Assignment 5: Satellite orbits

Introduction

Dr. Robert Heinkelmann

Two body problem

General case of gravitational interaction: **n-body problem**

simplification:
Just two bodies considered in an isolated system, the effects of all other bodies cancel out or are neglected.

→ **Two body problem**

A light test body and a heavy central body $M >> m$

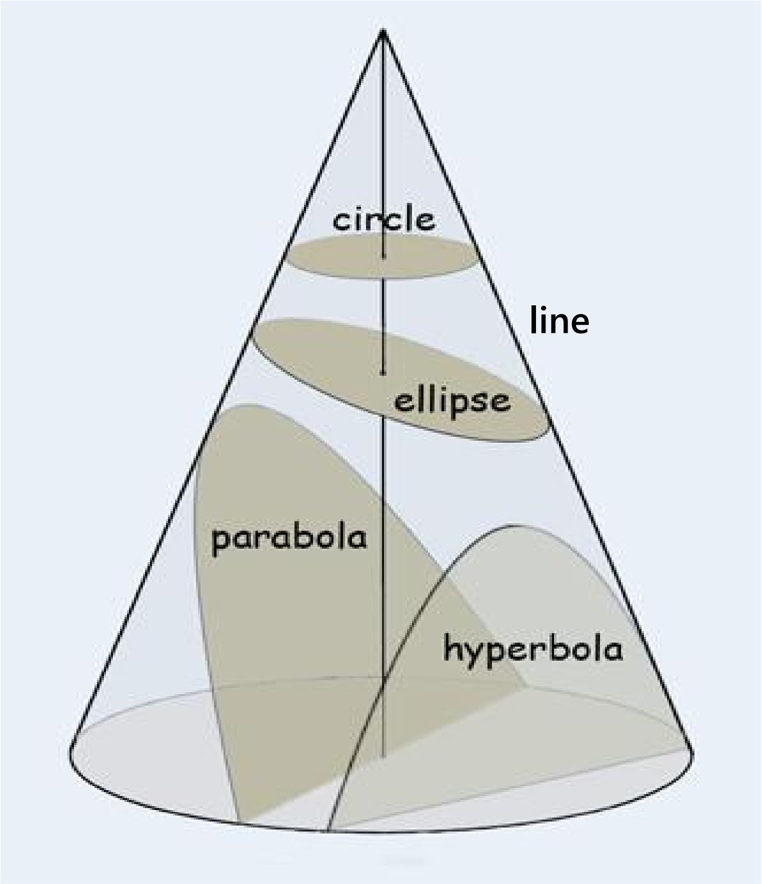
→ only the test body is "affected", while the central body stays at rest.

Example: Earth & artificial satellite:
$M = M_\oplus = 5.9722 \cdot 10^{24}$ kg
ISS (currently most heavy artificial orbiter):
$m = m_{ISS} = 4.44615 \cdot 10^5$ kg

Two body problem (cont.)

General solution:
The trajectory of the light test body takes the form of a conic section (see figure to the right).
Conic sections:

- line (very unlikely)
- hyperbola
- parabola
- ellipse
- circle (perfect circle also unlikely)



Two body problem (cont.)

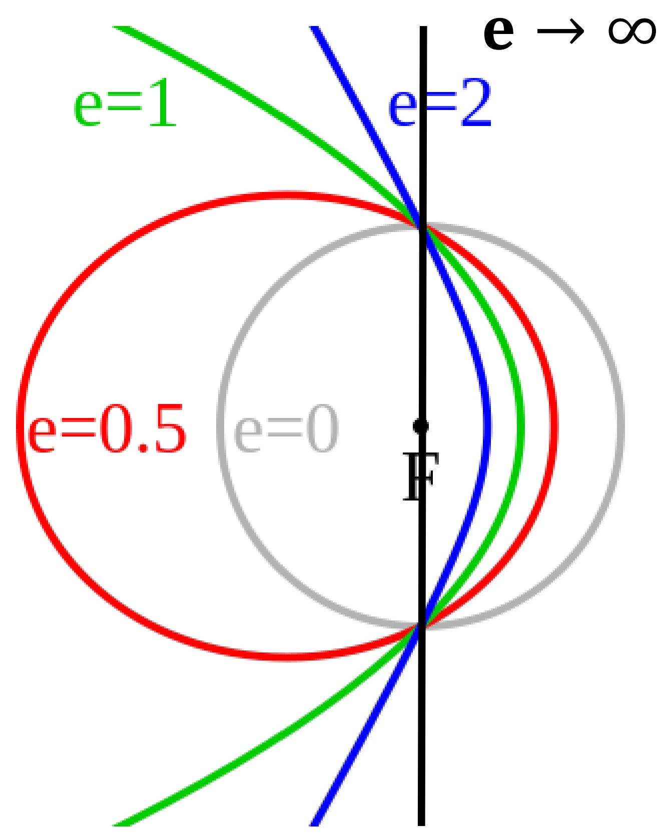
Problem:
Which trajectory does the satellite actually take?

Depends on the state of the satellite
(state = position and velocity)
and the mass of the central body.

Cases:
Decent or impact: $v < v_c$

Circular orbit:	$v_c = \sqrt{\frac{GM_\oplus}{r}}$	$\mathrm{e} = 0$
Ellipsoidal orbit:	$v_c < v < v_e$	$0 < \mathrm{e} < 1$
Parabolic orbit:	$v_e = \sqrt{\frac{2GM_\oplus}{r}}$	$\mathrm{e} = 1$
„Escape velocity"		
Hyperbolic orbit:	$v > v_e$	$\mathrm{e} > 1$
Straight line:	$r \to \infty$	$\mathrm{e} \to \infty$

Conic sections grouped by eccentricity:



Orbit definition

Orbit

General definition:
Any test mass that gravitationally interacts with a central mass.

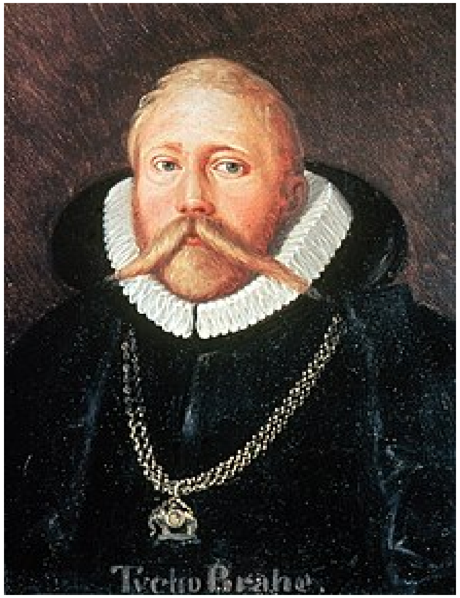
Narrow-sense definition:
(Customarily applied)
Gravitational two-body interaction that leads to a **closed trajectory**, i.e. **ellipse**
(circle is a special ellipse)

Orbit: the test mass "circumnavigates" the heavy central mass
⇒ orbit geometry: ellipse

Kepler Laws

Kepler inherited from **Tycho Brahe** (Danish Astronomer) a rich collection of very precise planetary positional time series. From the analyses of these observational data, he founded his well-known 3 laws.

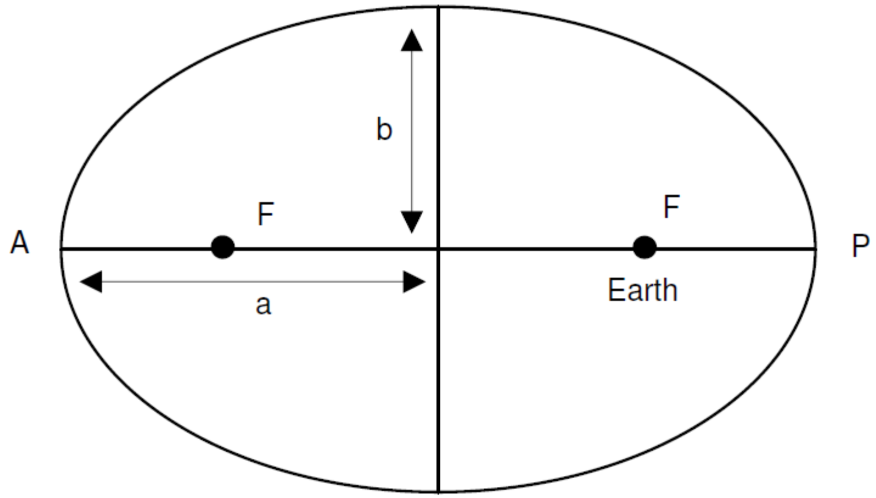
Johannes Kepler
27.12.1571 – 15.11.1630
German Generalist



Tycho Brahe
14.12.1546 – 24.10.1601
Danish Astronomer

Kepler Laws

Keplers 1st Law:
The orbit of a test mass is an **ellipse** with the central mass at one of the two foci.



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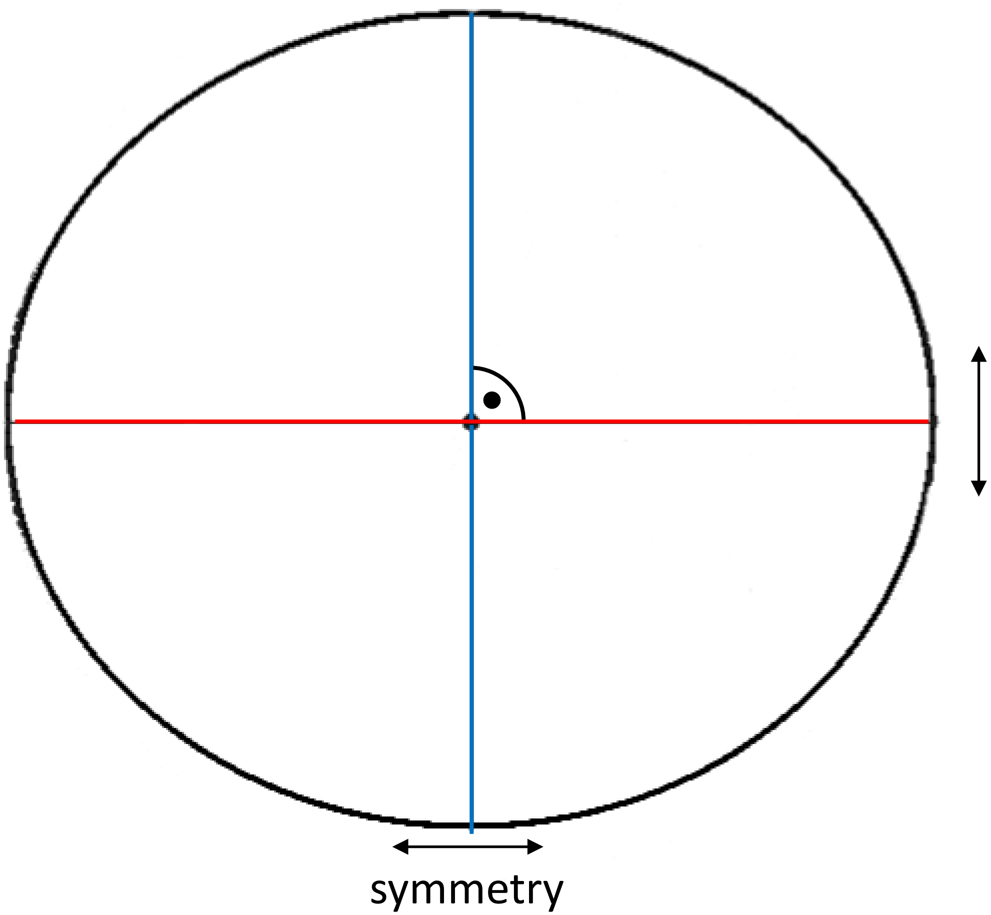
Ellipse terminology

