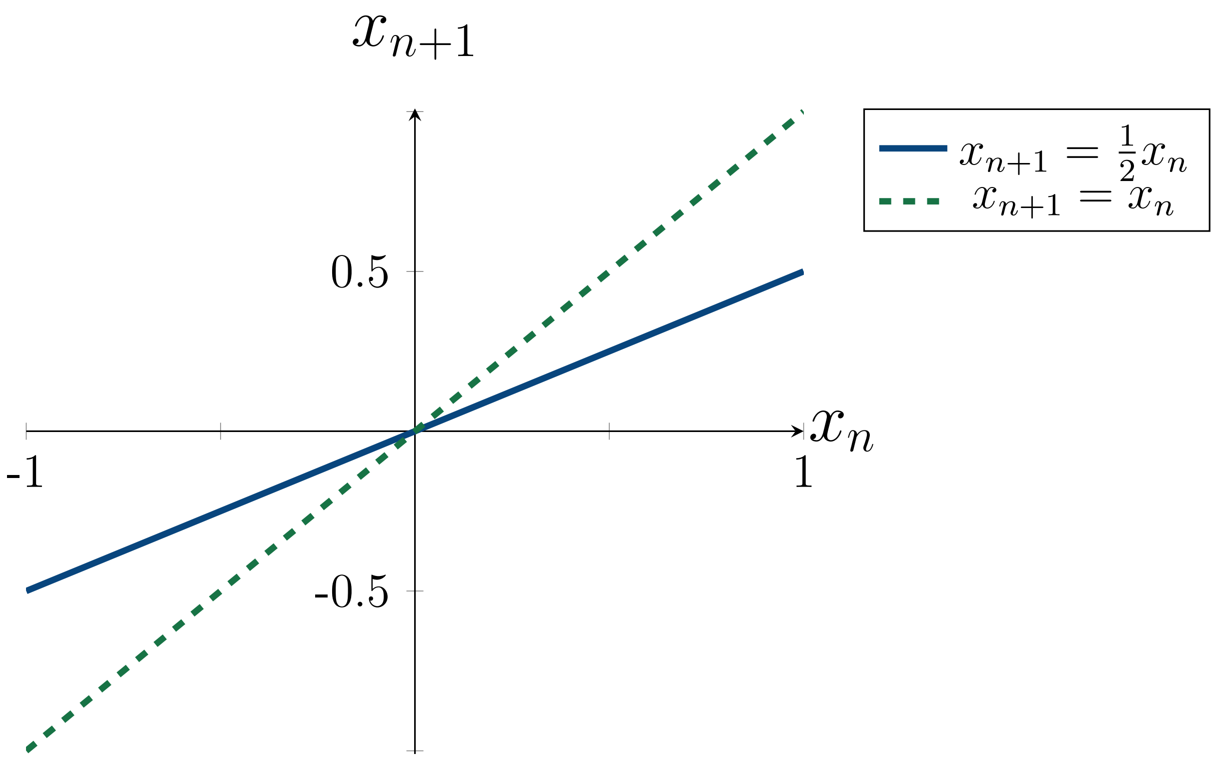

Figure 2: Setting up the cobweb diagram for $x_{n+1} = \frac{1}{2}x_n$.

Example 3.9.

