Abstract Algebras of Observables and their Concrete Models: The Gelfand-Naimark-Segal Construction

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Abstract

C^*-algebras are a widely studied class of algebras which represent bounded operators on physical systems in quantum and statistical mechanics. In quantum mechanics the most prominent C^*-algebra is the algebra of bounded operators on a Hilbert space, interpreted as a space of physical states. A natural question is whether every C^*-algebra may be interpreted as an algebra of bounded operators on some physical system, and how one may naturally construct such a concrete model. This question motivates the primary goal of this paper, which is an in-depth investigation of the Gelfand-Naimark-Segal (GNS) construction on an abstract C^*-algebra, relying on tools from abstract algebra, representation theory, and functional analysis. Motivated by physical intuition throughout, we analyze properties of C^*-algebras and how representations can be constructed for C^*-algebras. After this initial exploration we will dive into the abstract theory of states on a C^*-algebra before leading into how states characterize GNS constructions, with the notion of measurement motivating the whole classification.

1 Introduction

The introduction of the theory of operator algebras began in the early 1900s following the development of the theory of Hilbert spaces and the spectra of operators on them [II04]. In the same period the field of quantum physics was in its infancy, with physicists such as Werner Heisenberg and Erwin Schrödinger attempting to formulate mathematical models which could explain the experimental quantization of electron energies in atoms [Hel21, pp. 139, 167]. Schrödinger proposed a model known as wave mechanics which described atomic and subatomic particles using probability distributions, and reduced the quantization of electron energies in an atom to an eigenvalue problem. On the other hand Heisenberg suggested a model referred to as matrix mechanics, where matrices were used to encode observable quantities and the relations between them [Hel21, p. 167]. Motivated by these works Stone, von Neumann, and Murray continued the development of the theory of operator algebras, and in doing so laid the work of Heisenberg and Schrödinger on the solid framework of adjoint operators on Hilbert spaces of L^2 functions [BR87, p. 139].

In 1943, ten years following the work of Stone, von Neumann, and Murray, Gelfand and Naimark introduced what would later be referred to as C^*-algebras in a study of certain Banach spaces endowed with a multiplication structure [Dor94, pp. 3-19]. Gelfand and Naimark showed that all such algebraic structures can be identified with closed subspaces of the space of bounded operators on some Hilbert space. In 1947 Segal clarified this work in the context of the developing quantum theory, demonstrating the importance of not only C^*-algebras but also their concrete representations [Dor94, pp. 55-65]. In this paper we will investigate the work of Gelfand, Naimark, and Segal constituting the GNS construction which provides concrete models of C^*-algebras in terms of bounded operators on Hilbert spaces. To this end we begin by introducing the notion of C^*-algebras along with a number of important results in analogy with bounded operators. Following this initial characterization we will classify the representation theoretic structure of the theory of C^*-algebras. Finally, in analogy with physical system states we will introduce the notion of an abstract state on a C^*-algebra before developing the GNS construction and some of its important characterizations. Throughout the paper we will motivate definitions by examples and links to the underlying physics which C^*-algebras formalize.

2 Background Information

2.1 Abstract C^*-Algebras

The study of operator algebras allows for the combination of powerful tools from both analysis and abstract algebra in the characterization of well-known mathematical structures. One such ubiquitous structure, which is the primary focus of this paper, is the C^*-algebra. Intuitively, C^*-algebras are Banach spaces which behave like the collection of bounded operators on a Hilbert space.

Definition 2.1 (C^*-algebra)**.** A C^***-algebra** is a Banach space $\mathfrak{A}$ over $\mathbb{C}$ together with a multiplication operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ and a conjugate linear map $^* : \mathfrak{A} \to \mathfrak{A}$. These operations satisfy the following coherency conditions for all $A, B, C \in \mathfrak{A}$

C1. $A \cdot (B + C) = A \cdot B + A \cdot C$ (left-distributivity)
C2. $(B + C) \cdot A = B \cdot A + C \cdot A$ (right-distributivity)
C3. $||A \cdot B|| \leq ||A||\,||B||$ (sub-multiplicative)
C4. $(AB)^* = B^* A^*$
C5. $(A^*)^*$ (involution)
C6. $||A^* A|| = ||A||^2$ (C^*-condition)

$\mathfrak{A}$ is said to be **unital** if $\exists I \in \mathfrak{A}$ such that $AI = A = IA$ for all $A \in \mathfrak{A}$ [Dav96, p. 1].

The C^*-condition may be motivated by the same property that the adjoint operation satisfies on $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Indeed, given any Hilbert space $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$ is a C^*-algebra with operations of pointwise addition, composition, and the adjoint operation taking the place of an involution. Additionally, the conditions C3 and C6 give that $||A||^2 = ||A^*A|| \leq ||A^*||\,||A||$, so $||A|| \leq ||A^*|| \leq ||(A^*)^*|| = ||A||$. This implies * is continuous as $||A^* - B^*|| = ||(A - B)^*|| = ||A - B||$ implies that * is Lipschitz.

For the remainder of this paper $\mathfrak{A}$ will denote a unital C^*-algebra with unity I. For notational simplicity we will also use * to denote the involution on possibly different C^*-algebras throughout. The assumption that $\mathfrak{A}$ is unital may be removed using methods of approximate units or unitizations [Mur90, pp. 39, 78], but in this paper we restrict to the unital case for the sake of space. Under this assumption observe that $I^* = I^*I = (I^*I)^* = (I^*)^* = I$, which also implies $||I|| = ||I^*I|| = ||I||^2$ using C6, so $||I|| = 1$.

A first family of interesting examples of C^*-algebras arises from the study of compact Hausdorff spaces [KR97, p. 236].
Example 2.1. If Y is a compact Hausdorff space, then $C(Y) = \{f : Y \to \mathbb{C} : f \text{ is continuous}\}$ is a C^*-algebra under pointwise addition and multiplication, with the supremum norm, $||\cdot||_Y$, and an involution given by complex conjugation: $\forall f \in C(Y),\ f^*(y) := \overline{f(y)},\ \forall y \in$

Y. Indeed, for any $f \in C(Y)$ and $x, y \in Y$, $|f^*(x) - f^*(y)| = |\overline{f(x) - f(y)}| = |f(x) - f(y)|$ by definition of the complex modulus, so if $|f(x) - f(y)| < \epsilon$, for $\epsilon > 0$, $|f^*(x) - f^*(y)| < \epsilon$. This implies $f^* \in C(Y)$ is continuous as f is. Additionally, $(f^*)^*(y) = \overline{\overline{f(y)}} = f(y)$, so $(f^*)^* = f$, and for $g \in C(Y)$, $(fg)^*(y) = \overline{f(y)g(y)} = f^*(y)g^*(y)$, so $(fg)^* = f^*g^*$. Note the order doesn't matter since multiplication in $\mathbb{C}$ is commutative. Finally, for condition C3 $|(fg)(y)| = |f(y)||g(y)| \leq ||f||_Y \, ||g||_Y$, so by definition of the supremum $||fg||_Y \leq ||f||_Y \, ||g||_Y$, and for condition C6

$$||f^*f||_Y = \sup_{y \in Y}\{|(f^*f)(y)|\} = \sup_{y \in Y}\{|f^*(y)||f(y)|\} = \sup_{y \in Y}\{|f(y)|^2\} = ||f||_Y^2$$

Thus $(C(Y), \cdot, ^*)$ satisfies all the axioms of a C^*-algebra.

