THE PROJECTIVE STRUCTURE OF QUANTUM MECHANICS

EA E THOMPSON

Abstract. Quantum Mechanics is a physical theory of the early to mid-20th century, originally formalized mathematically largely in the language of Linear Algebra. However, this linear structure is not shared by the mathematical formulation of Classical Mechanics, which is instead founded on the principles of abstract spaces—**manifolds**—with special additional structures. In recent years there have been numerous attempts to rectify the gap between these core physical theories by introducing a geometric formalization of Quantum Mechanics. We aim to motivate and explore basic properties of this construction through an analysis of the topological and geometric structures endowed on the proper state space of quantum systems, which, according to modern literature, takes the form of a **Projective Space**. In doing so we emphasize the necessity of abstracting from Euclidean space to general manifolds. After establishing this point, we investigate how we can perform calculus on general manifolds by abstracting the classical notion of a tangent to a curve or surface to that of a tangent space on a manifold. We then introduce and analyze the metric structures on our manifolds. Finally, we explore the prototypical metric on Projective Space, and how it can be used to construct and interpret one of the central postulates of Quantum Mechanics.

1. Introduction

As physicists such as Max Planck (1858-1942), Niels Bohr (1885-1962), and Erwin Schrödinger (1887-1961) began to rectify their understanding of the sub-atomic world in the early to mid-20th century, disparities between their newly formed model for quantum systems and the pre-existing model for macroscopic systems utilizing Newton's laws of motion became evident. While the description and evolution of classical systems, such as a train riding along a track, required the use of abstract and often non-linear geometries and dynamics, the evolution of quantum systems appeared to be best described using linear operators and linear combinations of basic states modeled in some abstract vector space. This discrepancy has disturbed the physics community for nearly a century, largely due to the accuracy of both Quantum and Classical Mechanics in predicting phenomena on their respective scales. However, through the research of physicists such as Troy A. Schilling in his 1996 PhD Thesis [8], it is now understood that quantum mechanics can also be formalized using largely geometric arguments on the true state space of quantum systems, Projective Space.

To obtain a full appreciation for this geometric interpretation of Quantum mechanics, we shall first introduce the notion of a quotient topology, which we will use to produce the phase space for finite dimensional quantum systems—*Complex Projective Space*. As a recurring theme throughout this paper we will motivate the structure of Projective Space through actions of important symmetries on the original state space, which is the unit sphere, in line with the interpretations of classical spaces which often arise from physical symmetries. For this purpose we briefly recall the notion of a group [2].

Definition 1.1. A set G with a binary function $\cdot : G \times G \to G$ is said to be a group if for any $f, g, h \in G$ the following properties hold:

- (Associativity) $g \cdot (f \cdot h) = (g \cdot f) \cdot h$
- (Identity) there exists $\mathbb{1} \in G$ such that $\mathbb{1} \cdot g = g \cdot \mathbb{1} = g$
- (Inverses) for each $g \in G$ there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = \mathbb{1}$

After inducing the topology on Projective Space, we prove important properties of the resulting topological structure. Then, in order to analyze dynamics on our state space we take on the task of defining a calculi on Projective Space as well as a notion of distance, where again both structures are induced by a suitable collection of symmetries on the unit sphere. Finally, after providing examples of these structures on the unit sphere and hyperspheres, we use our notion of distance on the state space to construct a physical explanation for the probabilities inherent in quantum measurement.

2. The Art of Gluing

For $n+1$ dimensional quantum systems, the state space was historically taken to be the unit sphere in $\mathbb{C}^{n+1}$. However, for any state z on the unit sphere, it is the case that any phase multiple $e^{i\theta}z$, for $\theta \in \mathbb{R}$, produces identical measurement results, and hence is experimentally indistinguishable. Thus the physical system states are not the individual points on the unit sphere, but rather equivalence classes of points on the unit sphere, with

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two points identified if they differ by a phase factor $e^{i\theta}$. We now develop a suitable structure using tools from topology to topologize the true state space of our quantum system obtained from this identification.

Definition 2.1. Let (X,τ) be a topological space, and let $\sim$ be an equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes, $[x]$, for $x \in X$, and $\pi : X \to X/\sim$ the natural projection mapping $x \mapsto [x]$. Then we define a topology τ_q on $X/\sim$ by the statement that

$$U \subseteq X/\sim \implies (U \in \tau_q \iff \pi^{-1}(U) \in \tau)$$

This topology on $X/\sim$ is called the **quotient topology** induced by π [6].

This topology on the space of equivalence classes is the largest topology for which the projection mapping is continuous. We shall show that this topology satisfies a special Universal Property for maps out of the quotient space, which will uniquely characterize its topological structure [6].

Theorem 2.2 (Universal Property of the Quotient Topology)**.** *Let $f : (X,\tau) \to (Z,\tau')$ be a map and let $\sim$ be an equivalence relation on X such that for all $a, b \in X$, if $a \sim b$ then $f(a) = f(b)$. Then there exists a unique map $F : X/\sim\to Z$ such that $F \circ \pi = f$. The quotient topology τ_q is the unique topology on $X/\sim$ such that f is continuous if and only if F is continuous.*

Proof. We define $F : X/\sim\to Z$ by simply stating that $F([x]) = f(x)$ for all $[x] \in X/\sim$. To see that this map is well-defined, note that f is constant on equivalence classes. Then if a and b are in the same equivalence class $[x]$, $a \sim b$ so $F([a]) = f(a) = f(b) = F([b])$. Consequently this map is independent of our choice of representative. Additionally by construction $F \circ \pi(x) = F([x]) = f(x)$ for any point of X, so $F \circ \pi = f$.

To show uniqueness we observe that if $g : X/\sim\to Z$ is another map satisfying $g \circ \pi = f$, then for any $[x] \in X/\sim$, $g([x]) = g \circ \pi(x) = f(x) = F \circ \pi(x) = F([x])$. Consequently $F = g$, so the map is unique.

Now we demonstrate the universal property of the quotient. Observe that f is continuous if and only if for all $U \in \tau'$, $f^{-1}(U) \in \tau$. By definition of the quotient topology and the fact that π is a surjective map this holds if and only if $\pi(f^{-1}(U)) \in \tau_q$. We note that

$$F^{-1}(U) = \{[x] \in X/\sim : F([x]) = f(x) \in U\} = \pi(f^{-1}(U))$$

Thus when $X/\sim$ is equipped with the quotient topology, the continuity of f and F is equivalent.

To show this property uniquely characterizes the quotient topology, suppose τ_p is another topology on $X/\sim$ which satisfies it. Consider the choice of functions $\pi_q : (X,\tau) \to (X/\sim, \tau_q)$ and $\pi_p : (X,\tau) \to (X/\sim, \tau_p)$. First, observe that the identity $\mathbb{1}_{X/\sim,p} : (X/\sim, \tau_p) \to (X/\sim, \tau_p)$ is continuous as all $U \in \tau_p$ are sent to $\mathbb{1}^{-1}_{X/\sim,p}(U) = U \in \tau_p$. Then, as $\mathbb{1}_{X/\sim,p}$ is the unique map which satisfies $\mathbb{1}_{X/\sim,p} \circ \pi_p = \pi_p$ and τ_p exhibits our universal property by assumption, π_p must be continuous. Next, note that $\mathbb{1}_{X/\sim,pq} : (X/\sim, \tau_p) \to (X/\sim, \tau_q)$ and $\mathbb{1}_{X/\sim,qp} : (X/\sim, \tau_q) \to (X/\sim, \tau_p)$ are our unique maps satisfying $\mathbb{1}_{X/\sim,pq} \circ \pi_p = \pi_q$ and $\mathbb{1}_{X/\sim,qp} \circ \pi_q = \pi_p$, respectively. Now, since π_q is continuous by definition of the quotient topology, we have that $\mathbb{1}_{X/\sim,pq}$ must be continuous since τ_p satisfies our universal property. In particular, this implies that for all $U \in \tau_q$, $U = \mathbb{1}^{-1}_{X/\sim,pq}(U) \in \tau_p$, so $\tau_q \subseteq \tau_p$. Next, since we have shown that π_p is continuous and $\mathbb{1}_{X/\sim,qp} \circ \pi_q = \pi_p$, by the fact that τ_q satisfies our universal property we know that $\mathbb{1}_{X/\sim,qp}$ is continuous. This implies that for all $U \in \tau_p$, $U = \mathbb{1}^{-1}_{X/\sim,qp}(U) \in \tau_q$, so we have the reverse inclusion $\tau_p \subseteq \tau_q$. Hence indeed we find uniqueness of the topology on $X/\sim$ which satisfies our desired universal property. ☐

This universal property provides us with the tools to define continuous mappings on spaces which are described by equivalence classes by first defining the mapping on our base space for the projection. Although we have this property, often the quotient topology does not preserve many topological properties of the original space. For this reason we look for quotient topologies induced by continuous symmetries of our original space, which tend to preserve a greater number of topological features after the identification of points. This approach also aligns with the identification of points under structure-preserving symmetries on many physical spaces in classical and quantum mechanics.

Definition 2.3. A **topological group** $(G,\tau,\cdot)$ is a topological space (G,τ) and a group $(G,\cdot)$ such that the binary operation $\cdot$ makes the map from $G \times G$ to G defined by $(g,f) \mapsto g \cdot f^{-1}$ for all $g, f \in G$ continuous [6].

Using this structure we define a way for a topological group to act as a space of symmetries on particular topological spaces [6].

Definition 2.4. Let (X,τ) be a topological space and $(G,\tau_G,\cdot_G)$ a topological group. We define a left G-action on X by a continuous map $\cdot_A : G \times X \to X$ satisfying the following properties for all $x \in X$ and $g, h \in G$

(1) (identity) $\mathbb{1} \cdot_A x = x$
(2) (associativity) $(g \cdot_G h) \cdot_A x = g \cdot_A (h \cdot_A x)$

It is an elementary result that the action of a group on any set X defines an equivalence relation on X by $x \sim g \cdot_A x$ for any $x \in X$ and $g \in G$, where we call the equivalence classes **orbits** of the action, and denote them by $G \cdot_A x := \{g \cdot_A x \in X : g \in G\}$. We shall not prove this result here for brevity [2]. Additionally, each fixed $g \in G$ acts as a homeomorphism on X through $\cdot_A$ with inverse $g^{-1} \in G$ due to the associativity of the action [6]. It follows that if G and X are as in the above definition, we obtain a quotient space, which we denote X/G, where the equivalence classes are orbits of G. Assuming some additional structure on G we obtain a number of topological properties which are preserved via passing to the quotient space. We prove the following result in the case of metric spaces, which is sufficient for our applications, though an identical result holds for simple Hausdorff spaces [6].

Theorem 2.5. *Let $(G, \tau_G, \cdot_G)$ be a compact topological group acting continuously on a metric space (X, d) through an operation $\cdot_A$. Then the quotient space X/G is a Hausdorff space.*

In preparation for the proof of this result we require the notion of a product topology [6].