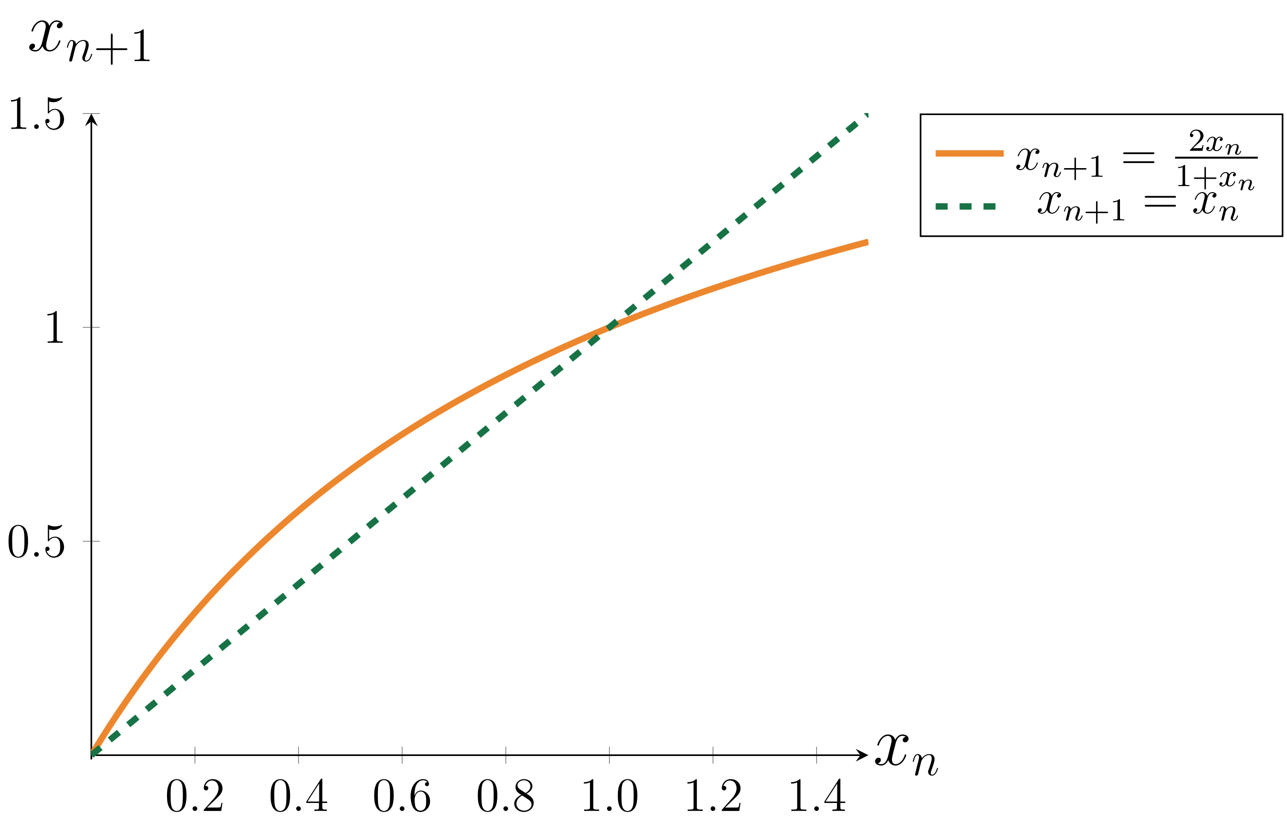

Figure 3: Setting up the cobweb diagram for the Beverton-Holt equation $x_{n+1} = \frac{2x_n}{1+x_n}$.

Example 3.10.

Exercise 3.9. Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ \\ 1 - e^{-x} & \text{if} \quad x \geq 0. \end{cases}$$

a) Sketch the graph of $f(x)$.

b) Find the unique equilibrium x_* of the dynamical system $\mathcal{S} = \{\mathbb{R}, f\}$.

c) Is x_* attracting?

d) Is x_* Lyapunov stable?

Exercise 3.10. Show that figure 3 is "generic" amongst all Beverton-Holt models in the sense that the shape of the picture is unchanged if we use a general $b > 1, L > 0$. *Hint: look at the sign of* $f(x) - x$*.*

3.4 Linearization

Cobwebbing is a nice, elegant, visual way to quickly determine the stability of an equilibrium point. However, sometimes the stability of an equilibrium depends sensitively on the values of any parameters appearing in the time-one map: we saw this in lemma 3.6. Since cobweb diagrams can only be drawn for fixed parameter values (unless you make some supplementary argument about "genericity" of the particular diagram you're using), an alternative method for establishing (in)stability is desirable. The most popular alternative method is **linearization**. Again, a full discussion of linearization would require more calculus than I'm prepared to assume or introduce, so I'll be sparing on complete details here.

As with cobwebbing, we learn by example:

Example 3.11. *Consider the DLM*

$$x_{n+1} = rx_n \left(1 - x_n\right). \tag{3.4}$$

As long as $r \in [0, 4]$, (3.4) *defines a dynamical system on the state space* $[0, 1]$*; that is,*

$$f(x) = rx(1 - x)$$

maps $[0, 1]$ *to itself. To see why, simply draw a graph of* $f(x)$ *for* $x \in [0, 1]$ *(see figure 4): the maximum of* $f(x)$ *on this domain is* 0 *and the maximum is is* $\frac{r}{4}$ *occurring at* $x = \frac{1}{2}$*, so* $f(x) \leq 1$ *if* $r \leq 4$*.*