Now, in the case that $\mathfrak{A}$ is unital, we have a generalized notion of eigenvalues which is important in the classification of C^*-algebras and their elements.

Definition 2.2 (Spectrum)**.** The **spectrum** of an element $A \in \mathfrak{A}$ is the set

$$\text{sp}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathfrak{A}^\times\}$$

where $\mathfrak{A}^\times := \{A \in \mathfrak{A} : \exists B \in \mathfrak{A}, AB = I = BA\}$ is the set of invertible elements [Str21, p. 2].

With the notion of a spectrum we can classify the notion of positive elements in a C^*-algebra, in analogy with positive bounded operators on a Hilbert space.

Definition 2.3 (Positivity)**.** Let $A \in \mathfrak{A}$. Then,

P1. A is **self-adjoint** if $A^* = A$. $\mathfrak{A}_{sa} \subseteq \mathfrak{A}$ denotes the subset of self-adjoint elements.

P2. A is **positive** if $A \in \mathfrak{A}_{sa}$ and $\text{sp}(A) \subseteq \mathbb{R}_{\geq 0}$ [Str21, p. 26]. $\mathfrak{A}_+ \subseteq \mathfrak{A}_{sa}$ denotes the subset of positive elements.

The proof of the following two results are beyond the scope of this paper as they require the Gelfand Transform and functional calculus of C^*-algebras, but they provide a useful classification of self-adjoint and positive elements which will assist in later proofs. The first result is the equality [Mur90, p. 46]

$$\mathfrak{A}_+ = \{A^*A \in \mathfrak{A} : A \in \mathfrak{A}\} \tag{2.1}$$

and the second result is the following proposition [Dav96, p. 9].

Proposition 2.4. *If $A \in \mathfrak{A}_{sa}$ then there exists $B, C \in \mathfrak{A}_+$ such that $A = B - C$ and $BC = 0$. In particular, $sp(A) \subseteq \mathbb{R}$.*

We can immediately apply Equation 2.1 to observe that $\mathfrak{A}_+$ is stable under *-conjugation. That is, for $A \in \mathfrak{A}_+$ we can write $A = C^*C$ for some $C \in \mathfrak{A}$, so for any $B \in \mathfrak{A}$, $B^*AB = B^*C^*CB = (CB)^*CB \in \mathfrak{A}_+$. This proves the following corollary which will be needed in the GNS construction.

Corollary 2.5. *If $A \in \mathfrak{A}_+$ and $B \in \mathfrak{A}$ then $B^*AB \in \mathfrak{A}_+$ [Str21, p. 29].*

Now that we have a mathematical object as well as some characterizations of elements it may contain, it is important to classify how two such mathematical objects communicate. A natural method of communication is to consider set functions which preserve the operations on our mathematical object.

Definition 2.6 ($*$-homomorphisms)**.** If $\mathfrak{A}, \mathfrak{B}$ are C^*-algebras, a $*$**-homomorphism** is a $\mathbb{C}$-linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ such that for any $A, A' \in \mathfrak{A}$,

$$\varphi(AA') = \varphi(A)\varphi(A') \text{ and } \varphi(A^*) = \varphi(A)^*$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are unital, we require $\varphi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$.

The preservation of the $*$-operation by a $*$-homomorphism is a very strong condition, and forces the $*$-homomorphism to be a bounded operator with norm at most 1 [KR97, p. 242].

Proposition 2.7. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism. Then* $||\varphi(A)|| \leq ||A||, \forall A \in \mathfrak{A}$.

In order to prove this claim we require one more piece of machinery related to the spectrum of an element A in $\mathfrak{A}$, which is its **spectral radius** [KR97, p. 180]

$$\mathrm{r}(A) := \sup\{|\lambda| : \lambda \in \mathrm{sp}(A)\}$$

This radius is always finite, and in fact bounded above by the norm of A.

Lemma 2.8. *If $\lambda \in sp(A)$ for $A \in \mathfrak{A}$,* $|\lambda| \leq ||A||$.

Proof. We argue the contrapositive. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > ||A|| \geq 0$. Let $B = \frac{1}{\lambda}A$, so $||B|| = \frac{||A||}{|\lambda|} < 1$. Then I claim $I - B$ has an inverse given by $\sum_{n=0}^{\infty} B^n$. Observe by the triangle inequality that for any $N > m \in \mathbb{N}$,

$$\left|\left|\sum_{n=m}^{N} B^n\right|\right| \leq \sum_{n=m}^{N} ||B^n|| \leq \sum_{n=m}^{N} ||B||^n$$

using the sub-multiplicative property of the norm inductively. The right hand side goes to zero as $m, N \to \infty$ since $\sum_{n=0}^{\infty} ||B||^n$ is a geometric series with ratio $||B|| < 1$. Thus $\sum_{n=0}^{k} B^n$ is Cauchy, and hence converges in $\mathfrak{A}$ as it is a Banach space. Finally, since multiplication by a fixed element is continuous, due to condition C3,

$$(I-B)\sum_{n=0}^{\infty} B^n = \lim_{N\to\infty} \sum_{n=0}^{N}(I-B)B^n = \lim_{N\to\infty}\left[I + \sum_{n=1}^{N} B^n - \sum_{n=1}^{N} B^n - B^{N+1}\right] = \lim_{N\to\infty}(I-B^{N+1})$$

As $||B|| < 1$, $\left|\left|B^{N+1}\right|\right| \leq ||B||^{N+1} \to 0$ as $N \to \infty$, so the limit on the right converges to I since addition is continuous in a Banach space. The multiplication of $(I - B)$ on the right follows identically. Thus $I - B$ is invertible, so $\lambda I - \lambda B = \lambda I - A$ is invertible, being a non-zero scalar multiple. In other words, $\lambda \notin \mathrm{sp}(A)$, and the claim follows. ■

Taking the supremum in Lemma 2.8 gives $r(A) \leq ||A||$. For the sake of space we do not argue it here, but another essential result is that in the case of A being self-adjoint we have equality [KR97, p. 238]. This is now sufficient to prove the proposition.