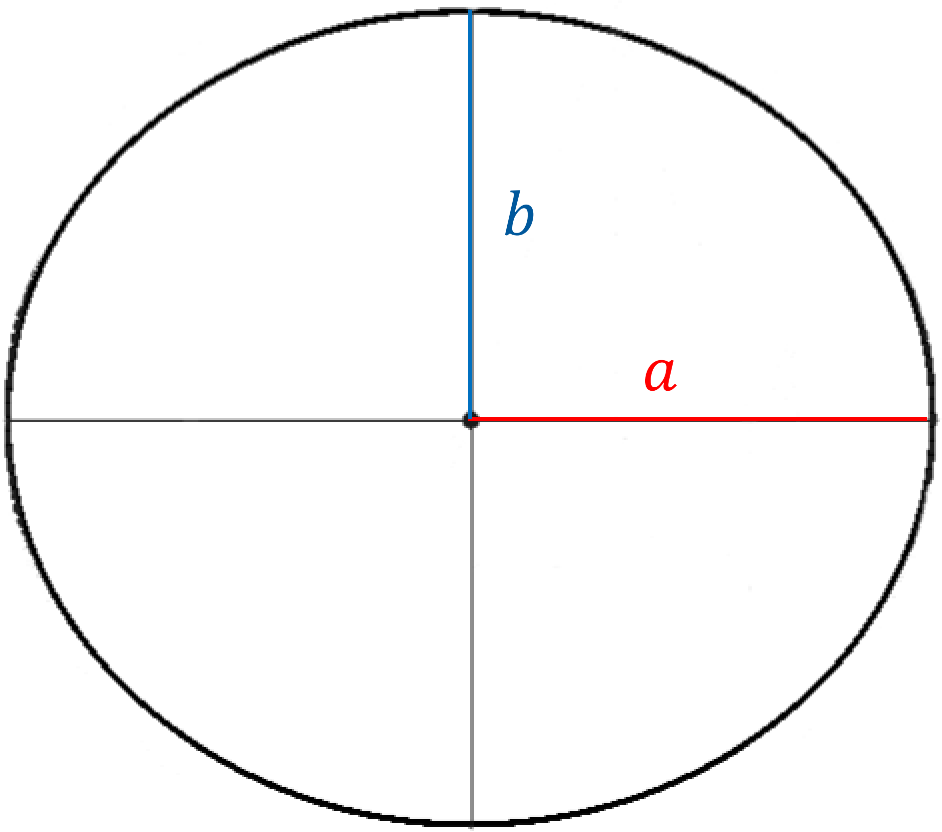


longest diameter of the ellipse:
major axis, line of apsides

shortest diameter of the ellipse:
minor axis

major axis and minor axis are perpendicular

Ellipse terminology



semi-major axis a

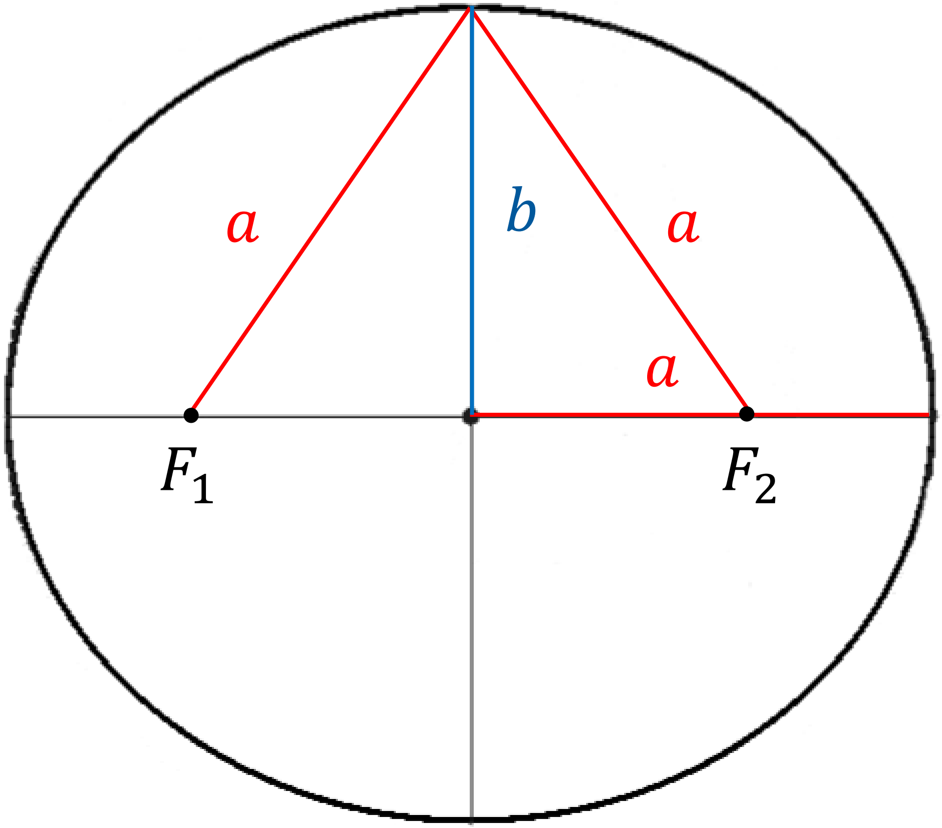
semi-minor axis b

definition interval:
$a, b \in \mathbb{R}^+$

with $0 < b < a$

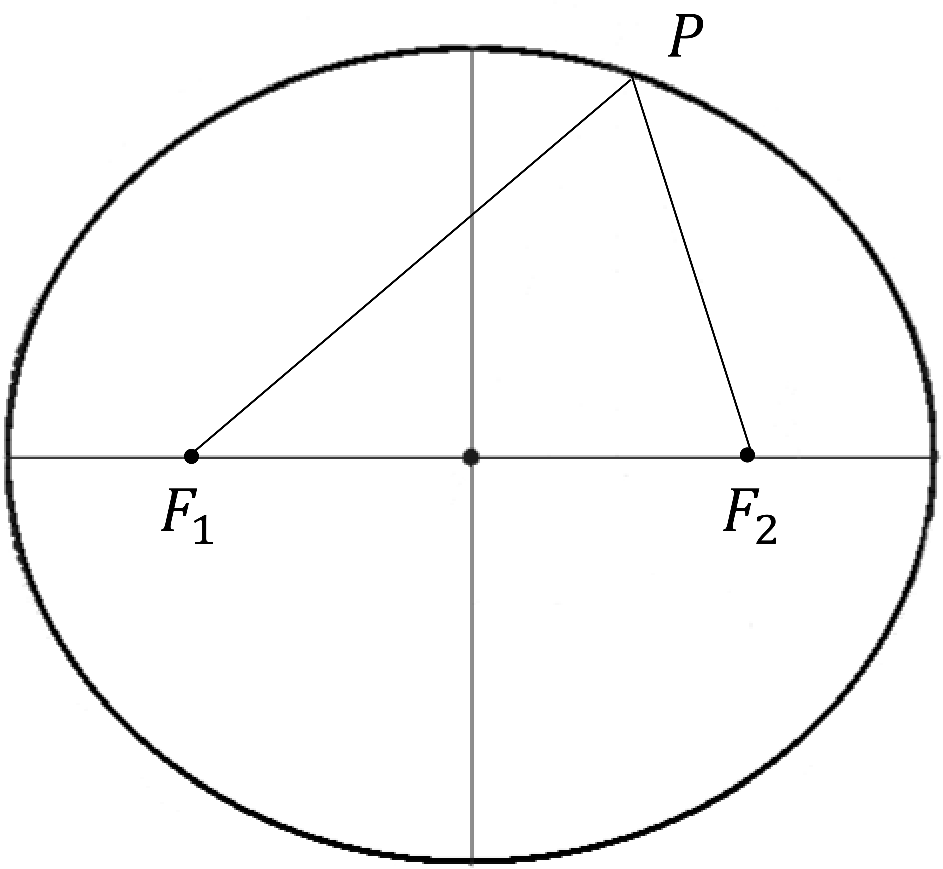
($0 < b \leq a$ circle as a special ellipse)

Ellipse terminology



F_1, F_2: focal points, foci

Ellipse terminology



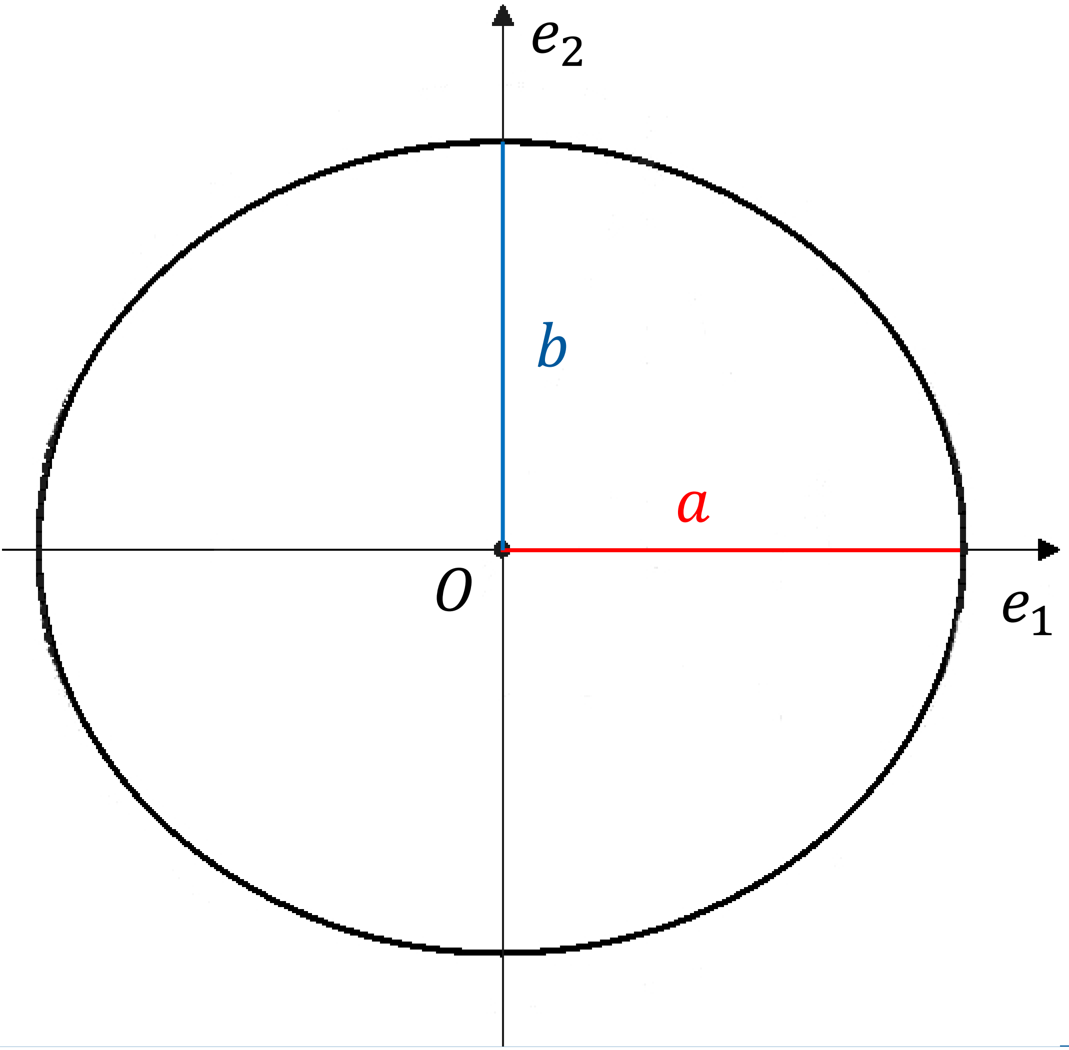
Ellipse in $\mathbb{R}^2$ as a set of points:
The ellipse as a **set of points** that share the characteristic that the sum of the distance to the two focal points of the ellipse $F_1, F_2 \in \mathbb{R}^2$ is constant:

$$E = \left\{P \in \mathbb{R}^2 \mid \left\|\overrightarrow{PF_1}\right\| + \left\|\overrightarrow{PF_2}\right\| = c\right\}$$

with the constant $c = 2a$.