Now, we know that the equilibria for (3.4) *are*

$$x_* = 0, \ \frac{r-1}{r}.$$

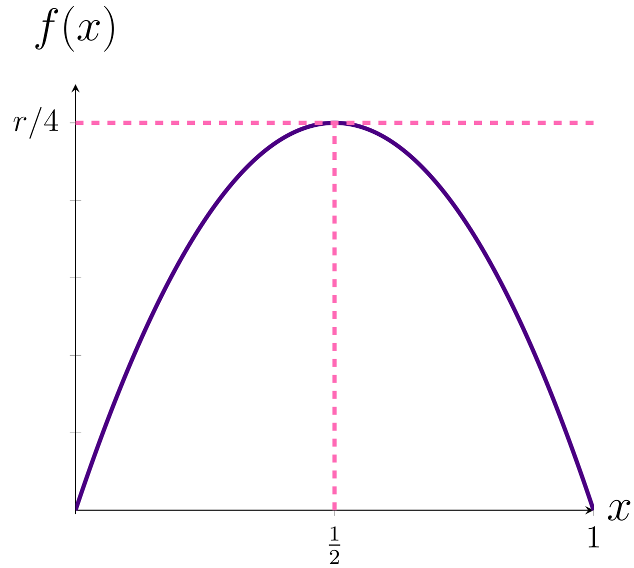

Figure 4: Plot of the time-one map for DLM. The maximum occurs at $x = \frac{1}{2}$.

Note that the nontrivial equilibrium only lives in $[0, 1]$ *if* $r \geq 1$*: accordingly, for* $r \leq 1$ *there is only one equilibrium. What are the stability characteristics of each equilibrium?*

Let's start with the trivial case $x_* = 0$*. The starting point in investigating Lyapunov stability is to pick* $x_0 \approx x_* = 0$*. What happens when we apply the time-one map to* x_0*? We obtain*

$$x_1 = f(x_0) = rx_0 - rx_0^2.$$

Now, since x_0 *is very close to* 0*, we know that* $x_0^2 \ll x_0$*. Accordingly, upon dropping the squared term in the above, we approximately have*

$$x_1 \approx rx_0.$$

Since r *can only be a relatively small number and* x_0 *is assumed to be very small, the above implies that* $x_1 \ll 1$*. Iterating this gives us the approximate difference equation*

$$x_{n+1} \approx rx_n, \tag{3.5}$$

which is a Malthus-type system. Lemma 3.6 tells us that the zero-equilibrium of (3.5) *is asymptotically stable if* $r \in [0, 1)$*, Lyapunov stable if* $r = 1$*, and unstable if* $r \in (1, 4]$*. Consequently, we have argued that* $x_* = 0$ *may or may not be stable! In particular, if* $r > 1$ *then this equilibrium is not stable (meaning that, if there is a sensible carrying capacity, then the zero-equilibrium cannot be stable)*

Next, let's look at the nontrivial equilibrium $x_* = \frac{r-1}{r}$ *with* $r > 1$*. Let's define a new variable by*

$$y_n = x_n - \frac{r-1}{r}.$$

Assume $x_0 \approx x_* = \frac{r-1}{r}$, so that $|y_0|$ is very small. y_n then approximately satisfies the difference equation

$$\begin{aligned} y_{n+1} &= x_{n+1} - x_* \\ &= r x_n \left(1 - x_n\right) - x_* \\ &= r \left(y_n + x_*\right)\left(1 - x_* - y_n\right) - x_* \\ &= r \left(1 - 2x_*\right) y_n + r x_* \left(1 - x_*\right) - x_* - r y_n^2 \\ &\approx r \left(1 - 2x_*\right) y_n, \end{aligned}$$

since $f(x_*) = x_*$. After a bit of simplification, this becomes

$$y_{n+1} \approx (2 - r)\, y_n. \tag{3.6}$$

So, y_n (but not x_n!) approximately obeys Malthus-type dynamics. Consequently, lemma 3.6 implies that the nontrivial equilibrium $x_* = \frac{r-1}{r}$ is asymptotically stable if $|2 - r| < 1$, Lyapunov stable if $|2 - r| = 1$ or unstable if $|2 - r| > 1$. After a bit of inequality-pushing, we can simplify these criteria: we obtain asymptotic stability if $r \in (1, 3)$, Lyapunov stability if $r = 3$, or instability if $r \in (3, 4]$.

Remark. Strictly speaking, the conditions on r guaranteeing stability or instablity only apply *approximately*. How do we guarantee that this "approximate (in)stability" obtained via linearization translates to the actual system? A complete answer is outside of the scope of this course, but the short version is that our argument is justified by the **Hartman-Grobman-Cushing theorem** [1, Appendix A], [2, Appendix D].

We summarize the key ideas from the above example with the following procedure.