Proof of Prop 2.7. If $A \in \mathfrak{A}$ and $\lambda \notin \mathrm{sp}(A)$, then $\lambda \notin \mathrm{sp}(\varphi(A))$. Indeed, if $\lambda I_{\mathfrak{A}} - A$ had inverse $B \in \mathfrak{A}$, then

$$(\lambda I_{\mathfrak{B}} - \varphi(A))\varphi(B) = (\lambda\varphi(I_{\mathfrak{A}}) - \varphi(A))\varphi(B) = \varphi(\lambda I_{\mathfrak{A}} - A)\varphi(B) = \varphi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$$

and similarly for multiplication of $\varphi(B)$ on the left, so $\lambda I_{\mathfrak{B}} - \varphi(A)$ has inverse in $\mathfrak{B}$. This implies $\mathrm{sp}(\varphi(A)) \subseteq \mathrm{sp}(A)$, so $r(\varphi(A)) \leq r(A)$ as the supremum is taken over a smaller set. But, by condition C6 and the fact A^*A and $\varphi(A)^*\varphi(A)$ are self-adjoint, $||A||^2 = ||A^*A|| = r(A^*A)$ and $||\varphi(A)||^2 = ||\varphi(A)^*\varphi(A)|| = r(\varphi(A)^*\varphi(A)) = r(\varphi(A^*A))$. Hence using our previous result

$$||\varphi(A)||^2 = r(\varphi(A^*A)) \leq r(A^*A) = ||A||^2$$

and taking square roots gives the desired inequality. ■

Having a source of a large number of positive elements in a C^*-algebra is important for the characterization of C^*-algebras in general. Consider $A \in \mathfrak{A}_{sa}$ and $r \geq ||A||$. Note $\lambda \in \mathrm{sp}(rI - A)$ if and only if $r - \lambda \in \mathrm{sp}(A)$. Lemma 2.8 implies that $|r - \lambda| \leq ||A||$, and by Proposition 2.4 $r - \lambda \in \mathbb{R}$, so $-||A|| \leq r - \lambda \leq ||A||$. It follows that $\lambda \in \mathbb{R}$ as $r \in \mathbb{R}$, and $0 \leq r - ||A|| \leq \lambda$. In other words, $\mathrm{sp}(rI - A) \subseteq \mathbb{R}_{\geq 0}$ as λ was an arbitrary spectral value. Hence we obtain $rI - A \in \mathfrak{A}_+$. This result is collected in the following Corollary.

Corollary 2.9. *If $A \in \mathfrak{A}_{sa}$, then $rI - A \in \mathfrak{A}_+$ for any $r \in \mathbb{R}$, with $r \geq ||A||$.*

2.2 Concrete Models of C^*-Algebras

Now that we have some understanding of C^*-algebras and their maps we can consider concrete models of C^*-algebras. Intuitively, a model, or *representation*, of an abstract C^*-algebra is an association of elements in the algebra to bounded operators on some Hilbert space in a way that preserves the structure on either side of the association [Str21, p. 46].

Definition 2.10 ($*$-representation)**.** A **$*$-representation** of a C^*-algebra $\mathfrak{A}$ is a pair $(\varphi, \mathscr{H})$ of a Hilbert space $\mathscr{H}$ together with a $*$-homomorphism $\varphi : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$.

A first example of a $*$-representation may be recalled from Conway's Functional Analysis [Con07, p. 28].

Example 2.2. Let (X, Ω, μ) be a σ-finite measure space. Then $L^\infty(X)$ is a C^*-algebra with pointwise operations and complex conjugation as an involution. The proof is analogous to the case of $C(Y)$ covered previously [KR97, pp. 276-277]. Additionally, the map $g \mapsto M_g$, where $M_g : L^2(X) \to L^2(X)$ is given by $M_g(\varphi)(x) = g(x)\varphi(x)$ for any $\varphi \in L^2(X)$ and $x \in X$, is a $*$-representation of $L^\infty(X)$ on the Hilbert space $L^2(X)$. First for $f, g \in L^\infty(X)$, and $\varphi \in L^2(X)$, $M_f(M_g(\varphi)) = fg\varphi = M_{fg}(\varphi)$, so $M_f \circ M_g = M_{fg}$. Additionally, the unit in $L^\infty(X)$ is the constant function 1, and $M_1(\varphi) = 1 \cdot \varphi = \varphi$, so M_1 is the identity on $L^2(X)$. $\mathbb{C}$-linearity follows by distributivity of multiplication in $\mathbb{C}$. Finally, if $\psi \in L^2(X)$ as well,

$$\langle M_f(\varphi), \psi \rangle = \int_X \overline{M_f(\varphi)} \psi d\mu = \int_X \overline{f\varphi} \psi d\mu = \int_X \overline{\varphi}\overline{f} \psi d\mu = \int_X \overline{\varphi} M_{f^*}(\psi) = \langle \varphi, M_{f^*}(\psi) \rangle$$

so by uniqueness of the adjoint for bounded operators $M_f^* = M_{f^*}$.

We have a number of important and useful ways of classifying representations of C^*-algebras. The primary classification in the case of the GNS construction is the notion of a cyclic representation.

Definition 2.11 (Cyclic Representations)**.** Let $(\varphi, \mathscr{H})$ be a representation of $\mathfrak{A}$. Then, $(\varphi, \mathscr{H})$ is **cyclic** if there exists a **cyclic vector** $v \in \mathscr{H}$ such that $\mathrm{cl}(\varphi(\mathfrak{A})v) = \mathscr{H}$, where $\varphi(\mathfrak{A})v = \{\varphi(A)v : A \in \mathfrak{A}\}$ [KR97, p. 276].

Note that $\varphi(\mathfrak{A})v$ is a linear subspace of $\mathscr{H}$ as φ is linear.

Now, as in the case of C^*-algebras alone it is important to ask how we can map between two representations of a C^*-algebra.

Definition 2.12 (Morphism of $*$-representations)**.** A map between representations $(\varphi, \mathscr{H})$ and $(\psi, \mathscr{K})$ for a C^*-algebra $\mathfrak{A}$ is a bounded linear operator $T : \mathscr{H} \to \mathscr{K}$ such that for any $A \in \mathfrak{A}$,

$$T \circ \varphi(A) = \psi(A) \circ T$$

T is also known as an **intertwiner** [Par18, p. 1126].

In the case that T is unitary, that is $T^* = T^{-1}$, we say that it is a **unitary equivalence**. This is the primary notion we use to discuss the equivalence of representations for a C^*-algebra, and we will discuss it further in the context of the GNS construction.

2.3 States and Construction of Hilbert Spaces

In order to construct representations in general we need a mechanism which allows us to create Hilbert spaces that our C^*-algebras can act on. We can obtain such a mechanism through our motivation in quantum mechanics. In von Neumann's formalization of quantum mechanics in terms Hilbert spaces and bounded operators we interpret elements of the Hilbert space as states and the self-adjoint bounded operators as observables which can be measured. Explicitly, given a Hilbert space $\mathscr{H}$ and a normalized vector $h \in \mathscr{H}$, the measurement of a self-adjoint operator $\mathcal{O} \in \mathscr{B}(\mathscr{H})$ with respect to the state h is given using the inner product: $\mathrm{Meas}_h(\mathcal{O}) := \langle \mathcal{O}h, h\rangle$. Since we want to realize the elements of an abstract C^*-algebra as bounded operators on a Hilbert space, we want to learn how to measure elements on some type of generalized state. Thinking about states in this way as families of measurements motivates the following definition [Str21, p. 42].

Definition 2.13 (State)**.** A linear functional $\rho : \mathfrak{A} \to \mathbb{C}$ is **positive** if $\rho(\mathfrak{A}_+) \subseteq \mathbb{R}_{\geq 0}$. If in addition $||\rho|| = 1$, ρ is said to be a **state**. We denote the collection of states by $\mathcal{S}(\mathfrak{A})$.

