# Grothendieck Topoi and the Étale Site

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## 1 Introduction

One of the most important structures in algebraic geometry, and modern geometry more generally, is the concept of a sheaf on a space. The primary reason for the prevalence of sheaves in geometry is the fact that they provide a framework for formalizing when locally defined objects on a space can be glued uniquely into a globally defined object. For example, in differential geometry we have a number of important sheaves including sheaves of $C^k$-functions for $k \in \mathbb{N} \cup \{\infty\}$, sheaves of analytic functions, and even sheaves of holomorphic functions on complex manifolds.

Since the introduction of categorical language into algebraic geometry, category theory has provided an elegant way of describing sheaves on a topological space. On the other hand, the equivalence of categories between the category of affine schemes and the category of commutative rings with identity suggests a more general approach to sheaf theory which includes sheaves on categories that aren't derived from the open sets on a topological space.

In these notes we provide an introduction to this theory, which has roots in the notion of a Grothendieck site on a category [MLM92]. We will begin with introducing the notion of covering families and grothendieck topologies on a category before formalizing a number of equivalent notions of sheaves on a site. We will also prove a number of important properties of sheaves on a site, including the fact that the category of sheaves is (co)complete. Following this initial treatment we will provide a brief introduction to one of the most important sites in algebraic geometry, the étale site as well as some of its motivation relation to analytic topologies, following the treatments in [Mil13, Voo23]. Necessary categorical preliminaries will be included in the appendices so as to avoid cluttering the main sections, including a brief treatment on fibrations in the context of base-change.

## 2 Sheaves on a Site

Although sheaves are initially introduced in the setting of point set topological spaces, their categorical formalizations have been drastically generalized by Grothendieck and his school. A first hint to this approach is that for any (small) category $\mathcal{C}$ we can considered the (set-valued) presheaf category $\mathsf{Ps}(\mathcal{C}) := [\mathcal{C}^{op}, \mathsf{Set}]$. Recall that if $X$ is a topological space, we consider the presheaf category $\mathsf{Ps}(\mathsf{Top}(X)) = [\mathsf{Top}(X)^{op}, \mathsf{Set}]$, in which case the sheaf condition is phrased in terms of open coverings on $X$. Explicitly, we state that a presheaf $F : \mathsf{Top}(X)^{op} \to \mathsf{Set}$ is a sheaf if for any open set $U \subseteq X$ with cover $U = \bigcup_{i\in I} U_i$, a collection of sections $s_i \in F(U_i)$ such that $F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j)$ has a unique gluing $s \in F(U)$ such that $F(U_i \subseteq U)(s) = s_i$ for all $i \in I$. In the language of Section A this is exactly the statement that we have the equalizer diagram

$$F(U) \xrightarrow{\langle F(U_i \subseteq U) : i\in I\rangle} \prod_{i\in I} F(U_i) \underset{\langle F(U_i\cap U_j\subseteq U_j)\circ\pi_j : i,j\in I\rangle}{\overset{\langle F(U_i\cap U_j\subseteq U_i)\circ\pi_i : i,j\in I\rangle}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \cap U_j)$$

In order to generalize this we introduce a suitable notion of topology not just on a set, but on a category. This can be done in terms a covering families and sieves.

**Definition 2.1** [MLM92] A **sieve** on an object $C \in \mathcal{C}_0$ is a presheaf $S : \mathcal{C}^{op} \to \mathsf{Set}$ such that $S(D) \subseteq y_C(D)$ for all $D \in \mathcal{C}_0$, and for all $f : D \to E$ in $\mathcal{C}$, $S(f)$ is $y_C(f)$ restricted to $S(E)$.

Note that a sieve $S$ on an object $C$ is equivalent to a subset of all maps into $C$ which is closed under pre-composition.

Next, for $C \in \mathcal{C}_0$, let $\mathsf{Sieve}(C)$ denote the poset of sieves on $C$ ordered by restriction. That is, if $S$ and $R$ are sieves on $C$, then $S \subseteq R$ if and only if $S(D) \subseteq R(D)$ for all $D \in \mathcal{C}_0$. Then for every $h : D \to C$ in $\mathcal{C}$ we have a natural map of posets $h^* : \mathsf{Sieve}(C) \to \mathsf{Sieve}(D)$ given on $S \in \mathsf{Sieve}(C)$ by

$$h^*(S)(E) = \{g : E \to D \,|\, h \circ g \in S(E)\}$$

These operations define a functor $y_{sv} : \mathcal{C}^{op} \to \mathsf{Poset}$ from $\mathcal{C}$ to the category of posets with monotone maps between them.

Sieves will now take the place of coverings by open sets in our definition of a sheaf. In this way we can define a topology on a category.

**Definition 2.2** [MLM92, p. 110] A **Grothendieck topology** on a category $\mathcal{C}$ consists of a functor $J : \mathcal{C} \to \mathsf{Poset}$ such that for each $C \in \mathcal{C}_0$, $J(C) \subseteq \mathsf{Sieve}(C)$ is a sub-poset of the sieve poset with induced maps such that the following hold:

(i) (Maximality) $y_C \in J(C)$ for all $C \in \mathcal{C}_0$

(ii) (Transitivity) if $S \in J(C)$ and $R \subseteq y_C$ such that for any $h : D \to C$ in $S$, $h^*(R) \in J(D)$, then $R \in J(C)$

Classically the condition that for $h : D \to C$, $h^*$ restricts to a map $J(C) \to J(D)$, is usually referred to as a third axiom called the stability axiom. A **site** is then a category $\mathcal{C}$ together with a Grothendieck Topology (for size considerations we often take $\mathcal{C}$ to be small).

Intuitively we think of sieves in a Grothendieck topology as “covering” the objects they lie over. In particular, for a topological space $X$ we have a natural site on $\mathsf{Top}(X)$ given by covers of open sets. A sieve on an open set $U$ in $\mathsf{Top}(X)$ is then exactly a families of open sets contained in $U$ which is closed downward, and a covering sieve is such a family with union equal to $U$. As in the case of point-set topology it is often more convenient to work with simpler structure which generate topologies than the topology in full. This brings us to the notion of a basis of a Grothendieck topology.

**Definition 2.3** [MLM92, p. 111] A basis for a Grothendieck topology on a category $\mathcal{C}$ is a set function $K$ such that $K(C) \subseteq \mathcal{P}(\mathcal{C}(-, C))$, where we view $\mathcal{C}(-, C)$ as the set of arrows into $C$. The function $K$ must satisfy the following conditions:

(i') $\{f : D \to C\} \in K(C)$ for any isomorphism $f : D \to C$ in $\mathcal{C}$

(ii') (Stability) If $A \in K(C)$, then for any $g : D \to C$ there exists a $B \in K(D)$ such that for every $h : E \to D$ in $B$ , there exists $f : D' \to C$ in $A$ and $h' : E \to D'$ such that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, dashed, "\exists h'"'] & D \arrow[r, "g"] & C \\ & D' \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

(iii') If $A \in K(C)$ and for each $f \in A$, $B_f \in K(\text{dom}(f))$, then the composite $A \circ \{B_f\}_{f\in A} \in K(C)$ where

$$A \circ \{B_f\}_{f\in A} := \{f \circ g : D \to C : f \in A, g \in B_f\}$$

In the case when $\mathcal{C}$ has pullbacks the stability axiom can be phrased slightly differently. In particular, if $A \in K(C)$, then for any $g : D \to C$ we want the base-change of $A$ by $g$, $\{\pi_f : E \times_C D \to D | f : E \to C \in A\}$, to be in $K(D)$. This begins to illuminate the naming convention of the stability axiom, since it says that covers are stable under base change.

From a basis $K$ we generate a topology $J$ by saying $S \in J(C)$ if and only if there exists $R \in K(C)$ such that $R \subseteq S$. Intuitively this says that a sieve covers $C$ if and only if it covers a basis element that covers $C$. This indeed generates a topology on $\mathcal{C}$.