Method 3.12 (Linearization Recipe)**.** *Consider the dynamical system*

$$x_{n+1} = f\left(x_n\right)$$

with an equilibrium x_*. To determine the stability of x_*, follow the procedure below:

1. *introduce the new variable $y = x - x_*$ and assume y is small;*
2. *perform a linear approximation of the time-one map; that is, find $f'(x_*) \in \mathbb{R}$ such that*

   $$f(x) \approx x_* + f'(x_*) y$$

   (this amounts to setting all terms that are quadratic, cubic, or higher-order in y to zero);
3. *assess the stability of the zero equilibrium for the new dynamical system*

   $$y_{n+1} = f'(x_*) y_{n+1}$$

   using lemma 3.6;

4. *the stability characteristics of the zero-equilibrium found in item 3 are the stability characteristics of x_* for the original dynamical system.*

Readers with a background in calculus should know that I did not call the coefficient of y in the linear approximation $f'(x_*)$ by accident! This number is indeed the **derivative of $f(x)$ at $x = x_*$**.

While linearization is pretty easy to implement, it does not work all the time. For instance, one may encounter functions that do not admit a linear approximation (see exercise 3.9).

Exercise 3.11. Consider the "usual" discrete logistic model

$$p_{n+1} = p_n + a\,(K - p_n)\,p_n, \quad p_n \in [0, \infty).$$

Implement a change of variables $p_n \to x_n$ so that the above difference equation becomes (3.4) for appropriate r (defined in terms of a, K).

Exercise 3.12. Consider the Ricker model

$$p_{n+1} = e^{r\left(1-\frac{p_n}{K}\right)} p_n$$

where $r \geq 0$ and $K > 0$.

a) Implement a change of variables $p_n \to x_n$ so that the above difference equation becomes

$$x_{n+1} = e^{r(1-x_n)} x_n.$$

b) Plot a graph of the time-one map for this dynamical system. Contrast the plot with that of the time-one map for the dynamical system in (3.4). *Hint: look at $x \gg 1$.*

c) Determine the stability of the two equilibria of the Ricker model as a function of r. *Hint: remember that, for every real number z,*

$$e^z = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \cdots.$$

d) Based on your answer in part b), do you think the Ricker model has any qualitative advantages over the discrete logistic model studied in the examples?

4 Invariant Sets

Let $\mathcal{S} = \{X, f\}$ be a dynamical system, and let $x_* \in X$ be an equilibrium of $\mathcal{S}$. We already know that, once a dynamical system touches an equilibrium, it always stays there. In other words, the set

$$\{x_*\} \subseteq X$$

is **invariant** under the dynamics. This naturally leads us to ask whether or not there are interesting invariant sets that are not just made of equilibria (of course, the entire state space itself is an invariant set, but this is a tautology!). For our purposes, there are two more types of invariant sets we need to define:

- **periodic orbits** and
- **attractors**.

4.1 Orbits and Periodicity

Before concluding our general theoretical discussion, we need to introduce an important class of subsets of state space. Throughout this brief section, let $\mathcal{S} = \{X, f\}$ be a dynamical system.

Definition 4.1. *Let $x_0 \in X$. Define the **orbit of** x_0 by*

$$\mathcal{O}_{x_0} = \{x \in X \mid x = \Phi_n(x_0) \quad \text{for some } n \in \mathbb{N}_0\}.$$

In plain language, $\mathcal{O}_{x_0}$ is just the set containing all the points visited by x_0 as it moves according to the prescribed dynamics. Note that a simulation of a dynamical system starting from a given x_0 is really just a computation of a subset of $\mathcal{O}_{x_0}$.

Example 4.2. *Suppose x_* is an equilibrium of $\mathcal{S}$. Then,*

$$\mathcal{O}_{x_*} = \{x_*\}.$$

In both pure and applied mathematics, one is often concerned with orbits that carrying a little more structure:

Definition 4.3. *Let $\mathcal{O} \subseteq X$ be an orbit. We say that $\mathcal{O}$ is a **periodic orbit with period** N (or simply an N**-periodic orbit**) if N is the smallest nonzero natural number such that*

$$\Phi_N(x) = x \quad \forall\, x \in \mathcal{O}.$$

Example 4.4. *If x_* is an equilibrium point, then $\mathcal{O}_{x_*} = \{x_*\}$ is a 1-periodic orbit.*

Example 4.5. *Consider the dynamical system $\{\mathbb{R}, x \mapsto -x\}$. The orbit of $x = +1$ is*

$$\mathcal{O}_{+1} = \{-1, +1\}.$$

This is clearly a periodic orbit with period 2, since

$$f(\pm 1) = \mp 1$$

$$f(f(\pm 1)) = \pm 1.1$$

Example 4.6. *Let $\mathcal{S} = \left\{S^1, \theta \mapsto \theta + \frac{\pi}{2}\right\}$. We have that*

$$\mathcal{O}_0 = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$

is a 4-periodic orbit.