We can obtain a natural example by looking back to the C^*-algebras of the form $C(Y)$, for Y a compact Hausdorff space [Str21, p. 43].

Example 2.3. If μ is a Borel measure on Y which is normalized, then the assignment

$$L : f \mapsto \int_Y f d\mu, \ \forall f \in C(Y)$$

is a state on $C(Y)$. Linearity of the integral of a Borel measure on continuous functions implies this is a linear functional. Additionally, by the triangle inequality for Lebesgue integrals $|L(f)| \leq ||f||_Y \int_Y d\mu = ||f||_Y$, and $|L(1)| = \int_Y d\mu = 1 = ||1||_Y$ since the measure is normalized, so we have that this functional is bounded with operator norm 1. Finally, $\lambda \in \mathrm{sp}(f)$ if and only if $\lambda - f(y)$ is zero for some $y \in Y$, as otherwise the reciprocal function $\frac{1}{\lambda - f(y)}$ is continuous as well and the product is the unit function 1. Thus, $\mathrm{sp}(f) = \mathrm{Im}(f)$, so f is positive if and only if $\mathrm{Im}(f) \subseteq \mathbb{R}_{\geq 0}$. This implies that if f is positive $\int_Y f d\mu \geq \int_Y 0 d\mu = 0$, so this linear functional is indeed a state.

In the case of our quantum-mechanical motivation, if $\mathscr{H}$ is a Hilbert space and $h \in \mathscr{H}$ is a non-zero vector, then Meas_h defines a state on $\mathscr{B}(\mathscr{H})$. Indeed it is a linear functional since the first component of an inner product is linear, it is positive by definition of positive operators, and for any $\mathcal{O} \in \mathscr{B}(\mathscr{H})$, by the Cauchy-Schwarz inequality

$$|\langle \mathcal{O}h, h\rangle| \leq \sqrt{\langle \mathcal{O}h, \mathcal{O}h\rangle \langle h, h\rangle} \leq ||\mathcal{O}|| \sqrt{\langle h, h\rangle} = ||\mathcal{O}||$$

since h is normalized. This implies $||\mathrm{Meas}_h|| \leq 1$, and as $|\langle \mathrm{Id}_{\mathscr{H}} h, h\rangle| = 1$, we have equality. In this way the notion of a state defined above generalizes the notion of a physical state as a family of measurements.

An important property of positive linear functionals, and hence states, is that they are always $*$-homomorphisms, and so in a rigorous way measurements on states vary continuously by Proposition 2.7.

Proposition 2.14. *If $\rho : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional, then $\rho(A^*) = \overline{\rho(A)}$ for any $A \in \mathfrak{A}$.*

Proof. First if $A \in \mathfrak{A}_{sa}$ then by Proposition 2.4 there exist $B, C \in \mathfrak{A}_+$ such that $A = B - C$. Then $\rho(B), \rho(C) \in \mathbb{R}_{\geq 0}$, so $\rho(A) = \rho(B) - \rho(C) \in \mathbb{R}$. Now if $a \in \mathfrak{A}$ is arbitrary, we can write $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$. Observe that $\left(\frac{a+a^*}{2}\right)^* = \frac{a^*+a}{2} = \frac{a+a^*}{2}$ and $\left(\frac{a-a^*}{2i}\right)^* = \frac{a^*-a}{-2i} = \frac{a-a^*}{2i}$, so $a = b + ic$, for $b, c \in \mathfrak{A}_{sa}$. It follows by conjugate linearity of $*$ that

$$\rho(a^*) = \rho((b+ic)^*) = \rho(b-ic) = \rho(b) - i\rho(c) = \overline{\rho(b) + i\rho(c)} = \overline{\rho(a)}$$

using the fact that $\rho(b), \rho(c) \in \mathbb{R}$. ■

A natural question at this point is how an abstract state can even begin providing a representation of a C^*-algebra. One key result in this direction is the following.

Lemma 2.15. *If ρ is a state on $\mathfrak{A}$, then the map $u_\rho : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ given by $u_\rho(A, B) = \rho(B^*A)$ is a semi-inner product on $\mathfrak{A}$.*

Proof. If $A, B, C \in \mathfrak{A}$ and $c \in \mathbb{C}$, we have by conjugate linearity of * and linearity of ρ that

$$u_\rho(cA+B, C) = \rho(C^*(cA+B)) = \rho(cC^*A+C^*B) = c\rho(C^*A)+\rho(C^*B) = cu_\rho(A, C)+u_\rho(B, C)$$

and

$$u_\rho(C, cA+B) = \rho((cA+B)^*C) = \rho(\overline{c}A^*C+B^*C) = \overline{c}\rho(A^*C)+\rho(B^*C) = \overline{c}u_\rho(C, A)+u_\rho(C, B)$$

Thus we have linearity and conjugate linearity. If $A \in \mathfrak{A}$, by Equation 2.1 $A^*A \in \mathfrak{A}_+$, so by definition $u_\rho(A, A) = \rho(A^*A) \geq 0$, and u_ρ is positive semi-definite. Finally, for $A, B \in \mathfrak{A}$ conjugate-symmetry follows by Proposition 2.14

$$u_\rho(B, A) = \rho(A^*B) = \rho((B^*A)^*) = \overline{\rho(B^*A)} = \overline{u_\rho(A, B)}$$

Thus u_ρ is a semi-inner product on $\mathfrak{A}$. ■

This result also allows us to further characterize states. Consider $\rho \in \mathcal{S}(\mathfrak{A})$ and $A \in \mathcal{A}$ with $||A|| \leq 1$. Then $||A^*A|| = ||A||^2 \leq 1$, so by Corollary 2.9, $I - A^*A \in \mathfrak{A}_+$, and as ρ is positive $\rho(I - A^*A) \geq 0$. Then using the Cauchy-Schwarz inequality for semi-inner products and the fact that $I^* = I$, $|\rho(A)|^2 = |\rho(I^*A)|^2 \leq \rho(I^*I)\rho(A^*A) \leq \rho(I)^2$, so $||\rho|| \leq \rho(I)$. The other inequality follows from $|\rho(I)| \leq ||\rho||\,||I|| = ||\rho||$ since $||I|| = 1$, so $\rho(I) = ||\rho||$.

With this characterization in hand we can also realize the topological structure of the state space $\mathcal{S}(\mathfrak{A})$ in the dual space $\mathfrak{A}^*$.

Corollary 2.16. *The state space $\mathcal{S}(\mathfrak{A})$ is a convex weak-$*$ compact subset of the closed unit ball in $\mathfrak{A}^*$ [Mur90, p. 146].*

Proof. The argument following Lemma 2.15 allows us to write the state space as

$$\mathcal{S}(\mathfrak{A}) = \{\rho \in \mathfrak{A}^* : \forall A \in \mathfrak{A}_+, \rho(A) \geq 0, \text{ and } \rho(I) = 1\}$$

If $\rho, \sigma \in \mathcal{S}(\mathfrak{A})$, and $s \in [0,1]$, then $s\rho + (1-s)\sigma \in \mathfrak{A}^*$, being a vector space, $s\rho(I) + (1-s)\sigma(I) = s+(1-s) = 1$, and for $A \in \mathfrak{A}_+$, $s\rho(A)+(1-s)\sigma(A) \geq 0$ as $s, 1-s, \rho(A), \sigma(A) \geq 0$. Thus $s\rho + (1-s)\sigma \in \mathcal{S}(\mathfrak{A})$, so the state space is convex.