Definition 2.6. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Then the product topology, $\tau_{1\times 2}$, on the set $X_1 \times X_2$ with projections $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ is generated by the basis

$$\mathcal{B} := \{p_i^{-1}(U) \subseteq X_1 \times X_2 : U \in \tau_i, 1 \leq i \leq 2\}$$

By construction this topology admits the useful property that the projection mappings are continuous. Additionally, any mapping $f : (Y, \tau_Y) \to (X_1 \times X_2, \tau_{1\times 2})$ can be written as $f = (p_1 \circ f, p_2 \circ f)$. Since the basic open sets of the product space are pre-images of the projection mappings, f is continuous if and only if both $p_1 \circ f$ and $p_2 \circ f$ are continuous [6]. This topology is the same which results from the product metric construction on two metric spaces (X_1, d_1) and (X_2, d_2) defined by

$$d_{1\times 2}((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2),\ \forall x_1, x_2 \in X_1, y_1, y_2 \in X_2$$

as seen in introductory Analysis courses. We recall from such courses that sequences in the product converge if and only if the component sequences converge in their respective topological spaces [9]. With this background we now have sufficient tools to prove Theorem 2.5.

Proof of Theorem 2.5. Let X/G have the quotient topology induced by $\pi : X \to X/G$. Let $x, y \in X$. Note that for all open sets $U \subseteq X$, $g \cdot_A U \subseteq X$ is open for all $g \in G$ as the group acts via homeomorphisms, which send open sets to open sets. Then $G \cdot_A U = \bigcup_{g\in G} g \cdot_A U \subseteq X$ is open as the union of open sets. Note that $\pi^{-1}(\pi(U)) = \bigcup_{x\in U} G \cdot_A x = G \cdot_A U$, since the points in X/G are precisely the orbits of the action. Now suppose every pair of neighborhoods of $\pi(x)$ and $\pi(y)$ intersect. To show X/G is Hausdorff it is sufficient to show that this assumption implies $\pi(x) = \pi(y)$. Then, for each $n \in \mathbb{N}$ consider the open balls $B_{1/n}(x), B_{1/n}(y) \subseteq X$. As shown above, $G \cdot_A B_{1/n}(x)$ and $G \cdot_A B_{1/n}(y)$ are also open and are complete pre-images of sets in X/G. Then by definition of the quotient topology $\pi(G \cdot_A B_{1/n}(x))$ and $\pi(G \cdot_A B_{1/n}(y))$ are open neighborhoods of $\pi(x)$ and $\pi(y)$, respectively. Thus, they must intersect at some point $\pi(q_n) \in X/G$, where consequently $q_n \in G \cdot_A B_{1/n}(x) \cap G \cdot_A B_{1/n}(y)$. By definition of these sets there exist $x_n \in B_{1/n}(x)$ and $y_n \in B_{1/n}(y)$, as well as $g_{n_x}, g_{n_y} \in G$ such that $g_{n_x} \cdot_A x_n = q_n = g_{n_y} \cdot_A y_n$, or in other words, $x_n = g_n \cdot_A y_n$, where we define $g_n = g_{n_x}^{-1} \cdot_G g_{n_y}$. Fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\varepsilon < \frac{1}{N}$. It follows that for $n \geq N$, $x_n \in B_{1/n}(x)$, so $d(x_n, x) < \frac{1}{n} < \varepsilon$, and identically $d(y_n, y) < \varepsilon$. Thus $x_n \to x$ and $y_n \to y$, so $(x_n, y_n) \to (x, y)$ in $X \times X$. Consider the map $\mathfrak{m} : G \times X \to X \times X$ defined by $(g, z) \mapsto (g \cdot_A z, z)$. As the component maps $(g, z) \mapsto g \cdot_A z$ and $(g, z) \mapsto z$ are continuous maps by assumption of our action and definition of the product topology, the map $\mathfrak{m}$ is continuous [6]. Since $y_n \to y$, I claim that $A := \{y_n : n \in \mathbb{N}\} \cup \{y\}$ is compact. Indeed, if $\{U_i\}_{i\in I}$ is an open cover of A, then some U_i contains y. By convergence of our sequence it follows that for some $M \in \mathbb{N}$, $y_m \in U_i$ for all $m \geq M$. Then for each y_j, with $1 \leq j < M$ there exists U_j containing y_j, so $A \subseteq U_i \bigcup_{j=1}^{M-1} U_j$. It follows that the product $G \times A$ is compact, being the finite product of compact spaces, which is an important result, but outside the scope of this paper, thus we shall not prove it here [6]. Then $\mathfrak{m}(G \times A)$ is compact, being the continuous image of a compact set, and hence it is closed since $X \times X$ is a Hausdorff space being a product metric space. Note $((g_n \cdot y_n, y_n))_{n\in\mathbb{N}} \subseteq \mathfrak{m}(G \times A)$. As $((g_n \cdot y_n, y_n))_{n\in\mathbb{N}} = ((x_n, y_n))_{n\in\mathbb{N}}$ converges in $X \times X$ to (x, y) and $\mathfrak{m}(G \times A)$ is closed, $(x, y) \in \mathfrak{m}(G \times A)$. Therefore there must exist $h \in G$ such that $x = h \cdot_A y$, which implies that $\pi(x) = \pi(y)$ since they are in the same orbit, completing the proof (inspired by the proof of Theorem 1.2 from Chapter 1 of [1]). □

Now that we understand the structure preserving properties of continuous actions we look at the symmetry group on the complex unit sphere which induces our desired quantum mechanical state space, the complex Projective Space. Although the theory of quantum mechanics is fundamentally complex, in this paper we restrict ourselves to structures associated with real Euclidean space. To this end we now make a topological identification of the spaces $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ for use in the remainder of the paper.

Theorem 2.7. *The map $T : \mathbb{C}^n \to \mathbb{R}^{2n}$ defined by*

$$T(x_1 + iy_1, ..., x_n + iy_n) = (x_1, y_1, ..., x_n, y_n)$$

is an $\mathbb{R}$-linear isometry with respect to the ℓ^2_{2n} norm $||\cdot||_2$ on $\mathbb{R}^{2n}$, and the standard norm $||\cdot||_{\mathbb{C}}$ on $\mathbb{C}^n$.

Proof. First for linearity let $x = (x_1 + iy_1, ..., x_n + iy_n), a = (a_1 + ib_1, ..., a_n + ib_n) \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned} T(x + \alpha a) &= T((x_1 + \alpha a_1) + i(y_1 + \alpha b_1), ..., (x_n + \alpha a_n) + i(y_n + \alpha b_n)) \\ &= (x_1 + \alpha a_1, y_1 + \alpha b_1, ..., x_n + \alpha a_n, y_n + \alpha b_n) \\ &= (x_1, y_1, ..., x_n, y_n) + \alpha(a_1, b_1, ..., a_n, b_n) = T(x) + \alpha T(a) \end{aligned}$$

so T is indeed linear. Next, to show T is an isometry observe that

$$||x||_{\mathbb{C}} = \sqrt{\sum_{j=1}^{n} |x_j + iy_j|^2} = \sqrt{\sum_{j=1}^{n} (x_j^2 + y_j^2)} = ||T(x)||_2$$

by definition of the 2-norm. Thus T is indeed a linear isometry between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ $\square$

We recall that isometries are injective, and that injective linear maps between vector spaces of the same dimension are linear isomorphisms by the Dimension Theorem [2]. Since $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are both $2n$ dimensional vector spaces over $\mathbb{R}$, T is a linear isomorphism. Additionally, if $x \in \mathbb{R}^{2n}$, we have that $\left|\left|T^{-1}(x)\right|\right|_{\mathbb{C}} = \left|\left|T(T^{-1}(x))\right|\right|_2 = ||x||_2$, so the inverse map T^{-1} is also an isometry. Consequently, since isometries are continuous, we have that T is a homeomorphism so $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ have identical linear, topological, and metric structures. Therefore, for the sake of this paper identifying the two spaces is reasonable, and for the remainder we implicitly identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, and by abuse of notation refer to $(x_1, y_1, ..., x_n, y_n)$ as an element of $\mathbb{C}^n$ on occasion.

2.1. **Projective Space as a Consequence of Symmetry.** Now that we have suitably set our foundations for quotient topologies and continuous actions, we can construct the complex projective n space, which we denote by $\mathbb{C}P^n$ [5].

Definition 2.8. Define an equivalence relation on $\mathbb{C}^{n+1}\backslash\{0\}$ by $x \sim y$ if and only if there exists $\alpha \in \mathbb{C}$ such that $x = \alpha y$. The quotient space under this equivalence relation on complex lines through the origin is defined to be the **Complex Projective n Space**:

$$\mathbb{C}P^n := \mathbb{C}^{n+1}\backslash\{0\}/\sim$$

Throughout we let $\pi_P : \mathbb{C}^{n+1} \to \mathbb{C}P^n$ denote the canonical projection. Note that the points in $\mathbb{C}P^n$ correspond with lines through the origin in $\mathbb{C}^{n+1}$. Now, we can realize $\mathbb{C}P^n$ as a space resulting from some action of symmetries on the original state space for quantum mechanics, the hypersphere in $\mathbb{C}^{n+1}$ which is defined by

$$\mathbb{S}(\mathbb{C}^{n+1}) := \{z \in \mathbb{C}^{n+1} : ||z||_{\mathbb{C}} = 1\}$$

Since on every line in $\mathbb{C}^{n+1}$ there is a point of norm 1, the restriction $\pi_P|_{\mathbb{S}(\mathbb{C}^{n+1})} : \mathbb{S}(\mathbb{C}^{n+1}) \to \mathbb{C}P^n$ is surjective, so now we need only find the appropriate action on $\mathbb{S}(\mathbb{C}^{n+1})$ which would induce this quotient. To ensure that the quotient space we move to is the appropriate space to describe our physical systems, we require that no points in the original space are fixed by non-trivial elements of our symmetry group. This property of group actions is known as being **free** [2].

Definition 2.9. Let G be a group acting on a set X. The action $\cdot_A$ is said to be free if for all $g \in G$ different from the identity, and all $x \in X$, $g \cdot_A x \neq x$.

The appropriate symmetry group for the unit sphere $\mathbb{S}(\mathbb{C}^{n+1})$ which will induce our desired phase space $\mathbb{C}P^n$ is the unitary group $U_1(\mathbb{C})$.

Definition 2.10. The unitary group $U_1(\mathbb{C})$ is the circle $\mathbb{S}(\mathbb{C}) = \{z \in \mathbb{C} : ||z||_{\mathbb{C}} = |z| = 1\}$ with the group operation of multiplication.

We recall that all elements in $\mathbb{S}(\mathbb{C})$ can be written as $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus, $U_1(\mathbb{C})$ is the group of phase-factors mentioned at the start of this section. Note that this is indeed a group due to the associativity of multiplication in $\mathbb{C}$, the existence of $1 \in \mathbb{S}(\mathbb{C})$, and the fact that $e^{i\theta}e^{-i\theta} = e^0 = 1$. In particular, it is a compact topological group, which we now explore using our identification in an example.