*Proof of Basis.* Let $K$ be a basis for a topology on $\mathcal{C}$, and we show that the described condition defines a topology $J$ on $\mathcal{C}$. First, since $K$ contains all singletons with isomorphisms, in particular $\{1_C : C \to C\} \in K(C)$, so the maximal sieve $y_C \in J(C)$ as it contains the identity on $C$.

Next, to show that $J : \mathcal{C} \to \mathsf{Poset}$ is a functor, we need to show that for $h : D \to C$ in $\mathcal{C}$ and for $S \in J(C)$, $h^*(S) \in J(D)$. Since $S \in J(C)$ there exists $A \in K(C)$ such that $A \subseteq S$. By the stability axiom for Grothendieck bases we have a $B \in K(D)$ such that for any $g : E \to D$ in $B$, we have $f : D' \to C$ in $A$ ad $h' : E \to D'$ such that $h \circ g = f \circ h'$. In terms of $S$ this says that for any $g : E \to D$ in $B$, $h \circ g = f \circ h' \in S$ since $A \subseteq S$ and $S$ is a sieve, so $g \in h^*(S)$. In other words, $B \subseteq h^*(S)$, so $h^*(S) \in J(D)$.

Finally, to show transitivity suppose $S \in J(C)$ is a covering sieve and $R \subseteq y_C$ such that for any $h : D \to C$ in $S$, $h^*(R) \in J(D)$. In terms of $K$ this says that there exists $A \in K(C)$ such that $A \subseteq S$, and for all $h : D \to C$ in $S$, there exists $B_h \in J(D)$ such that $B_h \subseteq h^*(R)$. Then by axiom (iii') for $K$ we have that

$$A \circ \{B_h\}_{h\in A} = \{h \circ g : E \to C \,|\, h \in A, g \in B_h\} \in K(C)$$

Since $A \subseteq S$, we have that for all $h \in A$ and all $g \in B_h$, $g \in h^*(R)$ so $h \circ g \in R$ by definition. In other words, $A \circ \{B_h\}_{h\in A} \subseteq R$, so $R \in J(C)$, as desired. ∎

A few examples of Grothendieck topologies and sites can now be given, including the famed Zariski site as a structure on a purely algebraic category [MLM92].

(a) For any category $\mathcal{C}$, there is a trivial topology where the only covering sieves are maximal sieves.

(b) On the other extreme, for any category $\mathcal{C}$ we have the atomic topology where all non-empty sieves are covering sieves. For this topology to exist we must require that two morphisms $f : D \to C$ and $g : E \to C$ can be completed into a commuting square (not necessarily uniquely).

(c) If $P$ is a partially ordered set viewed as a category, the **dense topology** is given by specifying the covering sieves of $p \in P$ to be those subsets $D \subseteq y_p$ such that for any $r \leq p$, there exists $q \leq r$ with $q \in D$. In general, $\mathcal{C}$ has a dense topology where $S \subseteq y_C$ is a covering sieve if and only if for any $f : D \to C$ there exists $g : E \to D$ such that $f \circ g \in S$.

(d) The **Zariski site** on $\mathsf{CRing}^{op}$ can be defined in terms of a basis. A cover of a cring $A$ is a finite family of the form

$$\{A \to A_{f_i} | 1 \leq i \leq n\}$$

in $\mathsf{CRing}$ up to possible isomorphism such that $A = \langle f_1, ..., f_n \rangle$.

*Proof.* Let $K$ be the proposed basis. The up to isomorphism condition in the definition ensures that all singleton isomorphisms are are in $K$, so condition (i') is satisfied.

Since $\mathsf{CRing}^{op}$ has pullbacks given by pushouts in $\mathsf{CRing}$, we can use the pullback version of the stability condition. To this effect let $\{A \to A_{f_i} | 1 \leq i \leq n\}$ be in $K(A)$ and let $g : A \to B$ be a map of crings. For each $i$ we can perform the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & A_{f_i} \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B \otimes_A A_{f_i} \end{tikzcd}$$

However, since $A_{f_i} \cong A[x]/\langle f_i x - 1\rangle$, we can use properties of the tensor to conclude that

$$B \otimes_A A_{f_i} \cong B \otimes_A A[x]/\langle f_i x - 1\rangle \cong (B \otimes_A A[x])/(B \otimes \langle f_i x - 1\rangle) \cong B[x]/\langle g(f_i)x - 1\rangle$$

This implies that the base changed set is isomorphic to

$$\{B \to B_{g(f_i)} : 1 \leq i \leq n\}$$

where $\langle g(f_1), ..., g(f_n)\rangle = g(\langle f_1, ..., f_n\rangle)B = g(A)B = B$, since we are working with unital rings. This shows that the base change of a cover is again a cover.

Finally, to show transitivity consider again a basic cover $\{A \to A_{f_i} : 1 \leq i \leq n\}$ along with for each $1 \leq i \leq n$, a basic cover $\{A_{f_i} \to (A_{f_i})_{g_{i,j}} : 1 \leq j \leq n_i\}$, where $\langle g_{i,1}, ..., g_{i,n_i}\rangle$ in $A_{f_i}$. Note that if $g_{i,j} = \frac{a_{i,j}}{f_i^k}$ for some integer $k \geq 0$ and some $a_{i,j} \in A$, then $(A_{f_i})_{g_{i,j}} \cong A_{f_i, a_{i,j}} \cong A_{f_i a_{i,j}}$ using properties of the localization using its universal property. This gives us isomorphic basic covers

$$\{A_{f_i} \to A_{f_i a_{i,j}} : 1 \leq j \leq n_i\}$$

Then the composite set is exactly the collection of localizations

$$\{A \to A_{f_i a_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq n_i\}$$

To see that this is a basic open cover, note that if $1 \leq i \leq n$, then by assumption we have $b_{i,j} \in A$ such that for a sufficiently large integer $K$,

$$f_i^K = \sum_{j=1}^{n_i} b_{i,j} a_{i,j}$$

Since we are dealing with a finite set of localizations we can choose a large enough $K$ that works for all $i$. Now since $\langle f_1, ..., f_n \rangle = A$, we also have that $\langle f_1^{K+1}, ..., f_n^{K+1} \rangle = A$. This fact follows by raising the expression of $1 \in A$ in terms of the $f_i$ to a suitably high power (e.g. $n(K+1)$). Then we have some $c_1, ..., c_n \in A$ such that

$$1 = \sum_{i=1}^{n} c_i f_i^{K+1} = \sum_{i=1}^{n} c_i f_i \sum_{j=1}^{n_i} b_{i,j} a_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} (c_i b_{i,j})(f_i a_{i,j})$$

as desired. ■

The Zariski site is our primary motivating example of a site, but as we will see soon we also have another extremely important site that in a rigourous sense gives a finer topology for studying schemes.

### 2.0.1 Sheaves

Now that we have a sensible notion of topology on a category we can start discussing sheaves. There are many equivalent formulations of sheaves, and we will provide a few to illuminate the concept.