P: any point on the ellipse E, $P \in E$, note that the sketch displays only one example

Ellipse terminology



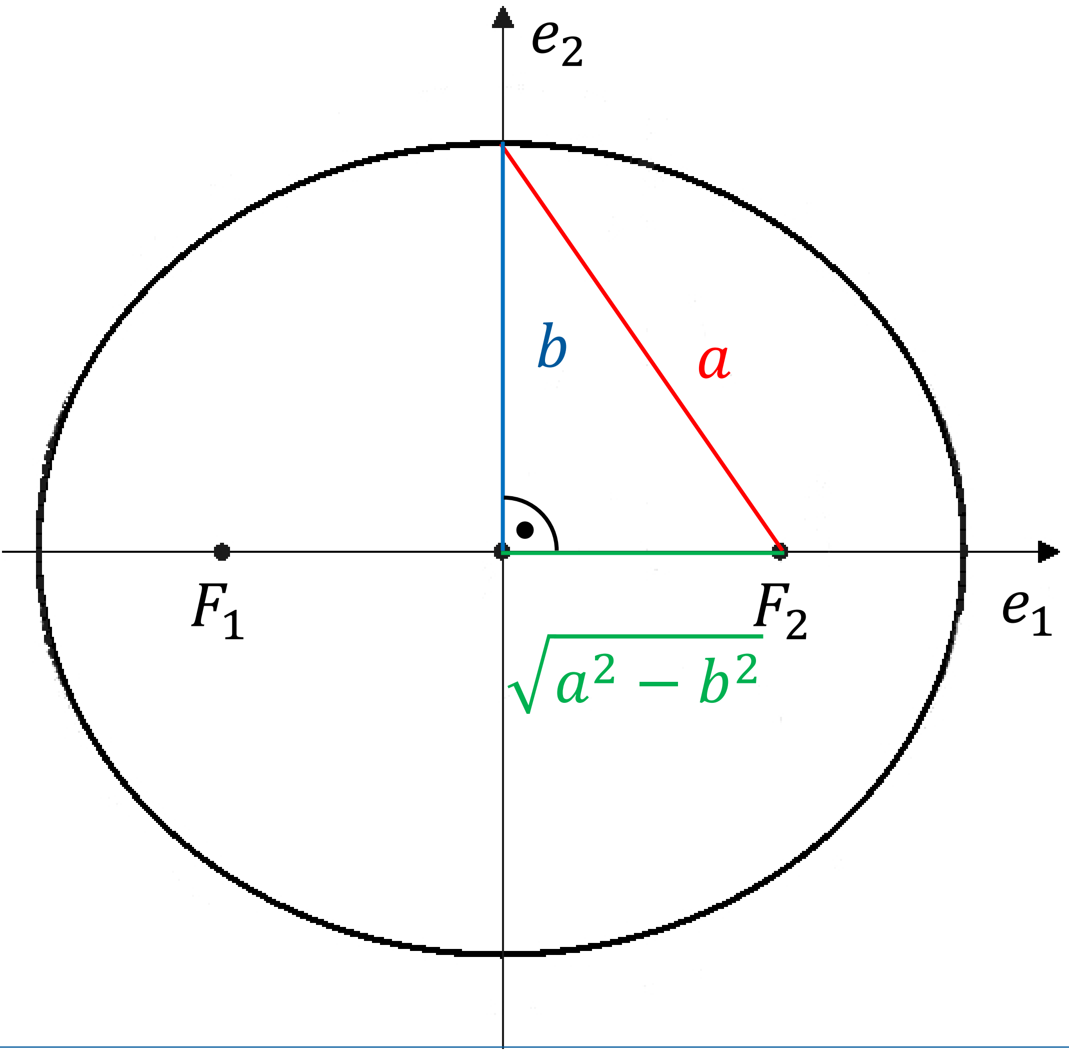
With the introduction of a Cartesian coordinate system as displayed in the figure, the ellipse in $\mathbb{R}^2$ can be specified in implicit form.

For all points on the ellipse w.r.t. to the $\{\boldsymbol{e_1}; \boldsymbol{e_2}\}$ - Cartesian coordinate system, the implicit **analytical equation** holds:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where (x,y) are the coordinates referring to the $\{\boldsymbol{e_1}; \boldsymbol{e_2}\}$ - Cartesian system.

Ellipse terminology

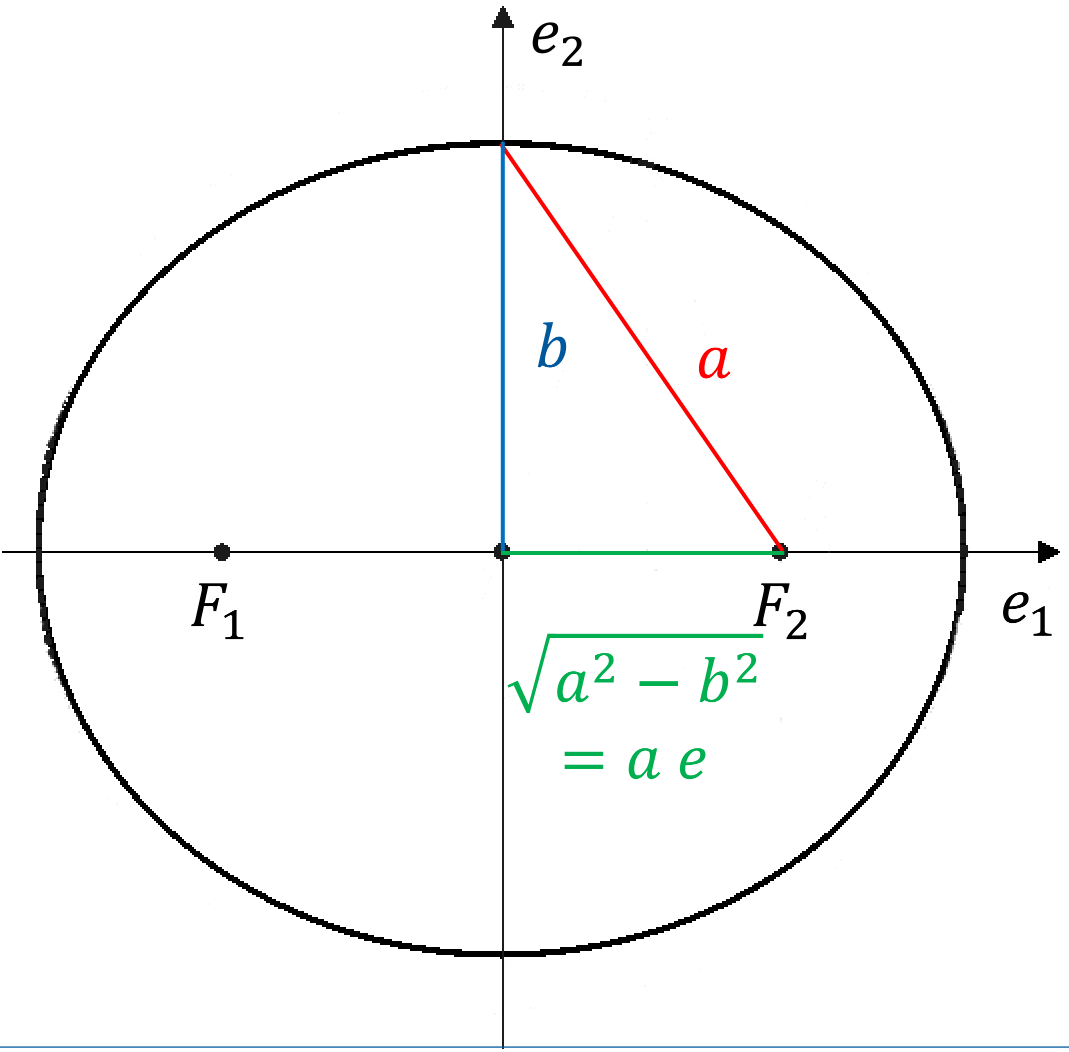


F_1, F_2: focal points, foci

Pythagorean theorem:

$$\left(\sqrt{a^2 - b^2}\right)^2 + b^2 = a^2$$

Ellipse terminology



F_1, F_2: focal points, foci

Pythagorean theorem:

$$\left(\sqrt{a^2 - b^2}\right)^2 + b^2 = a^2$$

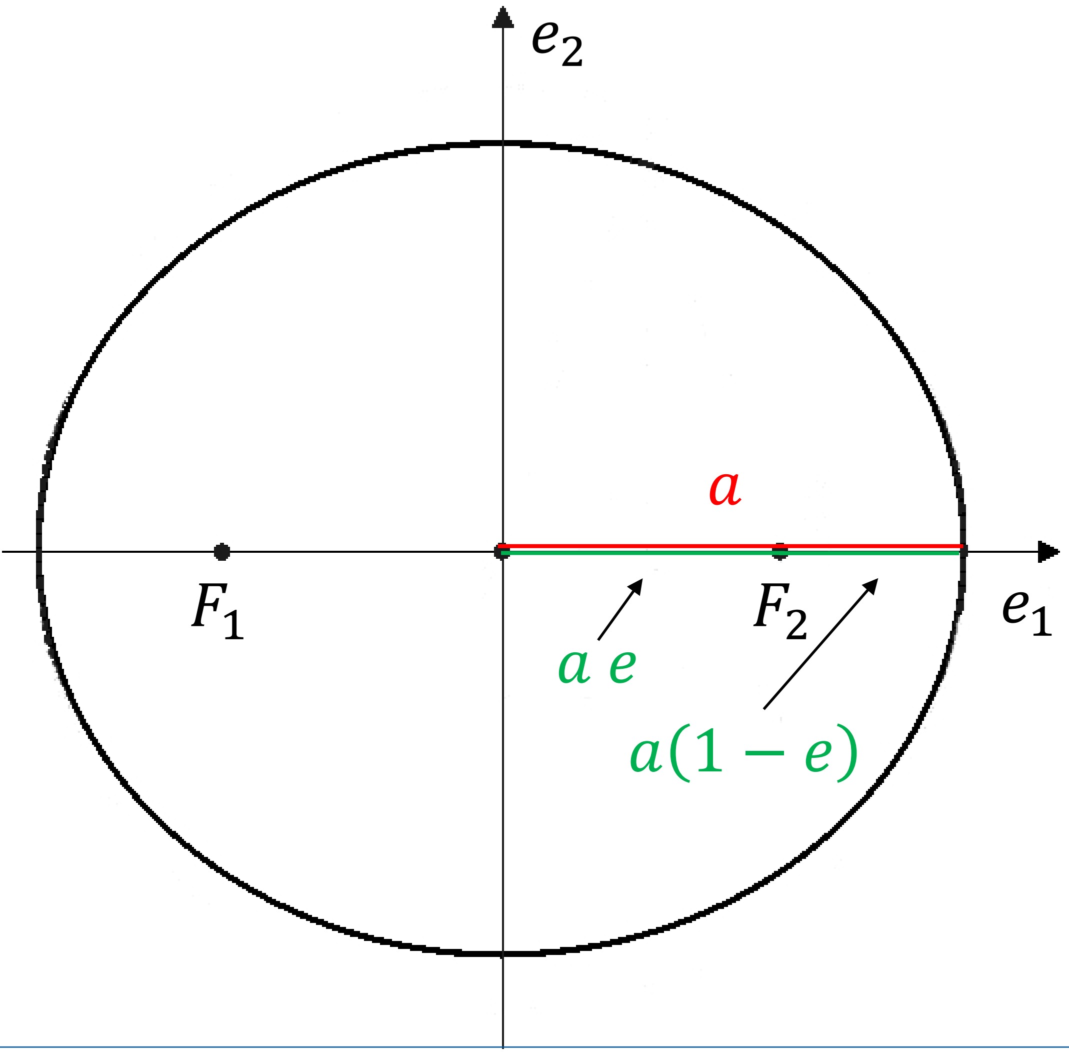
Definition:

$$a\, e \equiv \sqrt{a^2 - b^2}$$

Eccentricity:

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{\frac{a^2 - b^2}{a^2}}$$

Ellipse terminology



F_1, F_2: focal points, foci

Pythagorean theorem:

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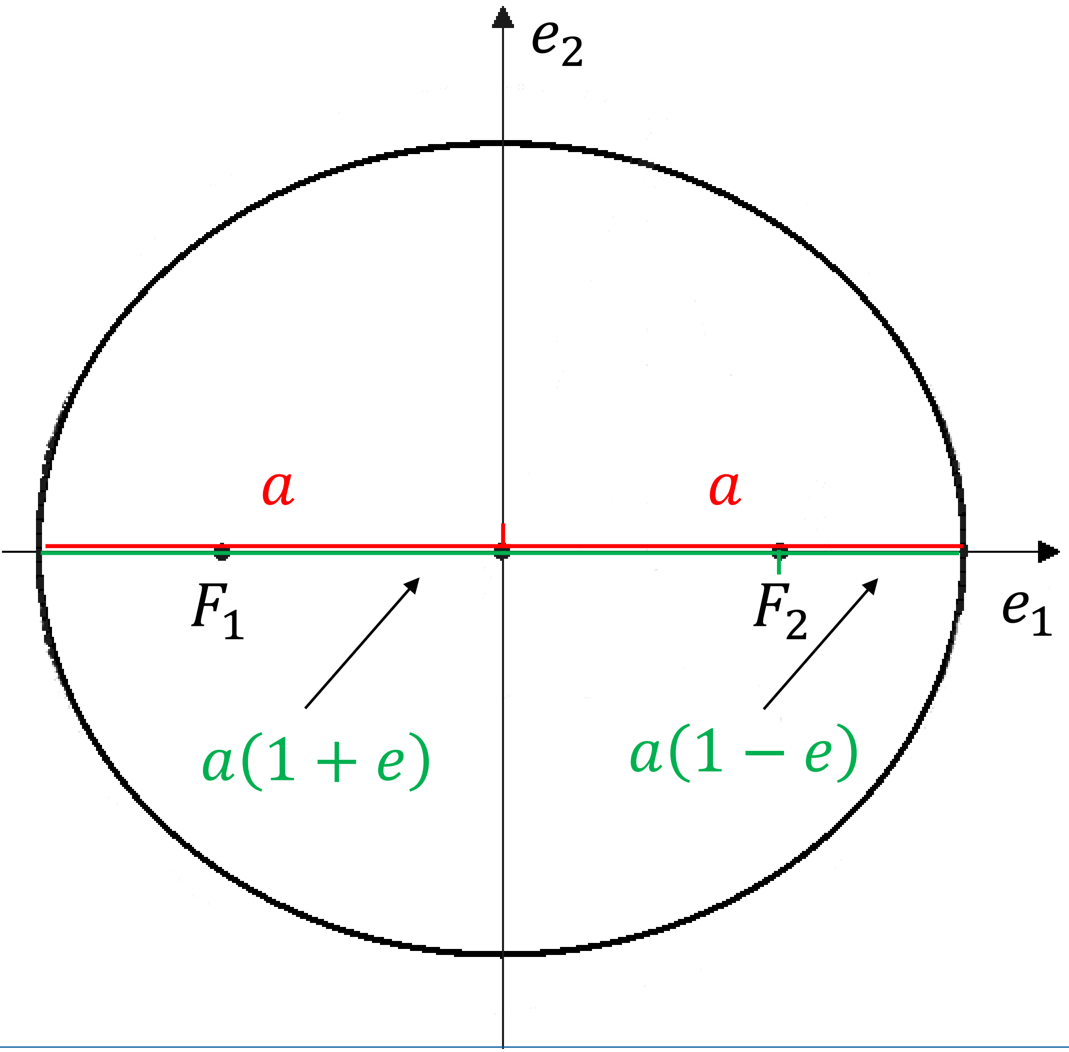
Periapsis:

$$a - a\ e = a(1-e)$$

Ellipse terminology



F_1, F_2: focal points, foci

Pythagorean theorem:

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$$e = \frac{\sqrt{a^2-b^2}}{a} = \sqrt{\frac{a^2-b^2}{a^2}}$$

Periapsis:

$$a - a\,e = a(1-e)$$

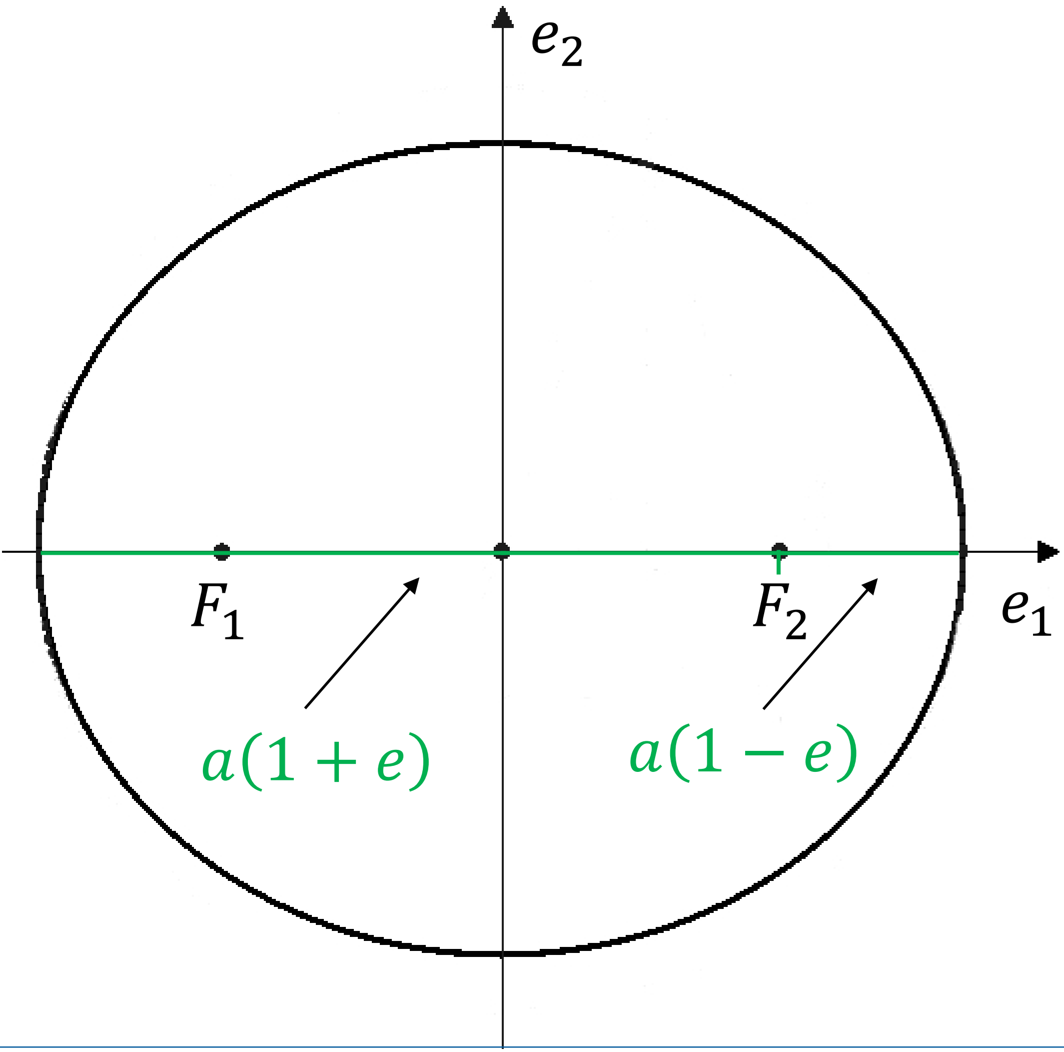
Apoapsis:

$$2a - (a - a\,e) = a + a\,e = a(1+e)$$

GFZ Helmholtz Centre Potsdam

TU berlin

Ellipse terminology



F_1, F_2: focal points, foci

<u>**Keplers 1st Law**:</u>
The orbit of a test mass is an **ellipse** with the central mass at F_2.

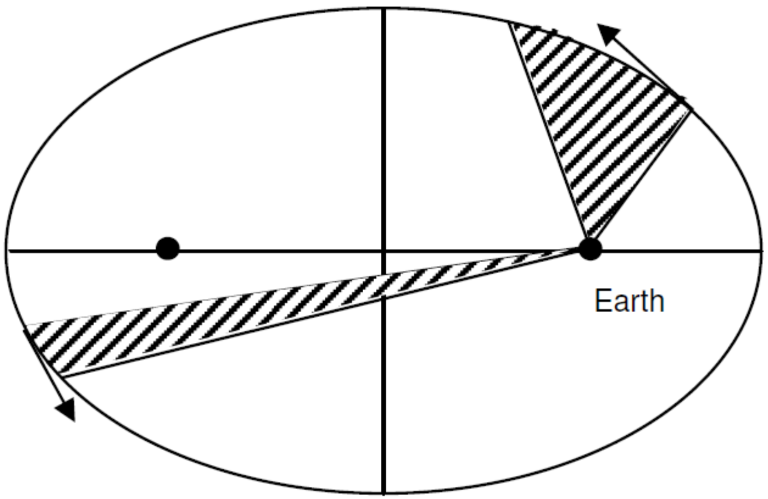
Pericenter distance
$r_{peri} = a(1-e)$
Apocenter distance
$r_{apo} = a(1+e)$

Case: central mass = Earth
Pericenter = „Perigee“
Apocenter = „Apogee“

Case: central mass = Sun
Pericenter = „Perihelion“
Apocenter = „Aphelion“

Keplerian Laws

Keplers 2nd Law:
A line segment joining the test mass and the central mass sweeps out equal areas during equal intervals of time.



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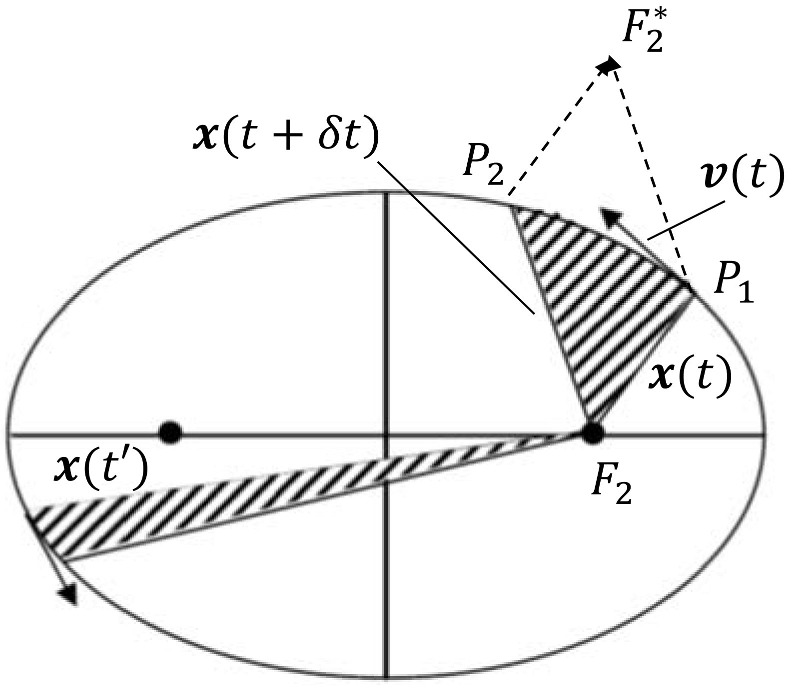
27.12.1571 – 15.11.1630
German Generalist

Keplerian Laws

Areal velocity $\boldsymbol{\omega_x}$ is a vector with its magnitude being associated to the rate of change of the area that is opened up by two points of the ellipse $P_1 \neq P_2 \in E$ and the focal point F_2 with the central mass, when the two points get asymptotically close to each other. With $\boldsymbol{x}$ as the instantaneous position and $\boldsymbol{v} = \dot{\boldsymbol{x}}$ as the instantaneous tangential velocity of the orbiter, we get:



$$\boldsymbol{\omega_x}(t) \equiv \lim_{\delta t \to 0} \frac{\boldsymbol{x}(t) \times \boldsymbol{x}(t+\delta t)}{2\,\delta t} =$$

$$= \lim_{\delta t \to 0} \frac{\boldsymbol{x}(t) \times (\boldsymbol{x}(t) + \boldsymbol{v}(t)\delta t)}{2\,\delta t} =$$

$$= \lim_{\delta t \to 0} \frac{\boldsymbol{x}(t) \times \boldsymbol{v}(t)}{2} \frac{\delta t}{\delta t} = \frac{1}{2}\big(\boldsymbol{x}(t) \times \boldsymbol{v}(t)\big)$$

<u>**Keplers 2nd Law:**</u>

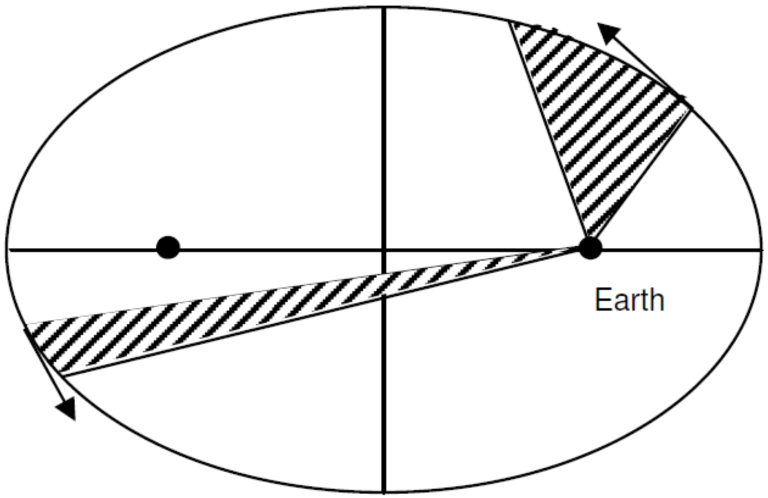
$$\boldsymbol{\omega_x}(t) = \boldsymbol{\omega_x}(t') \Rightarrow \boldsymbol{x} \times \boldsymbol{v} = const.$$

Keplerian Laws

<u>Keplers 2nd Law</u>:
A line segment joining the test mass and the central mass sweeps out equal areas during equal intervals of time.

$$\boldsymbol{x}(t) \times \boldsymbol{v}(t) = const.$$

⇒ *The **closer** the test mass is to the central mass, the **faster** it will travel and vice versa.*



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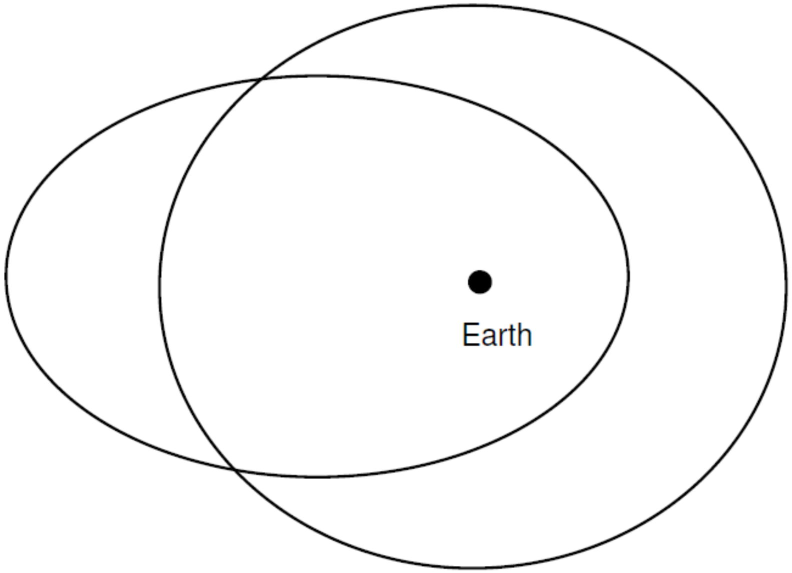
Keplerian Laws

Keplers 3rd Law:
The square of the **orbital period** is directly proportional to the cube of the semi-major axis.