Example 2.11. We first endow $U_1(\mathbb{C})$ with the subspace topology from $\mathbb{C} \cong \mathbb{R}^2$. We identify $U_1(\mathbb{C})$ with $\mathbb{S}(\mathbb{C}) \cong \mathbb{S}(\mathbb{R}^2)$, via the standard map $x + iy \mapsto (x, y)$. We consider the multiplication and inversion map $((x + iy), (a + ib)) \mapsto \frac{x+iy}{a+ib} = \frac{xa-by+i(xb+ay)}{a^2+b^2} = xa - by + i(xb + ay)$, where $x + iy, a + ib \in U_1(\mathbb{C})$ so $a^2 + b^2 = ||a + ib||_{\mathbb{C}}^2 = 1$. In our identification with $\mathbb{R}^2$ this can be written as $((x, y), (a, b)) \mapsto (xa - by, xb + ay)$. As the component maps $((x, y), (a, b)) \mapsto xa - by$ and $((x, y), (a, b)) \mapsto xb + ay$ are polynomials in the components, they are continuous functions. Hence the total function $((x, y), (a, b)) \mapsto (xa - by, xb + ay)$ is also continuous. Then by definition $U_1(\mathbb{C})$ is a topological group.

In addition, we recall that $\mathbb{S}(\mathbb{C})$ is the inverse image of the set $\{1\} \subseteq \mathbb{R}$ under the norm $||\cdot||_{\mathbb{C}}$. As norms are continuous and singletons are closed in $\mathbb{R}$, $\mathbb{S}(\mathbb{C})$ is a closed and bounded subset of $\mathbb{C}$. Since $\mathbb{C}$ and $\mathbb{R}^2$ are isometric, by the Heine-Borel Theorem we find that all closed and bounded subsets of $\mathbb{C}$ are compact, so $\mathbb{S}(\mathbb{C})$ is compact. Thus, since $U_1(\mathbb{C}) = \mathbb{S}(\mathbb{C})$ as sets, we find that $U_1(\mathbb{C})$ is indeed a compact topological group.

Now that we understand the structure of $U_1(\mathbb{C})$ to a greater degree, we turn to its action on the hypersphere $\mathbb{S}(\mathbb{C}^{n+1})$.

Proposition 2.12. *$U_1(\mathbb{C})$ acts linearly and continuously on $\mathbb{C}^{n+1}$ by scalar multiplication, and this action restricts to a free action on $\mathbb{S}(\mathbb{C}^{n+1})$.*

Proof. Let $e^{i\varphi} \in U_1(\mathbb{C})$ for $\varphi \in \mathbb{R}$, and let $A_{e^{i\varphi}}$ denote the associated map induced by the action. Now, using our identification of $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2(n+1)}$, we define the action of $e^{i\varphi}$ on an element $(x_1, y_1, ..., x_{n+1}, y_{n+1}) \in \mathbb{C}^{n+1}$ by

$$A_{e^{i\varphi}}(x_1, y_1, ..., x_{n+1}, y_{n+1}) = (x_1 \cos\varphi - y_1 \sin\varphi, x_1 \sin\varphi + y_1 \cos\varphi, ..., x_{n+1} \cos\varphi - y_{n+1} \sin\varphi, x_{n+1} \sin\varphi + y_{n+1} \cos\varphi)$$

Then, if the point lies in $\mathbb{S}(\mathbb{C}^{n+1})$ so $\sum_{i=1}^{n+1}(x_i^2 + y_i^2) = 1$, we have

$$\begin{aligned}(1)\quad \sum_{i=1}^{n+1}(x_i \cos\varphi - y_i \sin\varphi)^2 + (x_i \sin\varphi + y_i \cos\varphi)^2 &= \sum_{i=1}^{n+1}\Big[x_i^2 \cos^2\varphi - 2x_iy_i \cos\varphi \sin\varphi + y_i^2 \sin^2\varphi + x_i^2 \sin^2\varphi \\ &\quad + 2x_iy_i \cos\varphi \sin\varphi + y_i^2 \cos^2\varphi\Big] = \sum_{i=1}^{n+1} x_i^2 + y_i^2 = 1\end{aligned}$$

so the image of the action again lies in $\mathbb{S}(\mathbb{C}^{n+1})$. Thus the action can indeed be restricted to $\mathbb{S}(\mathbb{C}^{n+1})$. Additionally, this action is linear, with the following matrix representation in the canonical orthonormal coordinates of $\mathbb{R}^{2(n+1)}$, $[A_{e^{i\varphi}}] = \mathrm{diag}(B(\varphi), ..., B(\varphi))$ where $B(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$ occurs $n + 1$ times. Since the operator norm of $A_{e^{i\varphi}}$ is equivalent to the spectral norm of $[A_{e^{i\varphi}}]$ which necessarily finite, $A_{e^{i\varphi}}$ is a bounded operator and hence continuous on $\mathbb{C}^{n+1}$.

Next we show the restricted action on $\mathbb{S}(\mathbb{C}^{n+1})$ is free. If $z = (x_1, y_1, ..., x_{n+1}, y_{n+1}) \in \mathbb{S}(\mathbb{C}^{n+1})$ with $A_{e^{i\varphi}}(z) = (z)$, then for each i, $x_i = x_i \cos\varphi - y_i \sin\varphi$ and $y_i = x_i \sin\varphi + y_i \cos\varphi$. As $||z||_{\mathbb{C}} = 1$, at least one of the x_i or y_i is non-zero. We consider the case of some $x_i \neq 0$, while the case of $y_i \neq 0$ is analogous. Multiplying our expressions for x_i and y_i by $\cos\varphi$ and $\sin\varphi$, respectively, and then adding them we obtain

$$x_i \cos\varphi + y_i \sin\varphi = x_i \cos^2\varphi - y_i \sin\varphi \cos\varphi + x_i \sin^2\varphi + y_i \sin\varphi \cos\varphi = x_i$$

We can then add $x_i = x_i \cos\varphi + y_i \sin\varphi$ and $x_i = x_i \cos\varphi - y_i \sin\varphi$ to find that $2x_i = 2x_i \cos\varphi$, or that $\cos\varphi = 1$ since $x_i \neq 0$. This equality implies that $\varphi = 2\pi k$, for $k \in \mathbb{Z}$, which gives us $e^{i\varphi} = e^{2\pi ki} = 1$, so our element acting on z was the identity of $U_1(\mathbb{C})$. Thus the action of $U_1(\mathbb{C})$ on $\mathbb{S}(\mathbb{C}^{n+1})$ is indeed free. $\square$

As a consequence of this action we observe that the orbit of a point $p \in \mathbb{S}(\mathbb{C}^{n+1})$ is

$$\{q \in \mathbb{S}(\mathbb{C}^{n+1}) : zq = p, z \in U_1(\mathbb{C})\} = \{q \in \mathbb{S}(\mathbb{C}^{n+1}) : q \in \pi_P^{-1}(\{\pi_P(p)\})\} = \pi_P|_{\mathbb{S}(\mathbb{C}^{n+1})}^{-1}(\{\pi_P(p)\})$$

so the orbits coincide exactly with the fibers of our projection map $\pi_P|_{\mathbb{S}(\mathbb{C}^{n+1})}$ in $\mathbb{S}(\mathbb{C}^{n+1})$. Thus $\mathbb{C}P^n$ is the resulting quotient space $\mathbb{S}(\mathbb{C}^{n+1})/U_1(\mathbb{C})$. As $U_1(\mathbb{C})$ is a compact topological group from Example 2.11 and $\mathbb{S}(\mathbb{C}^{n+1})$ is Hausdorff, being a subspace of a normed linear space $\mathbb{C}^{n+1}$, we find that $\mathbb{C}P^n$ is also Hausdorff from Theorem 2.5. We shall now show that $\mathbb{S}(\mathbb{C}^{n+1})$ is compact which will imply $\mathbb{C}P^n$ is compact being the continuous image of a map with domain $\mathbb{S}(\mathbb{C}^{n+1})$.

Proposition 2.13. *The complex unit sphere $\mathbb{S}(\mathbb{C}^{n+1})$ is a compact topological space in the subspace topology.*

Proof. Observe that $\mathbb{S}(\mathbb{C}^{n+1})$ is the inverse image of the closed set $\{1\} \subseteq \mathbb{R}$ under the continuous map given by the norm $||\cdot||_{\mathbb{C}}$. Additionally, $\mathbb{S}(\mathbb{C}^{n+1}) = \{z \in \mathbb{C}^{n+1} : ||z||_{\mathbb{C}} = 1\}$ is bounded, so under our identification of $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2(n+1)}$ we find by the Heine-Borel Theorem that $\mathbb{S}(\mathbb{C}^{n+1})$ is a compact space. $\square$

Thus $\mathbb{C}P^n$ is indeed a compact topological space as well, and so it attains all of the structure that comes with compactness.

3. The Calculus of Abstract Spaces

Although we now have a topological structure for Projective Spaces, which allows us to determine what it means for points to be “close,” as of yet we have no way of evaluating dynamics on Projective Space which is needed if we wish to analyze our quantum systems in Projective Space. Thus, we move to the notion of smooth structures on abstract spaces which may allow us to rectify this current deficiency.

Definition 3.1. Let (M,τ) be a topological space. We say that (M,τ) is an **n-dimensional topological manifold** if M is a Hausdorff and second-countable space such that for every point $p \in M$ there exists an open neighborhood U of p and a homeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$ for some open subset V of $\mathbb{R}^n$. The pair (U,φ) is called a **coordinate chart** on M [5].

Here we say that a topological space is **second-countable** if it has a countable basis of open sets. Throughout this section we shall denote a general n-manifold by (M,τ). As topological manifolds are locally like Euclidean space we can use our local homeomorphisms to define notions of differentiation on the space. For instance, if $f : M \to \mathbb{R}$ is a continuous function, we can define differentiability by stating that f is differentiable at a point $p \in M$ if there is a coordinate chart (U,φ) about p such that $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(p)$. However, this definition may not be independent of the choice of chart, so we must first explicitly restrict our collection of charts so that this notion is well defined.

Definition 3.2. Let (U,φ) and (V,ψ) be charts on M. We say that the charts are C^k **compatible**, for $k \in \mathbb{N}$, if either $U \cap V = \varnothing$ or the **transition functions**

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V), \text{ and } \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are k-times continuously differentiable [5].

We collect our transition functions into a single structure, known as an **Atlas**.

Definition 3.3. An atlas $\mathcal{A}$ of M is a collection of charts on M, such that the open sets in the charts form an open cover on M. Then we say that an atlas $\mathcal{A}$ is C^k if all charts in it are C^k compatible. A C^k **structure** on M is a maximal atlas in the sense that it is not contained in any larger C^k atlas [5].

For the sake of this paper and many physical applications we are interested in C^∞ structures, also known as **smooth structures**. If $\mathcal{A}$ is a smooth structure on M, we say that the pair $(M,\mathcal{A})$ is a smooth manifold. The next result we cover gives us the existence of smooth structures on manifolds without requiring us to define all charts on the manifold [5].