Recall that by definition of a sieve $S$ on $C$, we have a natural inclusion $\iota_S : S \hookrightarrow y_C$ given on components by inclusion of subsets. In the setting of topological spaces we consider the unique gluing of sections defined over open coverings. If a sieve $S$ is supposed to be a covering of $C$, then for a presheaf $P$ on $\mathcal{C}$, we want to say that $P$ is a sheaf if for any assignment $f \mapsto x_f$ from $f \in S$ to $x_f \in P(\text{dom}(f))$ such that $P(g)(x_f) = x_{f \circ g}$, there exists a unique $x \in P(C)$ for which $P(f)(x) = x_f$ for $f \in S$. Categorically this can be phrased as an extension problem

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, hook, "\iota_S"'] & P \\ y_C \arrow[ur, dashed, "\exists! \widehat{\alpha}"'] & \end{tikzcd}$$

In other words we can describe sheaves as those presheaves $P$ which lift maps from covers uniquely into maps from maximal covers. As in the classical setting, this condition is equivalent to $P(C)$ being an equalizer of a diagram over the maps in a covering sieve:

$$P(C) \xrightarrow{\langle P(f) : f \in S \rangle} \prod_{f \in S} P(\text{dom} f) \underset{\langle \pi_{\text{dom} g} \mid f \circ g \in S \rangle}{\overset{\langle P(g) \circ \pi_{\text{dom} f} \mid f \circ g \in S \rangle}{\rightrightarrows}} \prod_{f \circ g \in S} P(\text{dom} g)$$

If a topology is generated by a basis $K$ we can also describe the sheaf condition in another form using this equalizer classification. This re-characterization can be seen to have a very geometric flavour when considering a category $\mathcal{C}$ with pullbacks. If $K$ is a basis on such a category, then the sheaf condition becomes the statement that for any basic cover $\{f_i : C_i \to C \mid i \in I\}$, $P(C)$ is an equalizer of the following diagram:

$$P(C) \xrightarrow{\langle P(f_i):i\in I\rangle} \prod_{i\in I} P(C_i) \underset{\langle P(\pi^2_{i,j})\circ\pi_j \mid i,j\in I\rangle}{\overset{\langle P(\pi^1_{i,j})\circ\pi_i \mid i,j\in I\rangle}{\rightrightarrows}} \prod_{i,j\in I} P(C_i \times_C C_j)$$

For any site $(\mathcal{C}, J)$, we obtain a natural subcategory $\mathsf{Sh}(\mathcal{C}, J)$ of the category of presheaves on $\mathcal{C}$, $\mathsf{Ps}(\mathcal{C})$, given by sheaves on the site. As we will see soon this category satisfies a number of important properties, including being complete and cocomplete. Note that $\mathsf{Ps}(\mathcal{C})$ is (co)complete since $\mathsf{Set}$ is, where all the (co)limits in $\mathsf{Ps}(\mathcal{C})$ can be computed term-wise in $\mathsf{Set}$. To show $\mathsf{Sh}(\mathcal{C})$ is (co)complete we will show that it is what is known as a reflective subcategory under the inclusion $\iota : \mathsf{Sh}(\mathcal{C}) \hookrightarrow \mathsf{Ps}(\mathcal{C})$. $\mathsf{Sh}(\mathcal{C})$ being a reflective subcategory means that $\iota$ has a left-adjoint, $\mathfrak{a}$, which we call the associated sheaf functor.

To construct $\mathfrak{a}$ we proceed in stages. First, a presheaf is said to be separated if it satisfies the uniqueness axiom of a sheaf, but not necessarily the gluing axiom. That is when a lift along a sieve inclusion exists, it is necessarily unique. We begin by defining a functor that produces a separated presheaf from a general presheaf and then we show that the functor applied to a separated presheaf produces a sheaf.

Define $(-)^+ : \mathsf{Ps}(\mathcal{C}) \to \mathsf{Ps}(\mathcal{C})$ by

$$P^+(C) := \lim_{\to R\in J(C)} \mathsf{Nat}(R, P)$$

where the diagram the limit is taken over has a map $\mathsf{Nat}(R, P) \to \mathsf{Nat}(S, P)$ for all $S \subseteq R$ given by pre-composition along the inclusion. This defines a presheaf since for a map $f : D \to C$ we have a natural transformation $\alpha_{f,R} : f^*R \to R$ given by sending $g \in f^*R$ to $f \circ g \in R$. Pulling back along this natural transformation gives maps of hom-sets which induce a map on the level of $P^+$ by the universal property of the colimit:

$$\begin{tikzcd} P^+(C) \arrow[r, dashed] & P^+(D) \\ \mathsf{Nat}(R, P) \arrow[u] \arrow[r, "(\alpha_{f,R})^*"'] & \mathsf{Nat}(f^*R, P) \arrow[u] \end{tikzcd}$$

Note that since we are considering set valued presheaves we can give an explicit realization of $P^+(C)$ as the set

$$P^+(C) \cong \coprod_{R\in J(C)} \mathsf{Nat}(R, P)/\sim$$

where $\alpha : R \to P$ and $\beta : T \to P$ are equivalent if there is a common refinement $S \subseteq R \cap T$ such that $\beta|_S = \alpha|_S$.

Associated to the functor $(-)^+$ we have a comparison natural transformation $\eta_P : P \to P^+$ for any presheaf $P$ where $(\eta_P)_C : P(C) \to P^+(C)$ factors through the yoneda isomorphism

$$\begin{tikzcd} P(C) \arrow[rr, "(\eta_P)_C"] \arrow[dr, "\cong"'] & & P^+(C) \\ & \mathsf{Nat}(y_C, P) \arrow[ur, hook] & \end{tikzcd}$$

where the inclusion is the natural map from the colimit.

As in [MLM92, p. 131] we now show that this construction with its associated natural transformation allows us to characterize sheaves and separated presheaves.

**Lemma 2.4** A presheaf $P$ is separated if and only if $\eta_P : P \to P^+$ is monic and is a sheaf if and only if it is an isomorphism.

*Proof.* To begin, let $P$ be a presheaf. We start with the claim on being separated, which can be proven in both directions simultaneously.

Note that as with limits, a map of presheaves is monic if and only if each component set function is monic in $\mathsf{Set}$ since $\mathsf{Set}$ is (co)complete. Then we must show that for each $C \in \mathcal{C}_0$, $(\eta_P)_C : P(C) \to P^+(C)$ is injective. Since this map can be written as yoneda followed by the inclusion $\mathsf{Nat}(y_C, P) \to P^+(C)$, this is equivalent to this later map being injective. However, this map being injective is precisely the statement that $P$ is a separated presheaf since it says that each map $y_C \to P$ can be the lift of at most one map $S \to P$ for each covering sieve $S$. Note that here we are using the classification of $P^+(C)$ in terms of its equivalence relation, which is given by equality upon restriction.

The second claim can be proven in an equally easy fashion from observing that the map $\mathsf{Nat}(y_C, P) \to P^+(C)$ being a isomorphism is exactly the lifting condition for being a sheaf. ■

This result allows us to easily show that the $(-)^+$ construction produces separated presheaves.

**Lemma 2.5** For any presheaf $P$, $P^+$ is separated.

*Proof.* To show $P^+$ is separated it is sufficient to show that $\eta_{P^+} : P^+ \to (P^+)^+$ is monic, or equivalently that for every $C \in \mathcal{C}_0$, $(\eta_{P^+})_C : P^+(C) \to P^{++}(C)$ is injective. Recall from the definition of $(-)^+$ that

$$P^{++}(C) = \lim_{\to R \in J(C)} \mathsf{Nat}(R, P^+)$$

and $(\eta_{P^+})_C$ is determined by the inclusion of $\mathsf{Nat}(y_C, P^+)$ in $P^{++}(C)$.

Let $S, R \in J(C)$ such that $\alpha : S \to P$ and $\beta : R \to P$ are maps of presheaves that get sent to the same equivalence class in $P^{++}(C)$. Let $\widehat{\alpha} : y_C \to P^+(C)$ and $\widehat{\beta} : y_C \to P^+(C)$ be the representing maps under yoneda. Then by definition there exists a sieve $T \in J(C)$ for which $\widehat{\alpha}|_T : T \to P^+(C)$ and $\widehat{\beta}|_T : T \to P^+(C)$ are equal. Note that this says that for any $f : D \to C$ in the covering sieve $T$, $P^+(f)(\alpha) = \widehat{\alpha}_D(f) = \widehat{\beta}_D(f) = P^+(f)(\beta)$. This says exactly that for all $f : D \to C$ in $T$, there exists a common refinement $T_f$ of $f^*(S)$ and $f^*(R)$ such that the pullbacks of $\alpha$ and $\beta$ agree on $T_f$. From the transitivity axiom these $T_f$ glue together to give a covering sieve $U$ of $C$ which refines both $S$ and $R$, and by construction of $U$, $\alpha$ and $\beta$ agree on $U$ so by definition of the equivalence relation defining $P^+(C)$, are equal in $P^+(C)$. ■

Next we can use our construction of $P^+$ to show that it is minimal with respect to certain maps of presheaves [MLM92, p. 131].