Of course, we can determine if a given point lives in a periodic orbit through numerical simulations. However, it would be nice to have some theory that tells us where to start looking! Fortunately, this is conceptually straightforward. Let $\mathcal{O}$ be an N-periodic orbit for the dynamical system $\mathcal{S}$, and let $x \in \mathcal{O}$. We then have

$$f^{(N)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{N \text{ times}}(x) = x. \tag{4.1}$$

Now, define a new dynamical system by

$$\mathcal{S}^{(N)} = \left\{X, f^{(N)}\right\}.$$

Using (4.1), we see that *each equilibrium of $\mathcal{S}^{(N)}$ is an element of an M-periodic orbit of $\mathcal{S}$, where $M \leq N$.* In other words, finding all N-periodic orbits of $\mathcal{S}$ can be accomplished by finding the equilibria of a related dynamical system. This also means that we can talk about stability of periodic orbits using the theory we've already built up for stability of equilibria!

Remark. If $\mathcal{S}$ has a unique N-periodic orbit, then $\mathcal{S}^{(N)}$ must have at least N equilibria.

Example 4.7. *So far, all of the examples of periodic orbits we've seen are "generic", in the sense that the underlying state space is full of periodic orbits. Here, we consider a 2-periodic orbit that is unique. Consider the state space $\mathbb{R}$ with dynamics generated by*

$$x_{n+1} = 1 - x_n^2.$$

The equilibria of this system are $\frac{-1\pm\sqrt{5}}{2}$. We may look for 2-periodic orbits by noticing that

$$f^{(2)}(x) = 1 - \left(1 - x^2\right)^2 = x^2(2 - x^2).$$

Then, we simply find the equilibria of $\{\mathbb{R}, x \mapsto x^2(2 - x^2)\}$:

$$x = f(x) \iff x = x^2(2 - x^2) \iff x = 0, 1, \frac{-1 \pm \sqrt{5}}{2} \quad \textit{via guess-and-check.}$$

Since we already know $\frac{-1\pm\sqrt{5}}{2}$ are equilibria, we conclude that the only 2-periodic orbit of our dynamical system is

$$\mathcal{O}_0 = \{0, 1\}.$$

This is easily verified via direct computation:

$$1 - 0^2 = 1, \quad 1 - 1^2 = 0.$$

Exercise 4.1. Show that the Malthus system $\{\mathbb{R}, x \mapsto ax\}$ has no periodic orbits of period greater than one unless $a = -1$.

Exercise 4.2. Suppose $\mathcal{O}$ is an N-periodic orbit for $\mathcal{S}$. Prove that

$$\Phi_{n+N}(x) = \Phi_n(x) \quad \forall\, x \in \mathcal{O}, \quad \forall\, n \in \mathbb{N}_0.$$

Exercise 4.3. Draw a cobweb diagram for the dynamical system in example 4.7, making sure to show the trajectories for $x_0 = 0$ and $x_0 = 0.5$. Use your diagram to determine whether or not the unique 2-periodic orbit is "stable".

4.2 Attractors (both Familiar and Strange)

Definition 4.8. *(Informal) Let $U \subseteq X$. We call U a **(stable) attractor** if*

1. *U is an invariant set: for all $n \in \mathbb{N}_0$ and all $x \in U$, $\Phi_n(x) \in U$,*
2. *U is Lyapunov stable: for all $x \in X$ that are "close to" U, $\Phi_n(x)$ is also "close to U" for all n (this means we could eventually have $\Phi_n(x) \in U$)*
3. *U is attracting: for all $x \in X$ that are "close to" U,*

$$\lim_{n\to\infty} \Phi_n(x) \in U.$$

Example 4.9. *Let x_* be an asymptotically stable equilibrium for $\mathcal{S} = \{X, f\}$. Then, $\mathcal{O}_{x_*}$ is an attractor.*

This definition allows us to sensibly discuss the "asymptotic stability" of a periodic orbit. We can also apply the definition to discuss more exotic invariant sets that are not so easy to write down. For instance, when we perform numerical experiments on the discrete logistic equation, we'll see **strange attractors** appear "from thin air": these complicated invariant sets cannot be drawn with pencil and paper!

Exercise 4.4. Is an attractor necessarily a single orbit?

References

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[3] Yuri A. Kuznetsov, *Elements of applied bifurcation theory*, third edition, Applied Mathematical Sciences, vol. 112, Springer-Verlag, New York, 2004.