Note $\mathcal{S}(\mathfrak{A})$ is a subset of the unit ball in $\mathfrak{A}^*$, which is weak-$*$ compact by Alaoglu's Theorem, so it is sufficient to show that $\mathcal{S}(\mathfrak{A})$ is weak-$*$ closed. To show this let $\{\rho_i\}_{i\in\mathscr{I}} \subseteq \mathcal{S}(\mathfrak{A})$ be a net such that $\rho_i \to \rho \in \mathfrak{A}^*$ in the weak-$*$ topology. Then for all $A \in \mathfrak{A}$, $\langle \rho_i, A\rangle \to \langle \rho, A\rangle$ in $\mathbb{C}$. First, since $\rho_i(I) = 1$ for all $i \in \mathscr{I}$, in the component-wise limit $\rho(I) = 1$. Next, if $A \in \mathfrak{A}_+$, $\rho_i(A) \geq 0$ for all $i \in \mathscr{I}$, so by monoticity of limits in $\mathbb{R}$, $\rho(A) \geq 0$. Thus $\rho \in \mathcal{S}(\mathfrak{A})$, completing the proof. ■

Corollary 2.16 Demonstrates that the positive functionals on a C^*-algebra have a natural geometric realization and structure. Further, as the state space is a convex weak-$*$ compact subset of the closed unit ball in $\mathfrak{A}^*$, it follows that any given non-trivial C^*-algebra has an abundance of available states. Following the GNS construction we will make this statement more precise in algebraic terms. This geometric perspective also allows us to interpret these abstract states through physical intuition.

Next, although Lemma 2.15 provides a first step at constructing a representation of a C^*-algebra, we need not just a semi-inner product space, but a Hilbert space. The step from semi-inner product space to inner product space will be shown in the GNS construction, but we argue the next required step from inner product space to Hilbert space here.

Lemma 2.17. *If $(V, \langle,\rangle)$ is an inner product space, there exists a Hilbert space completion $(\mathscr{H}, \langle,\rangle_{\mathscr{H}})$ together with an isometry $\iota : V \hookrightarrow \mathscr{H}$ such that $\iota(V)$ is dense in $\mathscr{H}$.*

Proof. Let $||\cdot||$ denote the norm induced by the inner product on V. From analysis on metric spaces we have a metric completion $\mathscr{H}$ of V with respect to the metric induced by $||\cdot||$, which is given by equivalence classes of Cauchy sequences [Sal13, p. 63]. Let $[A_n]$ denote the equivalence class of a Cauchy sequence $(A_n)_{n\in\mathbb{N}}$. We also have an isometry $\iota : V \hookrightarrow \mathscr{H}$ sending an element to the equivalence class of its constant sequence, which has dense image [Sal13, p. 63]. It remains to define an inner product $\langle,\rangle_{\mathscr{H}}$ on $\mathscr{H}$ extending the inner product on V. To this effect, for $[A_n], [B_n] \in \mathscr{H}$ we define

$$\langle [A_n], [B_n]\rangle_{\mathscr{H}_\rho} := \lim_{n\to\infty} \langle A_n, B_n\rangle$$

To show this limit exists observe that for $n, m \in \mathbb{N}$,

$$\begin{aligned} |\langle A_n, B_n\rangle - \langle A_m, B_m\rangle| &= |\langle A_n - A_m, B_n\rangle + \langle A_m, B_n - B_m\rangle| \\ &\leq |\langle A_n - A_m, B_n\rangle| + |\langle A_m, B_n - B_m\rangle| && \text{(triangle ineq.)} \\ &\leq ||A_n - A_m||\,||B_n|| + ||A_m||\,||B_n - B_m|| && \text{(Cauchy-Schwarz ineq.)} \end{aligned}$$

As the A_n and B_n sequences are Cauchy, and hence bounded, when $n, m \to \infty$ the right hand side will go to zero. This implies that the sequence $\langle A_n, B_n\rangle$ is Cauchy in $\mathbb{C}$, and so converges to a unique limit. Next to show the inner product is well defined suppose $[A_n] = [A'_n]$ and $[B_n] = [B'_n]$. Then observe that for all n,

$$|\langle A_n, B_n\rangle - \langle A'_n, B'_n\rangle| = |\langle (A_n - A'_n), B_n\rangle + \langle A'_n, (B_n - B'_n)\rangle|$$

$$\leq |\langle (A_n - A'_n), B_n\rangle| + |\langle A'_n, (B_n - B'_n)\rangle| \qquad \text{(triangle ineq.)}$$
$$\leq \left|\left|A_n - A'_n\right|\right| \, ||B_n|| + \left|\left|A'_n\right|\right| \, \left|\left|B_n - B'_n\right|\right| \quad \text{(Cauchy-Schwarz ineq.)}$$

From our previous work $\langle A'_n, A'_n\rangle = ||A'_n||^2$ and $\langle B_n, B_n\rangle = ||B_n||^2$ converge in $\mathbb{C}$, and hence $||A'_n||$ and $||B_n||$ are bounded. Further, by definition of the equivalence classes $||A_n - A'_n||$ and $||B_n - B'_n||$ converge to 0 in the limit as $n \to \infty$. Thus, both terms in the above inequality go to zero as $n \to \infty$ as they are products of bounded sequences and sequences which converge to zero. It follows that $|\langle A_n, B_n\rangle - \langle A'_n, B'_n\rangle| \to 0$ as $n \to \infty$, implying that

$$\lim_{n\to\infty} \langle A_n, B_n\rangle = \lim_{n\to\infty} \langle A'_n, B'_n\rangle$$

so the inner product is well-defined. The conjugate-symmetry and linearity properties of the inner product follow from those of the inner product on V as well as the additivity of limits in $\mathbb{C}$. Hence we need only show the positive-definiteness, so let $[A_n] \in \mathscr{H}$ such that $\langle [A_n], [A_n]\rangle_{\mathscr{H}} = 0$. However, this is equivalent to $\lim_{n\to\infty} \langle A_n, A_n\rangle = 0$, which is equivalent to $A_n \to 0$, so by definition of the equivalence classes $[A_n] = [0]$. ■

3 The Gelfand-Naimark-Segal Construction

We now have sufficient structure to proceed with the GNS construction on a state of a unital C^*-algebra [KR97, p. 278].