**Lemma 2.6** If $F$ is a sheaf and $P$ is a presheaf, then any map $\phi : P \to F$ of presheaves factors uniquely through $\eta$.

*Proof.* Suppose $\phi : P \to F$ is a map of presheaves with $F$ a sheaf. Then for any covering sieve $R \in J(C)$ we have a map $\phi_* : \mathsf{Nat}(R, P) \to \mathsf{Nat}(R, F)$ given by post-composition by $\phi_*$. Since $F$ is a sheaf we have a natural isomorphism $\mathsf{Nat}(R, F) \cong \mathsf{Nat}(y_C, F)$ given by the lifting criterion. Composing with the yoneda isomorphism then gives a map $\phi'_R : \mathsf{Nat}(R, P) \to F(C)$.

This map is evidently natural with respect to sieve inclusions, $S \subseteq R$, and so induces a unique map $\Phi_C : P^+(C) \to F(C)$ given by the colimit universal property. The naturality of $\Phi$ follows from the naturality of $\phi$ and the uniqueness of the $\Phi_C$ with respect to the colimit universal property. Further, $\phi$ factors through $\Phi$ since in the commutative diagram

$$\begin{tikzcd}
P(C) \arrow[r, dashed] \arrow[dr, "\cong"] & P^+(C) \arrow[r, dashed] & F(C) \\
& \mathsf{Nat}(y_C, P) \arrow[u] \arrow[r, "\phi_*"'] & \mathsf{Nat}(y_C, F) \arrow[u, dashed, "\cong"']
\end{tikzcd}$$

the bottom yoneda isomorphisms and pushforward by $\phi$ compose to give $\phi$ by naturality.

Uniqueness of this factorization follows from the uniqueness of the map out of the colimit which is determined by the $\phi_*$ maps. ■

Finally, we show that $(-)^+$ for separated presheaves produces full sheaves.

**Lemma 2.7** If $P$ is a separated presheaf, then $P^+$ is a sheaf.

*Proof.* From Lemma 2.5 we already have that $P^+$ is separated. Using Lemma 2.4 it remains only to show that for each $C \in \mathcal{C}_0$, the map $\mathsf{Nat}(y_C, P^+)$ into $P^{++}(C)$ is surjective.

To this effect we can consider a covering sieve $S \in J(C)$ along with a natural transformation $\alpha : S \to P^+$. We need only show that there exists a refinement of $S$ for which the restriction of $\alpha$ to the refinement is in our image. Note that for each $D \in \mathcal{C}_0$ and each $f : D \to C$ in $S$, $\alpha_D(f) \in P^+(D)$ is an equivalence class of natural transformations. For each $f$ in $S$ where $\mathrm{dom}(f) = D$ choose a representative from $\alpha_D(f)$, $\widetilde{\alpha}_f : S_f \to P$. Note that by definition of the equivalence relation we have for each $f : D \to C$ in $S$ and each $g : E \to D$, a refinement $S_{f,g}$ of $g^*(S_f)$ and $S_{f\circ g}$ on which the representatives agree.

By the transitivity part of the Grothendieck topology we have a covering sieve $T = \{f \circ h : f \in S, h \in S_f\}$. We then define $\widetilde{\alpha} : T \to P$ by $\widetilde{\alpha}_E(f \circ h) := \widetilde{\alpha}_{f,E}(h)$. To see this is well-defined suppose $f \circ h = k \circ g$ for $f, k \in S$ and $h \in S_f, g \in S_k$. Consider the to maps $y_E \to P$ represented by $\widetilde{\alpha}_E(f \circ h)$ and $\widetilde{\alpha}_E(k \circ g)$, respectively. Then let $S_{(f,h),(k,g)}$ be a common refinement of $S_{f,h}$ and $S_{k,g}$. It follows that $\widetilde{\alpha}_f$ and $\widetilde{\alpha}_k$ are equivalent when restricted to this cover, so in particular both of our maps $y_E \to P$ are lifts of these equivalent restrictions, so as $P$ is separated they must be the same. In other words, $\widetilde{\alpha}_E$ is well-defined, and since it is defined in terms of natural transformations it too is natural.

Finally, we consider the map $y_C \to P^+$ represented by $\widetilde{\alpha}$. For each $f : D \to C$ in $S$ we have that $\alpha_D(f)$ is represented by $\widetilde{\alpha}_f : S_f \to P$, while $P^+(f)(\widetilde{\alpha}) = \widetilde{\alpha} \circ \alpha_{f,T} : f^*T \to P$. Now, by construction $S_f$ is a refinement of $f^*T$, and on $S_f$, $\widetilde{\alpha}_f$ and $\widetilde{\alpha} \circ \alpha_{f,T}$ agree by construction. It follows that these objects are equal in $P^+$ since they agree upon refinement of the cover, and so $\widetilde{\alpha}$ is a lift of $\alpha$. ■

This collection of lemmas shows that not only do we have a functor $\mathfrak{a} := ((-)^+)^+ : \mathsf{Ps}(\mathcal{C}) \to \mathsf{Sh}(\mathcal{C})$, $a$ is left adjoint to $\iota$ using the universal property characterization of adjunctions. Indeed, applying Lemma 2.6 twice gives for any presheaf map $\alpha : P \to F$ into a sheaf $F$, a unique factorization

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[drr, "\alpha"'] & P^+ \arrow[r, "\eta_{P^+}"] \arrow[dr, dashed, "!"] & P^{++} = \mathfrak{a}(P) \arrow[d, dashed, "!"] \\ & & F \end{tikzcd}$$

It follows that $\mathsf{Sh}(\mathcal{C})$ is co-complete since $\mathsf{Ps}(\mathcal{C})$ is co-complete. Indeed, $\mathfrak{a}$ being a left-adjoint implies that it preserves colimits [Mac71, p. 118]. On the other hand, since $\iota$ is a full and faithful inclusion of categories, the co-unit of the adjunction, $\epsilon : \mathfrak{a} \circ \iota \to 1_{\mathsf{Sh}(\mathcal{C})}$, is an isomorphism. This implies that a colimit of a diagram can be obtained by including into presheaves, computing the colimit, and then applying the associated sheaf functor.

Additionally, this shows that $\mathsf{Sh}(\mathcal{C})$ is a reflective subcategory of $\mathsf{Ps}(\mathcal{C})$. Although the proof is beyond the scope of these notes, it is an important fact that $\mathsf{Sh}(\mathcal{C})$ being a reflective subcategory of $\mathsf{Ps}(\mathcal{C})$ implies that it is equivalent to the category of Eilenberg-Moore algebras $\mathsf{Ps}(\mathcal{C})^T$ for the monad $T = \iota \circ a$ on $\mathsf{Ps}(\mathcal{C})$. This algebra category is complete since the functor $\mathsf{Ps}(\mathcal{C})^T \to \mathsf{Ps}(\mathcal{C})$ reflects limits, and these limits are computed as in $\mathsf{Ps}(\mathcal{C})$ [Mac71]. This shows that $\mathsf{Sh}(\mathcal{C})$ is a complete category.

We also have an adjunction for the global sections functor $\Gamma : \mathsf{Ps}(\mathcal{C}) \to \mathsf{Set}$, just as in the case of sheaves on topological spaces, given by the constant presheaf $\Delta : \mathsf{Set} \to \mathsf{Ps}(\mathcal{C})$. In particular, $\Gamma$ is right-adjoint to $\Delta$. It follows that $\Gamma \circ \iota$ is right-adjoint to $\mathfrak{a} \circ \Delta$ by composition of adjunctions [Mac71]. The functor $\mathfrak{a} \circ \Delta$ is often called the constant sheaf functor.

