5/11/2023

Lec 1 Introduction + Modelling Population of Microorganism
Part 1

Gause's Experiments on Paramecia

Fast growth quickly levelling off = logistic growth

Interpretation:

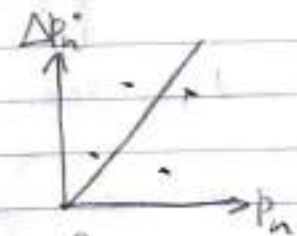
the change in population density from $n \to n+1$ is $r>0$ times at n

$$\Delta P_n = r P_n$$

$P_{n+1} \to P_n \Rightarrow P_{n+1} = (1+r)P_n \to$ Difference eqn (recursive)

(3) Compute P_n using only P_0?

$$P_n = (1+r)^n P_0$$



How to find r?

$$\Delta_P = P_{n+1} - P_n$$

obey $\Delta P_n = r P_n$

line of best fit slope $= r$

linear algebra

Malthusian growth $\neq$ logistic growth

$\Delta P_n = r(P_n) P_n$ $\quad r(P_n)$ vanish when P_n reach $K > 0$

$r(P_n) \geq 0$ $\quad r(P_n) = a(K - P_n)$

Param 1 = K

Param 2 = a

carrying capacity = Equilibrium = L

DLM = Discrete logistic model

$$r_{n+1} = P_n\left(1 + a(K - P_n)\right)$$

Use

For different a, K, DLM may give rise to <u>chaos</u>

Beverton-Holt model : Involve b, L

$$\Delta_{P_n} \quad r(\underbrace{P_n}) = \frac{b}{1 + \frac{b-1}{L}\underbrace{P_n}} - 1$$

$$r(L) = \frac{b}{1 + \frac{b-1}{L}L} - 1$$

As $n \to \infty$, BH solution with $P_0 > 0$

$\cdot \; P_{eq} = L$

P_{eq} is stable ~~equill~~ equilibrium

The ricker model $\qquad r(P_n) = e^{c\left(1 - \frac{P_n}{m}\right)} - 1$

$$\exp(x) = \lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$P_{n+1} = f(P_n)$$

$$f : [0,\infty) \to [0,\infty)$$

Malthus: $f(p) = (1+r)p$

DLM

BH

Ricker

All discrete-time dynamical-systems

Abstract Dynamical System Theory

Ref Kuznetsov Ch1

(1) State space

- Plain English X = set of all possible states some physical / biological the process can be in

info. require to complete understand

$X = \dot{x} = \dot{y}$

IMPORTANT: X can be any set!

Ex1: Population modelling State space?

$X = \mathbb{N}_0$ Round number

$X = \mathbb{R}$ or $[0,\infty)$ $[= \leq, \geq$ $(= <, >$

Ex2: Wasps + Spiders? (competing)

$$X = \left\{ (w,s) \;\middle|\; \begin{array}{l} w \in [0,\infty) \\ s \in [0,\infty) \end{array} \right\}$$ Def = Set $(S_1 \times S_2)$

$= [0,\infty) \times [0,\infty)$

Ex3: Bead on a wire sliding without F

$$\underline{X} = \overset{\text{position}}{(-\infty,\infty)} \times (-\infty,\infty)$$
$$= \mathbb{R}^{\text{position}} \times \mathbb{R}^{\text{velocity}}$$
$$= \mathbb{R}^2$$

Ex4: Bead sliding on a hoop?

$$X = \underset{\text{position}}{(\text{Angle around circle})} \times \mathbb{R}$$

Unit Circle $= S^1$

$$\therefore X = S^1 \times \mathbb{R} = \text{Cylinder (hollow)}$$

(2) Evolution Operators

Def: A family of functions

$$\Phi_n(x) : \underline{X} \to X$$
$$x \mapsto \Phi_{(x)}$$

Index by "n" $\in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is called a "fam." of evolution operators if

1) $\Phi_0(x) = x \quad \forall x \in X$

2) $\forall n, m \in \mathbb{N}_0$

$$\left.\Phi_{n+m}(x) = \Phi_n\left(\Phi_m(x)\right)\right\} \text{"Autonomy"}$$

Think of n = time

$\Phi_n(x)$ = result of moving x for time n

(3) Dynamical Systems Defined

Def: DS is a pair

$$S = \{X, \{\Phi_n\}_{n \in \mathbb{N}_0}\}$$

w/ X and $\{\Phi_n\}_{n \in \mathbb{N}_0}$ a fam. of eval. ops.