Theorem 3.1 (GNS Construction). *If $\rho : \mathfrak{A} \to \mathbb{C}$ is a state then there exists a representation $(\varphi_\rho, \mathscr{H}_\rho)$ of $\mathfrak{A}$ with a cyclic vector $\xi_\rho \in \mathscr{H}_\rho$ such that*

$$\rho(A) = \langle \varphi_\rho(A)\xi_\rho, \xi_\rho\rangle_{\mathscr{H}_\rho}, \ \forall A \in \mathfrak{A}$$

Proof. The proof of this theorem will follow in stages. Let u_ρ denote the semi-inner product from Lemma 2.15. We begin by constructing a vector space where this is an inner product. Consider the set $\mathfrak{K} = \{A \in \mathfrak{A} : u_\rho(A, A) = 0\}$. Since u_ρ is a semi-inner product we have the Cauchy-Schwarz inequality. In particular, if $A \in \mathfrak{K}$ and $B \in \mathfrak{A}$,

$$|u_\rho(A, B)|^2 \leq u_\rho(A, A)u_\rho(B, B) = 0 \tag{3.2}$$

so $u_\rho(A, B) = 0$. This implies that if $A, B \in \mathfrak{K}$ and $\alpha \in \mathbb{C}$,

$$u_\rho(A + \alpha B, A + \alpha B) = u_\rho(A, A) + \alpha u_\rho(B, A) + \overline{\alpha} u_\rho(A, B) + |\alpha|^2 u_\rho(B, B) = 0$$

so $A + \alpha B \in \mathfrak{K}$. Hence $\mathfrak{K}$ is a linear subspace, so $\mathfrak{A}/\mathfrak{K}$ is a vector space. Additionally, for $A, B \in \mathfrak{A}$, $\langle A + \mathfrak{K}, B + \mathfrak{K}\rangle := u_\rho(A, B)$ defines an inner product on the quotient. It is sufficient to show this mapping is well-defined and positive definite as the remaining properties follow from the corresponding properties for u_ρ. If $C = A - A', D = B - B' \in \mathfrak{K}$ then by the result in Equation 3.2

$$u_\rho(A, B) = u_\rho(A'+C, B'+D) = u_\rho(A', B')+u_\rho(A', D)+u_\rho(C, B')+u_\rho(C, D) = u_\rho(A', B')$$

so $\langle , \rangle$ is indeed well-defined as u_ρ gives the same output for inputs in the same coset. If $\langle A + \mathfrak{K}, A + \mathfrak{K}\rangle = 0$, then $u_\rho(A, A) = 0$ which implies $A \in \mathfrak{K}$ and hence $A + \mathfrak{K} = 0 + \mathfrak{K}$. Thus $\langle , \rangle$ indeed defines an inner product on $\mathfrak{A}/\mathfrak{K}$. Further, by Lemma 2.17 we have a natural Hilbert space completion $\mathscr{H}_\rho$ in terms of equivalence classes of Cauchy sequences, together with an isometry $\iota : \mathfrak{A}/\mathfrak{K} \hookrightarrow \mathscr{H}_\rho$ such that the image is dense.

Now that we have a Hilbert space all that remains is constructing a representation $\varphi_\rho : \mathfrak{A} \to \mathscr{H}_\rho$. Note that bounded maps are uniformly continuous, and uniformly continuous maps on $\mathfrak{A}/\mathfrak{K}$ into a complete metric space can be extended uniquely to its metric completion [DD02, p. 262]. Hence, it is sufficient to define bounded maps on $\mathfrak{A}/\mathfrak{K}$ since we can then post-compose with the isometry ι to obtain a uniformly continuous map into $\mathscr{H}_\rho$. To this effect we define for each $A \in \mathfrak{A}$ a map $\widetilde{\varphi_\rho}(A)$ on $\mathfrak{A}/\mathfrak{K}$ by $\widetilde{\varphi_\rho}(A)(B+\mathfrak{K}) = AB + \mathfrak{K}$ for all $B + \mathfrak{K} \in \mathfrak{A}/\mathfrak{K}$. To see that this mapping is well-defined first note that by Equation 3.2 if $A \in \mathfrak{A}$ and $K \in \mathfrak{K}$,

$$u_\rho(AK, AK) = \rho((AK)^*AK) = \rho((A^*AK)^*K) = u_\rho(K, A^*AK) = 0$$

so $AK \in \mathfrak{K}$. Then if $B + \mathfrak{K} = B' + \mathfrak{K}$, $B - B' \in \mathfrak{K}$, so $AB - AB' = A(B - B') \in \mathfrak{K}$ and hence $AB + \mathfrak{K} = AB' + \mathfrak{K}$. Thus $\widetilde{\varphi_\rho}(A)$ is well-defined.

Next we must show that $\widetilde{\varphi_\rho}(A)$ is continuous with respect to the topology on $\mathfrak{A}/\mathfrak{K}$ induced by its inner product. Let $A, B \in \mathfrak{A}$. By Equation 2.1 $A^*A \in \mathfrak{A}_+ \subseteq \mathfrak{A}_{sa}$, so $||A^*A||\,I - A^*A = ||A||^2\,I - A^*A \in \mathfrak{A}_+$ by Corollary 2.9 and the C6 condition. Then Corollary 2.5 gives that $||A||^2\,B^*B - (AB)^*AB = B^*(||A||^2 - A^*A)B \in \mathfrak{A}_+$, so $\rho(||A||^2\,B^*B - (AB)^*AB) \geq 0$. By linearity this implies $\rho((AB)^*AB) \leq ||A||^2\,\rho(B^*B)$ which translates upon taking square roots and using the definition of the norm from an inner product,

$$||\widetilde{\varphi_\rho}(A)(B+\mathfrak{K})|| = \sqrt{\langle \widetilde{\varphi_\rho}(A)(B+\mathfrak{K}), \widetilde{\varphi_\rho}(A)(B+\mathfrak{K})\rangle} \leq ||A||\,\sqrt{\langle B+\mathfrak{K}, B+\mathfrak{K}\rangle} = ||A||\,||B+\mathfrak{K}||$$

using the definition of the inner product on $\mathfrak{A}/\mathfrak{K}$. Thus $||\widetilde{\varphi_\rho}(A)|| \leq ||A||$, so $\widetilde{\varphi_\rho}(A)$ is bounded. Hence $\iota \circ \widetilde{\varphi_\rho}(A)$ extends to a uniformly continuous map $\varphi_\rho(A)$ on $\mathscr{H}_\rho$, where by extends we mean $\varphi_\rho(A)(\iota(B+\mathfrak{K})) = \iota(\widetilde{\varphi_\rho}(A)(B+\mathfrak{K}))$ for any $B+\mathfrak{K} \in \mathfrak{A}/\mathfrak{K}$. In particular, for $[A_n + \mathfrak{K}] \in \mathscr{H}_\rho$ we know that $\iota(A_m + \mathfrak{K}) \to [A_n]$ as $m \to \infty$ [Sal13, p. 262, Thm 9.5.4], so

$$\varphi_\rho(A)([A_n + \mathfrak{K}]) = \lim_{m\to\infty} \varphi_\rho(A)\iota(A_m + \mathfrak{K}) = \lim_{m\to\infty} \iota(\widetilde{\varphi_\rho}(A)(A_m + \mathfrak{K})) = [AA_n + \mathfrak{K}]$$