The sheaf category on a site is also what is known as a cartesian closed category. Explicitly this means that the product functor has a left adjoint, often called an exponential. Explicitly, a sheaf category inherits the exponential structure from presheaves. In particular, if $P \in \mathsf{Ps}(\mathcal{C})$ and $F \in \mathsf{Sh}(\mathcal{C})$, then we have a sheaf $F^P$ given by

$$F^P(C) := \mathsf{Nat}(y_C \times P, F)$$

This fact will be important for showing the existence of certain adjunctions in the section to follow.

Finally, an important property related to the theory of enriched categories is that the yoneda embedding for presheaves can also be extended to sheaves using the associated sheaf functor by considering

$$a \circ y : \mathcal{C} \to \mathsf{Sh}(\mathcal{C}, J)$$

This functor is isomorphic to the yoneda embedding if the topology $J$ is sub-canonical (i.e. all representable presheaves are sheaves). This extension of yoneda satisfies a number of analogous properties to yoneda due to the fact that $\mathfrak{a}$ is a left adjoint to the inclusion of sheaves into presheaves [MLM92, p. 139].

# 3 Grothendieck Topoi and Geometric Morphisms

Among others, the properties demonstrated for the category of sheaves over a site in the previous section illustrate that such a category is an object known as a topos. In particular, a category equivalent to a category of sheaves over a site is referred to as a **Grothendieck topos**. For the sake of space we will not delve into the generalizations of our current work to the setting of arbitrary topoi, but we refer the interested reader to sources such as [MLM92, Joh02, Joh14, Jac99].

An important step in understanding Grothendieck topoi comes from understanding what maps are appropriate for defining a category of Grothendieck topoi. In other words, what maps would preserve the structure of a Grothendieck topoi? To motivate our definition we can go back to where our story began, which was with sheaves on topological spaces. In this setting a map of topological spaces, $f : X \to Y$, induces an adjunction between sheaf categories

$$\begin{tikzcd} \mathsf{Sh}(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Sh}(Y) \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

where $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ is the familiar functor from lecture which takes a sheaf $\mathcal{F}$ on $X$ to the sheaf $f_*\mathcal{F}$ on $Y$ given at an open $U \subseteq Y$ by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)$. On the other hand, the functor $f^*$ is given by taking the associated sheaf functor of the following definition for a sheaf $\mathcal{G}$ on $Y$:

$$f^*_{pre}\mathcal{G}(V) := \lim_{\to, f(V)\subseteq U} \mathcal{G}(U), \ \forall V \in \mathsf{Top}(X)$$

In general this adjoint pair between sheaf categories preserves the desired data and so is appropriate for use as a map of sheaf categories. We generalize this notion to the case of sites to define what we call **geometric morphisms** [MLM92].

**Definition 3.1** A **geometric morphism** $f : \mathcal{B} \to \mathcal{E}$ between Grothendieck topoi is an adjoint pair $f^* : \mathcal{E} \to \mathcal{B} : f_*$ such that $f^*$ is left exact (i.e. $f^*$ preserves finite limits).

With geometric morphisms and composition of adjoint pairs, Grothendieck topoi form a category, $\mathsf{GrTopoi}$. In fact, $\mathsf{GrTopoi}$ has a natural 2-categorical structure given by taking a 2-cell between geometric morphisms $f, g : \mathcal{B} \to \mathcal{E}$ to be a natural transformation $\alpha^* : f^* \Rightarrow g^*$ (which is equivalent to a natural transformation $\alpha_* : g_* \Rightarrow f_*$ using the adjunction units and co-units).

Grothendieck topoi and geometric morphisms provide an excellent framework for studying base-change. In particular, although we will not prove it here as it goes beyond the scope of these notes, slice categories for Grothendieck topoi are themselves Grothendieck topoi [MLM92], and the associated base-change functors described in Section A.3 assemble into geometric morphisms between these Grothendieck topoi. Further, the slice categories for a Grothendieck topoi can be described in a particular elegant form using what is known as the Grothendieck construction.

Suppose $(\mathcal{C}, J)$ is a site, and $\mathcal{F} \in \mathsf{Sh}(\mathcal{C}, J)$ is a sheaf on the site. Then the slice category is equivalent to

$$\mathsf{Sh}(\mathcal{C}, J)/\mathcal{F} \simeq \mathsf{Sh}\left(\int_{\mathcal{C}} \mathcal{F}, J_{\mathcal{F}}\right)$$

where $\int_{\mathcal{C}} \mathcal{F}$ is the Grothendieck construction for $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Set}$, which is defined by the following data:

(Objs) Pairs $(C, X)$ for $C \in \mathcal{C}_0$ and $X \in \mathcal{F}(C)$

(Maps) A map $(C, X) \to (D, Y)$ is a map $f : C \to D$ such that the triangle

$$\begin{tikzcd} & * \arrow[dl, "X"'] \arrow[dr, "Y"] & \\ \mathcal{F}(C) & & \mathcal{F}(D) \arrow[ll, "\mathcal{F}(f)"] \end{tikzcd}$$

commutes.

There is a natural projection functor $\pi_0 : \int_{\mathcal{C}} \mathcal{F} \to \mathcal{C}$ given by forgetting the element in the pair. The Grothendieck topology $J_{\mathcal{F}}$ is then induced by this projection functor.

# 4 The Étale Site

We now shift gears from a discussion of sheaves on a general site to a particularly important site in algebraic geometry. We have previously seen that we can realize the Zariski site on the category $\mathsf{CRing}^{op}$ which realizes the classical construction of affine schemes from which schemes are glued. However, the Zariski topology that comes from the Zariski site is very coarse, with most resulting spaces having poor separation properties.

An incredibly important alternative perspective in modern algebraic geometry comes from the study of the Étale site. In this section we will provide a brief introduction to the Étale site following along with the notes of Milne [Mil13] and Vooys [Voo23]. We will only state and sketch certain results which illuminate the motivation and idea of the Étale site as a detailed treatment of this concept would be far beyond the scope of the current paper. As discussed in Vooys [Voo23], although the Zariski topology is too coarse for our purposes, we do not want to change the topology on the spaces we are considering as it may cause our equivalence of categories between affine schemes and commutative rings to be lost. Instead, we consider a new site on the category of schemes which allows us to refine our notion of coverings without altering the existing Zariski topologies on affine schemes.

As the name suggests, the Étale site involves replacing open covers by Étale morphisms of schemes. Recall that a morphism of schemes is Étale if and only if it is both flat and unramified. As shown in [Voo23], Étale morphisms satisfy a number of desirable properties including being closed under composition, closed under base change, and closed under left composition by unramified maps. Additionally, maps that would normally be coverings in the sense of the Zariski topology, such as isomorphisms of schemes and open immersions, are also Étale [Voo23]. This implies that looking at covers by Étale morphisms refines our covering by open sets when dealing with the topologies on schemes.

Now, given a base scheme $X$ we are interested in the slice category $\acute{\mathsf{E}}\mathsf{t}/X$ with objects Étale morphisms $f : Y \to X$, and morphisms commutative triangles

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

The above properties for Étale maps imply that the maps in this category are necessarily Étale. Additionally, since Étale maps are stable under pullback and isomorphisms, and consequently identities, are Étale, the category $\acute{\mathsf{E}}\mathsf{t}/X$ has a terminal object and pullbacks, and so is finitely complete by Proposition A.3. This implies we can define a Grothendieck topology on $\acute{\mathsf{E}}\mathsf{t}/X$ via a Grothendieck pre-topology using pullbacks.

**Definition 4.1** [Mil13, p. 39] The set function $K$ defined on $\acute{\mathsf{E}}\mathsf{t}/X$ by setting $K(Y \xrightarrow{p} X)$ to consist of families $\{\varphi_i : U_i \to Y \in \acute{\mathsf{E}}\mathsf{t}/X(U_i, Y) : i \in I\}$ such that $\bigcup_{i \in I} \varphi_i(U_i) = Y$ as topological spaces is a Grothendieck pre-topology on $\acute{\mathsf{E}}\mathsf{t}/X$.