Ex.1 (Malthus type growth/decay)

$$X = \mathbb{R}, \text{ Fix } a \in \mathbb{R}, a \neq 0$$

$$\Phi_0(x) = x; \Phi_1(x) = ax, \Phi_2(x) = a^2 x$$

$$\Phi_n(x) = a^n x$$

Check Autonomy: $\begin{cases} \Phi_{n+m}(x) = a^{n+m} x \\ \Phi_n(\Phi_m(x)) = a^n(\Phi_m(x)) = a^n a^m x = a^{n+m} x \end{cases}$

$\Rightarrow \{\mathbb{R}, \{\Phi_n(x)\}_{n \in \mathbb{N}}\}$ fam. of DS

$x_0 \in \mathbb{X}$ any real ~~#~~

$x_1 = \Phi_1(x_0) = ax_0$

$x_2 = \Phi_1(x_1) = a^2 x_0 = \Phi_2(x_0)$

$x_{n+1} = \Phi_1(x_n) = ax_n$

Malthus / gen.

In gen. D.S.,
def:
$f : \mathbb{X} \to \mathbb{X}$
by $f(x) = \Phi_1(x)$
Then $x_1 = f(x_0)$
$x_{n+1} = f(x_n)$
$\Rightarrow \Phi_{n+1}(x_0) = x_{n+1}$
Because off autonomy

Ex.2

$\mathbb{X} = [0, \infty)$

$f(x) =$ DLM, BH, or Rickers
$= x + ax(K-x)$

$\Phi_0(x) = x$, $\Phi_1(x) = f(x)$, $\Phi_2(x) = f(f(x))$, ... $\Phi_n = \overbrace{f(f(f(f(f \ldots f(x)}^{n \text{ times}}$

$\Rightarrow \{\mathbb{X}, \{\Phi_n\}\}$ give a D.S

Ex.3:

$\mathbb{X} = \mathbb{R}$

$f(n,x) : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$

Difference equation $x_{n+1} = f(n, x_n)$ (*)

Def. D.S

$\Phi_0(x) = x$ $\quad \Phi_2(x) = f(2,x)$ $\overset{?}{=} f(1,2)$ or 0

$\Phi_1(x) = f(1,x)$

Not always true

$\Phi_1(\Phi_1(x)) = f(1, f(1,x))$

Not a D.S., Autonomy fail!

(4) Equilibrium

Think about carrying capacity - K

$\mathcal{S} = \{X, \{\Phi_n\}_n\}$ be a D.S. Let $f(x) = \Phi_1(x)$

What pts. in X are special?

- Pts. that don't move in time

i.e. $\boxed{x_{eq}}$ s.t. $x_{eq} = f(x_{eq})$ $f(x_{eq}) = \Phi_1(x_{eq})$

Def. $x_{eq} \in X$ is called an "equilibrium" of $\mathcal{S}$ if $f(x_{eq}) = x_{eq}$. "fixed pt"

Lemma: $\forall n \in \mathbb{N}$, $\Phi_n(x_{eq}) = x_{eq}$

Pf: Use autonomy

$$X_{n+1} = a x_n \Rightarrow X_{eq} = 0$$

$$f(x) = x \to ax = x \begin{cases} x = 0 \\ x \neq 0 \Rightarrow a = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \text{ if } x = 0 \\ a = 1 \text{ if } x \neq 0 \end{cases}$$

So: if $a \neq 1$, ∞ is the unique equil.
if $a = 1$, every $x \in \mathbb{R}$ is an equil.

Ex 8: (DLM) $X_{n+1} = x + a\left(K - \cancel{x}^{\,x_n}\right) X_n$

$X_{n+1} = f(x_n)$ $\quad x = f(x) = x + a(K - x)x$

$$\Rightarrow 0 = a_x (K - x)$$

$$\Rightarrow X_{eq} = K, 0$$

(5) Stability of Equilibria

Q: 1) How to describe equil. being "repelling" or "attract"?
2) How does "attraction"/"repulsion" of X_{eq} depend on how far we start from it?