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3$$

$$T^2 = \frac{4\pi^2}{G(M+m)} \cdot a^3$$

T orbital period (s)
a semi-major axis (m)



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Orbital periods

What is **the orbital period** of an Earth satellite in orbit?

The mean motion can be computed as $n = \sqrt{\frac{G(M+m)}{a^3}} \approx \sqrt{\frac{GM}{a^3}}$

Sometimes the mean motion n of the satellite is specified in turns per day d^{-1}, e.g. in two line elements, or per second s^{-1}, in rad d^{-1} or rad s^{-1}, or in deg d^{-1} or deg s^{-1}.

From 2nd Keplerian law:

$$T = \frac{2\pi}{n} \text{ or } T = \frac{360°}{n} \text{ or } T = \frac{1}{n}$$

If the mean motion n is given in d^{-1}: $T = \frac{1}{n}$ [d]

Get T in hours $T[h] = T[d] \cdot 24$
Convert to hour, minute, second using your code from assignment 1 and round to integer minutes.

Get T in seconds $T[s] = T[h] \cdot 3600$

Helmholtz Centre
POTSDAM

Orbital velocity

Circular orbit:

$$T^2 = \frac{4\pi^2}{GM} \cdot a^3; \quad v = 2\pi \frac{a}{T} = \sqrt{\frac{GM}{a}} = n$$

The distance to the focal point is everywhere identical.
⇒ The velocity is constant.

Elliptical orbit:

$$v_{apo} = \sqrt{\frac{GM}{a}} \sqrt{\frac{1-e}{1+e}}; \quad v_{peri} = \sqrt{\frac{GM}{a}} \sqrt{\frac{1+e}{1-e}}$$

The velocity varies depending on the distance to the focal point.
⇒ differential equation

Maximal velocity at perigee, minimal velocity at apogee.

Orbit elements

Kepler elements:
To locate a test mass in the elliptical orbit around a central mass, **6 parameters** have to be specified:

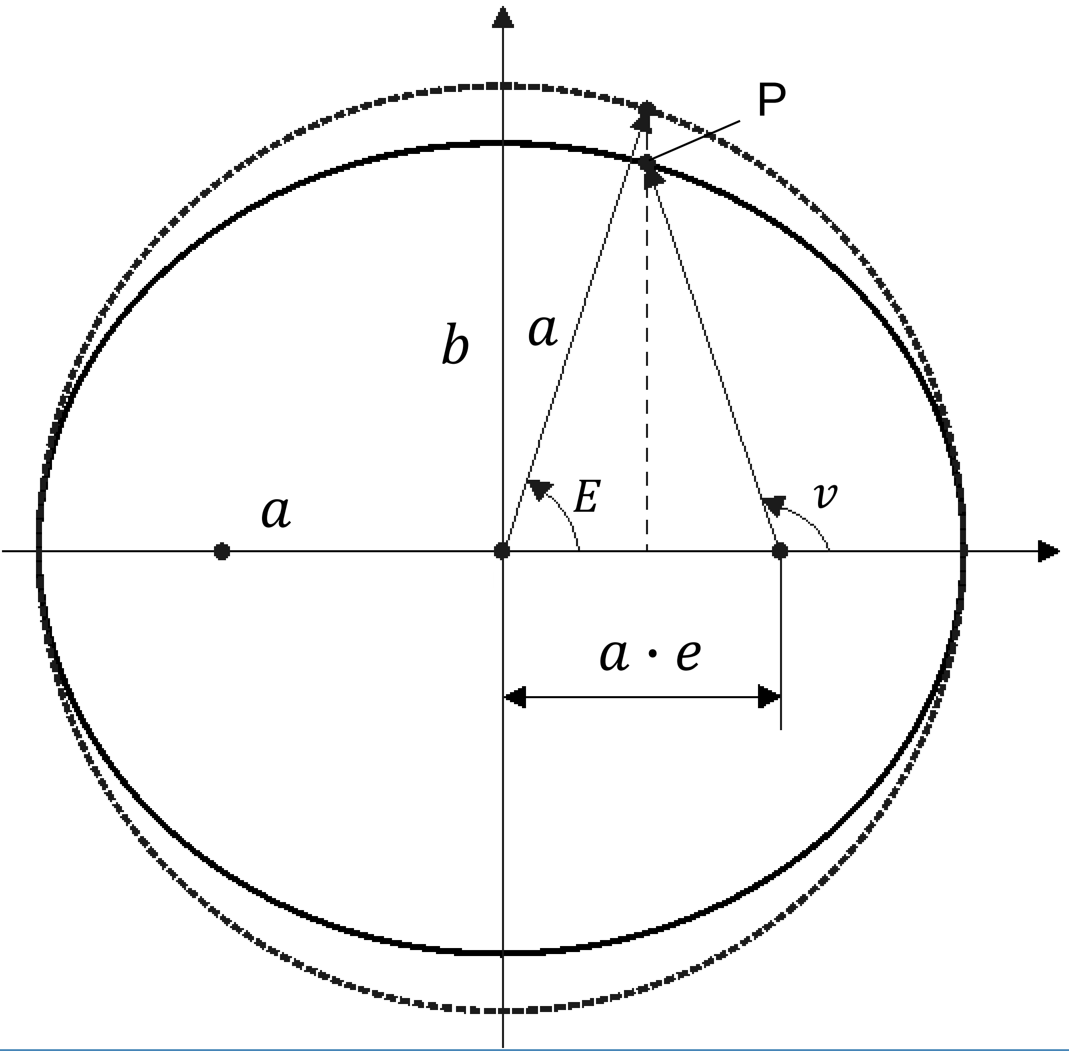
- **2 parameters** characterize the **shape and size of the ellipse** and

- **3 parameters** describe the **orientation** of the ellipse with respect to ECI, i.e. Geocentric Celestial Reference System (GCRS)

  then the orbit shape and orientation are fixed in space.

- **1 parameter** is necessary to specify the current **location** of the test mass **in the orbit**.

Orbit elements



2 elements describe the shape and size:

One element describes **the shape** of the ellipse:
***e*: eccentricity**, $e \in [0;1[$ or
***f*: flattening**, $f \in [0;1[$

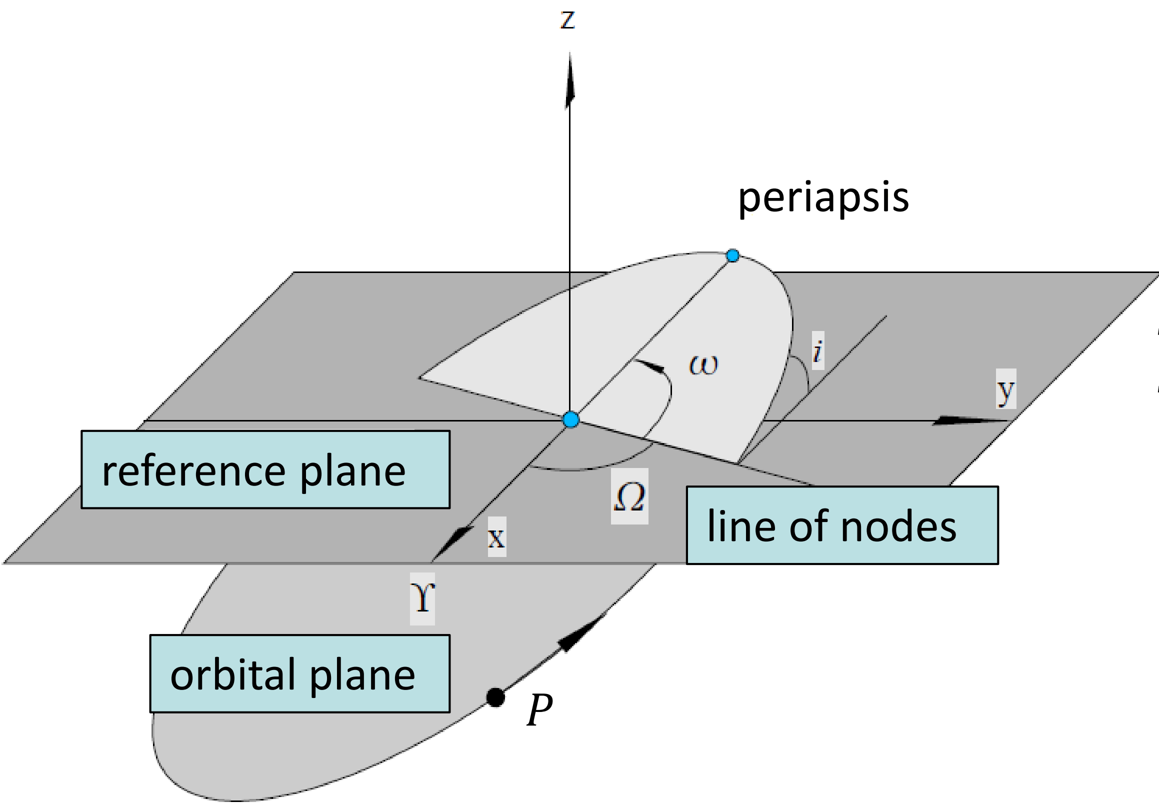
At least one axis is required to fix **the size / scale**:
***a*: semi-major axis** (m) or
***b*: semi-minor axis** (m)

The other elements can be derived:

$$f = \frac{a-b}{a} \qquad e^2 = 2f - f^2$$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} \qquad f = 1 - \sqrt{1-e^2}$$

Orbit elements



reference plane = celestial equator at J2000.0

♈: equinox at J2000.0

x, y, z are the coordinates ref. to GCRS axes

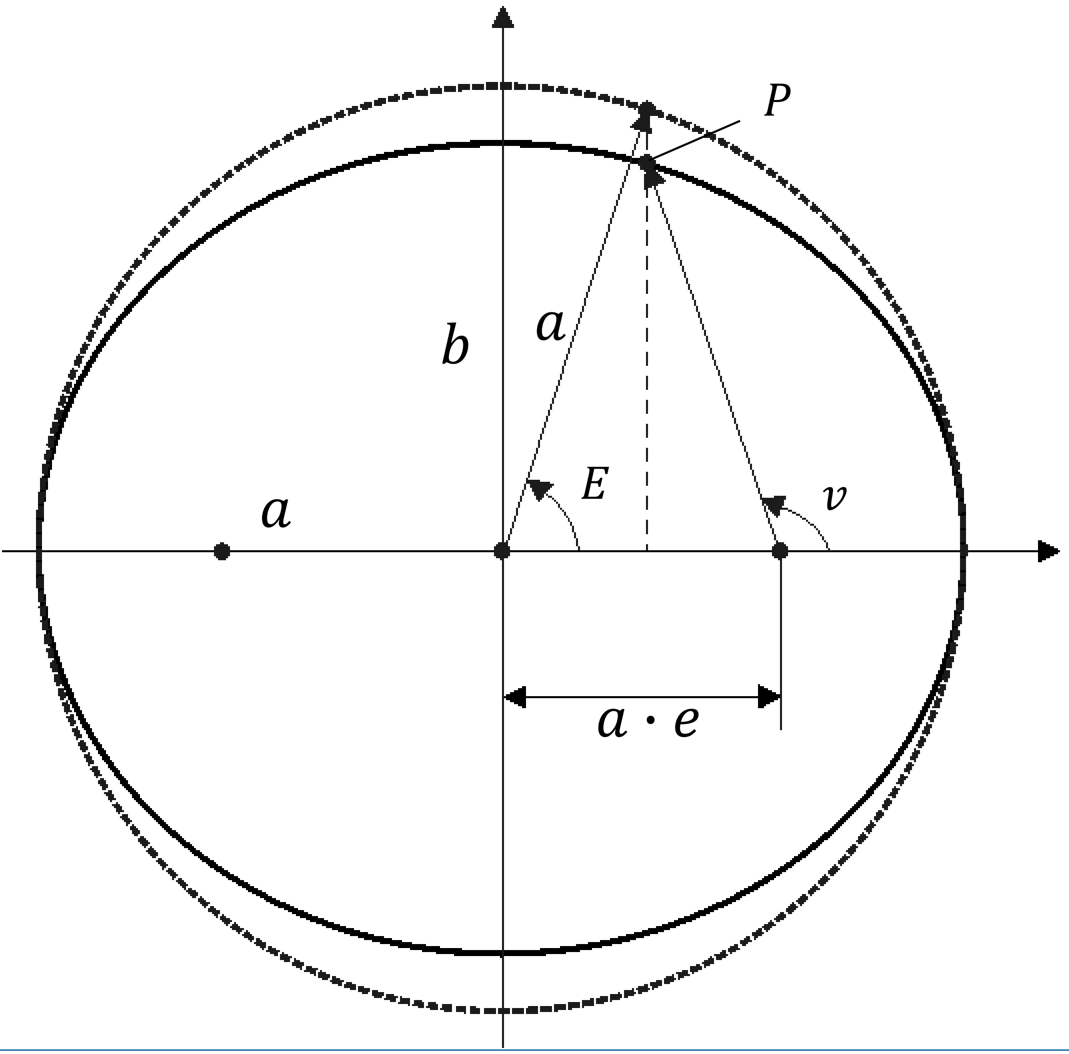
3 elements describe the **orientation** of the ellipse in space:

Ω**: longitude of ascending node**
$\Omega \in [0; 2\pi[$

ω**: argument of periapsis**
$\omega \in [0; 2\pi[$

i**: inclination**
$i \in [0; 2\pi[$
$i < \pi/2$ or $i > 3\pi/2$: prograde orbit
(prograde = in the direction of the rotation of the central body)
$\pi/2 < i < 3\pi/2$: retrograde orbit
$i = 0$ or π: equatorial orbit
$i = \pi/2$ or $3\pi/2$: polar orbit

Orbit elements



1 element describes the current location of the test mass in the orbit around the central mass:

v the true anomaly or

E the eccentric anomaly

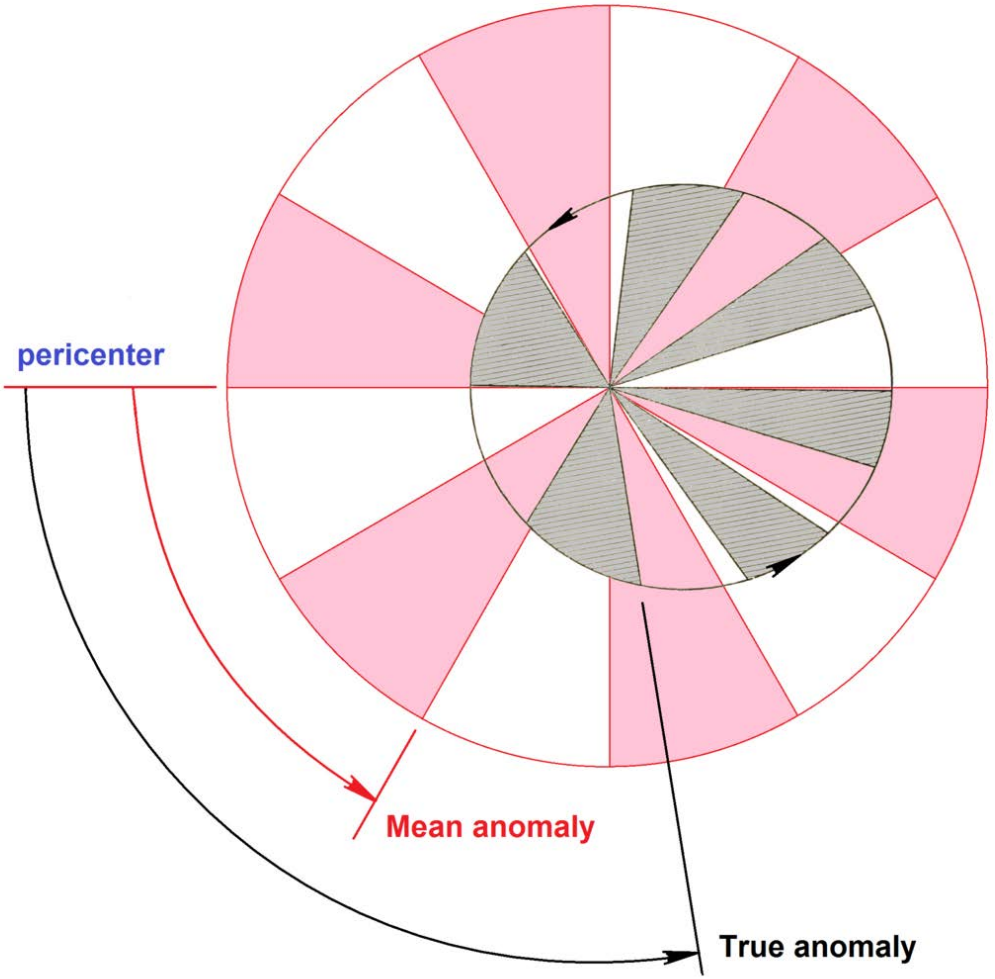
$\{v, E\} \in [0; 2\pi[$

$$\tan v = \frac{\sqrt{1-e^2}\sin E}{\cos E - e}$$

$$\tan E = \frac{\sqrt{1-e^2}\sin v}{\cos v + e}$$

Discussion of orbital time

Sketch showing relation of mean and true anomaly!



For the computation of the position of the satellite in orbit we need the true anomaly. True anomaly advances non-linear with time.

We are looking for an anomaly that is linear proportional with time!

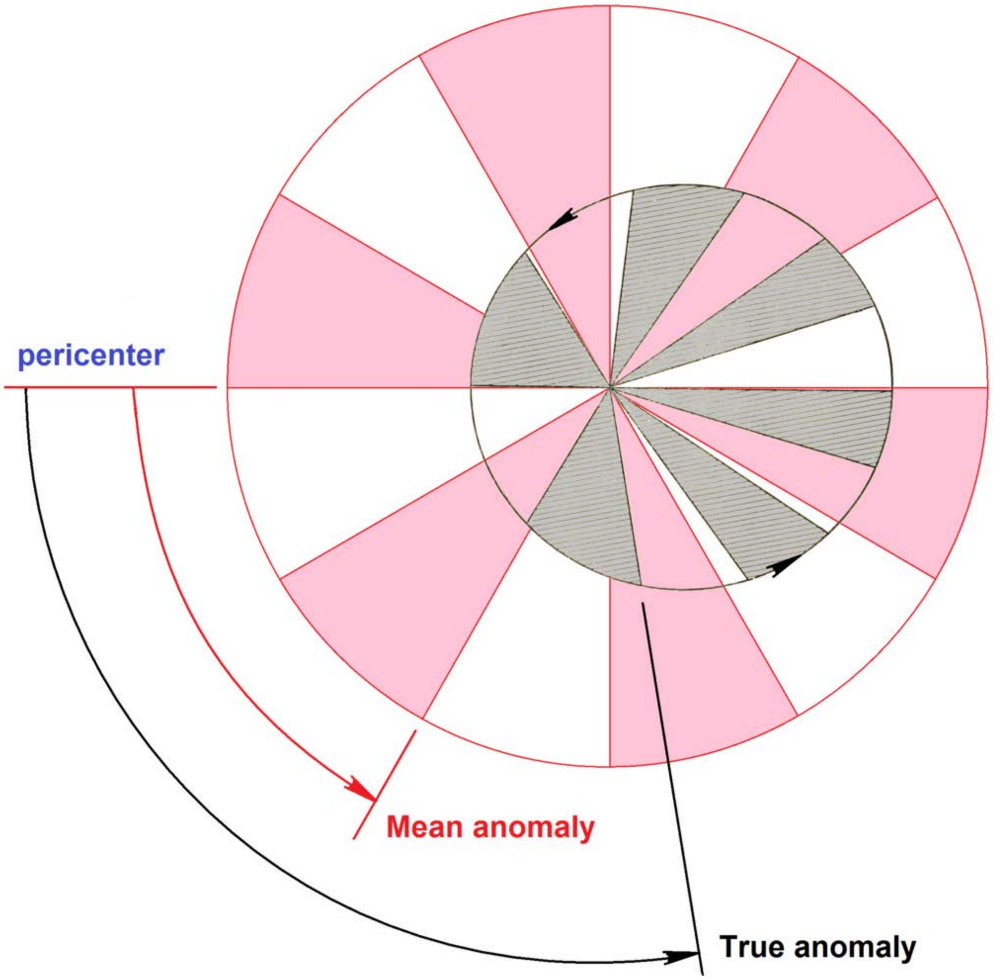
Good linear approximation via mean anomaly and mean motion:

$$n = \sqrt{\frac{GM_{\oplus}}{a^3}} = \text{const.}$$

Mean motion is the average angular velocity magnitude of the satellite.

Discussion of orbital time

Sketch showing relation of mean and true anomaly!



Knowing the orbital period T:

$$T = \frac{2\pi}{n}$$

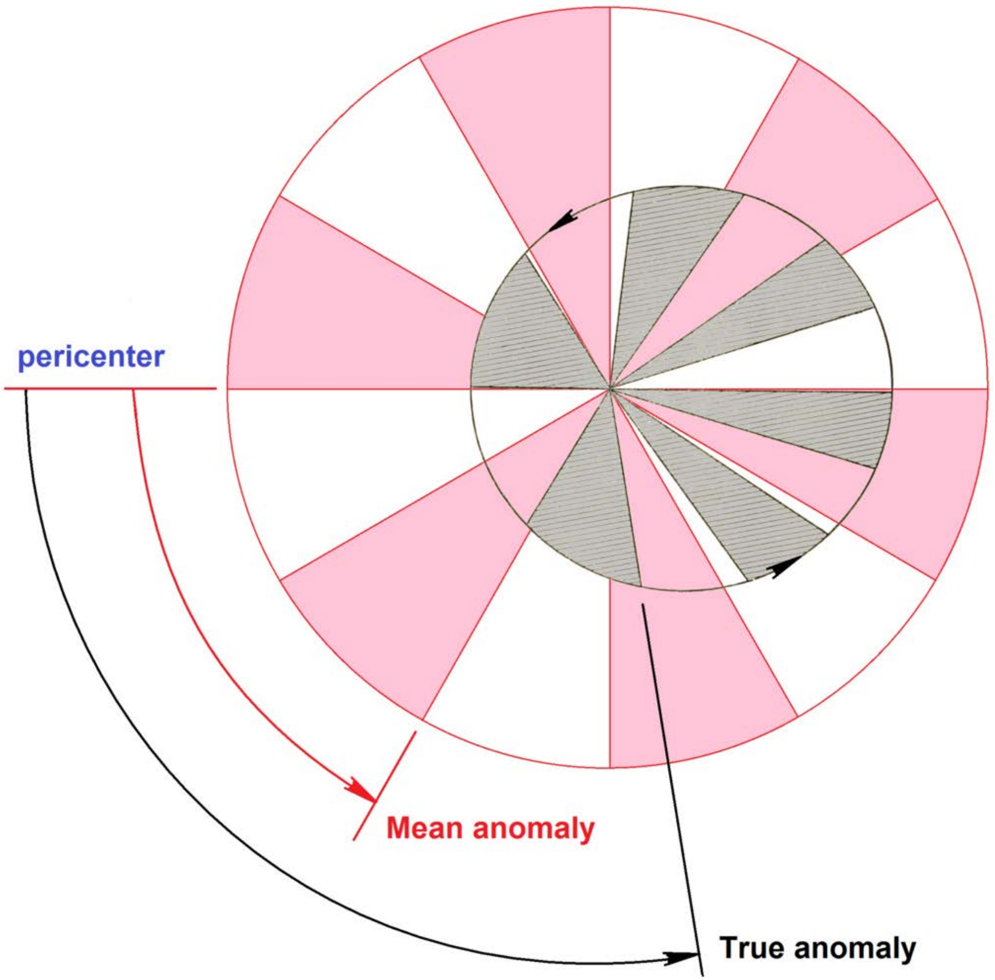
the corresponding rate of change is:

$$\Delta t = t - t_0 = \frac{M(t) - M(t_0)}{n} = \frac{\Delta M}{n}$$

result: $\Delta t \sim \Delta M$ (whereas $\Delta t \nsim \Delta E$!)

Discussion of orbital time

Sketch showing relation of mean and true anomaly!



How to advance in the orbit (anomaly) linear with time?