Theorem 3.4. *If $\mathcal{A}$ is a smooth atlas on M, then there exists a unique smooth structure $\mathcal{M}$ on M containing $\mathcal{A}$.*

Proof. Let $\mathcal{M}$ consist of all charts (U,φ) such that (U,φ) is smoothly compatible with every chart in $\mathcal{A}$. Since $\mathcal{A}$ is a smooth atlas, every chart in $\mathcal{A}$ is smoothly compatible with all others, so $\mathcal{A} \subseteq \mathcal{M}$. Thus the charts in $\mathcal{M}$ cover M since the charts in $\mathcal{A}$ form an atlas. We now need to show that all charts $(U,\varphi),(V,\psi) \in \mathcal{M}$ are smoothly compatible. If $U \cap V = \varnothing$ the charts are vacuously smoothly compatible. Otherwise let $p \in U \cap V$, and let $(W,\Phi) \in \mathcal{A}$ be a chart containing p. Then the maps $\varphi \circ \Phi^{-1} : \Phi(U \cap W) \to \varphi(U \cap W), \psi \circ \Phi^{-1} : \Phi(V \cap W) \to \psi(V \cap W), \Phi \circ \varphi^{-1} : \varphi(U \cap W) \to \Phi(U \cap W)$, and $\Phi \circ \psi^{-1} : \psi(V \cap W) \to \Phi(V \cap W)$ are all smooth by construction of $\mathcal{M}$. Since the composite of differentiable maps is differentiable, the composite of smooth maps is smooth [5], and consequently the maps

$$\varphi \circ \psi^{-1} = (\varphi \circ \Phi^{-1}) \circ (\Phi \circ \psi^{-1}) : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$
$$\psi \circ \varphi^{-1} = (\psi \circ \Phi^{-1}) \circ (\Phi \circ \varphi^{-1}) : \varphi(U \cap V \cap W) \to \psi(U \cap V \cap W)$$

are both smooth. Due to the fact that $p \in U \cap V \cap W$, $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are both smooth at p. Finally, since $p \in U \cap V$ was arbitrary the maps $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ and $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ are smooth. Thus $\mathcal{M}$ is a smooth atlas containing $\mathcal{A}$. Additionally, for any chart of M that is smoothly compatible with $\mathcal{M}$, it must be smoothly compatible with the subset $\mathcal{A}$ and hence already be in $\mathcal{M}$ by construction. Thus $\mathcal{M}$ is the unique smooth structure on M containing $\mathcal{A}$. ☐

A simple consequence of this result is that $\mathbb{R}^n$ has a unique canonical smooth structure containing the identity, since $\{(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})\}$ is a smooth atlas on $\mathbb{R}^n$. As a more constructive example which will be needed when we derive the structure on $\mathbb{C}P^n$, we consider the unit sphere [5].

Example 3.5. Let $\mathbb{S}(\mathbb{R}^n)$ denote the unit sphere in $\mathbb{R}^n$ with the subspace topology. Note $\mathbb{S}(\mathbb{R}^n)$ is second-countable and Hausdorff as it is a subspace of $\mathbb{R}^n$ which is both. We generalize the notion of the upper half-sphere by defining for each $i = 1,...,n$ the open set $\mathcal{U}_i^+ = \{(x_1,...,x_n) \in \mathbb{R}^n : x_i > 0\}$, which is open being the inverse image $\pi_i^{-1}((0,\infty))$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the ith projection mapping. Similarly we define the lower half-sphere $\mathcal{U}_i^- = \{(x_1,...,x_n) \in \mathbb{R}^n : x_i < 0\} = \pi_i^{-1}((-\infty,0))$. Then under the subspace topology $\widetilde{\mathcal{U}}_i^+ = \mathcal{U}_i^+ \cap \mathbb{S}(\mathbb{R}^n)$ and

$\widetilde{\mathcal{U}}_i^- = \mathcal{U}_i^- \cap \mathbb{S}(\mathbb{R}^n)$ are open. Further, for each $(x_1, ..., x_n) \in \mathbb{S}(\mathbb{R}^n)$, since the vector is of unit norm it must have some $x_i \neq 0$, in which case $(x_1, ..., x_n) \in \mathcal{U}_i^+ \cup \mathcal{U}_i^-$, so these open sets form a finite cover of $\mathbb{S}(\mathbb{R}^n)$. For each i we define maps $\varphi_i^+ : \widetilde{\mathcal{U}}_i^+ \to \mathbb{R}^n$ and $\varphi_i^- : \widetilde{\mathcal{U}}_i^- \to \mathbb{R}^n$ by

$$\varphi_i^{\pm}(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

Note that for any sequence $((x_{1,m}, ..., x_{n,m}))_{m\in\mathbb{N}} \subseteq \mathbb{S}(\mathbb{R}^n) \subseteq \mathbb{R}^n$ that converges, we know that each $(x_{j,m})_{m\in\mathbb{N}}$ converges, which implies $((x_{1,m}, ..., x_{i-1,m}, x_{i+1,m}, ..., x_{n,m}))_{m\in\mathbb{N}}$ converges in $\mathbb{R}^{n-1}$ [9], so all of these $\varphi_i^{\pm}$ maps are continuous. We observe that $\varphi_i^{\pm}(\widetilde{\mathcal{U}}_i^{\pm}) = B_1^{n-1}(0) := \{x \in \mathbb{R}^{n-1} : ||x||_2 < 1\}$. Indeed, for $(x_1, ..., x_n) \in \widetilde{\mathcal{U}}_i^{\pm}$, $\sum_{j=1}^n x_j^2 = 1$ so $1 - \sum_{j=1, j\neq i}^n x_j^2 = x_i^2 > 0$. On the other hand, for any $y = (y_1, ..., y_{n-1}) \in B_1^{n-1}(0)$, consider $\widetilde{y}_{\pm} = (y_1, ..., y_{i-1}, \pm\sqrt{1 - ||y||_2^2}, y_i, ..., y_{n-1})$, which exists as $||y||_2 < 1$. Then since $||\widetilde{y}_{\pm}||_2^2 = \sum_{j=1}^{n-1} y_j^2 + 1 - ||y||_2^2 = 1$ and $\sqrt{1 - ||y||_2^2} > 0$, we have $\widetilde{y}_+ \in \widetilde{\mathcal{U}}_i^+$ and $\widetilde{y}_- \in \widetilde{\mathcal{U}}_i^-$. Additionally, by construction $\varphi_i^{\pm}(\widetilde{y}_{\pm}) = y$. In particular we find that $\varphi_i^{\pm}$ is invertible with inverse defined by sending $y \in B_1^{n-1}(0)$ to $\widetilde{y}_{\pm}$ [5]. As the ℓ_2^{n-1} norm is continuous and the square root function is continuous on $(0, \infty)$ the components of $(\varphi_i^{\pm})^{-1}$ are again continuous maps, and so the $\varphi_i^{\pm}$ are homeomorphisms [5]. Thus $\mathbb{S}(\mathbb{R}^n)$ is a manifold of dimension $n-1$ with the atlas $\mathcal{A} = \{(\widetilde{\mathcal{U}}_i^{\pm}, \varphi_i^{\pm})\}_{1\leq i\leq n}$. We also observe that for any i and j

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}(x) = \varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}(x_1, ..., x_{n-1}) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, \pm\sqrt{1 - ||x||_2^2}, x_j, ..., x_{n-1})$$

where for convenience of notation we write the case of $i < j$ only. We note that all component functions of this transition map, excluding $\pm\sqrt{1 - ||x||_2^2}$, are projections onto components and hence infinitely differentiable. Additionally, as the product and sum of infinitely differentiable functions are infinitely differentiable, $x \mapsto 1 - ||x||_2^2 = 1 - \sum_{j=1}^n x_j^2$ is $\mathbb{C}^\infty$. Finally, as the square root function is infinitely differentiable on $(0, \infty)$, and the image of $1 - ||x||_2^2$ with domain $B_1^{n-1}(0)$ is contained in $(0, 1)$, we find that the composition $\pm\sqrt{1 - ||x||_2^2}$ is also infinitely differentiable [9]. Therefore as all component functions are $\mathbb{C}^\infty$ we observe that the transition function is $\mathbb{C}^\infty$, and an analogous argument holds for the case of $i > j$ and $i = j$. Similar arguments also hold for the transitions $(\varphi_i^{\pm})^{-1} \circ \varphi_j^{\pm}$, though we do not include them here for space considerations [5]. Thus $\mathcal{A}$ constitutes a smooth atlas on $\mathbb{S}(\mathbb{R}^n)$, so by Theorem 3.4 we have a smooth structure on $\mathbb{S}(\mathbb{R}^n)$, and this is the canonical smooth structure.

Next, using smooth structures we can now define the notion of a smooth map hinted at previously [5].

Definition 3.6. Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be smooth manifolds of dimensions n and m, respectively. We say a map $F : M \to N$ is smooth at $p \in M$ if there exists a chart $(U, \varphi) \in \mathcal{M}$ containing p and a chart $(V, \psi) \in \mathcal{N}$ containing $F(U)$ such that the map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \psi(F(U)) \subseteq \mathbb{R}^n$$

is C^∞ in the classical sense. If F is bijective with smooth inverse we say that F is a **Diffeomorphism**.

An important example of a diffeomorphism for any smooth manifold $(M, \mathcal{M})$ is any coordinate chart $(U, \varphi) \in \mathcal{M}$. Indeed for any $p \in U$ we can take the chart (U, φ) on U and the chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$ which of course contains $\varphi(U)$, in which case our map in local coordinates is simply

$$\mathbb{1}_{\mathbb{R}^n} \circ \varphi \circ \varphi^{-1} = \mathbb{1}_{\mathbb{R}^n} : \varphi(U) \to \varphi(U)$$

which is bijective, smooth, and of smooth inverse (being itself). Thus the charts of a smooth structure on a manifold form the basic building blocks for smooth maps. Additionally smooth maps not only preserve the smooth structure of the manifolds, but also the topological structure [5].

Proposition 3.7. *If $F : M \to N$ is a smooth map between smooth manifolds, then F is continuous.*

Proof. Let $p \in M$ and let (U, φ) and (V, ψ) be charts in the smooth structures on M and N, respectively, such that $p \in U$ and $F(U) \subseteq V$. Then the map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(F(U))$ is smooth by assumption, and consequently continuous by definition of smoothness on maps between Euclidean spaces. Then, we observe that restricted to U, we can write F as

$$F = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to F(U)$$

But, $\psi \circ F \circ \varphi^{-1}$ is continuous and the coordinate maps ψ^{-1} and φ are homeomorphisms, so the composite F is continuous on U. Additionally, as $p \in U$, F is continuous at p, so F is continuous on all of M since p was arbitrary. ∎

Corollary 3.8. *If $F : M \to N$ is a diffeomorphism, then F is a homeomorphism*

This corollary immediately follows from our Proposition 3.7 using the fact that diffeomorphisms are bijective smooth maps with smooth inverses. Although we have a notion of structure preserving maps on our smooth manifolds, we are still unable to perform calculus. From analysis we know that in $\mathbb{R}^n$ derivatives are synonymous with linear maps which provide a local linear approximation to a function. Thus, in order to have a notion of differentiation, we first need a notion of local linearity on a manifold. In $\mathbb{R}^n$ the vectors which derivatives act on can be thought of as being attached to the point where the derivative is taken, so we have a copy of $\mathbb{R}^n$ at each point $a \in \mathbb{R}^n$. We want to replicate this structure on an abstract manifold.