Def: (informal) (not precise)
Let $\vec{S} = \left\{ \mathbb{X}, \{\Phi_n\}_{n \in \mathbb{N}_0} \right\}$ be a D.S, w/ $X_{eq} \in \mathbb{X}$ an equili. for $\vec{S}$

1) we say X_{eq} is "Lyapunov Stable" if, for $x \approx X_{eq}$ we have $\Phi_n(x) \approx X_{eq}$ $\forall n$

$(X_0 \approx X_{eq} \Rightarrow X_n \approx X_{eq})$

2) We say X_{eq} is attracting if for all $X_0 \in \mathbb{X}$,
$\lim_{n \to \infty} \Phi_n(x) = X_{eq}$ (ie. $\Phi_n(x) \approx X_{eq}$ for ~~no~~ $n >> 1$)

3) We say X_{eq} is "assymptotically stabe" if it's Lyapunov Stable + attracting.

4) We say X_{eq} is unstable if it isn't 1) or 2)

Lyapunov stable $D.S \{T, X, \varphi^t\}$ with $\dot{X}$

$U \supset S_0$

$\dot{X}$: complete metric

$V \supset S_0$

S_0: clo

Review last week

Single-Species pop. models

Recursion
Difference
~~equ~~ eq'n

$P_{n+1} = f(P_n)$ P_n = Pop. density at day n

(Time-one map $= \Phi_1$)

Discrete-time
Dynamical
System Dynamic

Different f's: + DLM = Discrete logistic model
+ Ricker
$\boxed{\text{+ Malthusian}}$ - Can't capture carrying capacity
+ Bev-holt

DLM & Bev-holt : Stable carrying capacity

Dynamical System Theory

Two ingredients in a dynamical systems:

1) State space

2) Evolution Operators $\Phi_n(x)$
(rules for moving "n" time steps)

$x_0 \in X$, $x_1 = \Phi_1(x_0)$, $x_2 = \Phi_2(x_0)$ 2 step from 0

$x_n = \Phi_n(x_0) = \Phi_n(x_{n-1})$ $= \Phi_1(x_1)$ 1 step from 1

Pop. Model : $P_0 = 0$
Should have $P_n = 0 \; \forall n$
"Equilibrium"

Autonomy
independant

$x_{eq} \in X$ if St $\{X, f\}$ $x_{eq} = f(x_{eq})$ $= \Phi_1(x_{eq})$

$\{X, \Phi_n\}$ $\Phi_2(x_{eq}) = f(f(x_{eq}))$

$x_{eq} = f(x_{eq})$

$= x_{eq}$

More sophisticated ex carrying capacity

Stability of ~~eqat~~ equilibria tell us if equilibria attract or repel nearby points

Ex. $x_{n+1} = a x_n$ $\quad a \in \mathbb{R},\ a \neq 0$

Equilibrium $x_* = a = 1$
$x_* = 0$

$f(x) = ax$
$x_* = a x_*$
if $x_* = 0$, works
if $x_* \neq 0$, $a = 1$
$\Rightarrow$ Any x_* works

~~STATE SPACE~~ STABILITY of x_*?

Unstable bc. exp. growth. IF $|a| > 1$

$|a| < 1 \rightarrow$ Decay attracting, asymptotically stable

$|a| = \cancel{0}\ 1 \Rightarrow$ Lyapunov stable

1) $a = 1$ $\quad x_{n+1} = x_n$
if $x_0 \approx x_*$, $x_n = x_0 \approx x_*$
$\Rightarrow$ Lyapunov stable

2) $a = -1$ $\quad x_{n+1} = -x_n$

$x_1 = -x_0 \Rightarrow x_2 = -x_1 = x_0$
So $x_n = (-1)^n x_0$

$x_0 \approx 0 = x_* \Rightarrow x_n \approx (-1)^n x_0 \approx 0 \Rightarrow$ Still Lyapunov

$x_n = (a)^n x_0$ i.e. $\Phi_n(x) = a^n x$

iii) $|a| < 1$, $x_0 \approx x_* = 0$

Want to show asymp. stable:

a) Lyapunov
b) Attracting

a) Lyap.?

$$|x_n| = \underbrace{|a|^n}_{<1} |x_0| < |x_0| << 1$$

$\Rightarrow$ Lyap.

b)
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a^n x_0$$
$$= \cancel{\lim_{n\to\infty}} \left(\lim_{n\to\infty} a^n\right) x_0$$
$$= 0\, x_0$$
$$= 0$$
$$= x_*$$

c) $|a| > 1$, $x_0 \approx x_* = 0$

want to show <u>unstable</u>

$\lim_{n\to\infty} |x_n| = +\infty$ bcs $|a^n|^n \to \infty$

Malthus System $\Rightarrow$ stability was change to discuss

Harder, general problems, need some theory ...