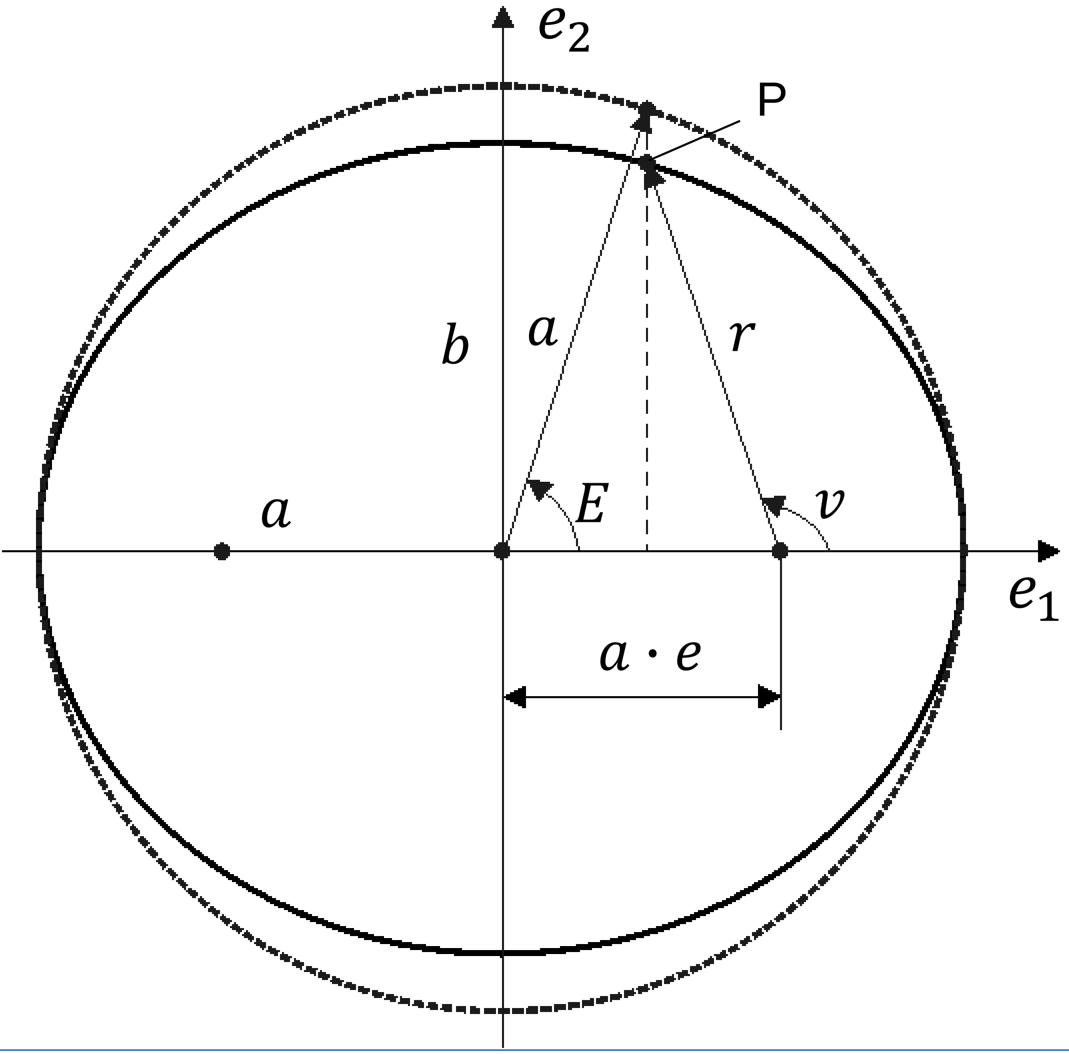
Realizing time increments, e.g. minute steps:
$t_{inc} = 0,1,2, \ldots \text{(min)}$

$M = M_0 + t_{inc}(d) \cdot n_{TLE} \cdot 2\pi$

Calculation of eccentric anomaly from mean anomaly, Kepler equation:
$M = E - e \sin E$

Solution of the inverse Kepler equation (assignment#2).

Determination of the position vector



True anomaly v from eccentric anomaly E:

$$v = \operatorname{atan2}\left(\frac{\sqrt{1-e^2}\sin E}{\cos E - e}\right)$$

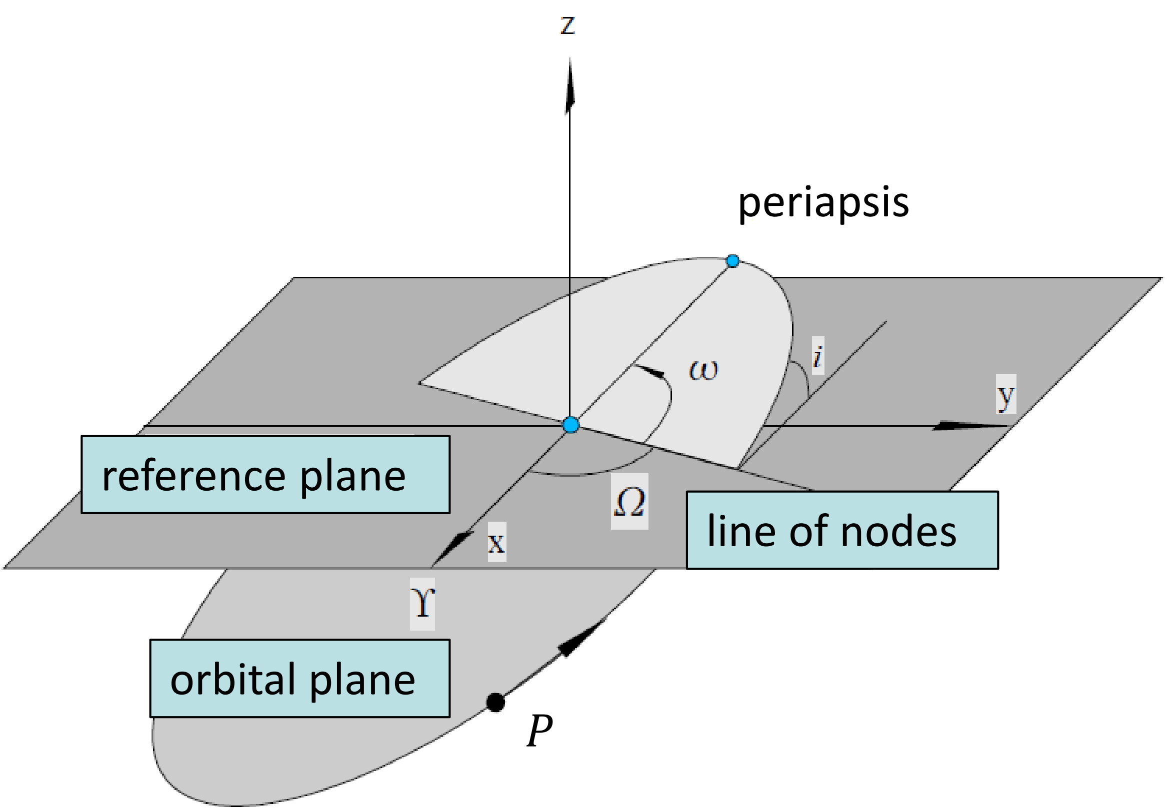
Distance of the satellite P to the central body (located at the right of the two focal points F_2):

$$r = a \cdot (1 - e \cdot \cos E)$$

Position vector in orbital system e_1, e_2, e_3 (e_3 out of orbital plane)

$$\boldsymbol{r_{orb}} = r \cdot \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}$$

Position in ECI



Rotation from the orbital plain to the reference plain (ECI):

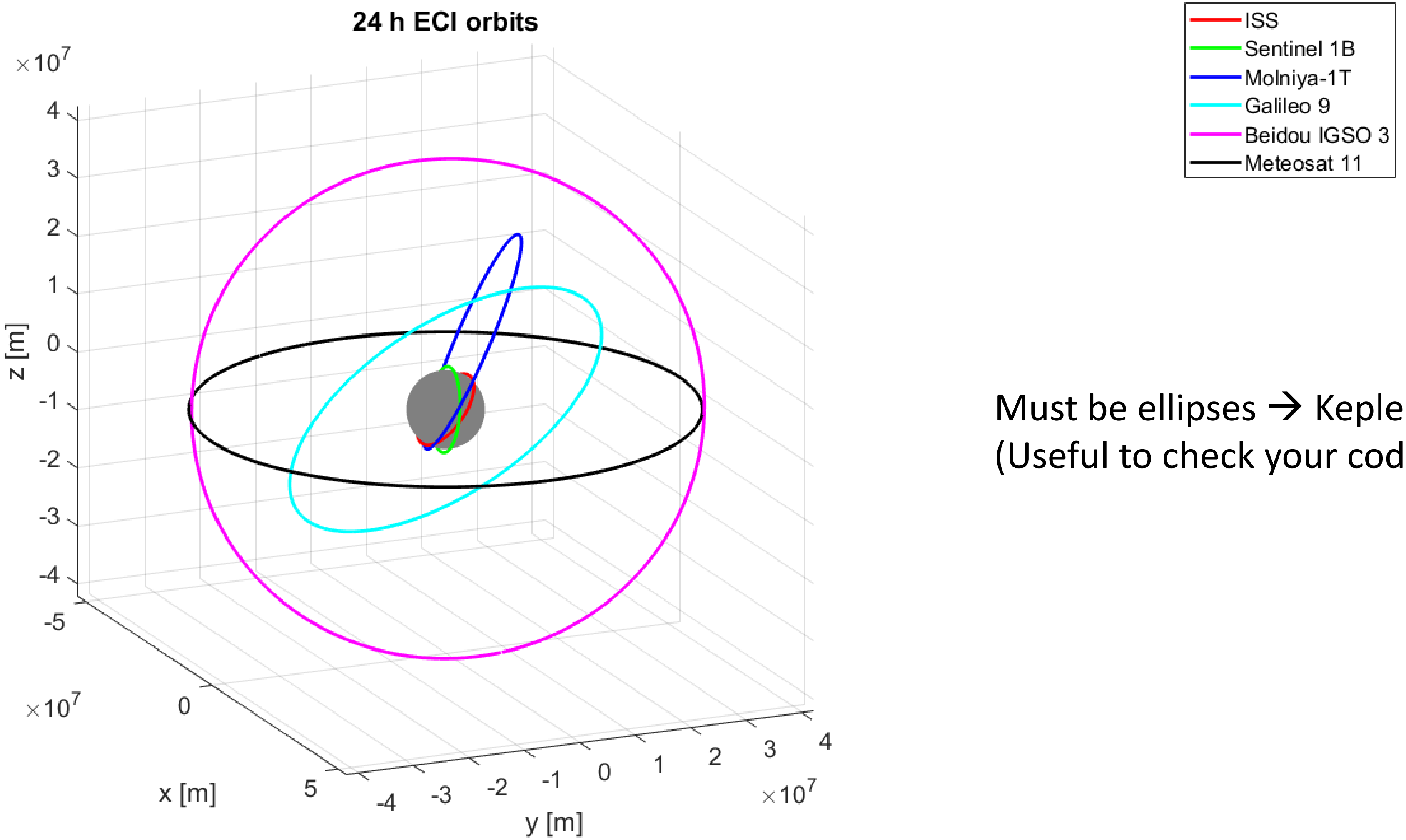
- rotate by Ω around the e_3-axis
- rotate by i around the e_1-axis
- rotate by ω around the e_3-axis

Three consecutive 3-dim rotations

$$\boldsymbol{r}_{\boldsymbol{ECI}} = R_3(-\Omega)R_1(-i)R_3(-\omega)\boldsymbol{r}_{\boldsymbol{orb}}$$

$$\boldsymbol{r}_{\boldsymbol{ECI}} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\cos\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \boldsymbol{r}_{\boldsymbol{orb}}$$

Orbits in ECI



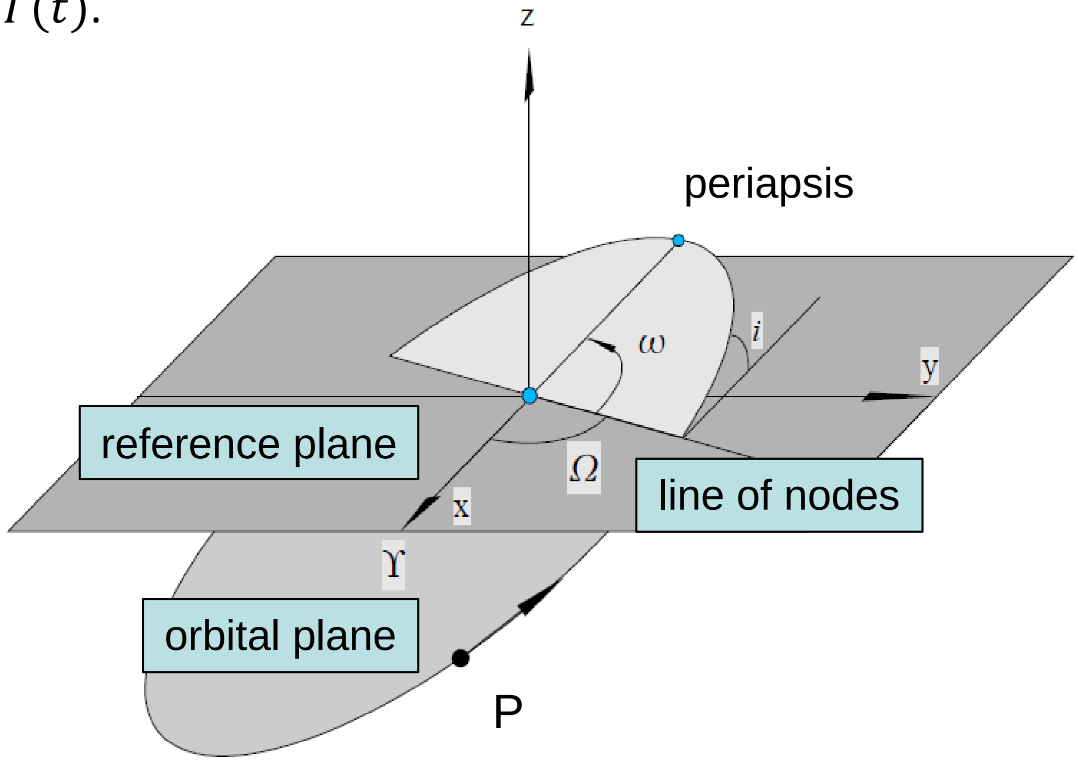
Must be ellipses → Kepler 1st law.
(Useful to check your code!)