We note that all we have on smooth manifolds so far are the smooth functions, maps, and coordinate charts giving the manifolds their structure. But from $\mathbb{R}^n$ we also have a vector like quantity which acts on smooth functions: the directional derivatives! For a given vector $v \in \mathbb{R}^n$ and point a, we denote the directional derivative by $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, and recall that for $f, g \in C^\infty(\mathbb{R}^n)$ the directional derivative satisfies the **liebniz rule**:

$$D_v|_a(fg) = g(a)D_v f(a) + f(a)D_v g(a)$$

This motivates the following generalization [5].

Definition 3.9. If $p \in M$ is a point, a map $D : C^\infty(M) \to \mathbb{R}$ is called a **derivation** on M if it is a linear functional on $C^\infty(M)$ and D satisfies the liebniz rule with respect to p. Let T_pM denote the collection of all derivations at p, called the ***tangent space at*** p for M.

Observe that if $f \in C^\infty(M)$ is the constant function 1, then for any derivation D at a point p of M, $D(f) = D(f \cdot f) = f(p)D(f) + f(p)D(f) = 2D(f)$, so $D(f) = 0$. As any other constant function is just αf, for $\alpha \in \mathbb{R}$, and derivations are linear, all constant functions are sent to zero under derivations, in analogy with classical derivatives.

We recall that the space of linear functionals on a vector space is again a vector space. Additionally, we observe that for any $f, g \in C^\infty(M)$, $D, R \in T_pM$, and $\alpha \in \mathbb{R}$,

$$\begin{aligned}(D + \alpha R)(fg) &= D(fg) + \alpha R(fg)\\ &= [f(p)D(g) + g(p)D(f)] + \alpha[f(p)R(g) + g(p)R(f)] = f(p)(D + \alpha R)(g) + g(p)(D + \alpha R)(f)\end{aligned}$$

so $D + \alpha R \in T_pM$ is also a derivation. Thus as T_pM is a closed subset of a vector space it is itself a vector space, and hence a candidate for our local linear structure. To show that this is the correct candidate, we demonstrate that as a vector space $T_a\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ for any $a \in \mathbb{R}$. To reduce clutter, in the following proof and the rest of the paper we use Einstein notation to write $\sum_{i=1}^n v_i e_i = v^i e_i$, where a superscript followed by a subscript of the same index indicates we are summing over the available range of the index which is indicated from context.

Proposition 3.10. *Let $a \in \mathbb{R}^n$. Then the map $\varphi : \mathbb{R}^n \to T_a\mathbb{R}^n$ defined by $v \mapsto D_v|_a$ is a linear isomorphism.*

Proof. Recall that since all elements of $f \in C^\infty(\mathbb{R}^n)$ are smooth, for any $v \in \mathbb{R}^n$, $D_v|_a(f) = D_v f(a) = Df(a)v$. Then for all $v, w \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$D_{v+\alpha w}|_a(f) = Df(a)(v + \alpha w) = Df(a)v + \alpha Df(a)w = D_v|_a(f) + \alpha D_w|_a(f) = (D_v|_a + \alpha D_w|_a)(f)$$

Thus $D_{v+\alpha w}|_a = D_v|_a + \alpha D_w|_a$ so the map is indeed linear. To show injectivity, suppose v is in the kernel of the map, so $D_v|_a(f) = 0$ for all $f \in C^\infty(\mathbb{R}^n)$. For each $1 \leq j \leq n$ the projection π_j is smooth, being a linear map on a finite dimensional vector space, and $D\pi_j = \pi_j$. Consequently, $0 = D_v|_a(\pi_j) = \pi_j v = v_j$, where $v = (v_1, ..., v_n)$. Thus $v = 0$, the zero vector, so the kernel of the map is $\{0\}$. Finally to show surjectivity consider $\omega \in T_a\mathbb{R}^n$ and $u = [\omega(\pi_1), ..., \omega(\pi_n)]^T \in \mathbb{R}^n$, which will be our candidate vector. By Taylor's Theorem with Remainder [10], for each $f \in C^\infty(\mathbb{R}^n)$ as well as any open ball $U \subseteq \mathbb{R}^n$ containing a, there exist $C^\infty(U)$ functions $g_1(x), ..., g_n(x)$ such that $f(x) = f(a) + (x^i - a^i)g_i(x)$, in Einstein notation, where $g_i(a) = D_{e_i} f(a)$ for $\{e_1, ..., e_n\}$ the canonical orthonormal basis of $\mathbb{R}^n$. Then using the liebniz rule and linearity we can write

$$\begin{aligned}\omega(f) &= \omega(f(a)) + \omega((x^i - a^i)g_i(x)) = 0 + \omega(x^i - a^i)g_i(a) + (a^i - a^i)\omega(g_i(x))\\ &= \omega(x^i)D_{e_i} f(a) - \omega(a^i)g_i(a) = \omega(x^i)D_{e_i} f(a)\end{aligned}$$

But, as a function x^i is the i-th projection operator, π_i, so $\omega(f) = \omega(\pi^i)D_{e_i} f(a) = Df(a)[\omega(\pi_1), ..., \omega(\pi_n)]^T = D_u|_a(f)$. Thus $D_u|_a = \omega$, and hence our map is surjective. Since the $D_{e_i}|_a$ form a basis for the space of directional derivatives, it follows that it is also a basis of $T_a\mathbb{R}^n$ [5]. ☐

We shall soon show that an analogous result holds in the case of a general smooth manifold, further enforcing how this is the correct choice for a local linear structure on manifolds. Before we proceed further we finally define what it means to differentiate a smooth map [5].

Definition 3.11. Let $F : M \to N$ be a smooth map between smooth manifolds. For each $p \in M$ and each $v \in T_pM$ we define a map $dF_p : T_pM \to T_{F(p)}N$ by pre-composition, which is to say that for all $f \in C^\infty(N)$,

$$dF_p(v)(f) = v(f \circ F)$$

We call dF_p the **differential of** F **at** p.

For brevity and in order to arrive at our main results we don't prove the routine check that dF_p is a well-defined map from T_pM to $T_{F(p)}N$ [5]. We now briefly show that the differential of a smooth map at a point satisfies many of the well-known properties of classical derivatives on Euclidean space [5].

Proposition 3.12. *Let $F : M \to N$ and $G : N \to K$ be smooth maps and let $p \in M$. Then the following properties hold:*

1. *the differential of the identity map is the identity: $d(\mathbb{1}_M)_p = \mathbb{1}_{T_pM}$*
2. *the map $dF_p : T_pM \to T_{F(p)}N$ is linear*
3. *the chain rule holds: $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}K$*
4. *if F is a diffeomorphism then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and $d(F^{-1})_{F(p)} = (dF_p)^{-1}$*

Proof. Throughout let $v, w \in T_pM$, $\alpha \in \mathbb{R}$, $h \in C^\infty(M)$, $f \in C^\infty(N)$, and $g \in C^\infty(K)$ all be arbitrary.

(1) Applying the differential of the identity we observe

$$d(\mathbb{1}_M)_p(v)(h) = v(h \circ \mathbb{1}_M) = v(h)$$

so $d(\mathbb{1}_M)_p(v) = v = \mathbb{1}_{T_pM}(v)$ as h was arbitrary. Thus, as v was also arbitrary $d(\mathbb{1}_M)_p = \mathbb{1}_{T_pM}$.

(2) Observe that

$$dF_p(v + \alpha w)(f) = (v + \alpha w)(f \circ F) = v(f \circ F) + \alpha w(f \circ F) = dF_p(v)(f) + \alpha dF_p(w)(f) = (dF_p(v) + \alpha dF_p(w))(f)$$

Then as f was arbitrary, $dF_p(v + \alpha w) = dF_p(v) + \alpha dF_p(w)$.

(3) We can write the following for the composition of differentials:

$$d(G \circ F)_p(v)(g) = v(g \circ G \circ F) = dF_p(v)(g \circ G) = dG_{F(p)}(dF_p(v))(g)$$

where in the last equality the appropriate point of evaluation for dG is at $F(p)$ since $dF_p(v) \in T_{F(p)}N$. Thus as g was arbitrary $(dG_{F(p)} \circ dF_p)(v) = d(G \circ F)_p(v)$, and as v is also arbitrary $dG_{F(p)} \circ dF_p = d(G \circ F)_p$.

(4) We now consider the case of F being a diffeomorphism. Then $dF_p : T_pM \to T_{F(p)}N$ and $d(F^{-1})_{F(p)} : T_{F(p)}N \to T_pM$ are linear maps by part (2). Additionally by part (3) and part (1) the composites are

$$dF_p \circ d(F^{-1})_{F(p)} = dF_{F^{-1}(F(p))} \circ d(F^{-1})_{F(p)} = d(F \circ F^{-1})_{F(p)} = d(\mathbb{1}_N)_{F(p)} = \mathbb{1}_{T_{F(p)}N}$$

and

$$d(F^{-1})_{F(p)} \circ dF_p = d(F^{-1} \circ F)_p = d(\mathbb{1}_M)_p = \mathbb{1}_{T_pM}$$

by parts (1) and (3). Thus $d(F^{-1})_{F(p)} = (dF_p)^{-1}$, so the two maps are linear isomorphisms.

□

With these results on differentials we can now illustrate an analogous result to Theorem 3.10 as follows.

Theorem 3.13. *Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Then for every $p \in M$ the linear space T_pM is n-dimensional.*

Proof. Let $(U, \varphi) \in \mathcal{A}$ be a chart. Then we know that $\varphi : U \to \varphi(U)$ is a diffeomorphism, so by part (3) of Proposition 3.12 we have that $d\varphi_p : T_pU \to T_{\varphi(p)}\varphi(U)$ is an isomorphism. For brevity we do not prove it here, but we use the result that $T_pU \cong T_pM$ are isomorphic as vector spaces under the differential of the inclusion map $\iota : U \to M$ [5]. Then we have the chain $T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n$. Since $T_{\varphi(p)}\mathbb{R}^n$ is n-dimensional by Proposition 3.10, so is T_pM. Further, $d(\varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^n \to T_pM$ is an isomorphism using the identification of $T_{\varphi(p)}\mathbb{R}^n \cong T_{\varphi(p)}\varphi(U)$ and $T_pM \cong T_pU$. It follows that the image of the basis $\{D_{e_i}|_{\varphi(p)} : 1 \leq i \leq n\}$ of $T_{\varphi(p)}\mathbb{R}^n$ in $d(\varphi^{-1})_{\varphi(p)}$ is a basis of T_pM, which we denote by $d(\varphi^{-1})_{\varphi(p)}(D_{e_i}|_{\varphi(p)}) =: D_{e_i^\varphi}|_p$. □

