Finite Difference Method - Time Domain Project
EERF 6351

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Contents

1 Differential Equations

Write a system of linear differential equations for Ampere's law and Faraday's law.

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\nabla \times \overrightarrow{H} = \overrightarrow{J} + \frac{\partial \overrightarrow{D}}{\partial t}$$

For Faraday's law, we have:

$$\nabla \times \overrightarrow{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \langle \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \rangle = -\langle \frac{\partial B_x}{\partial t}, \frac{\partial B_y}{\partial t}, \frac{\partial B_z}{\partial t} \rangle$$

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \tag{1}$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \tag{2}$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \tag{3}$$

$$\nabla \times \overrightarrow{H} = \overrightarrow{J} + \frac{\partial \overrightarrow{D}}{\partial t}$$

$$\nabla \times \overrightarrow{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \langle \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \rangle = \langle \frac{\partial D_x}{\partial t}, \frac{\partial D_y}{\partial t}, \frac{\partial D_z}{\partial t} \rangle$$

$$\frac{\partial D_x}{\partial t} + \sigma E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \tag{4}$$

$$\frac{\partial D_y}{\partial t} + \sigma E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \tag{5}$$

$$\frac{\partial D_z}{\partial t} + \sigma E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \tag{6}$$

Changes with respect to y or z are set to zero. The electric field in the z direction is also set to zero:

$$
\begin{aligned}
-\mu \frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\
-\mu \frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\
-\mu \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\
\frac{\partial D_x}{\partial t} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x \\
\frac{\partial D_y}{\partial t} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y \\
\frac{\partial D_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z
\end{aligned}
$$

$$
\begin{aligned}
-\mu \frac{\partial H_x}{\partial t} &= 0 - 0 \\
-\mu \frac{\partial H_y}{\partial t} &= 0 - \frac{\partial}{\partial x} 0 \\
-\mu \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - 0 \\
\epsilon \frac{\partial E_x}{\partial t} &= 0 - 0 - \sigma E_x \\
\epsilon \frac{\partial E_y}{\partial t} &= 0 - \frac{\partial H_z}{\partial x} - \sigma E_y \\
\epsilon \frac{\partial}{\partial t} 0 &= \frac{\partial H_y}{\partial x} - 0 - \sigma E_z
\end{aligned}
$$

$$
\begin{aligned}
-\mu \frac{\partial H_x}{\partial t} &= 0 \\
-\mu \frac{\partial H_y}{\partial t} &= 0 \\
-\mu \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} \\
\epsilon \frac{\partial E_x}{\partial t} &= -\sigma E_x \\
\epsilon \frac{\partial E_y}{\partial t} &= -\frac{\partial H_z}{\partial x} - \sigma E_y \\
0 &= \frac{\partial H_y}{\partial x} - 0
\end{aligned}
$$

$$
\begin{aligned}
-\mu \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} \\
\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y &= -\frac{\partial H_z}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H_z}{\partial t} &= -\frac{1}{\mu} \frac{\partial E_y}{\partial x} \\
\frac{\partial E_y}{\partial t} &= -\frac{1}{\epsilon} \left[\frac{\partial H_z}{\partial x} + \sigma E_y \right]
\end{aligned}
$$

2 Central Difference Equations

2.1 Faraday's Law

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\frac{\partial E_y}{\partial x}$$

The magnetic field and the electric field will be interleaved in time and space. To make the Yee grid work effectively, the forward difference equations will be used in space, and the central difference equations will be used in time.

$$\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n-\frac{1}{2}}\left(i+\frac{1}{2}\right)}{\Delta t} = -\frac{1}{u(i)}\frac{E_y^n(i+1) - E_y^n(i)}{2\Delta x}$$

Rearranging gives:

$$H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) = H_z^{n-\frac{1}{2}}\left(i+\frac{1}{2}\right) - \frac{\Delta t}{u(i)}\frac{E_y^n(i+1) - E_y^n(i)}{2\Delta x}$$

2.2 Ampere's Law

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\left[\frac{\partial H_z}{\partial x} + \sigma E_y\right]$$

$$\frac{E_y^{n+1}(i) - E_y^n(i)}{\Delta t} = -\frac{1}{\epsilon}\left[\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)}{\Delta x} + \sigma(i)E_y^n(i)\right]$$

$$E_y^{n+1}(i) - E_y^n(i) = -\frac{\Delta t}{\epsilon}\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)}{\Delta x} - \frac{\Delta t}{\epsilon}\sigma(i)E_y^n(i)$$

$$E_y^{n+1}(i) = -\frac{\Delta t}{\epsilon}\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)}{\Delta x} - \frac{\Delta t}{\epsilon}\sigma(i)E_y^n(i) + E_y^n(i)$$

$$E_y^{n+1}(i) = -\frac{\Delta t}{\epsilon}\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)}{\Delta x} + E_y^n(i)\left[1 - \frac{\Delta t\sigma(i)}{\epsilon}\right]$$

This gives the final system of recurrence relations:

$$E_y^{n+1}(i) = \left[1 - \frac{\Delta t\sigma(i)}{\epsilon(i)}\right]E_y^n(i) - \frac{\Delta t}{\Delta x\epsilon(i)}\left[H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)\right] \tag{7}$$

The above is coupled to the equation below, and the below must be evaluated first:

$$H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) = H_z^{n-\frac{1}{2}}\left(i+\frac{1}{2}\right) - \frac{\Delta t}{u(i)}\frac{E_y^n(i+1) - E_y^n(i)}{2\Delta x} \tag{8}$$

3 Programming

For the purposes of programming, the equations must be rewritten to be more digestible. Let:

$$A = \frac{\Delta t}{2\Delta x u(i)}$$

$$B = \left[1 - \frac{\Delta t\sigma(i)}{\epsilon(i)}\right]$$

$$C = \frac{\Delta t}{\Delta x\epsilon(i)}$$

This gives the following system to program:

$$H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) = H_z^{n-\frac{1}{2}}\left(i+\frac{1}{2}\right) - A(i)\left(E_y^n(i+1) - E_y^n(i)\right) \tag{9}$$

$$E_y^{n+1}(i) = B(i)E_y^n(i) - C(i)\left[H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}\right) - H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2}\right)\right] \tag{10}$$