again using [Sal13, p. 262, Thm 9.5.4]. Thus $\varphi_\rho(A)$ acts componentwise on equivalence classes of sequences, and so is linear by distributivity, C1. As $\varphi_\rho(A)$ is linear and continuous it is bounded. It remains to show that φ_ρ is a $*$-homomorphism. As the action is component-wise φ_ρ preserves the multiplicative and vector space structures of $\mathfrak{A}$, so it is sufficient to check that it preserves the involution structure. For $A \in \mathfrak{A}$ and $[B_n + \mathfrak{K}], [C_n + \mathfrak{K}] \in \mathscr{H}_\rho$, we can use the classification of the Hilbert space inner product in Lemma 2.17 and our definition of the inner product on $\mathfrak{A}/\mathfrak{K}$ to write

$$\langle [AB_n + \mathfrak{K}], [C_n + \mathfrak{K}]\rangle_{\mathscr{H}_\rho} = \lim_{n\to\infty} \rho(C_n^*AB_n) = \lim_{n\to\infty} \rho((A^*C_n)^*B_n) = \langle [B_n + \mathfrak{K}], [A^*C_n + \mathfrak{K}]\rangle_{\mathscr{H}_\rho}$$

so by uniqueness of the adjoint of an operator on a Hilbert space, $\varphi_\rho(A^*) = \varphi_\rho(A)^*$. Thus $(\varphi_\rho, \mathscr{H}_\rho)$ is a representation of $\mathfrak{A}$.

Finally we must show that $\mathscr{H}_\rho$ has a cyclic vector. Since $\mathfrak{A}$ is assumed unital we may consider the vector $\xi_\rho := [I + \mathfrak{K}] \in \mathscr{H}_\rho$. Then $\varphi_\rho(\mathfrak{A})\xi_\rho = \{[A + \mathfrak{K}] : A \in \mathfrak{A}\}$ is precisely the image of ι in $\mathscr{H}_\rho$, which is dense by construction of the original metric completion. Thus ξ_ρ is indeed a cyclic vector. Finally, for $A \in \mathscr{H}_\rho$ we have

$$\rho(A) = \langle A + \mathfrak{K}, I + \mathfrak{K}\rangle = \lim_{n\to\infty} \langle A + \mathfrak{K}, I + \mathfrak{K}\rangle = \langle [A(I + \mathfrak{K})], [I + \mathfrak{K}]\rangle_{\mathscr{H}_\rho} = \langle \varphi_\rho(A)\xi_\rho, \xi_\rho\rangle_{\mathscr{H}_\rho}$$

completing the construction. ■

3.1 Existence and Uniqueness of GNS Constructions

Although we now have this powerful constructive result for C^*-algebras, its use is limited unless we have the existence of a sufficient number of states on any given C^*-algebra. Not only can we demonstrate such an existence, we can in fact give a stronger result related to measuring the spectra of elements in a C^*-algebra.

Proposition 3.2. *If $A \in \mathfrak{A}$ and $a \in sp(A)$, then there exists $\rho \in \mathcal{S}(\mathfrak{A})$ such that $\rho(A) = a$ [KR97, p. 257].*

Proof. Observe that if $b, c \in \mathbb{C}$, $baI - bA = b(aI - A) \notin \mathfrak{A}^\times$ as $aI - A \notin \mathfrak{A}^\times$, so $(ba + c)I - (bA + cI) = baI - bA \notin \mathfrak{A}^\times$. This implies $ba + c \in \mathrm{sp}(bA + cI)$, so from Lemma 2.8 $|ba + c| \leq ||bA + cI||$ for any $b, c \in \mathbb{C}$. Now we may define a linear functional $\psi : \mathrm{span}\{A, I\} \to \mathbb{C}$ by $\psi(A) = a$ and $\psi(I) = 1$, and extending by linearity. Then $|\psi(bA + cI)| = |ba + c| \leq ||bA + cI||$ for any b, c, so $||\psi|| \leq 1$. As $|\psi(I)| = 1 = ||I||$ equality follows, $||\psi|| = 1$. By the Hahn-Banach Theorem there exists a linear functional $\rho : \mathfrak{A} \to \mathbb{C}$ such that $||\rho|| = ||\psi|| = 1$, and $\rho|_{\mathrm{span}\{A,I\}} = \psi$, so in particular $\rho(A) = \psi(A) = a$.

To show ρ is positive let $B \in \mathfrak{A}_+$, and let $\rho(B) = x + iy$, $x, y \in \mathbb{R}$. Without loss of generality we may suppose B is non-zero as ρ is linear. Then let $0 < s < \frac{1}{||B||}$. Note that $\mathrm{sp}(sB) = s \cdot \mathrm{sp}(B)$ since $\lambda \in \mathrm{sp}(sB)$ if and only if $s(\lambda/sI - B) = \lambda I - sB \notin \mathfrak{A}^\times$, if and only if $\lambda/s \in \mathrm{sp}(B)$, since $s \neq 0$. Further, $\mathrm{sp}(sB) = s \cdot \mathrm{sp}(B) \subseteq [0, 1]$ since $0 < s\,||B|| < 1$ and $\mathrm{sp}(B) \subseteq [0, ||B||]$ by Lemma 2.8 along with the fact B is positive. It follows that $\mathrm{sp}(I - sB) = \{1 - \alpha : \alpha \in \mathrm{sp}(sB)\} \subseteq [0, 1]$. Hence $0 \leq \lambda \leq 1$ for all $\lambda \in \mathrm{sp}(I - sB)$, so $r(I - sB) \leq 1$ taking the supremum. Since B is self-adjoint and s is real, $I - sB$ is self-adjonit so $||I - sB|| = r(I - sB) \leq 1$ by the remark following Lemma 2.8. Observe

$$1 - sx \leq \sqrt{(1 - sx)^2 + s^2y^2} = |1 - s(x + iy)| = |\rho(I - sB)| \leq ||I - sB|| \leq 1$$

using the fact that $||\rho|| = 1$. It follows that $x \geq 0$ since s is positive. Finally, to show $y = 0$ consider for $n \in \mathbb{N}$, $C_n = B + (iny - x)I = (B - xI) + inyI$. Using condition C6 and the fact $B^* = B$,

$$||C_n||^2 = ||C_n^* C_n|| = \left|\left|(B - xI)^2 + iny(B - xI) - iny(B - xI) + n^2y^2 I\right|\right| \leq \left|\left|(B - xI)^2\right|\right| + n^2y^2$$

using the triangle inequality and the fact that $||I|| = 1$ in the last inequality. On the other hand $|\rho(C_n)|^2 = |x + iy + iny - x|^2 = (n^2 + 2n + 1)y^2$, so as $|\rho(C_n)| \leq ||C_n||$,

$$(n^2 + 2n + 1)y^2 \leq \left|\left|(B - xI)^2\right|\right| + n^2y^2, \text{ or } (2n + 1)y^2 \leq ||B - xI||^2$$

As this holds for all $n \in \mathbb{N}$ the left hand side will grow without bound unless $y = 0$. Hence $\rho(B) = x \geq 0$, so ρ is positive, implying $\rho \in \mathcal{S}(\mathfrak{A})$. ■

Proposition 3.2 together with the GNS construction implies that we have representations which correspond to measurements of each spectral value of a given element $A \in \mathfrak{A}$. In fact, not only do GNS representations identify measurements of C^*-algebra elements, they are fully characterized by them, up to unitary equivalence [Str21, p. 48].

