Local to Global: An Introduction to Sheaves

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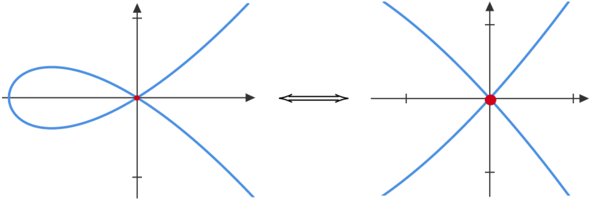
Math 511 Presentation

Motivation

Motivating Question

How can we study the relation between local and global properties of geometric spaces algebraically?



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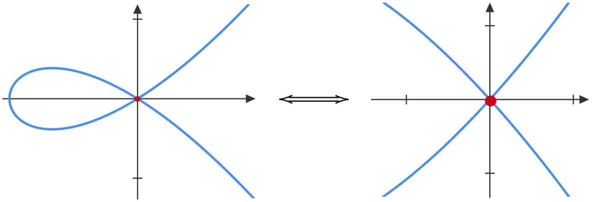

One Answer: Sheaves and sheaf cohomology!

What is a sheaf?

- Throughout let $(X, \tau) \in$ **Top**.

Def^{n}: (Sheaves)

A **pre-sheaf** on X with values in $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{C}$$

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If $\forall U \in \mathcal{O}(X)$ $\mathcal{F}$ satisfies

- $\forall U = \bigcup_{i \in I} U_i, \forall s_i \in \mathcal{F}(U_i),$

$$\forall i, j \in I(s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}) \implies \exists! s \in \mathcal{F}(U),\ \forall i \in I(s|_{U_i} = s_i)$$

it is called a **sheaf**

Example: Smooth Manifolds

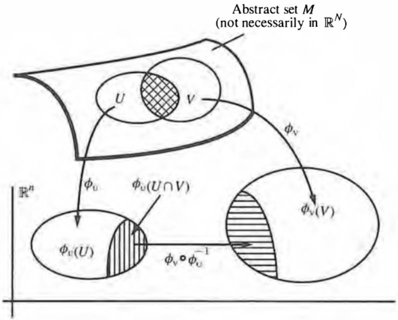
Eg: Smooth Manifolds

A smooth manifold is a pair $(M, \mathcal{O}_M)$, with $M \in$ **Top** and $\forall U \in M$, $\mathcal{O}_M(U) =$ smooth real-valued functions, satisfying

- $\forall p \in M, \exists U, p \in U$, such that

$$(U, \mathcal{O}_M|_U) \cong (\mathbb{R}^n, \mathcal{O}_{C^\infty})$$

for some $n \in \mathbb{N}$



Maps of sheaves

Def^{n}: (Sheaf Map)

A map between sheaves $\mathcal{F}, \mathcal{G} : \mathcal{O}(X)^{op} \to \mathcal{C}$ is a collection

$$(\eta_U \in \mathsf{Hom}_{\mathcal{C}}(\mathcal{F}(U), \mathcal{G}(U)))_{U \in \mathcal{O}(X)}$$

such that the diagram commutes for any $U \subseteq V \in \mathcal{O}(X)$.

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "|_U"] \arrow[d, "\eta_V"'] & \mathcal{F}(U) \arrow[d, "\eta_U"] \\ \mathcal{G}(V) \arrow[r, "|_U"'] & \mathcal{G}(U) \end{tikzcd}$$

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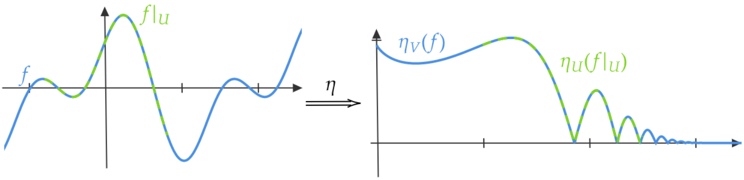

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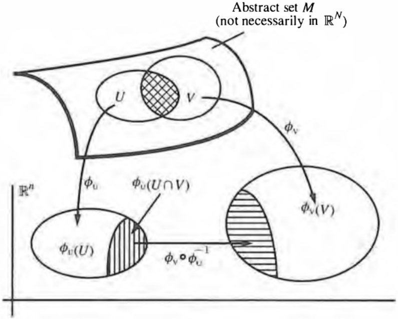