1) Cobweb Diagrams (visually)

2) Linearization (Symbolic) requires calc.

Ex 38 on sheet $x_{n+1} = 2x_n$ ie $f(x) = 2x$

Intersect @ Equilibrium pt. $x_* = 0$

$x_n = f(x_n)$

Up to solid over dash (+)

Down solid over dash (−)

$\Rightarrow$ unstable

Linearization

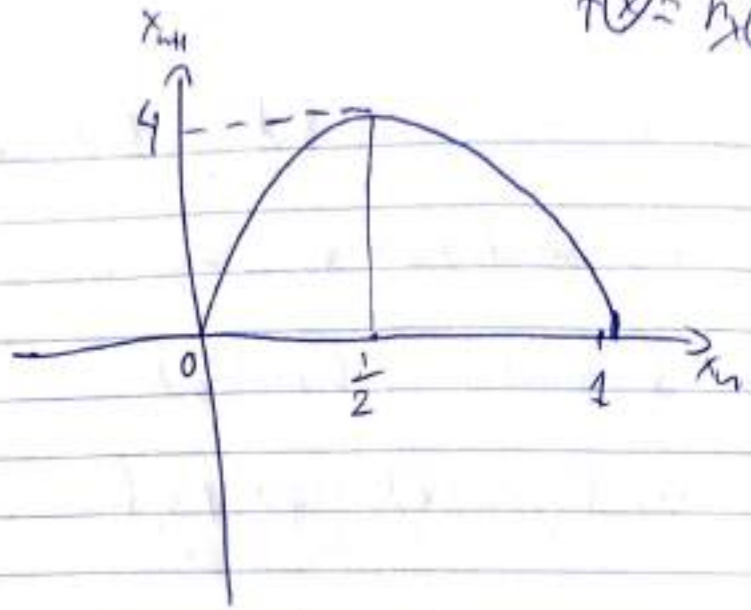
Ex: DLM $x_{n+1} = r x_n (1 - x_n)$

r : perimeter

when can we take $[0,1]$ as our state space?

Sf: Time-one map take $[0,1]$ to itselfs.

$f(x) = rx(1-x)$



need: $f(x) \le 1$

$f(\frac{1}{2}) = r/4$

$\Rightarrow r \in [0, \cancel{1}\, 4]$

Equilibria? $x_{*,0} = 0$

other: $x = f(x) = rx(1-x) \Rightarrow 1 = r(1-x) \Rightarrow x = 1 - \frac{1}{r}$

$\frac{1}{r} = 1 - x$

$x = 1 - \frac{1}{r}$

Consider $x_{*,0} = 0$. first we look @ its stability

Pick smth small $\quad x_0 \approx 0$ so $|x_0| \ll 1 = \frac{1}{10^6}$

$x_1 = rx_0 - \underbrace{rx_0^2}$

$\approx 10^{-5} \quad \approx 10^{-11}$ } upper bounds

$\approx 10^{6-5} \approx rx_0$ take smallest one

$\boxed{x_{n+1} \approx rx_n}$ $\Rightarrow$ • Asymptote if $r < 1$

• unstable if $r > 1$

$$x_{*,1} = 1 - \frac{1}{r} = \frac{r-1}{r}$$

$$y_1 = x_n - \left(\frac{r-1}{r}\right) \text{ i.e. } x_n \approx x_{*,1} \iff |y_n| \ll 1$$

$$\begin{aligned} y_{n+1} &= x_{n+1} - x_* \\ &= r x_n (1 - x_n) - x_* \\ &= r(y_n + x_{*,1})(1 - x_{*,1} - y_n) - x_{*,1} \\ &= r(1 - 2x_{*,1}) y_n + r x_{*,1}(1 - x_{*,1}) - x_{*,1} - r y_n^2 \\ &\approx r(1 - 2x_{*,1}) y_n \end{aligned}$$