Thus, using the notation of Theorem 3.13 we have that any vector $v \in T_pM$ can be expressed uniquely as

$$v = v^i D_{e_i^\varphi}|_p$$

Not only does the tangent space of a smooth manifold reflect its local Euclidean structure, even the differentials of smooth maps can be related to Euclidean derivatives. This is done using the Jacobian matrix of the differential in local coordinates. First, for a smooth Euclidean map $G : \mathbb{R}^n \to \mathbb{R}^m$, we can determine how its differential at $p \in \mathbb{R}^n$ acts on the basis vectors $D_{e_i}|_p \in T_p\mathbb{R}^n$ by evaluating the following expression for arbitrary $f \in C^\infty(\mathbb{R}^m)$ using the chain rule and writing $\{r_1, ..., r_m\}$ for the canonical orthonormal basis on $\mathbb{R}^m$:

$$dG_p\left(D_{e_i}|_p\right)(f) = D_{e_i}|_p(f \circ G) = D_{r^j} f(G(p)) D_{e_i} G_j(p) = \left(D_{e_i} G^j(p) D_{r_j}|_{G(p)}\right)(f)$$

Note the last two terms in the above sequence of equalities are sums over j in Einstein notation. Consequently we have that

$$dG_p\left(D_{e_i}|_p\right) = D_{e_i} G^j(p) D_{r_j}|_{G(p)} \tag{2}$$

We now consider the case of a smooth map $F : M \to N$ for M and N, n and m dimensional smooth manifolds, respectively. Let $p \in M$, and let (U, φ) and (V, ψ) be charts for M and N such that $p \in U$ and $F(p) \in V$. Then we can write the action of the differential of F on a basis vector of T_pM in the (U, φ) local coordinates as follows:

$$\begin{aligned}
dF_p\left(D_{e_i^{\varphi}}|_p\right) &= dF_p\left(d(\varphi^{-1})_{\varphi(p)}\left(D_{e_i}|_{\varphi(p)}\right)\right) \\
\text{(by part (3) of Proposition 3.12)} \quad &= d(F \circ \varphi^{-1})_{\varphi(p)}\left(D_{e_i}|_{\varphi(p)}\right) \\
\text{(by part (3) of Proposition 3.12)} \quad &= d(\psi^{-1})_{(\psi\circ F\circ\varphi^{-1})(\varphi(p))}\left(d(\psi \circ F \circ \varphi^{-1})_{\varphi(p)}\left(D_{e_i}|_{\varphi(p)}\right)\right) \\
\text{(using Equation 2)} \quad &= d(\psi^{-1})_{(\psi\circ F)(p)}\left(D_{e_i}(\psi \circ F \circ \varphi^{-1})^j(\varphi(p))D_{r_j}|_{(\psi\circ F)(p)}\right) \\
\text{(3)} \quad &= D_{e_i}\widetilde{F}^j(\varphi(p))D_{r_j^{\psi}}|_{F(p)}
\end{aligned}$$

writing $\widetilde{F} := \psi \circ F \circ \varphi^{-1}$ and using linearity of differentials in the last equality as well as our definition of $D_{r_j^{\psi}}|_{F(p)}$ in Theorem 3.13. Thus, in the standard bases of T_pM and $T_{F(p)}N$ with respect to the local coordinates (U, φ) and (V, ψ) used here, we have the matrix representation $[dF_p] = \left[D_{e_i}\widetilde{F}_j(\varphi(p))\right]_{1\le i\le n}^{1\le j\le m}$, which is the Jacobian of the map $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$.

3.1. **The Smooth Structure of Projective Space.** Now that we have a sensible notion of differentiation on smooth manifolds, we can look at imbuing a smooth structure on complex Projective Space. Analogous to our approach in the topological case, we show that under the action of $U_1(\mathbb{C})$, $\mathbb{S}(\mathbb{C}^{n+1})$ induces a smooth structure on $\mathbb{C}P^n$. First we need a group structure which will represent the symmetries of smooth spaces [5].

Definition 3.14. Let $(G, \mathcal{G}, \cdot)$ be a triple such that $(G, \mathcal{G})$ is a smooth manifold and $(G, \cdot)$ is a group. If the map $(g, f) \mapsto g \cdot f^{-1}$ is a smooth map from the product manifold $G \times G$ to the manifold G, then $(G, \mathcal{G}, \cdot)$ is said to be a ***Lie group***.

An action of a Lie group on a smooth manifold $(M, \mathcal{A})$ is identical to that in the case of a topological group, with the continuous action replaced by a smooth action $\cdot_A : M \times G \to M$. Additionally, the action of each element g of the group is a diffeomorphism since it is a smooth map with smooth inverse given by the action of g^{-1} [5]. As an example of a simple Lie group action we consider the group $U_1(\mathbb{C})$.

Example 3.15. As shown previously in Example 2.11, $U_1(\mathbb{C})$ is a compact topological group. Recall that $U_1(\mathbb{C}) = \mathbb{S}(\mathbb{C}) \cong \mathbb{S}(\mathbb{R}^2)$ under our isometric identification, so from Example 3.5 we have that $U_1(\mathbb{C})$ is a smooth manifold of dimension 1. Additionally, consider the multiplication and inversion map from Example 2.11, written as $((x, y), (a, b)) \mapsto (xa - by, xb + ay)$ under our identification. As the component maps $((x, y), (a, b)) \mapsto xa - by$ and $((x, y), (a, b)) \mapsto xb + ay$ are polynomials in the components, they are not only continuous but C^∞ functions [9]. Hence the total function $((x, y), (a, b)) \mapsto (xa - by, xb + ay)$ is also $\mathbb{C}^\infty$, so its composition with the smooth coordinate maps on $U_1(\mathbb{C})$ from Example 3.5 will also be smooth. Thus, the smoothness of this map tells us that $U_1(\mathbb{C})$ is indeed a Lie group of dimension 1.

Recall from Proposition 2.12 that the action of $U_1(\mathbb{C})$ on $\mathbb{S}(\mathbb{C}^{n+1})$ for an arbitrary element $e^{i\theta} \in U_1(\mathbb{C})$ gives the component maps $(x_1, y_1, ..., x_{n+1}, y_{n+1}) \mapsto x_i \cos\theta - y_i \sin\theta$ and $(x_1, y_1, ..., x_{n+1}, y_{n+1}) \mapsto x_i \sin\theta + y_i \cos\theta$ under our identification, which are linear combinations of the projection maps. Recall we write the associated map for the action as $A_{e^{i\theta}}$. Then, as the projection maps are smooth, $A_{e^{i\theta}}$ is a smooth map on $\mathbb{C}^{n+1}$, or equivalently $\mathbb{R}^{2(n+1)}$ under our identification. Additionally, identifying $\mathbb{S}(\mathbb{C}^{n+1}) \cong \mathbb{S}(\mathbb{R}^{2(n+1)})$ and using our smooth coordinate charts $\varphi_j^{\pm}$, for $1 \le j \le 2(n+1)$, on $\mathbb{S}(\mathbb{R}^{2(n+1)})$ from Example 3.5, we have that $\varphi_k^{\pm} \circ A_{e^{i\theta}} \circ (\varphi_j^{\pm})^{-1}$ are smooth for all $1 \le k, j \le 2(n+1)$ and all configurations of $\pm$, being the composition of smooth maps [5]. Hence $U_1(\mathbb{C})$ acts smoothly on $\mathbb{S}(\mathbb{C}^{n+1})$.

Analogously to the topological case, actions by compact Lie groups which are free and smooth can be used to induce smooth structures on the corresponding topological quotient space [5].

Theorem 3.16. *If $(G, \mathcal{G}, \cdot)$ is a compact Lie group which acts smoothly and freely on a smooth manifold $(M, \mathcal{A})$, then there exists a unique smooth structure on the quotient space M/G such that the projection $p : M \to M/G$ is smooth and for each $x \in M$ the differential $dp_x : T_pM \to T_{p(x)}M/G$ is surjective.*

Proof. The proof of existence is beyond the scope of this paper and hence will not be given here [5]. □

As an immediate consequence from Example 3.15 we find that $\mathbb{C}P^n$ is a smooth manifold with structure induced by the action of $U_1(\mathbb{C})$ on $\mathbb{S}(\mathbb{C}^{n+1})$ as required. Thus we can perform calculus on our new state space, $\mathbb{C}P^n$, with its calculus being determined by the calculus on $\mathbb{S}(\mathbb{C}^{n+1})$.

Although notions of smoothness and differentiability are essential for studying the dynamics on an abstract space, in order to assess the evolution of points in state space we need a suitable notion of distance on our space. In particular, in quantum mechanics a notion of distance between states is essential for describing the probabilistic features of the theory. Throughout the next section we aim to define a suitable notion of distance on a smooth manifold, and in doing so a distance on our projective phase space.

4. The Geometry of Abstract Spaces

The initial naive approach may be to define a local metric on an n-dimensional smooth manifold $(M, \mathcal{A})$ which is defined in a neighborhood of each point on M. However, we want our notion of distance to preserve the linear structure of our tangent spaces on M, so instead we look at defining local inner products [4].

Definition 4.1. A **Riemannian metric** g on a manifold M is a map defined such that at each point $p \in M$, we have a map $g_p : T_pM \times T_pM \to \mathbb{R}$ satisfying the following properties:

(1) (linearity) $g_p(\cdot, v)$ and $g_p(v, \cdot)$ are linear functionals on T_pM, $\forall v \in T_pM$
(2) (symmetry) $g_p(v, w) = g_p(w, v)$, $\forall v, w \in T_pM$
(3) (positive-definiteness) $g_p(v, v) \geq 0$ $\forall v \in T_pM$ and $g_p(v, v) = 0 \iff v = 0$

We call M a **Riemannian manifold**, and denote it by the pair (M, g).

Although this structure doesn't immediately give a metric in the topological sense, we can use it to induce a metric on each tangent space through a norm at each point $p \in M$ which we define by $||v||_p = \sqrt{g_p(v, v)}$ for all $v \in T_pM$. Additionally, although this definition is quite abstract, we can represent g in local coordinates and consequently construct a matrix for it at each point $p \in M$. First, if $(U, \varphi) = (U, (\varphi_1, ..., \varphi_n))$ is a chart around p, we can consider Equation 3 applied to the map $\varphi_j : M \to \mathbb{R}$ for each $1 \leq j \leq n$, where on $\mathbb{R}$ we take the canonical chart $(\mathbb{R}, \mathbb{1}_\mathbb{R})$. Note that for each $q \in \mathbb{R}$, $T_q\mathbb{R} = \text{span}\left\{ \frac{d}{dx}\Big|_q \right\}$. Then the differential of $\varphi_j = \pi_j \circ \varphi$ at a basis element $D_{e_i^\varphi}|_p \in T_pM$ is

$$d(\varphi_j)_p \left(D_{e_i^\varphi}|_p \right) = D_{e_i}(\mathbb{1}_\mathbb{R} \circ \varphi_j \circ \varphi^{-1})(\varphi(p)) \frac{d}{dx}\Big|_{\varphi_j(p)} = D_{e_i}\pi_j(\varphi(p)) \frac{d}{dx}\Big|_{\varphi_j(p)} = \delta_{ij} \frac{d}{dx}\Big|_{\varphi_j(p)} \tag{4}$$

since $D\pi_j(x) = \pi_j(x)$ for any $x \in \mathbb{R}^n$ since it is a linear map, and where δ_{ij} is the kronecker delta. We use Theorem 3.13 to identify $T_p\mathbb{R}$ with $\mathbb{R}$ by the map $a\frac{d}{dx}\Big|_p \mapsto a$, and, moving forward, we shall identify $d(\varphi_j)_p$ as a linear functional in this sense. Now let $v = v^i D_{e_i^\varphi}|_p, w = w^i D_{e_i^\varphi}|_p \in T_pM$ be arbitrary vectors, and observe that in this case a metric g at p acts as

$$g_p(v, w) = g_p \left(D_{e^i_\varphi}|_p, D_{e^j_\varphi}|_p \right) v_i w_j$$

using the linearity in each component of g_p, where this is a sum over both i and j in Einstein notation. But from Equation 4 and the linearity of differential maps, $v_i = d(\pi_i)_p(v)$ and $w_j = d(\pi_j)_p(w)$, so we can write $g_p = g_p \left(D_{e^i_\varphi}|_p, D_{e^j_\varphi}|_p \right) d(\pi_i)_p d(\pi_j)_p$. Then in the coordinate basis $\left\{ D_{e_1^\varphi}|_p, ..., D_{e_n^\varphi}|_p \right\}$ we can realize g as a matrix at each $p \in M$

$$[g_p] := \begin{bmatrix} g\left(D_{e_1^\varphi}|_p, D_{e_1^\varphi}|_p\right) & \cdots & g\left(D_{e_1^\varphi}|_p, D_{e_n^\varphi}|_p\right) \\ \vdots & \ddots & \vdots \\ g\left(D_{e_n^\varphi}|_p, D_{e_1^\varphi}|_p\right) & \cdots & \left(D_{e_n^\varphi}|_p, D_{e_n^\varphi}|_p\right) \end{bmatrix}$$