Transformation from ECI to ECEF

Analog assignment#4!
If transformation simplified to GMST, i.e. no precession, nutation, polar motion.

Let $GMST$ be given at time t: $GMST(t)$.

$$r_{ECEF} = R_3\big(GMST(t)\big)\, r_{ECI}$$

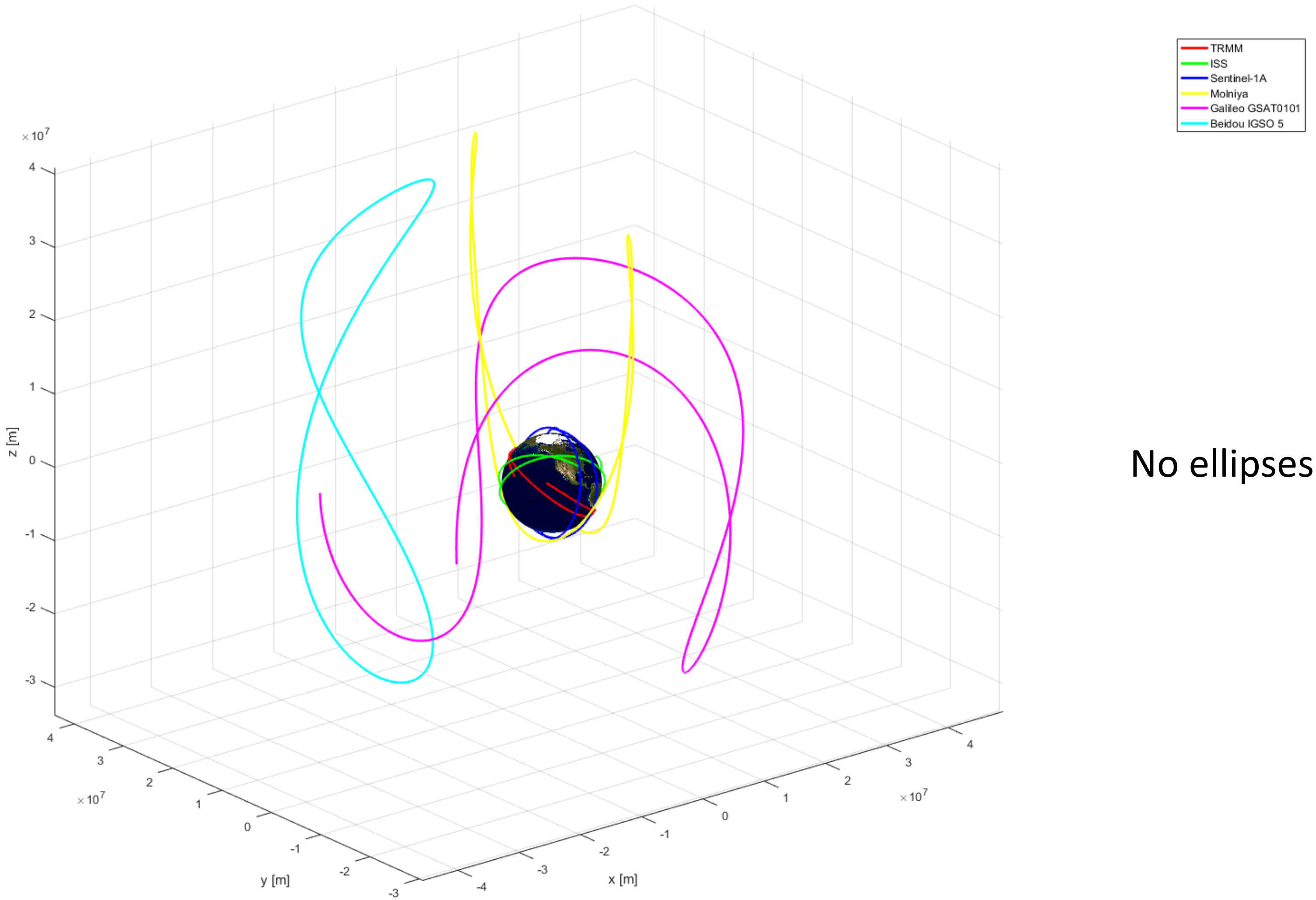


Synchronization of anomaly and Earth rotation

Most simple approach (useful for the assignment):

- Start epoch t is given (yyyy-mm-dd hh:mm:ss)
- Define d$t = 0$ to 1 d in minute steps → dt (minute-resolution for one day)
- Calculate $M(t+dt) = M_{TLE}(t)\ [rad] + dt\ [d] \cdot n\ [d^{-1}] \cdot 2\pi$
  where M is in $[rad]$
- Calculate the corresponding eccentric anomaly via Kepler equation
- Calculate the Earth rotation angle $GMST$: G$MST(t+dt)$ and use it for the transformation $r_{ECEF} = R_3\big(GMST(t+dt)\big)\, r_{ECI}$

Orbits in ECEF system

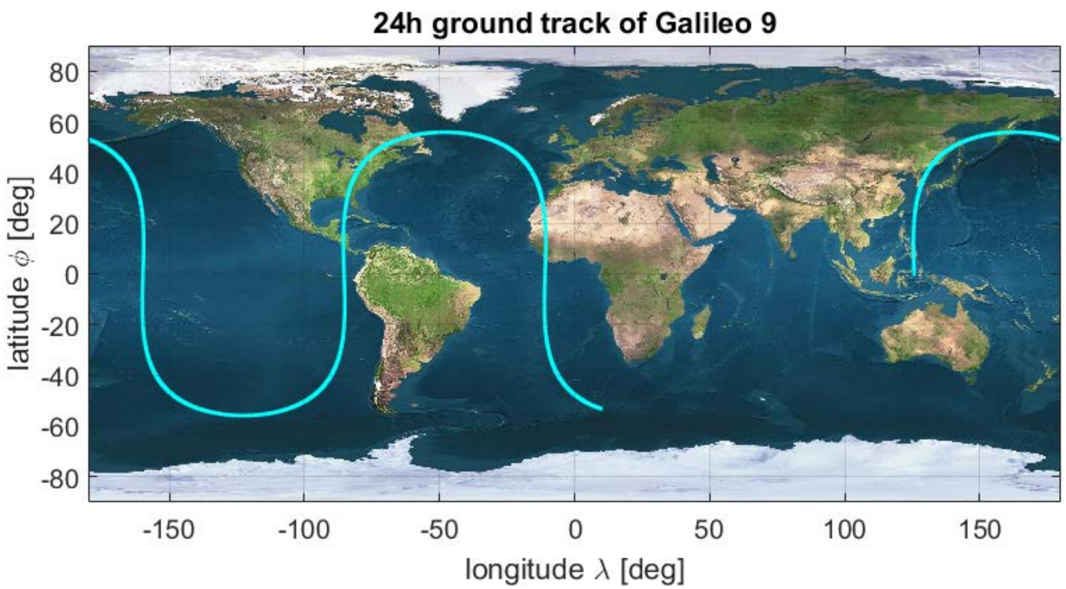


No ellipses in ECEF!

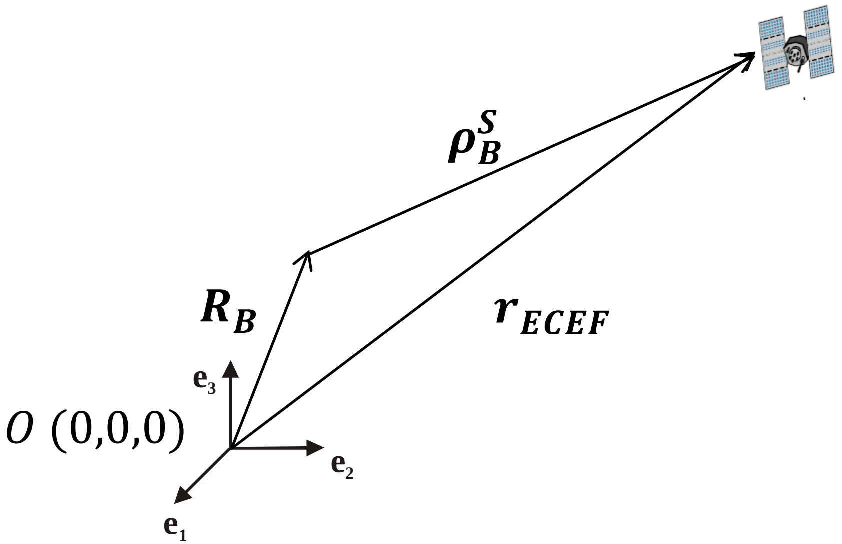
Ground tracks

Computation
1) Orbit elements -> position in ECI
2) transformation from ECI to ECEF (simplified)
3) conversion from Cartesian to geodetic coordinates (e.g. GRS80)
4) plot geodetic lat/lon on top of an Earth image or together with coastlines

Example (outdated)



Distance between satellite and observer



Earth centered Earth fixed (ECEF) frame

<u>Given/known:</u>

r_{ECEF} (inst. position of satellite in ECEF)

R_B (position of observer near Berlin)

<u>Unknown:</u>

$\|\rho_B^S\|$

Vector addition (commutative):

Vector:

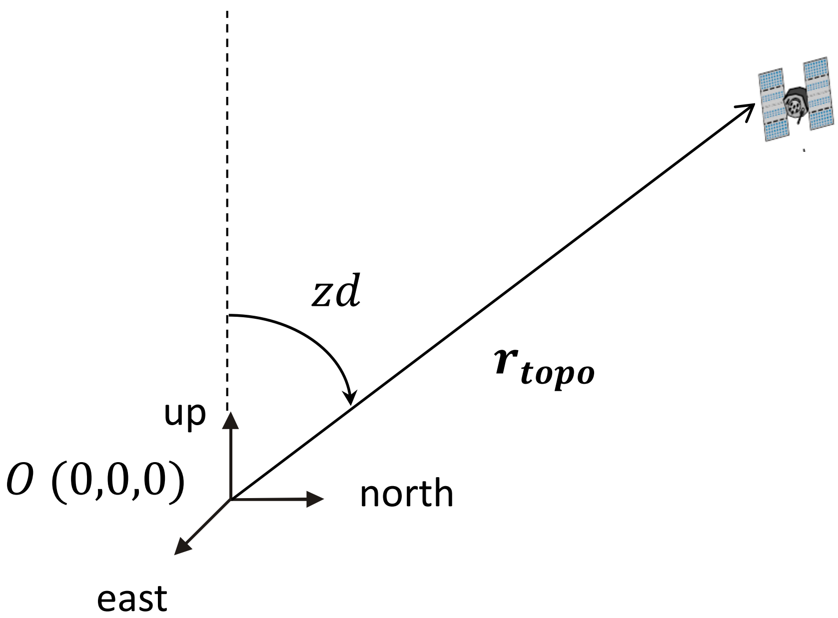
$$r_{ECEF} = R_B + \rho_B^S$$

$$\rho_B^S = r_{ECEF} - R_B$$

Distance = magnitude:

$$\|\rho_B^S\| = \|r_{ECEF} - R_B\|$$

Zenith distance of the satellite



Topocentric horizontal system

Given/known:

$\boldsymbol{r_{ECEF}}$

L, B (geodetic coordinates of an observer near Belin)

Unknown:

zd

$$\boldsymbol{r}_{\mathbf{topo}} = M_1 R_2\left(\frac{\pi}{2} - B\right) R_3(L)\,\boldsymbol{r}_{\mathbf{ECEF}}$$

$$zd = \frac{\pi}{2} - E$$

$$E = \arctan\left(r_{topo}(3) / \sqrt{r_{topo}(1)^2 + r_{topo}(2)^2} \right)$$