Thrown away $\square^2$

$\boxed{y_{n+1} \approx (2-r) y_n}$ Asymptote. Stable if $|2-r| < 1$

if $r < 3$

UNSTABLE if $r \in [3, 4]$

Last time

Linearization of Ricker model

$$e^x = 1 + x + \frac{1}{2}x^2 \ldots$$

Orbits, Periodicity, "Attractors"

$$O_{x_0} = \{x \in X \mid x = \Phi_n(x_0), n \in \mathbb{N}\}$$

Periodic Orbits

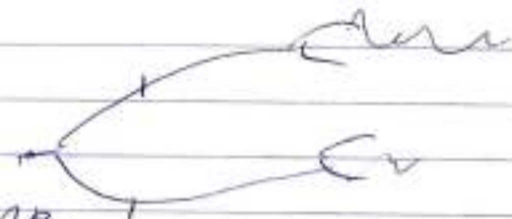
$x_{n+1} \neq x_n$

$x_0 = 1$ $\quad x_n = (-1)^n \quad x_{n+1} = f(x_n)$

$x_1 = -1$ $\quad$ Periodic $\quad$ Eq: $x = f(x)$

$x_2 = +1$ $\quad$ w/ Period 2 $\quad$ Period 2: $x = f(f(x))$

$x_3 = -1$



1PO

2PO

Equilibria of the LV model

$$\begin{cases} H_{n+1} = rH_n(1-aP_n) \quad (1) \\ P_n = ac\,H_nP_n \quad (2) \end{cases}$$

$$(H_*, P_*) = (0, 0)$$

$$= \left(\frac{1}{ac}, \frac{r-1}{ar}\right)$$

if $P_n \neq 0$

$(2) \Rightarrow 1 = a_c H_n$

$H_n \neq 0 \quad (1) \Rightarrow 1 = r(1-aP_n)$

Model 2: Nicholson - Bailey

Poisson-distributed

$H_{n+1} = rH_n e^{-aP_n}$ (coexistence on High co.)

= logistic growth P_n = unchange

Hybrid NB-B-H Model

equilibrium or oscillation.

LPA model of Intraspecies Predation

Single-Species & Host-Parasitoid

Age-structured population

DEM for interaction b/w larva, pupae, adults in a flour beetle population

$n = 2$ weeks
$4n = 8$ weeks

c = consuming

c_{el} ($L \subset E$) c_{ea} c_{pa}

Reproduction rate b and larva + adult

~~μ_e, μ_{el}~~ μ_l, μ_a

$$\boxed{\begin{aligned} L_{n+1} &= bA_n e^{-c_{el}L_n - c_{ea}A_n} \\ P_{n+1} &= (1-\mu_l) L_n \\ A_{n+1} &= P_n e^{-c_{pa}A_n} + (1-\mu_a) A_n \end{aligned}}$$

circadian rhythm

linked to mother-time

all likely divide by certain time of day

Modular ~~Arith~~ Arithmetic (Periodic on $\mathbb{Z}$)

Def'n Fix an $\mathbb{Z}$ "a"

We say 2 $\mathbb{Z}$ p,q are "equivalent mod a"

($p \equiv q$, mod a) if there's an int m such that

$$p = ma + q$$

1) ~~1→7~~ $1 \equiv 3 \bmod 2$

2) $16 \equiv 1 \bmod 3$

$$16 = 5 \times 3 + 1$$

$$m = 5$$

3) $p \equiv ? \bmod p$

$? = 0$

Def'n Let x, y be real ~~#~~'s. We say that $x \equiv y \bmod 1$

if $x - y$ is an int

≈ roughly, dec. parts equal

1) $1 \bmod 1 = 0$ 2) $p \bmod 1 = \underset{0}{?}$ for p int

3) $1.51 \bmod 1 = \boxed{0.51} \in [0,1)$

i.e. b/c $1.51 - 0.51 = 1$, an int

4) $-1.51 \bmod 1 = -0.49$

$-1.51 - 0.49 = -2$

Ex 2:

a) $1.1 \bmod 1 = 0.1$

b) $-0.1 \bmod 1 = +0.9$

c) $11.55 \bmod 1 = 0.55$

d) $34.17 \bmod 24 = 10.17$

10.17 AM $\approx 10:10$

Ex 3: $a, b \in \mathbb{R}$ with $a \neq 0$

$ab = 0 \bmod 1$

Is it true that $b = 0 \bmod 1$?

Not always $b \in \mathbb{R} \Rightarrow b = q$ rational number

Circle Geometry $\quad 1 \geq 0$

Know $\bigcirc \approx [a, b]$

What intervals to choose?