*Proof.* We need to prove the three properties of a pre-topology for $K$. The first property that all isomorphisms form singleton covers follows from the previous observation that isomorphisms are Étale. Similarly, condition (iii') for a pre-topology follows from the observation that the composition of Étale maps is Étale.

Finally, since Étale maps are stable under base-change (i.e. pullback), $K$ satisfies the stability axiom for a pre-topology. ■

Now that we have a pre-topology (i.e. a basis) on the category $\acute{\mathsf{E}}\mathsf{t}/X$ of Étale morphisms over $X$, we can consider sheaves on this category. Explicitly, from our previous work, a sheaf is a contravariant functor $\mathcal{F} : \acute{\mathsf{E}}\mathsf{t}/X^{op} \to \mathsf{Set}$ such that for any Étale map $f : U \to X$ and for any basic Étale cover $\{g_i : U_i \to U \,|\, i \in I\}$ of $U$ in $\acute{\mathsf{E}}\mathsf{t}/X$, $\mathcal{F}(U)$ is an equalizer for the diagram

$$\mathcal{F}(U) \xrightarrow{\langle \mathcal{F}(f_i): i \in I\rangle} \prod_{i\in I} \mathcal{F}(U_i) \underset{\langle \mathcal{F}(\pi^2_{i,j})\circ \pi_{U_j} : i,j\in I\rangle}{\overset{\langle \mathcal{F}(\pi^1_{i,j})\circ \pi_{U_i} : i,j\in I\rangle}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

From this definition and setup we could go on a long treatment of the theory of sheaves on the Étale site, their cohomology theory, and a number of their properties. However, instead for the sake of space we will spend the remainder of the section motivating why the Étale site is beneficial, referring the reader interested in the above details to [Mil13].

One of the most glaring problems with the Zariski topology is when looking at non-singular varieties over the complex numbers, their cohomology groups and other invariants in the context of schemes need not agree with the corresponding invariants when the variety is given its natural complex topology [Voo23]. On the other hand, the Étale site perspective fills many of these gaps.

To illustrate this point, lets consider the classical topological invariant on real and complex spaces given by the first fundamental group. In its original definition in terms of paths, it is difficult to generalize the fundamental group to the setting of schemes. However, a central result in algebraic topology says that when dealing with spaces with suitable connectedness assumptions, the fundamental group can instead be described entirely in terms of covering spaces. This covering space perspective cleanly generalizes to the setting of schemes and Étale coverings, allowing for the construction of the invariant known as the Étale fundamental group. Unlike in the Zariski case, when considering the Étale fundamental group on non-singular varieties over $\mathbb{C}$ we recover the analytic fundamental group [Mil13]. Explicitly, the Riemann Existence Theorem implies that the Étale fundamental group is equivalent to a topological completion of the analytic fundamental group.

# Appendices

## A Basic Category Theory

In this section we cover some basic category theory necessary to talk about sheaves on a site and topoi more generally. In the first section we will introduce the Yoneda lemma and a number of its consequences. Following this section we introduce general limits and colimits, including the special case of equalizers, products, and pullbacks.. Finally, we provide a section on internal objects in a category which allow us to define sheaves of groups, rings, algebras, etc.

To avoid size issues, throughout we assume that all categories are locally small in the sense that the maps between two objects form a set.

## A.1 The Yoneda Embedding

A central philosophy of category theory is that the content of objects is contained not in their internal structure but rather the maps into and out of them. This philosophy can be formally realized by the Yoneda Embedding. The Yoneda Embedding is a special case of a result known as the Yoneda Lemma [Mac71] which describes elements of a pre-sheaf in terms of certain natural transformations into it. In order to state this result we first must introduce a special class of pre-sheaves.

**Definition A.1** For a category $\mathcal{C}$ and an object $C \in \mathcal{C}_0$, the pre-sheaf represented by $C$ is the functor $y_C : \mathcal{C}^{op} \to \mathsf{Set}$ given on an object $D \in \mathcal{C}_0$ by

$$y_C(D) := \mathcal{C}(D, C)$$

and on an arrow $f : E \to D$ in $\mathcal{C}$ by

$$y_C(f) : \mathcal{C}(D, C) \to \mathcal{C}(E, C),\ (g : D \to C) \mapsto (g \circ f : E \to C)$$

The Yoneda lemma then states that for a pre-sheaf $F : \mathcal{C}^{op} \to \mathsf{Set}$, we have a natural bijection [Mac71]

$$[\mathcal{C}^{op}, \mathsf{Set}](y_C, F) \cong F(C)$$

for any $C \in \mathcal{C}_0$, which is given by sending $\alpha : y_C \Rightarrow F$ to $\alpha_C(1_C)$. The inverse to this map is given by sending $x \in F(C)$ to the natural transformation $\widetilde{x} : y_C \to F$ which it represents. At an object $D \in \mathcal{C}_0$ the component map $\widetilde{x}_D : y_C(D) \to F(D)$ is given by evaluation in the sense that a map $f : D \to C$ is sent to $F(f)(x)$.

A key aspect of the definition of representable functors given above is that it is functorial in $C \in \mathcal{C}_0$. In other words, $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathsf{Set}]$ given by $C \mapsto y_C$ is a functor, where $f : C \to D$ is sent to the natural transformation $y_f : y_C \to y_D$ given by post-composition. That is for $B \in \mathcal{C}_0$,

$$(y_f)_B : y_C(B) \to y_D(B), (g : B \to C) \mapsto (f \circ g : B \to D)$$

The Yoneda lemma tells us that this functor is an embedding in the sense that for any $C, D \in \mathcal{C}_0$, the map from $\mathcal{C}(C, D)$ to $[\mathcal{C}^{op}, \mathsf{Set}](y_C, y_D)$ is a bijection. Indeed, by the Yoneda lemma

$$[\mathcal{C}^{op}, \mathsf{Set}](y_C, y_D) \cong y_D(C) = \mathcal{C}(C, D)$$

These results provide an incredibly powerful tool for arguing about objects in a category by first embedding them in a category of pre-sheaves. Additionally, as we work through in Section 2, representable functors provide a natural way of encoding the sheaf condition on a general site.

## A.2 (co)Limits

One of the most important and prolific concepts in category theory is that of universal properties and their realizations in the form of limits and colimits in a category. Informally, limits (respectively colimits) provide a universal object over (respectively under) a diagram which represents that diagram. In lecture we were introduced to a number of examples for the categories of sheaves on a topological space and the category of schemes, including pullbacks, pushouts, products, coproducts, and filtered (co)limits.

In general a diagram in a category $\mathcal{C}$ of shape $I$, where $I$ is a small category, is simply a functor $D : I \to \mathcal{C}$. Given a diagram $D : I \to \mathcal{C}$, a cone over the diagram $D$ with vertex $A \in \mathcal{C}_0$ is a natural transformation $\alpha : \Delta_A \to D$. Here $\Delta_A : I \to \mathcal{C}$ is the constant, or diagonal, functor at $A$, so $\Delta_A(i) = A$ for all $i \in I_0$, and $\Delta_A(i \xrightarrow{f} j) = 1_A$ for any $f : i \to j$ in $I$. This data naturally assembles into a functor $\Delta_- : \mathcal{C} \to [I, \mathcal{C}]$, where for a map $f : A \to B$ in $\mathcal{C}$, $\Delta_f : \Delta_A \to \Delta_B$ is given by $f$ at all components. Since all maps are identities, naturality of $\Delta_f$ is trivial, while functoriality of $\Delta_-$ follows since on maps it is defined by just giving the map back on each component.

