Reflection 2

Math 525

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Prompt

Please tell me which of the four definitions of braids from the in-class activity on Friday February 2 is your favourite, and why. Please tell me if the definition is mathematically precise or not (and why), and describe your plan for upgrading the definition if it is needed. Note that you will not be graded on the accuracy of your proposed plan to make the definition precise - I just want to know your thought process.

My favourite definition of braid from the in-class activity on Friday was the "braids as particle dances" definition. Explicitly, the definition stated that a braid with n strings was the "time history" of n non-colliding particles, $\beta_1, ..., \beta_n$. The definition makes this intuition explicit by stating that each particle is represented by a "trajectory" $\beta_j : I \to \mathbb{C}$ such that the trajectories start and end at the integer points $\{1, ..., n\}$, possibly with permutations, and such that $\beta_i(t) \neq \beta_j(t)$ for $i \neq j$. The reason that I found this definition to be my favourite of those given is that it provides a good balance of precision in the definition and intuition in the visualization of the braids. Although the first definition of strings in 3D space is the most intuitive to see visually, the process of making the definition precise felt somewhat unnatural or inelegant to me. On the other hand, the artin group formulation of braids is completely formal, being defined explicitly as a group in terms of generators and relations, but requires more work to connect to the physical realization of braids.

Although the connection between physical braids and particles is not as immediate as the "braids as strings in 3D space" definition, I believe that the perspective of particles provides an important point of view for strings and braids. Explicitly, the dancing particle definition suggests the visualization of walking along strings as they are built up into a braid. This perspective provides a possible exercise for gaining understanding on the dancing particle definition at a level which is accessible to k-12 students:

(i) Students are given strings or ribbons of different colours and start at a node for their given colour, taping one end of the string/ribbon to the node.

(ii) Students walk in a path while unraveling the string or ribbon behind themselves before returning to one of the starting points, the whole time avoiding colliding with other students.

This exercise would allow students to create a braid, and at the end be able to easily visualize the orientation of crossings, as in Figure 1, depending on the order that the students went over the crossing. Additionally, as seen in the transition between the left and right diagrams in Figure 1 the students can then pull their strings/ribbons tight to understand the equivalence relation between particle paths in practice. Thus, although initial intuition for the dancing particle definition is lacking in comparison to the 3D strings, there are simple ways in which students of all levels can engage in and understand how the definitions behave in practice.

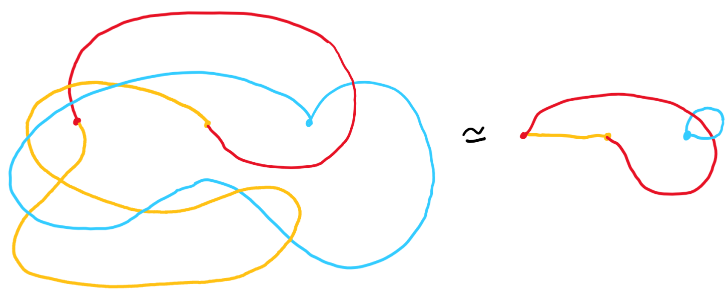

Figure 1: Braids as particle dance histories, where the left side is the original paths and the right side is homotopy equivalent.

The definition of braids in terms of particle dances requires some slight additions and modifications in order to be formalized rigorously. Explicitly, the notion of particles "moving in trajectories", the composition of braids, and the equivalence of the braids must be made precise. The solution to the first imprecision is simply the specification that trajectories are **continuous** functions $\beta : I \to \mathbb{C}^n$ such that if $p_i : \mathbb{C}^n \to \mathbb{C}$ is the ith projection

$$p_i \circ \beta(t) \neq p_j \circ \beta(t), \quad \text{for all } \; t \in [0,1] \; \text{ if } \; i \neq j$$

Additionally, we must require that such continuous functions satisfy $\beta(0) = (1, ..., n)$ and $\{p_1(\beta(1)), ..., p_n(\beta(1))\} = \{1, ..., n\}$, which ensures that the particles return to the spots they began, possibly with a permutation. One way to simplify this formalization is to use the nth configuration space, $\text{Conf}_n(\mathbb{C}) = \{(x_1, ..., x_n) \in \mathbb{C}^n : x_i \neq x_j, \text{ if } i \neq j\}$, in which case trajectories are continuous maps $\beta : I \to \text{Conf}_n(\mathbb{C})$ with the above start and end condition. However, in order to compose braids we must be careful about the possible permutation of the endpoint.

In order to be able to make the notion of equivalence of braids and the composition of braids precise we need another observation. Since permuting coefficients in $\text{Conf}_n(\mathbb{C})$ is a homeomorphism, we can think of a permutation $\sigma \in S_n$ as its associated homeomorphism $\varphi_\sigma : \text{Conf}_n(\mathbb{C}) \to \text{Conf}_n(\mathbb{C})$ where $\varphi_\sigma(x_1, ..., x_n) = (x_{\sigma(1)}, ..., x_{\sigma(n)})$. In particular, this defines a group action of S_n on $\text{Conf}_n(\mathbb{C})$ by homeomorphisms. This action gives an equivalence relation on $\text{Conf}_n(\mathbb{C})$, where two points are equivalent if they are in the same orbit for the action by S_n. Then braids in the dancing particle definition can be formalized as paths in the quotient space $\text{Conf}_n(\mathbb{C})/S_n$ where points are orbits of the action of S_n on $\text{Conf}_n(\mathbb{C})$. This is a reasonable characterization since our choice of enumeration of the particles is independent of the braids that are made in the end. With this formalization the composition of braids is precisely the composition of paths $\alpha, \beta : I \to \text{Conf}_n(\mathbb{C})/S_n$. Additionally, braid equivalence is exactly path homotopy equivalence for paths in $\text{Conf}_n(\mathbb{C})/S_n$. This procedure also realizes the particle dances definition of braids as the first fundamental group for the quotient space $\text{Conf}_n(\mathbb{C})/S_n$.