1) Measure θ° in degree $= [0, 360]$

2) Radian $: [0, 2\pi]$

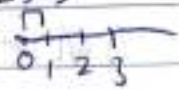
3) Circle has circumference $= 1$? $\quad C = 2\pi r$

$[0, 1]$

$S^1 = [0,1] / \{0 = 1\}$

Impose equivalence

$\cong \mathbb{R} \bmod 1$



Trig function Basics

Periodic Phenomena in Biology

- ~~Sleep~~ Sleep cycle
- Heart beat $\boxed{DOIP}$
- Cell Division ? (Coupled to night/day (circadian rhythm)
- Breathing

"Periodic motion coupled to Periodic motion"

Synchronization

Phase locking

$M_{forcing} : N_{spontaneous\ oscillations}$

T_n = time from start of experiments when gen. "n" was born

manipulation

b = gen. 0

Circle Dynamics

$\{\mathbb{R}, f\}$. $T_{n+1} = f(T_n)$

$\{S^1, \tilde{f}\}$ $T_{n+1} \bmod 1 = \tilde{f}(T_n) \bmod 1$

Def'n

$\tilde{f} : S^1 \to S^1$ by
$\tilde{f}(T \bmod 1) = f(T) \bmod 1$ ✱

if $T_n \bmod 1 = \varphi_n \to \varphi_{n+1} = \tilde{f}(\varphi_n)$

Ex. $f(T) = T$ $T_{n+1} = T_n$

Descended Circle Sys.

$\boxed{\varphi_{n+1} = \varphi_n}$

$\tilde{f}(\varphi) = f(T) \bmod 1$

$T \bmod 1 = T \bmod 1$
$= \varphi$

ONLY WORKS FOR SPECIAL f!!!

Ex. Consider $T_{n+1} = T_n^3$ i.e. $f(T) = T^3$

$-0.1 = 0.9 \bmod 1$
$\tilde{f}(-0.1 \bmod 1) = \tilde{f}(0.9 \bmod 1)$?

$= f(0.1 \bmod 1) = -(0.1)^3 \bmod 1 = 0.999$

(2)

$= (0.9)^3 \bmod 1$
$= 0.729$

$\Rightarrow$ Circle Dynamics is not well defined

"Diagram commutes"

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "\bmod 1"'] & \mathbb{R} \arrow[d, "\bmod 1"] \\ S^1 \arrow[r, "F"'] & S^1 \end{tikzcd}$$

Shift System

$T_{n+1} = T_n + T$

$T \in \mathbb{R}$ fixed

$\Rightarrow T_n = T_0 + nT$

Descended to a circle system

$\varphi_{n+1} = (\varphi_n + \tau) \bmod 1$

Possibly have periodic orbits!

Lemma Shift sys. has a periodic orbit iff and only if τ is rational

Pf Assume $\exists$ a P-periodic orbit O_{φ_0}

$\varphi_p = \varphi_0$
$= \varphi_0 + p\tau \bmod 1$

ab=0
b=0 or 1
Not always

$\Rightarrow 0 = p\tau \bmod 1$ So $p\tau = M$ an int
$\Rightarrow \tau = m/p$

11) Assume $\tau = \frac{M}{P}$ for M, P relatively prime

$$\varphi_n = \varphi_0 + n\frac{M}{P} \mod 1$$

Choose $n = P$

$$\Rightarrow \varphi_0 + M \mod 1$$
$$= \varphi_0$$

$\Rightarrow$ Periodic orbits of period $\leq P$

Rotation #

Def'n Given a sys. $T_{n+1} = f(T_n)$ that descends to well-defined circle dynamics, the ROTATION #'s ρ at φ_0 is $\rho = \lim_{n \to \infty} \frac{T_n - T_0}{n}$

Ex 1. T_* an equilib.

$$\rho = 0$$

Ex 2. $T_n = T_0 + n\tau$

$$\frac{T_n - T_0}{n} = \tau \Rightarrow \boxed{\rho = \tau}$$

Arnol'd

Arnold System

$\tau \geq$ shift $\in \mathbb{R}$

$b >$ Coupling $\in \mathbb{R}$
strength

$$T_{n+1} = T_n + \tau + b\sin(2\pi T_n) \quad \text{(A)}$$

T_n = time of cell division

Prop 7 i) If $|\tau/b| > 1$, (A) has no equilibria

ii) If $|\tau/b| < 1$, (A) has ∞ many ~~equil~~ equilibria that decends to two distinct eq. on the circle!

Prop 7 If a circle orbit for (A) is p-periodic

$\rightarrow$ ρ is rational!

In particular, $\rho = \frac{T_p - T_0}{p}$

ρ rational

$\approx$ Periodic exists

$\approx$ Phase-locking / Synchro. occurs