Pictorially the cone $\alpha$ can be considered as a collection of commuting diagrams of the following form:

$$\begin{tikzcd} & A \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ D(i) \arrow[rr, "D(f)"'] & & D(j) \end{tikzcd}$$

Cocones are completely dual, and so we will restrict to considering cones for the sake of space.

Given a fixed diagram $D : I \to \mathcal{C}$ we obtain a contravariant functor $\mathrm{Cone}_{/D} : \mathcal{C}^{op} \to \mathsf{Set}$. $\mathrm{Cone}_{/D}$ is defined by sending an object $A \in \mathcal{C}_0$ to the set of cones over $D$ with vertex $A$, $\mathrm{Cone}_{/D}(A) = [I, \mathcal{C}](\Delta_A, D)$, and sending a map $f : A \to B$ in $\mathcal{C}$ to the map on cones given by pre-composition:

$$\Delta_f^* : [I, \mathcal{C}](\Delta_B, D) \to [I, \mathcal{C}](\Delta_A, D), (\alpha : \Delta_B \to D) \mapsto (\alpha \circ \Delta_f : \Delta_A \to D)$$

Then to say that the diagram $D$ has a limit is to say that the functor $\mathrm{Cone}_{/D}$ is representable. A limit for the diagram is then a representing object which consists of a pair of an object $A \in \mathcal{C}_0$ together with a natural transformation $\alpha : \Delta_A \to D$ such that

$$\mathrm{Cone}_{/D} \cong \mathcal{C}(-, A)$$

where $1_A : A \to A$ is sent to $\alpha$ under the natural isomorphism.

We end the appendix by giving sufficient conditions for a category to be (finitely) complete in terms of well-known limits.

**Proposition A.2** If $\mathcal{C}$ has (finite) products and equalizers, then $\mathcal{C}$ has (finite) limits.

*Proof.* Let $J : \mathcal{D} \to \mathcal{C}$ be a (finite) diagram in $\mathcal{C}$. If $\mathcal{C}$ has (finite) products and equalizers, then we can form the equalizer diagram

$$\mathfrak{D} \xrightarrow{e} \prod_{d \in \mathcal{D}_0} J(d) \underset{\langle \pi_{\mathrm{cod}(f)} : f \in \mathcal{D}_1 \rangle}{\overset{\langle J(f) \circ \pi_{\mathrm{dom}(f)} : f \in \mathcal{D}_1 \rangle}{\rightrightarrows}} \prod_{f \in \mathcal{D}_1} J(\mathrm{cod}(f))$$

Using the projections from the product we have that $\mathfrak{D}$ is a cone over $J$ since equalizing the above diagram is equivalent to requiring that all triangles in the cone commute. This observation also implies that the universal property for the equalizer $\mathfrak{D}$ is exactly that of a limit of $J$, and so $\mathfrak{D} \cong \varprojlim J$. ■

For just finite completeness, we have an even simpler sufficient condition in terms of pullbacks.

**Proposition A.3** If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ is finitely complete.

*Proof.* By Proposition A.2 it is sufficient to show that if $\mathcal{C}$ has pullbacks and a terminal object it has binary products and equalizers. First, for objects $A, B \in \mathcal{C}_0$, the pullback along the terminal object is exactly the product of $A$ and $B$

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ A \arrow[r] & * \end{tikzcd}$$

Indeed, any maps to $A$ and $B$ will make the square commute since maps into the terminal object are unique, so the universal property of the pullback is the same as that of the product.

Next, consider a pair of maps $f, g : A \to B$ in $\mathcal{C}$. We can form the product $B \times B$ now, and take the pullback of $\langle f, g \rangle : A \to B \times B$ along the diagonal of $B$.

$$\begin{tikzcd} \langle f, g \rangle^{-1}(\Delta_B) \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\langle f, g \rangle"] \\ B \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$

Observe then that

$$\langle f, g \rangle^{-1}(\Delta_B) \xrightarrow{p_2} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is an equalizer, and being a cone for the pullback is equivalent to being a cone for the pair $f, g : A \to B$, so in fact our pullback gives the desired equalizer. ■

## A.3 Base-change and fibrations

As evidenced by the prevalence of relative schemes and integral models in algebraic and arithmetic geometry, a tool of central importance is that of base-change. However, base-change is an operation that exists on slice categories in a far greater level of generality than geometric categories, like the category of relative schemes over some base.

In this section we will prove that if a category has pullbacks, then it has a notion of base-change between slice categories. Additionally, we will prove that base change always has a left-adjoint, and that in the case of categories such as Grothendieck topoi which have exponential objects, the base change also has a right-adjoint. This fact, together with remarks in Section 3 that slice categories of Grothendieck topoi are again Grothendieck topoi implies that base-change induces geometric morphisms between relative Grothendieck topoi.

We will begin by constructing base-change functors out of the theory of fibered categories [Jac99]. To begin we must define the notion of a cartesian lift.

**Definition A.4** Let $p : \mathcal{E} \to \mathcal{B}$ be a functor. A **(strong) cartesian lift** of an arrow $f : B \to B'$ in $\mathcal{B}$ is an arrow $f^*_X : f^*X \to X$ in $\mathcal{E}$ above $f$ such that for any $g : Y \to X$ in $\mathcal{E}$ and any filling $h : p(Y) \to B$ of the triangle in $\mathcal{B}$, we have a unique filling of the triangle in $\mathcal{E}$:

$$\begin{tikzcd}
Y \arrow[d, dashed, "\exists! \widetilde{h}"'] \arrow[dr, "g"] & \\
f^*X \arrow[r, "f^*_X"'] \arrow[d, dotted, mapsto] & X \\
p(Y) \arrow[d, "h"'] \arrow[dr, "p(g)"] & \\
B \arrow[r, "f"'] & B'
\end{tikzcd}$$

Cartesian lifts now let us characterize fibrations of categories.

**Definition A.5** A **fibration** of categories is a functor $p : \mathcal{E} \to \mathcal{B}$ such that for every $f : B \to B'$ in $\mathcal{B}$, and every $X \in p^{-1}(B')$, there exists a cartesian lift of $f$ with codomain $X$, $f^*_X : f^*X \to X$.

One of the many reasons for studying fibrations, also known as fibered categories, is that the provide a sensible way of producing functors between fibers above objects in the base category $\mathcal{B}$. Here, the fiber $p^{-1}(B)$ above an object $B \in \mathcal{B}_0$ is the subcategory of $\mathcal{E}$ consisting of objects $X \in \mathcal{E}_0$ such that $p(X) = B$ and arrows $f : X \to Y$ between such objects such that $p(f) = 1_B$.

Provided we have choice of cartesian lift above each arrow $f : B \to B'$ in $\mathcal{B}$ and each $X \in p^{-1}(B')_0$, these cartesian lifts provide base-change functors between the fibers. Classically a choice of cartesian lifts for a fibration is called a **cleavage**.

**Proposition A.6** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a cleavage. Then each $f : A \to B$ in $\mathcal{B}$ induces a functor $f^* : p^{-1}(B) \to p^{-1}(A)$.