If we write $[v], [w]$ for the vectors in local coordinates, we have by definition of matrix multiplication that

$$g_p(v, w) = g_p \left(D_{e^i_\varphi}|_p, D_{e^j_\varphi}|_p \right) v_i w_j = [v]^T [g_p][w] \tag{5}$$

Concretely let us consider the example of the **round metric** on the unit sphere $\mathbb{S}(\mathbb{R}^n)$ [4].

Example 4.2. First the standard Euclidean metric E on $\mathbb{R}^n$ corresponds with the dot product relative to the standard coordinate basis $\{D_{e_i}|_p\}_{1 \leq i \leq n}$ on $T_p\mathbb{R}^n$. Consequently its matrix representation is given by $[E_p] = I_n$ for any $p \in \mathbb{R}^n$. Thus we can also write $E_p = \delta^{ii} d(\pi_i)_p^2$. The round metric on $\mathbb{S}(\mathbb{R}^n)$ is then the metric r induced by the inclusion $\iota : \mathbb{S}(\mathbb{R}^n) \hookrightarrow \mathbb{R}^n$. Explicitly for $p \in \mathbb{S}(\mathbb{R}^n)$ and $v, w \in T_p\mathbb{S}(\mathbb{R}^n)$,

$$r_p(v, w) := E_p(d\iota_p(v), d\iota_p(w))$$

which when considering $d\iota_p$ as an inclusion of $T_p\mathbb{S}(\mathbb{R}^n)$ into $T_p\mathbb{R}^n$, the metric r equates to a restriction of the metric E down to $\mathbb{S}(\mathbb{R}^n)$ [4]. As a concrete example we consider $\mathbb{S}(\mathbb{R}^3)$, and compute the metric in the chart $(\widetilde{\mathcal{U}}_3^+, \varphi_3^+)$ from Example 3.5, with $\varphi_3^+(x) = (x_1, x_2)$ and $(\varphi_3^+)^{-1}(\widetilde{x}_1, \widetilde{x}_2) = \left(\widetilde{x}_1, \widetilde{x}_2, \sqrt{1 - ||(\widetilde{x}_1, \widetilde{x}_2)||_2^2} \right)$ for $x = (x_1, x_2, x_3) \in \widetilde{\mathcal{U}}_3^+$ and $(\widetilde{x}_1, \widetilde{x}_2) \in B_1^2(0)$. Let $\{f_1 = (1, 0), f_2 = (0, 1)\}$ denote the canonical orthonormal basis on $\mathbb{R}^2$. In order to evaluate r at the basis derivations of $T_p\mathbb{S}(\mathbb{R}^3)$, $\left\{D_{f_1^{\varphi_3^+}}\Big|_p, D_{f_2^{\varphi_3^+}}\Big|_p\right\}$, we need $\mathbb{1}_{\mathbb{R}^3}\circ\pi_i\circ(\varphi_3^+)^{-1}$ for each projection π_i, $1\leq i\leq 3$. These components of our chart are given by $\pi_1\circ(\varphi_3^+)^{-1}(\widetilde{x}_1,\widetilde{x}_2)=\widetilde{x}_1, \pi_2\circ(\varphi_3^+)^{-1}(\widetilde{x}_1,\widetilde{x}_2)=\widetilde{x}_2$, and $\pi_3\circ(\varphi_3^+)^{-1}(\widetilde{x}_1,\widetilde{x}_2)=\sqrt{1-\widetilde{x}_1^2-\widetilde{x}_2^2}$. Note that in this notation, D_{f_i} will equate to the partial derivative $\frac{\partial}{\partial \widetilde{x}_i}$, for $1\leq i\leq 2$, in our calculations. Then we can compute r_p applied to our specified basis of $T_p\mathbb{S}(\mathbb{R}^3)$ by evaluating it at the basis elements and using its definition as a restriction of E_p to $T_p\mathbb{S}(\mathbb{R}^3)$:

$$\begin{aligned}
r_p\left(D_{f_1^{\varphi_3^+}}\Big|_p, D_{f_1^{\varphi_3^+}}\Big|_p\right) &= \delta^{ii} d(\pi_i)_p\left(D_{f_1^{\varphi_3^+}}\Big|_p\right) d(\pi_i)_p\left(D_{f_1^{\varphi_3^+}}\Big|_p\right) = \delta^{ii}\left(D_{f_1}(\pi_i\circ(\varphi_3^+)^{-1})\Big|_{\varphi(p)}\right)^2\\
&= D_{f_1}(\tilde{x}_1)^2 + D_{f_1}(\tilde{x}_2)^2 + D_{f_1}\left(\sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}\right)^2 = 1+\frac{\widetilde{x}_1^2}{1-\widetilde{x}_1^2-\widetilde{x}_2^2} = \frac{1-\widetilde{x}_2^2}{1-||(\widetilde{x}_1,\widetilde{x}_2)||_2^2}\\
r_p\left(D_{f_1^{\varphi_3^+}}\Big|_p, D_{f_2^{\varphi_3^+}}\Big|_p\right) &= \delta^{ii} d(\pi_i)_p\left(D_{f_1^{\varphi_3^+}}\Big|_p\right) d(\pi_i)_p\left(D_{f_2^{\varphi_3^+}}\Big|_p\right)\\
&= \delta^{ii}\left(D_{f_1}(\pi_i\circ(\varphi_3^+)^{-1})\Big|_{\varphi(p)}\right)\left(D_{f_2}(\pi_i\circ(\varphi_3^+)^{-1})\Big|_{\varphi(p)}\right)\\
&= D_{f_1}(\tilde{x}_1)D_{f_2}(\tilde{x}_1) + D_{f_1}(\tilde{x}_2)D_{f_2}(\tilde{x}_2) + D_{f_1}\left(\sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}\right)D_{f_2}\left(\sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}\right)\\
&= \frac{\widetilde{x}_1\widetilde{x}_2}{1-\widetilde{x}_1^2-\widetilde{x}_2^2} = \frac{\widetilde{x}_1\widetilde{x}_2}{1-||(\widetilde{x}_1,\widetilde{x}_2)||_2^2}\\
r_p\left(D_{f_2^{\varphi_3^+}}\Big|_p, D_{f_2^{\varphi_3^+}}\Big|_p\right) &= \delta^{ii} d(\pi_i)_p\left(D_{f_2^{\varphi_3^+}}\Big|_p\right) d(\pi_i)_p\left(D_{f_2^{\varphi_3^+}}\Big|_p\right) = \delta^{ii}\left(D_{f_2}(\pi_i\circ(\varphi_3^+)^{-1})\Big|_{\varphi(p)}\right)^2\\
&= D_{f_2}(\tilde{x}_1)^2 + D_{f_2}(\tilde{x}_2)^2 + D_{f_2}\left(\sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}\right)^2 = 1+\frac{\widetilde{x}_2^2}{1-\widetilde{x}_1^2-\widetilde{x}_2^2} = \frac{1-\widetilde{x}_1^2}{1-||(\widetilde{x}_1,\widetilde{x}_2)||_2^2}
\end{aligned}$$

Thus in matrix notation the round metric at any point $p\in\mathbb{S}(\mathbb{R}^3)$ with respect to this choice of coordinates is

$$[r_p] = \frac{1}{1-\left|\left|\varphi_3^+(p)\right|\right|_2^2}\begin{bmatrix} 1-(\varphi_3^+)_1(p)^2 & (\varphi_3^+)_1(p)(\varphi_3^+)_2(p)\\ (\varphi_3^+)_1(p)(\varphi_3^+)_2(p) & 1-(\varphi_3^+)_1(p)^2\end{bmatrix}$$

where we have used $(\varphi_3^+)_1$ and $(\varphi_3^+)_2$ to denote the components of the function φ_3^+. Note that the metric here can change drastically in appearance based on our choice of local coordinates, although, of course, the underlying abstract map does not change.

4.1. **The Geometry of Projective Space.** We now complete our analogy of structured group actions inducing properties on the resulting quotient space with our final theorem. In the case of Riemannian manifolds the appropriate structure preservation is the following [4].

Definition 4.3. Let $(M,g),(N,h)$ be Riemannian manifolds and $F: M\to N$ a smooth map. We say that F is a **Riemannian isometry** if for all $p\in M$ and $v,w\in T_pM$,

$$g_p(v,w) = h_{F(p)}(dF_p(v), dF_p(w))$$

We say that a Lie group $(G,\mathcal{G},\cdot)$ acts **isometrically** on a Riemannian manifold (M,g) if each $\chi\in G$ gives a map $p\mapsto \chi\cdot_A p$ which is an isometry on (M,g) [4].