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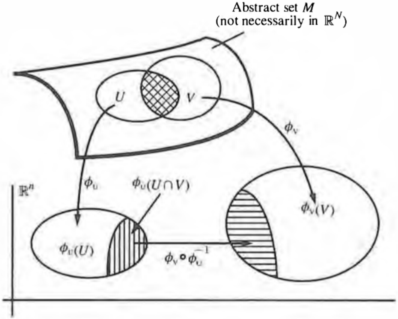

Observation: Differentiation and other operations on functions depend only on local behaviour

Characterizing Locality Through Universality: Stalks

- Fix a sheaf $\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{C}$

Def^n: (Stalks)

The **stalk** of $\mathcal{F}$ at $x \in X$ is **colimit**

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

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\mathcal{F}(U) \arrow[dr, "p_{U,x}"] \arrow[drr, bend left, "\forall f_U"] & & \\
& \mathcal{F}_x \arrow[r, dashed, "\exists!"] & \forall L \\
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$$(s \in \mathcal{F}(V)) \sim (r \in \mathcal{F}(U)) \implies
\begin{tikzcd}
\mathcal{F}(U) \arrow[dr, "|_{U \cap V}"] \arrow[drr, bend left, "|_W"] & & \\
& \mathcal{F}(U \cap V) \arrow[r, dashed] & \begin{matrix} \exists \mathcal{F}(W) \\ s|_W = r|_W \end{matrix} \\
\mathcal{F}(V) \arrow[ur, "|_{U \cap V}"'] \arrow[urr, bend right, "|_W"'] & &
\end{tikzcd}$$

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Exact

A sequence of sheaves on X, $0 \to \mathcal{H} \xrightarrow{\eta} \mathcal{F} \xrightarrow{\mu} \mathcal{G} \to 0$, induces a sequence

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Remark. The original sequence is exact if and only if

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Note

Surjectivity is local!

The Structure of Geometric Spaces: Category of Ringed Spaces

Def^n: (Ringed Space)

A **ringed space** is a pair $(X, \mathcal{O}_X)$ of $X \in$ **Top**, and $\mathcal{O}_X : \mathcal{O}(X)^{op} \to$ **Ring** a sheaf of rings

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Def^{n}: (Maps of Ringed Spaces)

A map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair of maps $\varphi : X \to Y$ and $\varphi^{\#} : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$

Inducing Sheaves

- Fix a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X over $\mathcal{C}$

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Def^{n}: (Push-forward)

The push-forward of $\mathcal{F}$ along f is the pre-sheaf

$$f_*\mathcal{F} : \mathcal{O}(Y)^{op} \to \mathcal{C}$$

given by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$

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Eg: Smooth Manifolds Revisited

- Let $(M, \mathcal{O}_M)$ be a smooth manifold
- Let $TM = \coprod_{p \in M} T_pM$ denote the tangent bundle
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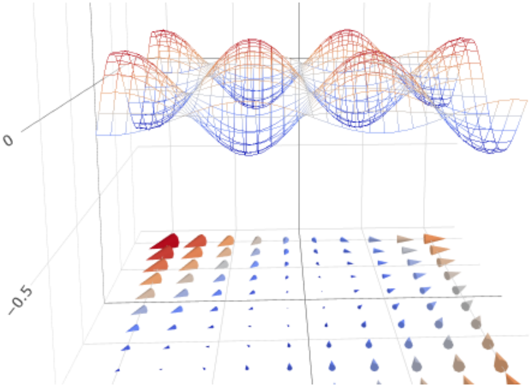

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- Let $\mathcal{A}_0, \mathcal{A}_\infty, \mathcal{A} : \mathcal{O}(X)^{op} \to$ **Ab** be sheaves of analytic functions, with the ones in $\mathcal{A}_0$ vanishing at 0 and the ones in $\mathcal{A}_\infty$ vanishing at ∞

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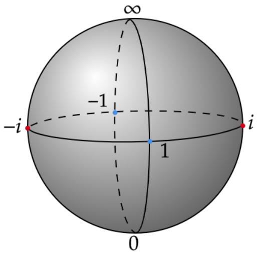