*Proof.* Fix $f : A \to B$ in $\mathcal{B}$. For each $X \in p^{-1}(B)_0$ we have a specified cartesian lift $f^*_X : f^*X \to X$, where $f^*X$ lies above $A$. We define $f^* : p^{-1}(B) \to p^{-1}(A)$ on objects by $f^*(X) := \mathrm{dom}(f^*_X)$. Next, let $g : Y \to X$ be a map in $p^{-1}(B)$. By the defining property of a cartesian lift we obtain a unique filling above the identity

$$
\begin{tikzcd}
f^*Y \arrow[r, "f^*_Y"] \arrow[dr, dashed, "\exists! f^*g"'] & Y \arrow[dr, "g"] & {} \arrow[r, <->, dotted] & {} & A \arrow[r, "f"] \arrow[dr, equal] & B \arrow[dr, equal] & \\
& f^*X \arrow[r, "f^*_X"'] & X & & & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

We define $f^*(g)$ to be this unique filling. By uniqueness, $f^*1_X = 1_{f^*X}$ since it fills the diagram

$$
\begin{tikzcd}
f^*X \arrow[r, "f^*_X"] \arrow[dr, equal] & X \arrow[dr, equal] & {} \arrow[r, <->, dotted] & {} & A \arrow[r, "f"] \arrow[dr, equal] & B \arrow[dr, equal] & \\
& f^*X \arrow[r, "f^*_X"'] & X & & & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Finally, if $h : Z \to Y$ is another map in the fiber $p^{-1}(B)$, then we have the composite diagram

$$
\begin{tikzcd}
f^*Z \arrow[r, "f^*_Z"] \arrow[dr, dashed, "\exists! f^*h"'] \arrow[ddrr, dashed, bend right, "\exists! f^*(g\circ h)"'] & Z \arrow[dr, "h"] & & & A \arrow[r, "f"] \arrow[dr, equal] & B \arrow[dr, equal] & & \\
& f^*Y \arrow[r, "f^*_Y"] \arrow[dr, dashed, "\exists! f^*g"'] & Y \arrow[dr, "g"] & {} \arrow[r, <->, dotted] & {} & A \arrow[r, "f"] \arrow[dr, equal] & B \arrow[dr, equal] & \\
& & f^*X \arrow[r, "f^*_X"'] & X & & & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

where by uniqueness of the filling $f^*(g \circ h) = f^*g \circ f^*h$. Thus $f^* : p^{-1}(B) \to p^{-1}(A)$ is a functor. ∎

The primary example of interest for us is called the standard fibration. For a category $\mathcal{C}$, we always have a category $\mathcal{C}^2$ which has arrows, $f : A \to B$, as objects, and commutative squares between arrows

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

as maps. Additionally, we have a natural projection functor $\pi_1 : \mathcal{C}^2 \to \mathcal{C}$ given by sending arrows to their codomains and squares to the maps between codomains on the right edge of the square.

When $\mathcal{C}$ has pullbacks the functor $\pi_1$ is a fibration of categories, where the cartesian lifts are pullback squares. Indeed, going through our definition of a cartesian arrow we obtain that a filling corresponds to the closing of a wedge, where the closing map is the unique one induced by the pullback

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, "h"] & & & M \arrow[dd, "t"'] \arrow[ddr, "\ell"] & \\
Y \arrow[d, "k"'] & A \arrow[dl, dashed, "g"] & N \arrow[l, "r"'] \arrow[ul, dashed, "\exists ! \widetilde{t}"'] \arrow[dl, "s"] \arrow[r, dotted, leftrightarrow] & {} & & \\
B & M \arrow[l, "\ell"] \arrow[ul, "t"] & & & Y \arrow[r, "k"'] & B
\end{tikzcd}$$

A cleavage is simply a choice of pullback for each diagram. Then for $C \in \mathcal{C}_0$, the fiber category $\pi_1^{-1}(C)$ is isomorphic to the slice category $\mathcal{C}/C$. It follows that given a cleavage for $\pi_1$, Proposition A.3 implies that we have well-defined base-change functors $f^* : \mathcal{C}/B \to \mathcal{C}/A$ for $f : A \to B$ in $\mathcal{C}$ induced by pullback.

Dually to all of the definitions and results so far we have a notion of opfibrations of categories where we consider opcartesian lifts which have the dual property. Given an opfibration $p : \mathcal{E} \to \mathcal{B}$ with opcleavage we denote the induced functor on fibers for a map $f : A \to B$ by $f_! : p^{-1}(A) \to p^{-1}(B)$. We will now show that a fibration is also an opfibration, and vice-versa, if and only if its pullback functors have left-adjoints [Shu07].

**Proposition A.7** A fibration $p : \mathcal{E} \to \mathcal{B}$ is an opfibration if and only if all pullback functors $f^* : p^{-1}(B) \to p^{-1}(A)$, for $f : A \to B$, have left adjoints.

*Proof.* Let $f : A \to B$ be a map in $\mathcal{B}$. Note that by the classification of cartesian lifts applied to identity fillings, we have a natural bijection between maps $Y \to X$ in $\mathcal{E}$ lying over $f$ and maps $Y \to f^*X$ in $p^{-1}(A)$. On the other hand, the dual property for opcartesian lifts gives a bijection between maps $Y \to X$ over $f$ and maps $f_!Y \to X$ in $p^{-1}(B)$. In other words, we have the natural bijection

$$p^{-1}(B)(f_!Y, X) \cong p^{-1}(B)(Y, f^*X)$$

so the existence of opcartesian lifts is equivalent to the existence of right adjoints, and vice-versa if we start with an opfibration. ■

The standard fibration $\pi_1 : \mathcal{C}^2 \to \mathcal{C}$ has left-adjoints for all of its base-change functors, and so itself is also an opfibration. Indeed, given $f : A \to B$, the left adjoint to $f^*$, $f_! : \mathcal{C}/A \to \mathcal{C}/B$ is given by post-composition. Indeed, the universal property of the pullback tells us that the two-dashed maps in the diagram

$$\begin{tikzcd}
A \times_B X \arrow[d] \arrow[dr] & & Y \arrow[ll, dashed] \arrow[dll, "h"'] \arrow[dl, dashed] \\
A \arrow[dr, "f"'] & X \arrow[d, "g"] & \\
& B &
\end{tikzcd}$$

are equivalent, so having one immediately implies having the other.

Thus, $f_! \dashv f^*$, so in this context $f^*$ always preserves limits since it has a left adjoint. In order to obtain a geometric morphism, however, we still need a right adjoint for $f^*$. This doesn't exist in general, but as we will now show, it will for Grothendieck topoi, since the standard fibration has right adjoints for its base-change functors if the base category is locally cartesian closed.

**Definition A.8** A category $\mathcal{C}$ with finite products is said to be **cartesian closed** if for every object $C \in \mathcal{C}$, the product functor $- \times C : \mathcal{C} \to \mathcal{C}$ has a right adjoint, $(-)^C$. $\mathcal{C}$ is said to be **locally cartesian closed** if for every $C \in \mathcal{C}_0$, $\mathcal{C}/C$ is cartesian closed.

With locally cartesian closedness we can now construct right adjoints for base-change functors.

**Proposition A.9** Let $\mathcal{C}$ be a locally cartesian closed category with pullbacks. Then the base-change functors for the standard fibration $\pi_1 : \mathcal{C}^2 \to \mathcal{C}$ have right adjoints.

*Proof.* Let $f : A \to B$, and let $f^* : \mathcal{C}/B \to \mathcal{C}/A$ be its base-change functor. For a right adjoint $f_* : \mathcal{C}/A \to \mathcal{C}/B$ we must have a natural correspondence between the dashed maps in the diagram

$$\begin{tikzcd}
f^*X \arrow[d] \arrow[dr] \arrow[rr, dashed] & & Y \arrow[dll, "h"'] & \\
A \arrow[dr, "f"'] & X \arrow[d, "g"] \arrow[rr, dashed] & & f_*Y \arrow[dll, "f_*h"] \\
& B & &
\end{tikzcd}$$

Note that products in a slice category are exactly pullbacks. Then since $\mathcal{C}/B$ is cartesian closed, the equivalence between the arrows in the diagram is exactly given by the exponential adjunction, where $f_*h : f_*Y \to B$ is exactly the exponential functor, $(-)^A$, associated with $f : A \to B$ applied to $f \circ h : Y \to B$. ■

Therefore for any cartesian closed category with pullbacks, $\mathcal{C}$, and in particular any Grothendieck topoi, we have for each $f : A \to B$ in $\mathcal{C}$ a triple adjunction

$$\begin{tikzcd} \mathcal{C}/B \arrow[r, "f^*" description] & \mathcal{C}/A \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_*"] \end{tikzcd}$$

which in the case of Grothendieck topoi gives our desired geometric morphisms associated with base-change.

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