Theorem 4.4. *If $(G,\mathcal{G},\cdot)$ is a compact Lie group which acts smoothly, freely, and isometrically on a Riemannian manifold (M,g), then the smooth manifold M/G (Theorem 3.16) has a unique metric $\widetilde{g}$ such that at each $x\in M$, the differential of the projection $p: M\to M/G$, dp_x, is a linear isometry from $(\ker dp_x)^\perp$ onto $T_{p(x)}M/G$ [4].*

Proof. Let $p(q)\in M/G$ and $\nu,\eta\in T_{p(q)}M/G$, where $q\in M$. From Theorem 3.16 $dp_q: T_qM\to T_{p(q)}M/G$ is surjective. We recall from linear algebra that we can decompose $T_qM = \ker dp_q\oplus(\ker dp_q)^\perp$ as a sum of a subspace and its orthogonal complement. Now there exists $v\in T_qM$ such that $dp_q(v)=\nu$, and we can decompose $v = v_{\ker}+v_\perp$ for unique $v_{\ker}\in\ker dp_q$ and $v_\perp\in(\ker dp_q)^\perp$. As $dp_q(v_{\ker})=0$ it follows that $dp_q(v_\perp)=\nu$ by linearity, so by uniqueness of $v_\perp$ we have that the restriction $dp_q|_{(\ker dp_q)^\perp}: (\ker dp_q)^\perp\to T_{p(q)}M/G$ is a linear isomorphism. Let $u_\perp\in(\ker dp_q)^\perp$ be the unique vector such that $dp_q(u_\perp)=\eta$. For an induced metric $\widetilde{g}$ on M/G to make dp_q into a linear isometry from $(\ker dp_q)^\perp$ to $T_{p(q)}M/G$ we require that

$$\widetilde{g}_{p(q)}(\nu,\eta) = \widetilde{g}_{p(q)}(dp_q(v_\perp), dp_q(u_\perp)) = g_q(v_\perp,u_\perp)$$

By uniqueness of $u_\perp$ and $v_\perp$ this requirement uniquely defines $\widetilde{g}_{p(q)}(\nu,\eta)$. Now we need to show this is independent of the choice of representative $q\in p(q)$ in our orbit. Let $q'\in p(q)$, so there exists $h\in G$ such that $q' = h\cdot_A q$.

To show $\widetilde{g}_{p(q)}$ is well defined we must show that for the unique $u'_\perp, v'_\perp \in (\ker dp_{q'})^\perp$ such that $dp_{q'}(u'_\perp) = \nu$ and $dp_{q'}(v'_\perp) = \eta$, $g_{q'}(v'_\perp, u'_\perp)$ equals $g_q(v_\perp, u_\perp)$. Now I claim $dp_{q'} \circ dh_q = dp_q$, where $dh_q : T_qM \to T_{q'}M$ is the differential of the action induced by h. By part (3) of Proposition 3.12 $dp_{q'} \circ dh_q = d(p \circ h)_q$. But since the image of a point in M under p is its orbit, $p \circ h = p$, as for any $x \in M$, $h \cdot_A x \in p(x)$ by definition so $p(x) = p(h \cdot_A x)$. Hence $dp_{q'} \circ dh_q = d(p \circ h)_q = dp_q$. Next we show $dh_q(w) \in (\ker dp_{q'})^\perp$ if $w \in (\ker dp_q)^\perp$, so let $\xi \in \ker dp_{q'}$. Since G acts by diffeomorphisms dh_q is surjective by part (4) of Proposition 3.12, so there exists $\zeta \in T_qM$ such that $dh_q(\zeta) = \xi$. Then $0 = dp_{q'}(\xi) = dp_q(\zeta)$, since $dp_q = dp_{q'} \circ dh_q$, so $\zeta \in \ker dp_q$. It follows by the fact that the action is isometric that

$$g_{q'}(\xi, dh_q(w)) = g_{q'}(dh_q(\zeta), dh_q(w)) = g_q(\zeta, w) = 0$$

since $\zeta \in \ker dp_q$ and $w \in (\ker dp_q)^\perp$. Hence as $\xi \in \ker dp_{q'}$ was arbitrary, $dh_q(w) \in (\ker dp_{q'})^\perp$. Consequently $dh_q(u_\perp), dh_q(v_\perp) \in (\ker dp_{q'})^\perp$ with $dp_{q'}(dh_q(u_\perp)) = dp_q(u_\perp) = \nu$ and $dp_{q'}(dh_q(v_\perp)) = dp_q(v_\perp) = \eta$. By uniqueness it follows that $dh_q(u_\perp) = u'_\perp$ and $dh_q(v_\perp) = v'_\perp$. Since G acts isometrically on M, we have

$$g_q(u_\perp, v_\perp) = g_{q'}(dh_q(u_\perp), dh_q(v_\perp)) = g_{q'}(u'_\perp, v'_\perp)$$

Thus $\widetilde{g}_{p(q)}$ is indeed well-defined, as desired. To see that $\widetilde{g}$ is a Riemannian metric, observe that since dp_q is linear $\widetilde{g}_{p(q)}$ will be linear in each component and symmetric as g_q is a Riemannian metric [4]. Additionally, $\widetilde{g}_{p(q)}(\nu, \nu) = g_q(v_\perp, v_\perp) \geq 0$, and is zero if and only if $v_\perp = 0$. If $\nu = 0$ then $v_\perp = 0$ by uniqueness, so $\widetilde{g}_{p(q)}(\nu, \nu) = g_q(v_\perp, v_\perp) = 0$. Otherwise, if $\nu \neq 0$ then we must have $v_\perp \neq 0$ since $dp_q(v_\perp) = \nu$, so $\widetilde{g}_{p(q)}(\nu, \nu) = g_q(v_\perp, v_\perp) > 0$. Thus $\widetilde{g}$ is a Riemannian metric on M/G, and by construction is the unique such one for which dp_q is a linear isometry from $(\ker dp_q)^\perp$ to $T_{p(q)}M/G$ for each $q \in M$. □

We once again consider the action of $U_1(\mathbb{C})$ on $\mathbb{S}(\mathbb{C}^{n+1})$ which was shown to be free in Proposition 2.12 and smooth in Example 3.15, and now give a sketch for why it is also isometric [4]. First we equip $\mathbb{C}^{n+1}$ with our Euclidean metric E under our identification with $\mathbb{R}^{2(n+1)}$. In doing so we also equip $\mathbb{S}(\mathbb{C}^{n+1})$ with the induced round metric. Let $p \in \mathbb{S}(\mathbb{C}^{n+1}) \subseteq \mathbb{C}^{n+1}$, $e^{i\varphi} \in U_1(\mathbb{C})$, and let $A_{e^{i\varphi}} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ denote the smooth map defined by the action. From our proof of Proposition 2.12 we know that $A_{e^{i\varphi}}$ is linear. Additionally, using the chart $(\mathbb{R}^{2(n+1)}, \mathbb{1}_{\mathbb{R}^{2(n+1)}})$ on $\mathbb{R}^{2(n+1)}$ under our identification as well as Equation 3 we have that in local coordinates

$$[d(A_{e^{i\varphi}})_p] = [D_{e_k}(A_{e^{i\varphi}})_j(p)]_{1 \leq k, j \leq 2(n+1)}$$

But the right side is the Jacobian of a linear map on Euclidean space, which produces the matrix representation of the map in the canonical orthonormal coordinates on $\mathbb{R}^{2(n+1)}$. In particular, if $(x_1, y_1, ..., x_{n+1}, y_{n+1}) = [v]$ and $(a_1, b_1, ..., a_{n+1}, b_{n+1}) = [w]$ are coordinate representations of two arbitrary vectors $v, w \in T_p\mathbb{S}(\mathbb{C}^{n+1})$, then $[d(A_{e^{i\varphi}})_p][v] = (x_1 \cos\varphi - y_1 \sin\varphi, x_1 \sin\varphi + y_1 \cos\varphi, ..., x_{n+1} \cos\varphi - y_{n+1} \sin\varphi, x_{n+1} \sin\varphi + y_{n+1} \cos\varphi)$ and similarly $[d(A_{e^{i\varphi}})_p][w] = (a_1 \cos\varphi - b_1 \sin\varphi, a_1 \sin\varphi + b_1 \cos\varphi, ..., a_{n+1} \cos\varphi - b_{n+1} \sin\varphi, a_{n+1} \sin\varphi + b_{n+1} \cos\varphi)$. Hence, from Equation 5, $E_{A_{e^{i\varphi}}(p)}(d(A_{e^{i\varphi}})_p(v), d(A_{e^{i\varphi}})_p(w))$ can be written as

$$\begin{aligned}
E_{A_{e^{i\varphi}}(p)}(d(A_{e^{i\varphi}})_p(v), d(A_{e^{i\varphi}})_p(w)) &= ([d(A_{e^{i\varphi}})_p][v])^T [E_{A_{e^{i\varphi}}(p)}]([d(A_{e^{i\varphi}})_p][w]) \\
&= ([d(A_{e^{i\varphi}})_p][v])^T I_{2n+2}([d(A_{e^{i\varphi}})_p][w]) \\
&= (x^j \cos\varphi - y^j \sin\varphi)(a_j \cos\varphi - b_j \sin\varphi) + (x^j \sin\varphi + y^j \cos\varphi)(a_j \sin\varphi + b_j \cos\varphi) \\
&= x^j a_j + y^j b_j + \cos\varphi \sin\varphi(-x^j b_j - y^j a_j + x^j b_j + y^j a_j) \\
&= x^j a_j + y^j b_j = [v]^T [E_p][w] = E_p(v, w)
\end{aligned}$$

Thus Theorem 4.4 tells us that $\mathbb{C}P^n$ has a unique Riemannian metric $\overline{r}$ induced by the action of $U_1(\mathbb{C})$ on $\mathbb{S}(\mathbb{C}^{n+1})$ and its round metric r. This induced metric is known as the **Fubini-Study Metric**. We are able to define notions of length on our state space $\mathbb{C}P^n$ using the metric $\overline{r}$, which in turn allows us to study the probabilistic dynamics of our quantum state space in full generality. In particular, suppose our quantum system is in a state $p \in \mathbb{C}P^n$. Let $\hat{O} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a linear operator, with eigenvector z_k and eigenvalue $a_k \in \mathbb{R}$. Then to obtain the probability of the system collapsing to the state $\pi(z_k) \in \mathbb{C}P^n$ and producing the value a_k after the measurement of $\hat{O}$ we take the shortest path $C : [0, 1] \to \mathbb{C}P^n$ connecting p and $\pi(z_k)$, known as a **Geodesic**. The probability of measurement of a_k is then given by

$$\text{Prob}(p \mapsto p_k) = \cos^2 \left[\frac{\int_0^1 \sqrt{\overline{r}_{C(t)}(\dot{C}(t), \dot{C}(t))} dt}{\sqrt{2\hbar}} \right] \qquad \text{(by [3])}$$

where $\dot{C}(t) := dC_t \left(\left. \frac{d}{dt} \right|_t \right) \in T_{C(t)}\mathbb{C}P^n$, the numerator of the expression is the length of the path C, and $\hbar = \frac{h}{2\pi}$ is a multiple of Planck's fundamental constant $h = 6.626 \times 10^{-34}\ m^2\ kg/s$. This result gives us a physical interpretation of probabilities in quantum systems in terms of the lengths of geodesics in complex Projective Space.

5. Conclusion

Throughout this paper we have explored a number of key features of the geometrical formulation of quantum mechanics. We derived notions of openness, closedness, and continuity, formulations of abstract calculi, and even a form of distance on the state space for finite dimensional quantum systems. At each step along the way we motivated our construction as a result of symmetries on our original linear quantum state space, with the structure preserving properties of the symmetries inducing associated structures on our new state space. This formulation allowed us to develop a deep insight into the probabilistic structure of quantum measurement by identifying it with the geodesics on the complex Projective Space. Although we have derived many powerful structures on our quantum state space, this work is only the tip of the mathematical iceberg of properties for which the Projective Space encapsulates, with many more being teased out from the complex structure of our space and analogies with the formulation of classical mechanics in differential geometry.

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Email address: elijah.thompson@ucalgary.ca