Theorem 3.3. *Let $(\pi, \mathscr{H}_\pi)$ be another cyclic representation of $\mathfrak{A}$ with cyclic vector $\xi_\pi \in \mathscr{H}_\pi$. Then $(\pi, \mathscr{H}_\pi)$ is unitarily equivalent to $(\varphi_\rho, \mathscr{H}_\rho)$, for $\rho \in \mathcal{S}(\mathfrak{A})$, with cyclic vectors mapped to each other, if and only if*

$$\langle \pi(A)\xi_\pi, \xi_\pi \rangle_{\mathscr{H}_\pi} = \langle \varphi_\rho(A)\xi_\rho, \xi_\rho \rangle_{\mathscr{H}_\rho}, \ \forall A \in \mathfrak{A}$$

Proof. First suppose $U : \mathscr{H}_\pi \to \mathscr{H}_\rho$ is a unitary equivalence such that $U\xi_\pi = \xi_\rho$. Then for $A \in \mathfrak{A}$ $U\pi(A) = \varphi_\rho(A)U$, so

$$\langle \pi(A)\xi_\pi, \xi_\pi \rangle_{\mathscr{H}_\pi} = \langle U^*\varphi_\rho(A)U\xi_\pi, \xi_\pi \rangle_{\mathscr{H}_\pi} = \langle \varphi_\rho(A)\xi_\rho, U\xi_\pi \rangle_{\mathscr{H}_\rho} = \langle \varphi_\rho(A)\xi_\rho, \xi_\rho \rangle_{\mathscr{H}_\rho}$$

Conversely, if we have the equality for any $A \in \mathfrak{A}$, define a map $U : \pi(\mathfrak{A})\xi_\pi \to \mathscr{H}_\rho$ by $U(\pi(A)\xi_\pi) = \varphi_\rho(A)\xi_\rho$. This map is linear since π and φ_ρ are. Observe that for $A \in \mathfrak{A}$,

$$\begin{aligned}
\langle U(\pi(A)\xi_\pi), U(\pi(A)\xi_\pi) \rangle_{\mathscr{H}_\rho} &= \langle \varphi_\rho(A)\xi_\rho, \varphi_\rho(A)\xi_\rho \rangle_{\mathscr{H}_\rho} \\
&= \langle \varphi_\rho(A)^*\varphi_\rho(A)\xi_\rho, \xi_\rho \rangle_{\mathscr{H}_\rho} \\
&= \langle \varphi_\rho(A^*A)\xi_\rho, \xi_\rho \rangle_{\mathscr{H}_\rho} \\
&= \langle \pi(A^*A)\xi_\pi, \xi_\pi \rangle_{\mathscr{H}_\pi} = \langle \pi(A)\xi_\pi, \pi(A)\xi_\pi \rangle_{\mathscr{H}_\pi}
\end{aligned}$$

using the fact that representations are $*$-homomorphisms. Thus U is an isometry, and in particular bounded. Let $x \in \mathscr{H}_\pi$, so $x = \lim_{n\to\infty} \pi(A_n)\xi_\pi$ for some $A_n \in \mathfrak{A}$ by cyclicity. Then define $\widetilde{U} : \mathscr{H}_\pi \to \mathscr{H}_\rho$ by $\widetilde{U}x = \lim_{n\to\infty} U\pi(A_n)\xi_\pi = \lim_{n\to\infty} \varphi_\rho(A_n)\xi_\rho$. Since $\pi(A_n)\xi_\pi$ is convergent and hence Cauchy in $\mathscr{H}_\pi$, and U is an isometry, $U\pi(A_n)\xi_\pi = \varphi_\rho(A_n)\xi_\rho$ is Cauchy in $\mathscr{H}_\rho$. Consequently $U\pi(A_n)\xi_\pi$ converges since $\mathscr{H}_\rho$ is complete. Thus $\widetilde{U}$ is well-defined. $\widetilde{U}$ is also linear since U is linear and addition and scalar multiplication are continuous in a Hilbert space, and so will commute with the limits defining $\widetilde{U}$. Further, since the norm on a Hilbert space is continuous, and using the fact that U is an isometry on $\pi(\mathfrak{A})\xi_\pi$, we have

$$\left|\left|\widetilde{U}x\right|\right|_{\mathscr{H}_\rho} = \lim_{n\to\infty} ||U\pi(A_n)\xi_\pi||_{\mathscr{H}_\rho} = \lim_{n\to\infty} ||\pi(A_n)\xi_\pi||_{\mathscr{H}_\pi} = ||x||_{\mathscr{H}_\pi}$$

which implies $\widetilde{U}$ is an isometry [Con07, p. 19, Prop. 5.2]. Thus $\widetilde{U}$'s image is closed in $\mathscr{H}_\rho$, and also contains the dense subspace $\varphi_\rho(\mathfrak{A})\xi_\rho$, so it equals $\mathscr{H}_\rho$. Thus $\widetilde{U}$ is an isometric isomorphism, which implies $\widetilde{U}^{-1} = \widetilde{U}^*$ [Con07, p. 32, Prop 2.5]. Additionally, observe that $\widetilde{U}\pi(A)\pi(B)\xi_\pi = \varphi_\rho(AB)\xi_\rho = \varphi_\rho(A)\widetilde{U}\pi(B)\xi_\pi$ for any $A, B \in \mathfrak{A}$ since $\widetilde{U}$ extends U, so $\widetilde{U}\pi(A) = \varphi_\rho(A)\widetilde{U}$ on $\pi(\mathfrak{A})\xi_\pi$, which is dense. Hence $\widetilde{U}\pi(A) - \varphi_\rho(A)\widetilde{U}$ and the 0 map agree on a dense subset, so we must have $\widetilde{U}\pi(A) - \varphi_\rho(A)\widetilde{U} = 0$ on all of $\mathscr{H}_\pi$ as both functions are continuous, implying $\widetilde{U}\pi(A) = \varphi_\rho(A)\widetilde{U}$. Thus $\widetilde{U}$ is a unitary equivalence being an isometric isomorphism and an intertwiner, and finally $\widetilde{U}\xi_\pi = \widetilde{U}\pi(I)\xi_\pi = \varphi_\rho(I)\xi_\rho = \xi_\rho$. ■

4 Conclusion

Throughout this paper we have investigated the structure of abstract C^*-algebras while motivating our study physically through the lens of quantum mechanics. In this process we have illustrated the importance of element spectra, which provide a first step in assigning measurements to elements of a C^*-algebra in a way which generalizes physical measurements. We then further extended this notion of measurement, in analogy to the case of bounded operators on Hilbert spaces, by introducing abstract states on a C^*-algebra and demonstrating that they possess a well-defined geometric structure. Finally, through the GNS construction we not only showed the existence of concrete models of a C^*-algebra attached to any given state, we demonstrated that such models, or representations, are uniquely characterized by their measurement values. As Segal discusses in his seminal work [Dor94, pp. 55-65], these characterizations of C^*-algebras illustrate their use as models for quantum mechanical systems, as well as their inherent connection to the scientific notion of measurement.

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