Figure: Riemann Sphere

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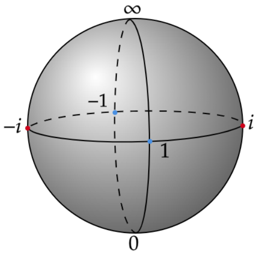
- Take the map $\mathcal{A}_0 \oplus \mathcal{A}_\infty \to \mathcal{A}$ given by addition
- By Liouville's Theorem $\mathcal{A}(X)$ consists of all constant functions



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Prop

Γ is a left-exact functor

Proof Idea: Let $0 \to \mathcal{H} \xrightarrow{\eta} \mathcal{F} \xrightarrow{\mu} \mathcal{G} \to 0$ be a SES. This induces a diagram

$$\begin{tikzcd} & \Gamma(\mathcal{H}) \arrow[r, "\eta_X"] \arrow[d] & \Gamma(\mathcal{F}) \arrow[r, "\mu_X"] \arrow[d] & \Gamma(\mathcal{G}) \arrow[d] & \\ 0 \arrow[r] & \mathcal{H}_x \arrow[r, dashed, "\eta_x"] & \mathcal{F}_x \arrow[r, dashed, "\mu_x"] & \mathcal{G}_x \arrow[r] & 0 \end{tikzcd}$$

Remark. We want to measure the failure of Γ to be right-exact.

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Construction. To extend Γ, for each $\mathcal{F} \in \mathcal{O}_X$-**Mod** we "take an injective resolution" $0 \to \mathcal{F} \to \mathcal{I}_\bullet$ and set

$$R^n\Gamma(\mathcal{F}) = H^n(\Gamma(\mathcal{I}_\bullet))$$

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Question. Does every $\mathcal{F} \in \mathcal{O}_X$-**Mod** have an injective resolution?

Existence of Sheaf Cohomology

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The category $\mathcal{O}_X$-**Mod** has enough injectives.

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- $\forall x \in X$, $\mathcal{O}_{X,x}$-**Mod** has enough injectives.

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- $\forall x \in X$, $\mathcal{O}_{X,x}$-**Mod** has enough injectives.
- $\implies \forall x \in X, \exists \iota_x : \mathcal{F}_x \hookrightarrow \mathcal{I}(x)$ in $\mathcal{O}_{X,x}$-**Mod**

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- Define $\mathcal{I} : \mathcal{O}(X)^{op} \to$ **Ab** by $\mathcal{I}(U) = \prod_{x \in U} \mathcal{I}(x)$

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- Define $\mathcal{I} : \mathcal{O}(X)^{op} \to$ **Ab** by $\mathcal{I}(U) = \prod_{x \in U} \mathcal{I}(x)$
- It can be shown $\mathcal{I} \in \mathcal{O}_X$-**Mod** is injective, and the induced map $\iota : \mathcal{F} \hookrightarrow \mathcal{I}$ is a monomorphism

Cor

A SES of $\mathcal{O}_X$-modules, $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$, induces a long-exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{F}) \arrow[r] & \Gamma(\mathcal{H}) \arrow[r] & \Gamma(\mathcal{G}) \arrow[dll, "\delta^0"] \\
& R^1\Gamma(\mathcal{F}) \arrow[r] & R^1\Gamma(\mathcal{H}) \arrow[r] & R^1\Gamma(\mathcal{G}) \arrow[dll, dashed] \\
& R^n\Gamma(\mathcal{F}) \arrow[r] & R^n\Gamma(\mathcal{H}) \arrow[r] & R^n\Gamma(\mathcal{G})
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Applications

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Canonical Example:

- Studying global properties of the complex logarithm

Thank you

Thank you for your time!

Any questions?

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{F}) \arrow[r] & \Gamma(\mathcal{H}) \arrow[r] & \Gamma(\mathcal{G}) \arrow[dll, "\delta^0"] \\
& R^1\Gamma(\mathcal{F}) \arrow[r] & R^1\Gamma(\mathcal{H}) \arrow[r] & R^1\Gamma(\mathcal{G}) \arrow[dll, dashed] \\
& R^n\Gamma(\mathcal{F}) \arrow[r] & R^n\Gamma(\mathcal{H}) \arrow[r] & R^n\Gamma(\mathcal{G})
\end{tikzcd}
$$

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